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Research article

On quadratic character sums of multivariate symmetric polynomials modulo p

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Abstract: In this paper, we use the elementary and analytic methods to investigate the computational problem of quadratic character sums related to quaternary and quintuple symmetric polynomials modulo a prime p, respectively. By applying properties of Gauss sums and those of third-order characters modulo p, we establish some identities for such quadratic character sums in the cases of $(3, p-1) = 1$ and $(3, p-1) = 3$.

Keywords: quadratic character sums; classical Gauss sums; multivariate symmetric polynomials; third-order characters

1. Introduction and main results

The study of character sums occupies a central position in analytic number theory, with profound connections to L-functions, exponential sums, and the distribution of prime numbers. Among these, quadratic character sums, intimately related to Legendre's symbol, serve as fundamental objects of investigation.

For an odd prime p and any integer n, Legendre's symbol is defined as follows:

$$\left(\frac{n}{p}\right) = \begin{cases} 1, & \text{if } n \text{ is a quadratic residue modulo } p, \\ -1, & \text{if } n \text{ is a quadratic non-residue modulo } p, \\ 0, & \text{if } n \equiv 0 \bmod p. \end{cases}$$

It's well-known that Legendre's symbol $\left(\frac{n}{p}\right)$ is a completely multiplicative function of n that is periodic with period p and vanishes when $p \mid n$. Consequently, it is usually expressed as $\left(\frac{n}{p}\right) = \chi_2(n)$, where χ_2 is a Dirichlet character modulo p. For this reason, Legendre's symbol is called the quadratic character modulo p.

Legendre's symbol plays an indispensable role in number theory. Up to now, the theoretical system of Legendre's symbol has been well-established, such as the Euler's criterion, Gauss's lemma, law of quadratic reciprocity, and so on. For more properties of Legendre's symbol, one can refer to References [1–7].

A particularly rich line of inquiry involves estimating and explicitly calculating character sums involving polynomials in multiple variables. For the quadratic character sum modulo a prime p, A. Weil [8] first estimated that

$$\sum_{b=1}^{p-1}\left(\frac{b^2-a^2}{p}\right)\left(\frac{b^2-1}{p}\right) \le 3\sqrt{p}, \quad a^2 \not\equiv 1 \bmod p.$$

Y. He and Q. Y. Liao [9] investigated two variable quadratic character sums and presented the following identities:

$$\sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\left(\frac{a^2-b^2}{p}\right)\left(\frac{b^2-1}{p}\right) = \begin{cases} 10-2p, & \text{if } p \equiv 1 \bmod 4, \\ 2p-6, & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

B. Nilanjan et al. [10] generalized above results to four variables and they got

$$\sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\sum_{c=2}^{p-2}\sum_{d=1}^{p-1}\left(\frac{a^2-b^2}{p}\right)\left(\frac{b^2-1}{p}\right)\left(\frac{c^2-d^2}{p}\right)\left(\frac{d^2-1}{p}\right)\left(\frac{a^2c^2-1}{p}\right) = O(p^2).$$

It is noteworthy that quadratic character sums associated with symmetric polynomials often yield superior results, which are attributed to their intrinsic symmetry. A notable advancement in this direction was made by Y. Y. Meng [11], who considered quadratic character sums related to ternary symmetric polynomials modulo p. By the elementary and analytic methods, she drew the following conclusions: If p is an odd prime with $(3, p-1) = 1$, then

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\left(\frac{a^4bc+b^4ac+c^4ab+abc}{p}\right) = -\left(\frac{-1}{p}\right)\cdot p.$$

If p is an odd prime with $p \equiv 1 \bmod 6$, and 2 is a cubic residue modulo p, then

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\left(\frac{a^4bc+b^4ac+c^4ab+abc}{p}\right) = \left(\frac{-1}{p}\right)\cdot(9pd-5p-d^2).$$

If p is an odd prime with $p \equiv 1 \bmod 6$, and 2 is a cubic non-residue modulo p, then

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\left(\frac{a^4bc+b^4ac+c^4ab+abc}{p}\right) = -\frac{1}{2}\left(\frac{-1}{p}\right)\cdot(9pd+9pb+10p+2d^2),$$

or

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\left(\frac{a^4bc+b^4ac+c^4ab+abc}{p}\right) = -\frac{1}{2}\left(\frac{-1}{p}\right)\cdot(9pd-9pb+10p+2d^2),$$

where

$$4p = d^2 + 27b^2 \ (d, b \in \mathbb{Z}), \quad d \text{ is uniquely determined by } d \equiv 1 \bmod 3. \tag{1.1}$$

The validity of (1.1) can be found in Reference [12].

While Meng's work provided a breakthrough for the ternary case and noted the potential extensibility of the methods to multivariate symmetric polynomials $f(x_1, \cdots, x_k)$, the explicit calculation for higher-dimensional cases remained an open challenge. The primary obstacle lies in the rapidly escalating computational complexity with the addition of more variables, which has prevented the derivation of concrete identities.

This work substantially generalizes Meng's results on ternary polynomials and extends the study of quadratic character sums to higher-dimensional symmetric polynomials. We overcome the computational challenges in multivariate polynomials, providing the explicit identities for quaternary and quintuple cases.

In this paper, we focus on the calculating problem of the quadratic character sums related to the quaternary and quintuple symmetric polynomials modulo a prime p. By applying the properties of classical Gauss sums and those of third-order characters modulo p, we obtain the following results.

Theorem 1.1. *Let p be an odd prime, then we have*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\left(\frac{a^4bcd + b^4acd + c^4abd + d^4abc + abcd}{p}\right) = \begin{cases} p^2, & \text{if } p \equiv 5 \bmod 6, \\ 3p \cdot (7p + 5d^2), & \text{if } p \equiv 1 \bmod 6, \end{cases}$$

where d is defined by (1.1).

Theorem 1.2. *Let p be an odd prime with* $(3, p-1) = 1$*, then we have*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{f=1}^{p-1}\left(\frac{a^4bcdf + b^4acdf + c^4abdf + d^4abcf + f^4abcd + abcdf}{p}\right) = -p^2.$$

Theorem 1.3. *Let p be an odd prime with* $(3, p-1) = 3$*. If* 2 *is a cubic residue modulo p, then we have*

$$\begin{aligned}&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{f=1}^{p-1}\left(\frac{a^4bcdf + b^4acdf + c^4abdf + d^4abcf + f^4abcd + abcdf}{p}\right)\\ =&3p \cdot (-7p + 19dp + 2d^3 - 5d^2),\end{aligned}$$

where d is defined by (1.1).

Theorem 1.4. *Let p be an odd prime with* $(3, p-1) = 3$*. If* 2 *is a cubic non-residue modulo p, then we have*

$$\begin{aligned}&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{f=1}^{p-1}\left(\frac{a^4bcdf + b^4acdf + c^4abdf + d^4abcf + f^4abcd + abcdf}{p}\right)\\ =&3p \cdot \left(-7p - \frac{19}{2}dp + \frac{81}{2}bp - 162b^3 - d^3 - 5d^2\right),\end{aligned}$$

or

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{f=1}^{p-1}\left(\frac{a^4bcdf+b^4acdf+c^4abdf+d^4abcf+f^4abcd+abcdf}{p}\right)$$
$$=3p\cdot\left(-7p-\frac{19}{2}dp-\frac{81}{2}bp+162b^3-d^3-5d^2\right),$$

where d and b are defined by (1.1).

Remark. The identities established in Theorem 1.4 contain a sign ambiguity. This is due to the explicit value of $\tau^3(\psi)-\tau^3(\overline{\psi})$, whose sign cannot be uniformly determined by the methods employed in this paper. Consequently, Theorem 1.4 yields two possible values.

Notations. Throughout the paper, the symbol (a,b) always denotes the greatest common divisor of a and b. The letters d and b are always defined by (1.1). We let χ_2 denote the Legendre's symbol modulo p; that is, $\chi_2(n)=\left(\frac{n}{p}\right)$.

2. Preliminary lemmas

To complete the proofs of our theorems, several simple lemmas are necessary. In this section, we will use many properties of the classical Gauss sums and those of third-order character modulo p. All of these can be found in References [1, 13], hence the related details will not be repeated here. To begin, we have the following lemmas.

Lemma 2.1. *Let p be an odd prime with $p\equiv 1 \bmod 3$. Then for any third–order character ψ modulo p, we have the identity*

$$\tau^3(\psi)+\tau^3(\overline{\psi})=dp,$$

where $\tau(\chi)=\sum_{a=1}^{p-1}\chi(a)e(\frac{a}{p})$ denotes the classical Gauss sum with any Dirichlet character χ modulo p, and d is defined by (1.1).

Proof. See W. P. Zhang and J. Y. Hu [14] or B. Berndt and R. Evans [15]. □

Lemma 2.2. *Let p be an odd prime with $p\equiv 1 \bmod 3$. Then for any third-order character ψ modulo p, we have*

$$\tau^3(\psi)-\tau^3(\overline{\psi})=\pm 3\sqrt{3}pb\cdot i, \tag{2.1}$$

where $i^2=-1$. Moreover, if 2 is a cubic non-residue modulo p, then

$$\psi(2)\tau^3(\overline{\psi})+\overline{\psi}(2)\tau^3(\psi)=-\frac{dp}{2}\pm\frac{9bp}{2}, \tag{2.2}$$

where $\tau(\chi)=\sum_{a=1}^{p-1}\chi(a)e(\frac{a}{p})$ denotes the classical Gauss sum with any Dirichlet character χ modulo p, and d and b are defined by (1.1).

Proof. Identities (2.1) and (2.2) are given by Eqs (47) and (48) of Y. Y. Meng [11], respectively. □

Lemma 2.3. *If p is a prime with $(3, p-1) = 3$, then for the quadratic character χ_2 and any third-order character ψ modulo p, we have*

$$\tau(\chi_2\overline{\psi}) = \frac{\overline{\psi}(2) \cdot \tau^2(\psi) \cdot \tau(\chi_2)}{p}.$$

Proof. See G. H. Chen and W. P. Zhang [16]. □

3. Proof of Theorem 1.1

In this section, we consider two cases: $(3, p-1) = 1$ and $(3, p-1) = 3$. First, by the properties of classical Gauss sums, we have

$$\begin{aligned}
&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\left(\frac{a^4bcd + b^4acd + c^4abd + d^4abc + abcd}{p}\right)\\
=&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\left(\frac{abcd}{p}\right)\left(\frac{a^3 + b^3 + c^3 + d^3 + 1}{p}\right)\\
=&\frac{1}{\tau(\chi_2)}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\left(\frac{abcd}{p}\right)\cdot\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m(a^3 + b^3 + c^3 + d^3 + 1)}{p}\right)\\
=&\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left(\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma^3}{p}\right)\right)^4\\
=&\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left|\sum_{a=1}^{p-1}\chi_2(a^3)e\left(\frac{ma^3}{p}\right)\right|^4. \qquad (3.1)
\end{aligned}$$

3.1. On the case of $(3, p-1) = 1$

If p is an odd prime with $(3, p-1) = 1$, then $p \equiv 5 \bmod 6$. Noting that when a passes through a reduced residue modulo p, a^3 also passes through a reduced residue modulo p, so we can deduce

$$\sum_{a=1}^{p-1}\chi_2(a^3)e\left(\frac{ma^3}{p}\right) = \sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma}{p}\right) = \tau(\chi_2)\cdot\chi_2(m). \qquad (3.2)$$

Combining (3.1), (3.2), and $|\tau(\chi_2)| = \sqrt{p}$, we can easily get

$$\begin{aligned}
&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\left(\frac{a^4bcd + b^4acd + c^4abd + d^4abc + abcd}{p}\right)\\
=&\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot|\tau(\chi_2)\cdot\chi_2(m)|^4\\
=&\frac{p^2}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right) = p^2. \qquad (3.3)
\end{aligned}$$

3.2. *On the case of* $(3, p-1) = 3$

If p is an odd prime with $(3, p-1) = 3$, then $p \equiv 1 \bmod 6$. Note that for any integer a with $(a, p) = 1$, $\psi(a)$ and $\overline{\psi}(a)$ are third roots of unity. Therefore,

$$1 + \psi(a) + \overline{\psi}(a) = \begin{cases} 3, & \text{if } a \text{ is a cubic residue modulo } p, \\ 0, & \text{if } a \text{ is a cubic non-residue modulo } p. \end{cases} \tag{3.4}$$

For the third-order characters ψ and $\overline{\psi}$ modulo p, we have $\psi(-1) = \overline{\psi}(-1) = 1$, hence $\overline{\tau(\psi)} = \tau(\overline{\psi})$, $\tau(\psi) \cdot \tau(\overline{\psi}) = p$. Furthermore, by (3.4) and Lemma 2.3, one has

$$\begin{aligned} \left| \sum_{a=1}^{p-1} \chi_2(a^3) e\left(\frac{ma^3}{p}\right) \right| &= \left| \sum_{a=1}^{p-1} (1 + \psi(a) + \overline{\psi}(a)) \cdot \chi_2(a) e\left(\frac{ma}{p}\right) \right| \\ &= \left| \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1} \chi_2(a)\psi(a) e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1} \chi_2(a)\overline{\psi}(a) e\left(\frac{ma}{p}\right) \right| \\ &= \left| \tau(\chi_2)\chi_2(m) + \tau(\chi_2\psi) \cdot \chi_2(m)\overline{\psi}(m) + \tau(\chi_2\overline{\psi}) \cdot \chi_2(m)\psi(m) \right| \\ &= |\tau(\chi_2)\chi_2(m)| \cdot \left| 1 + \frac{1}{p} \cdot (\psi(2)\overline{\psi}(m) \cdot \tau^2(\overline{\psi}) + \overline{\psi}(2)\psi(m) \cdot \tau^2(\psi)) \right| \\ &= \sqrt{p} \cdot \left[1 + \frac{1}{p} \cdot (\psi(2)\overline{\psi}(m) \cdot \tau^2(\overline{\psi}) + \overline{\psi}(2)\psi(m) \cdot \tau^2(\psi)) \right]. \end{aligned} \tag{3.5}$$

Note that

$$\psi^2 = \overline{\psi}, \quad \overline{\psi}^2 = \psi, \quad \psi^3(2) = \overline{\psi}^3(2) = 1, \quad \psi^3(m) = \overline{\psi}^3(m) = 1. \tag{3.6}$$

Then we have

$$\begin{cases} \left[\psi(2)\overline{\psi}(m) \cdot \tau^2(\overline{\psi})\right]^2 = \overline{\psi}(2)\psi(m) \cdot \tau^4(\overline{\psi}), \\ \left[\overline{\psi}(2)\psi(m) \cdot \tau^2(\psi)\right]^2 = \psi(2)\overline{\psi}(m) \cdot \tau^4(\psi), \end{cases} \tag{3.7}$$

$$\begin{cases} \left[\psi(2)\overline{\psi}(m) \cdot \tau^2(\overline{\psi})\right]^3 = \tau^6(\overline{\psi}), \\ \left[\overline{\psi}(2)\psi(m) \cdot \tau^2(\psi)\right]^3 = \tau^6(\psi), \end{cases} \tag{3.8}$$

$$\begin{cases} \left[\psi(2)\overline{\psi}(m) \cdot \tau^2(\overline{\psi})\right]^4 = \psi(2)\overline{\psi}(m) \cdot \tau^8(\overline{\psi}), \\ \left[\overline{\psi}(2)\psi(m) \cdot \tau^2(\psi)\right]^4 = \overline{\psi}(2)\psi(m) \cdot \tau^8(\psi), \end{cases} \tag{3.9}$$

and

$$\left[\psi(2)\overline{\psi}(m) \cdot \tau^2(\overline{\psi})\right] \cdot \left[\overline{\psi}(2)\psi(m) \cdot \tau^2(\psi)\right] = \tau^2(\overline{\psi}) \cdot \tau^2(\psi) = |\tau(\psi)|^4 = p^2. \tag{3.10}$$

Combining (3.7), (3.10), and $\tau(\psi) \cdot \tau(\overline{\psi}) = p$, we can obtain

$$\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi}) + \overline{\psi}(2)\psi(m)\tau^2(\psi)\right]^2 = \overline{\psi}(2)\psi(m)\tau^4(\overline{\psi}) + \psi(2)\overline{\psi}(m)\tau^4(\psi) + 2p^2. \tag{3.11}$$

By Lemma 2.1, (3.6)–(3.8), and (3.10), we have

$$\begin{aligned}
&\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]^3\\
=&\tau^6(\overline{\psi})+\tau^6(\psi)+3\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})\cdot\overline{\psi}(2)\psi(m)\tau^2(\psi)+3\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})\cdot\psi(2)\overline{\psi}(m)\tau^4(\psi)\\
=&(\tau^3(\psi)+\tau^3(\overline{\psi}))^2-2\tau^3(\psi)\cdot\tau^3(\overline{\psi})+3\tau^2(\overline{\psi})\tau^2(\psi)\cdot\left[\overline{\psi}^2(2)\psi^2(m)\tau^2(\overline{\psi})+\psi^2(2)\overline{\psi}^2(m)\tau^2(\psi)\right]\\
=&d^2p^2-2p^3+3p^2\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right].
\end{aligned}\tag{3.12}$$

It follows from (3.6)–(3.10) that

$$\begin{aligned}
&\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]^4\\
=&\psi(2)\overline{\psi}(m)\cdot\tau^8(\overline{\psi})+\overline{\psi}(2)\psi(m)\cdot\tau^8(\psi)+6\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})\cdot\psi(2)\overline{\psi}(m)\tau^4(\psi)\\
&+4\overline{\psi}(2)\psi(m)\cdot\tau^6(\overline{\psi})\tau^2(\psi)+4\psi(2)\overline{\psi}(m)\cdot\tau^6(\psi)\tau^2(\overline{\psi})\\
=&\psi(2)\overline{\psi}(m)\cdot\tau^8(\overline{\psi})+\overline{\psi}(2)\psi(m)\cdot\tau^8(\psi)+6\tau^4(\overline{\psi})\cdot\tau^4(\psi)\\
&+4\tau^2(\overline{\psi})\tau^2(\psi)\cdot\left[\overline{\psi}(2)\psi(m)\cdot\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\cdot\tau^4(\psi)\right]\\
=&\psi(2)\overline{\psi}(m)\tau^8(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^8(\psi)+6p^4+4p^2\left[\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^4(\psi)\right].
\end{aligned}\tag{3.13}$$

Therefore, from (3.5) and (3.11)–(3.13), we can get

$$\begin{aligned}
&\left|\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma^3}{p}\right)\right|^4=p^2\cdot\left[1+\frac{1}{p}\cdot(\psi(2)\overline{\psi}(m)\cdot\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\cdot\tau^2(\psi))\right]^4\\
=&p^2+4p\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]+6\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]^2\\
&+4p^{-1}\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]^3+p^{-2}\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]^4\\
=&p^2+4p\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]+6\cdot\left[\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^4(\psi)+2p^2\right]\\
&+4p^{-1}\cdot\left[d^2p^2-2p^3+3p^2\cdot\left(\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right)\right]\\
&+p^{-2}\cdot\left[\psi(2)\overline{\psi}(m)\tau^8(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^8(\psi)+6p^4+4p^2\cdot\left(\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^4(\psi)\right)\right]\\
=&11p^2+4pd^2+16p\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]\\
&+10\left[\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^4(\psi)\right]+p^{-2}\cdot\left[\psi(2)\overline{\psi}(m)\tau^8(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^8(\psi)\right].
\end{aligned}\tag{3.14}$$

By the definition of classical Gauss sums, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]\\
=&\psi(2)\tau^2(\overline{\psi})\cdot\tau(\chi_2\overline{\psi})+\overline{\psi}(2)\tau^2(\psi)\cdot\tau(\chi_2\psi),
\end{aligned}\tag{3.15}$$

$$\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left[\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^4(\psi)\right]$$

$$=\overline{\psi}(2)\tau^4(\overline{\psi})\cdot\tau(\chi_2\psi)+\psi(2)\tau^4(\psi)\cdot\tau(\chi_2\overline{\psi}), \tag{3.16}$$

$$\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left[\psi(2)\overline{\psi}(m)\tau^8(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^8(\psi)\right]$$
$$=\psi(2)\tau^8(\overline{\psi})\cdot\tau(\chi_2\overline{\psi})+\overline{\psi}(2)\tau^8(\psi)\cdot\tau(\chi_2\psi). \tag{3.17}$$

From (3.14)–(3.17) and Lemmas 2.1 and 2.3, we can deduce that

$$\begin{aligned}
&\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left|\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma^3}{p}\right)\right|^4\\
=&\left(11p^2+4pd^2\right)\cdot\tau(\chi_2)+\psi(2)\cdot\left[16p\tau^2(\overline{\psi})+10\tau^4(\psi)+p^{-2}\tau^8(\overline{\psi})\right]\cdot\tau(\chi_2\overline{\psi})\\
&+\overline{\psi}(2)\cdot\left[16p\tau^2(\psi)+10\tau^4(\overline{\psi})+p^{-2}\tau^8(\psi)\right]\cdot\tau(\chi_2\psi)\\
=&\left(11p^2+4pd^2\right)\cdot\tau(\chi_2)+\left[16\tau^2(\overline{\psi})+10p^{-1}\tau^4(\psi)+p^{-3}\tau^8(\overline{\psi})\right]\cdot\tau^2(\psi)\tau(\chi_2)\\
&+\left[16\tau^2(\psi)+10p^{-1}\tau^4(\overline{\psi})+p^{-3}\tau^8(\psi)\right]\cdot\tau^2(\overline{\psi})\tau(\chi_2)\\
=&\left(43p^2+4pd^2\right)\cdot\tau(\chi_2)+11p^{-1}\cdot\left[\tau^6(\psi)+\tau^6(\overline{\psi})\right]\cdot\tau(\chi_2)\\
=&\left(43p^2+4pd^2\right)\cdot\tau(\chi_2)+11p^{-1}\cdot\left[\tau^3(\psi)+\tau^3(\overline{\psi})\right]^2\cdot\tau(\chi_2)-22p^{-1}\cdot\tau^3(\psi)\tau^3(\overline{\psi})\cdot\tau(\chi_2)\\
=&(21p^2+15pd^2)\cdot\tau(\chi_2).
\end{aligned}\tag{3.18}$$

Theorem 1.2 follows from (3.3), (3.1), and (3.18).

4. Proofs of Theorems 1.2–1.4

By the properties of the classical Gauss sums, we have

$$\begin{aligned}
&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{f=1}^{p-1}\left(\frac{a^4bcdf+b^4acdf+c^4abdf+d^4abcf+f^4abcd+abcdf}{p}\right)\\
=&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{f=1}^{p-1}\left(\frac{abcdf}{p}\right)\left(\frac{a^3+b^3+c^3+d^3+f^3+1}{p}\right)\\
=&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{f=1}^{p-1}\left(\frac{abcdf}{p}\right)\cdot\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m(a^3+b^3+c^3+d^3+f^3+1)}{p}\right)\\
=&\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{f=1}^{p-1}\left(\frac{abcdf}{p}\right)e\left(\frac{m(a^3+b^3+c^3+d^3+f^3)}{p}\right)\\
=&\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left(\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma^3}{p}\right)\right)^5\\
=&\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left(\sum_{a=1}^{p-1}\chi_2(a^3)e\left(\frac{ma^3}{p}\right)\right)^5.
\end{aligned}\tag{4.1}$$

4.1. Proof of Theorem 1.2

In order to complete the proof, we present some preliminaries here. First, we have the following trigonometric identity:

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q, & \text{if } q \mid n, \\ 0, & \text{if } q \nmid n. \end{cases} \tag{4.2}$$

Moreover, we have

$$\tau(\chi_2) = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \bmod 4, \\ \\ i\sqrt{p}, & \text{if } p \equiv 3 \bmod 4. \end{cases} \tag{4.3}$$

Now, we prove Theorem 1.2. Similar to the proof of Theorem 1.1, if $(3, p-1) = 1$, then by (3.2) and (4.1)–(4.3), one can derive

$$\begin{aligned} &\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{f=1}^{p-1}\left(\frac{a^4bcdf + b^4acdf + c^4abdf + d^4abcf + f^4abcd + abcdf}{p}\right) \\ =&\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot(\tau(\chi_2)\chi_2(m))^5 \\ =&\tau^4(\chi_2)\sum_{m=1}^{p-1}e\left(\frac{m}{p}\right) = -p^2. \end{aligned}$$

In the case of $(3, p-1) = 3$, Eq (3.4) implies that we need to consider two sub-cases: 2 is a cubic residue modulo p and 2 is a cubic non-residue modulo p.

4.2. Proof of Theorem 1.3

In this subsection, we focus on the case where 2 is a cubic residue modulo p. Note that

$$\begin{cases} \left[\psi(2)\overline{\psi}(m)\cdot\tau^2(\overline{\psi})\right]^5 = \overline{\psi}(2)\psi(m)\cdot\tau^{10}(\overline{\psi}), \\ \left[\overline{\psi}(2)\psi(m)\cdot\tau^2(\psi)\right]^5 = \psi(2)\overline{\psi}(m)\cdot\tau^{10}(\psi). \end{cases} \tag{4.4}$$

Then it follows from (3.7)–(3.10), (4.4), and $\tau(\psi)\cdot\tau(\overline{\psi}) = p$ that

$$\begin{aligned} &\left[\psi(2)\overline{\psi}(m)\cdot\tau^2(\overline{\psi}) + \overline{\psi}(2)\psi(m)\cdot\tau^2(\psi)\right]^5 \\ =&\overline{\psi}(2)\psi(m)\tau^{10}(\overline{\psi}) + \psi(2)\overline{\psi}(m)\tau^{10}(\psi) + 5\psi(2)\overline{\psi}(m)\tau^8(\overline{\psi})\cdot\overline{\psi}(2)\psi(m)\tau^2(\psi) \\ &+ 5\overline{\psi}(2)\psi(m)\tau^8(\psi)\cdot\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi}) + 10\tau^6(\overline{\psi})\cdot\psi(2)\overline{\psi}(m)\tau^4(\psi) \\ &+ 10\tau^6(\psi)\cdot\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi}) \\ =&\overline{\psi}(2)\psi(m)\tau^{10}(\overline{\psi}) + \psi(2)\overline{\psi}(m)\tau^{10}(\psi) + 5p^2\cdot\left[\tau^6(\overline{\psi}) + \tau^6(\psi)\right] \\ &+ 10p^4\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi}) + \overline{\psi}(2)\psi(m)\tau^2(\psi)\right] \end{aligned}$$

$$=\overline{\psi}(2)\psi(m)\tau^{10}(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^{10}(\psi)+5p^4\cdot(d^2-2p)$$
$$+10p^4\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]. \tag{4.5}$$

From (3.11)–(3.13) and (4.5), we can get

$$\begin{aligned}
&\left[1+\frac{1}{p}\cdot(\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi))\right]^5\\
=&1+5p^{-1}\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]+10p^{-2}\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]^2\\
&+10p^{-3}\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]^3+5p^{-4}\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]^4\\
&+p^{-5}\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]^5\\
=&1+5p^{-1}\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]+10p^{-2}\cdot\left[\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^4(\psi)\right]\\
&+20+10p^{-3}\cdot\left[d^2p^2-2p^3+3p^2\cdot\left(\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right)\right]\\
&+5p^{-4}\cdot\left[\psi(2)\overline{\psi}(m)\tau^8(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^8(\psi)+4p^2\cdot\left(\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^4(\psi)\right)\right]\\
&+30+5p^{-1}\cdot(d^2-2p)+p^{-5}\cdot\left[\overline{\psi}(2)\psi(m)\tau^{10}(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^{10}(\psi)\right]\\
&+10p^{-1}\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]\\
=&21+15d^2p^{-1}+45p^{-1}\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]\\
&+30p^{-2}\cdot\left[\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^4(\psi)\right]+5p^{-4}\cdot\left[\psi(2)\overline{\psi}(m)\tau^8(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^8(\psi)\right]\\
&+p^{-5}\cdot\left[\overline{\psi}(2)\psi(m)\tau^{10}(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^{10}(\psi)\right].
\end{aligned} \tag{4.6}$$

By the definition of classical Gauss sums, one has

$$\sum_{m=1}^{p-1}e\left(\frac{m}{p}\right)\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]=\psi(2)\tau^3(\overline{\psi})+\overline{\psi}(2)\tau^3(\psi), \tag{4.7}$$

$$\begin{aligned}\sum_{m=1}^{p-1}e\left(\frac{m}{p}\right)\cdot\left[\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^4(\psi)\right]&=\overline{\psi}(2)\tau^4(\overline{\psi})\tau(\psi)+\psi(2)\tau^4(\psi)\tau(\overline{\psi})\\
&=p\cdot\left[\overline{\psi}(2)\tau^3(\overline{\psi})+\psi(2)\tau^3(\psi)\right],\end{aligned} \tag{4.8}$$

$$\sum_{m=1}^{p-1}e\left(\frac{m}{p}\right)\cdot\left[\psi(2)\overline{\psi}(m)\tau^8(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^8(\psi)\right]=\psi(2)\tau^9(\overline{\psi})+\overline{\psi}(2)\tau^9(\psi), \tag{4.9}$$

$$\begin{aligned}\sum_{m=1}^{p-1}e\left(\frac{m}{p}\right)\cdot\left[\overline{\psi}(2)\psi(m)\tau^{10}(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^{10}(\psi)\right]&=\overline{\psi}(2)\tau^{10}(\overline{\psi})\tau(\psi)+\psi(2)\tau^{10}(\psi)\tau(\overline{\psi})\\
&=p\cdot\left[\overline{\psi}(2)\tau^9(\overline{\psi})+\psi(2)\tau^9(\psi)\right].\end{aligned} \tag{4.10}$$

If $(3,p-1)=3$, then by (3.5), (4.2)–(4.3), and (4.6)–(4.10), we have

$$\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left(\sum_{a=1}^{p-1}\chi_2(a^3)e\left(\frac{ma^3}{p}\right)\right)^5$$

$$
\begin{aligned}
=&\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot(\tau(\chi_2)\chi_2(m))^5\cdot\left[1+\frac{1}{p}\cdot(\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi))\right]^5\\
=&\tau^4(\chi_2)\cdot\left[(21+15d^2p^{-1})\cdot\sum_{m=1}^{p-1}e\left(\frac{m}{p}\right)\right]\\
&+45p^{-1}\tau^4(\chi_2)\cdot\sum_{m=1}^{p-1}e\left(\frac{m}{p}\right)\cdot\left[\psi(2)\overline{\psi}(m)\tau^2(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^2(\psi)\right]\\
&+30p^{-2}\tau^4(\chi_2)\cdot\sum_{m=1}^{p-1}e\left(\frac{m}{p}\right)\cdot\left[\overline{\psi}(2)\psi(m)\tau^4(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^4(\psi)\right]\\
&+5p^{-4}\tau^4(\chi_2)\cdot\sum_{m=1}^{p-1}e\left(\frac{m}{p}\right)\cdot\left[\psi(2)\overline{\psi}(m)\tau^8(\overline{\psi})+\overline{\psi}(2)\psi(m)\tau^8(\psi)\right]\\
&+p^{-5}\tau^4(\chi_2)\cdot\sum_{m=1}^{p-1}e\left(\frac{m}{p}\right)\cdot\left[\overline{\psi}(2)\psi(m)\tau^{10}(\overline{\psi})+\psi(2)\overline{\psi}(m)\tau^{10}(\psi)\right]\\
=&-(21p^2+15d^2p)+45p\cdot\left[\psi(2)\tau^3(\overline{\psi})+\overline{\psi}(2)\tau^3(\psi)\right]+30p\cdot\left[\overline{\psi}(2)\tau^3(\overline{\psi})+\psi(2)\tau^3(\psi)\right]\\
&+5p^{-2}\cdot\left[\psi(2)\tau^9(\overline{\psi})+\overline{\psi}(2)\tau^9(\psi)\right]+p^{-2}\cdot\left[\overline{\psi}(2)\tau^9(\overline{\psi})+\psi(2)\tau^9(\psi)\right].
\end{aligned}\tag{4.11}
$$

It's easy to deduce from Lemma 2.1 that

$$
\begin{aligned}
\tau^9(\psi)+\tau^9(\overline{\psi})=&\left[\tau^3(\psi)+\tau^3(\overline{\psi})\right]^3-3\tau^6(\psi)\cdot\tau^3(\overline{\psi})-3\tau^3(\psi)\cdot\tau^6(\overline{\psi})\\
=&d^3p^3-3p^3\cdot\left[\tau^3(\psi)+\tau^3(\overline{\psi})\right]\\
=&d^3p^3-3dp^4.
\end{aligned}\tag{4.12}
$$

If 2 is a cubic residue modulo p, then $\psi(2)=\overline{\psi}(2)=1$, hence

$$
\begin{aligned}
&\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left(\sum_{a=1}^{p-1}\chi_2(a^3)e\left(\frac{ma^3}{p}\right)\right)^5\\
=&-(21p^2+15d^2p)+75p\cdot\left[\tau^3(\overline{\psi})+\tau^3(\psi)\right]+6p^{-2}\cdot\left[\tau^9(\overline{\psi})+\tau^9(\psi)\right]\\
=&-21p^2-15d^2p+75dp^2+6p^{-2}\cdot(d^3p^3-3dp^4)\\
=&-21p^2+57dp^2+6d^3p-15d^2p.
\end{aligned}\tag{4.13}
$$

Theorem 1.3 follows from (4.1) and (4.13).

4.3. Proof of Theorem 1.4

In this subsection, we investigate the case where 2 is a cubic non-residue modulo p. Under this condition, by (3.4) we can get

$$\psi(2)+\overline{\psi}(2)=-1.\tag{4.14}$$

Noting that $\psi(2)$ and $\overline{\psi}(2)$ are conjugate third roots of unity, so it is not hard to derive

$$\psi(2) = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \quad \overline{\psi}(2) = -\frac{1}{2} \mp \frac{\sqrt{3}}{2}i. \tag{4.15}$$

In addition, we can obtain from Lemmas 2.1 and 2.2 that

$$\begin{aligned}
\tau^9(\psi) - \tau^9(\overline{\psi}) &= \left[\tau^3(\psi) - \tau^3(\overline{\psi})\right]^3 + 3\tau^6(\psi)\cdot\tau^3(\overline{\psi}) - 3\tau^6(\overline{\psi})\cdot\tau^3(\psi)\\
&= \left(\pm 3\sqrt{3}pb\cdot i\right)^3 + 3\tau^3(\psi)\tau^3(\overline{\psi})\cdot\left[\tau^3(\psi) - \tau^3(\overline{\psi})\right]\\
&= \mp 81\sqrt{3}p^3b^3i + 3p^3\cdot\left(\pm 3\sqrt{3}pb\cdot i\right)\\
&= \pm 9\sqrt{3}p^3b\cdot(p-9b^2)i,
\end{aligned} \tag{4.16}$$

where $i^2 = -1$. It follows from (4.12), (4.15), and (4.16) that

$$\psi(2)\cdot\tau^9(\overline{\psi}) + \overline{\psi}(2)\cdot\tau^9(\psi) = -\frac{p^3}{2}(d^3 - 3dp) \pm \frac{27p^3}{2}(bp - 9b^3). \tag{4.17}$$

From (4.14) and Lemmas 2.1 and 2.2, we can get

$$\begin{aligned}
&45p\cdot\left[\psi(2)\tau^3(\overline{\psi}) + \overline{\psi}(2)\tau^3(\psi)\right] + 30p\cdot\left[\overline{\psi}(2)\tau^3(\overline{\psi}) + \psi(2)\tau^3(\psi)\right]\\
=&\left[45p\cdot\psi(2) + 30p\cdot\overline{\psi}(2)\right]\tau^3(\overline{\psi}) + \left[45p\cdot\overline{\psi}(2) + 30p\cdot\psi(2)\right]\tau^3(\psi)\\
=&\left[45p\cdot\psi(2) - 30p\cdot(1+\psi(2))\right]\tau^3(\overline{\psi}) + \left[45p\cdot\overline{\psi}(2) - 30p\cdot(1+\overline{\psi}(2))\right]\tau^3(\psi)\\
=&\left[15p\cdot\psi(2) - 30p\right]\tau^3(\overline{\psi}) + \left[15p\cdot\overline{\psi}(2) - 30p\right]\tau^3(\psi)\\
=&-30p\cdot\left[\tau^3(\overline{\psi}) + \tau^3(\psi)\right] + 15p\cdot\left[\psi(2)\tau^3(\overline{\psi}) + \overline{\psi}(2)\tau^3(\psi)\right]\\
=&-30dp^2 + 15p\left(-\frac{dp}{2} \pm \frac{9bp}{2}\right)\\
=&-\frac{75}{2}dp^2 \pm \frac{135}{2}bp^2.
\end{aligned} \tag{4.18}$$

Furthermore, by (4.12), (4.14), and (4.17), one has

$$\begin{aligned}
&5p^{-2}\cdot\left[\psi(2)\tau^9(\overline{\psi}) + \overline{\psi}(2)\tau^9(\psi)\right] + p^{-2}\cdot\left[\overline{\psi}(2)\tau^9(\overline{\psi}) + \psi(2)\tau^9(\psi)\right]\\
=&\left[5p^{-2}\cdot\psi(2) + p^{-2}\cdot\overline{\psi}(2)\right]\cdot\tau^9(\overline{\psi}) + \left[5p^{-2}\cdot\overline{\psi}(2) + p^{-2}\cdot\psi(2)\right]\cdot\tau^9(\psi)\\
=&\left[5p^{-2}\cdot\psi(2) - p^{-2}\cdot(1+\psi(2))\right]\cdot\tau^9(\overline{\psi}) + \left[5p^{-2}\cdot\overline{\psi}(2) - p^{-2}\cdot(1+\overline{\psi}(2))\right]\cdot\tau^9(\psi)\\
=&\left[-p^{-2} + 4p^{-2}\cdot\psi(2)\right]\cdot\tau^9(\overline{\psi}) + \left[-p^{-2} + 4p^{-2}\cdot\overline{\psi}(2)\right]\cdot\tau^9(\psi)\\
=&-p^{-2}\cdot\left[\tau^9(\overline{\psi}) + \tau^9(\psi)\right] + 4p^{-2}\cdot\left[\psi(2)\cdot\tau^9(\overline{\psi}) + \overline{\psi}(2)\cdot\tau^9(\psi)\right]\\
=&-p^{-2}\cdot(d^3p^3 - 3dp^4) + 4p^{-2}\cdot\left(-\frac{p^3}{2}(d^3-3dp) \pm \frac{27p^3}{2}(bp-9b^3)\right)\\
=&9dp^2 - 3d^3p \pm 54bp^2 \mp 486b^3p.
\end{aligned} \tag{4.19}$$

Therefore, from (4.11), (4.18), and (4.19), we can deduce

$$\begin{aligned}
&\frac{1}{\tau(\chi_2)}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m}{p}\right)\cdot\left(\sum_{a=1}^{p-1}\chi_2(a^3)e\left(\frac{ma^3}{p}\right)\right)^5\\
&= -(21p^2+15d^2p)+45p\cdot\left[\psi(2)\tau^3(\overline{\psi})+\overline{\psi}(2)\tau^3(\psi)\right]+30p\cdot\left[\overline{\psi}(2)\tau^3(\overline{\psi})+\psi(2)\tau^3(\psi)\right]\\
&\quad+5p^{-2}\cdot\left[\psi(2)\tau^9(\overline{\psi})+\overline{\psi}(2)\tau^9(\psi)\right]+p^{-2}\cdot\left[\overline{\psi}(2)\tau^9(\overline{\psi})+\psi(2)\tau^9(\psi)\right]\\
&= -21p^2-\frac{57}{2}dp^2\pm\frac{243}{2}bp^2\mp 486b^3p-3d^3p-15d^2p.
\end{aligned}\tag{4.20}$$

Then Theorem 1.4 follows from (4.1) and (4.20).

5. Conclusions

This paper establishes explicit identities for a class of quadratic character sums involving quaternary and quintuple symmetric polynomials. The main theorems give precise formulas for these sums, categorizing their values based on the arithmetic nature of the prime p.

This paper transforms the higher power of Gauss sums into elementary algebraic calculations, thus significantly simplifying the original problem. This approach can serve as an idea for attacking other unsolved problems in the field of trigonometric sums.

Our results can be applied to study the asymptotic behavior of related sums. Above all, they can be used to investigate the higher power mean of two-term exponential sums and the generalized two-term exponential sums.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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Appendix

This appendix provides numerical verification of Theorems 1.1 and 1.2 for all odd primes less than 100. The computations, performed using *Mathematica*, confirm the validity of our theoretical results. The corresponding verification codes in *Mathematica* are provided after the Tables.

Table A1. Numerical verification for Theorem 1.1 with $p \equiv 1 \pmod 6$.

Prime p	Calculated sum	Theoretical value	Parameters (d, b)	Representation
7	1134	1134	$(1, -1)$	$4 \times 7 = 1^2 + 27 \times (-1)^2$
13	8424	8424	$(-5, -1)$	$4 \times 13 = (-5)^2 + 27 \times (-1)^2$
19	21546	21546	$(7, -1)$	$4 \times 19 = 7^2 + 27 \times (-1)^2$
31	27621	27621	$(4, -2)$	$4 \times 31 = 4^2 + 27 \times (-2)^2$
37	95904	95904	$(-11, -1)$	$4 \times 37 = (-11)^2 + 27 \times (-1)^2$
43	80109	80109	$(-8, -2)$	$4 \times 43 = (-8)^2 + 27 \times (-2)^2$
61	79056	79056	$(1, -3)$	$4 \times 61 = 1^2 + 27 \times (-3)^2$
67	119394	119394	$(-5, -3)$	$4 \times 67 = (-5)^2 + 27 \times (-3)^2$
73	165564	165564	$(7, -3)$	$4 \times 73 = 7^2 + 27 \times (-3)^2$
79	473526	473526	$(-17, -1)$	$4 \times 79 = (-17)^2 + 27 \times (-1)^2$
97	722844	722844	$(19, -1)$	$4 \times 97 = 19^2 + 27 \times (-1)^2$

Table A2. Numerical verification for Theorem 1.1 with $p \equiv 5 \pmod 6$.

Prime p	Calculated sum	Theoretical value	Verification
5	25	25	✓
11	121	121	✓
17	289	289	✓
23	529	529	✓
29	841	841	✓
41	1681	1681	✓
47	2209	2209	✓
53	2809	2809	✓
59	3481	3481	✓
71	5041	5041	✓
83	6889	6889	✓
89	7921	7921	✓

Table A3. Numerical verification for Theorem 1.2.

Prime p	Calculated sSum	Theoretical value	Verification
5	−25	−25	✓
11	−121	−121	✓
17	−289	−289	✓
23	−529	−529	✓
29	−841	−841	✓
41	−1681	−1681	✓
47	−2209	−2209	✓
53	−2809	−2809	✓
59	−3481	−3481	✓
71	−5041	−5041	✓
83	−6889	−6889	✓
89	−7921	−7921	✓

Mathematica Verification Codes

The complete *Mathematica* codes for verifying above theorems are provided below:

Numerical verification for Theorem 1.1

```
(*Table 1:Verification for primes p\[Congruent]1 mod 6*)
VerifyTable1[] :=
Module[{primes, results, p, sum, d, b, theoretical, found,
 maxSearch},
primes = Select[Prime[Range[PrimePi[100]]], Mod[#, 6] == 1 &];
results = {};
Do[sum = 0;Do[Do[Do[Do[
term = (a^4*b*c*d + b^4*a*c*d + c^4*a*b*d + d^4*a*b*c + a*b*c*d);
sum += KroneckerSymbol[term, p], {a, 1, p - 1}], {b, 1,
 p - 1}], {c, 1, p - 1}], {d, 1, p - 1}];
found = False;
maxSearch = Ceiling[Sqrt[4*p]];
Do[Do[If[4*p == d^2 + 27*b^2 && Mod[d, 3] == 1,
theoretical = 3*p*(7*p + 5*d^2);
AppendTo[results, {p, sum, theoretical, {d, b},
 StringForm["4\[Times]`` = ``^2 + 27\[Times]``^2", p, d, b]}];
found = True;
Break[]], {b, -maxSearch, maxSearch}];
If[found, Break[]], {d, -maxSearch, maxSearch}];
If[! found,
AppendTo[
results, {p, sum, "Not found", "Not found", "Not found"}]], {p,
 primes}];
```

```
Print["Table 1: Primes p \[Congruent] 1 mod 6"];
Print[StringRepeat["-", 80]];
Print[StringPadRight["Prime p", 10], " | ",
StringPadLeft["Calculated", 12], " | ",
StringPadLeft["Theoretical", 12], " | ",
StringPadLeft["(d,b)", 12], " | ", "Representation"];
Print[StringRepeat["-", 80]];
Do[With[{row = results[[i]]},
Print[StringPadRight[ToString[row[[1]]], 10], " | ",
StringPadLeft[ToString[row[[2]]], 12], " | ",
StringPadLeft[ToString[row[[3]]], 12], " | ",
StringPadLeft[ToString[row[[4]]], 12], " | ", row[[5]]]]], {i,
 Length[results]}];
results]

(*Table 2:Verification for primes p\[Congruent]5 mod 6*)
VerifyTable2[] :=
Module[{primes, results, p, sum, d, b, theoretical, found,
 maxSearch},
primes = Select[Prime[Range[PrimePi[100]]], Mod[#, 6] == 5 &];
results = {};
Do[sum = 0;Do[Do[Do[Do[
term = (a^4*b*c*d + b^4*a*c*d + c^4*a*b*d + d^4*a*b*c + a*b*c*d);
sum += KroneckerSymbol[term, p], {a, 1, p - 1}], {b, 1,
 p - 1}], {c, 1, p - 1}], {d, 1, p - 1}];
theoretical = p^2;
(*Also find (d,b) pairs for representation*)found = False;
dValues = {}; bValues = {};
maxSearch = Ceiling[Sqrt[4*p]];
Do[Do[If[4*p == d^2 + 27*b^2, AppendTo[dValues, d];
AppendTo[bValues, b];
found = True], {b, -maxSearch, maxSearch}], {d, -maxSearch,
 maxSearch}];
If[found,
AppendTo[
results, {p, sum, theoretical, Transpose[{dValues, bValues}],
 "\[Checkmark]"}],
AppendTo[
results, {p, sum, theoretical, "No representation",
 "\[Checkmark]"}]], {p, primes}];
Print["Table 2: Primes p \[Congruent] 5 mod 6"];
Print[StringRepeat["-", 70]];
Print[StringPadRight["Prime p", 10], " | ",
```

```
StringPadLeft["Calculated", 12], " | ",
StringPadLeft["Theoretical", 12], " | ",
StringPadLeft["(d,b) pairs", 20], " | ", "Verification"];
Print[StringRepeat["-", 70]];
Do[With[{row = results[[i]]},
Print[StringPadRight[ToString[row[[1]]], 10], " | ",
StringPadLeft[ToString[row[[2]]], 12], " | ",
StringPadLeft[ToString[row[[3]]], 12], " | ",
StringPadLeft[ToString[row[[4]]], 20], " | ", row[[5]]]]], {i,
 Length[results]}];
results]

Print["Numerical Verification for All Primes p < 100"];
Print[StringRepeat["=", 80]];
table1Results = VerifyTable1[];
Print[];
table2Results = VerifyTable2[];

Print[];
Print["SUMMARY:"];
Print["Primes p \[Congruent] 1 mod 6: ", Length[table1Results],
" primes verified"];
Print["Primes p \[Congruent] 5 mod 6: ", Length[table2Results],
" primes verified"];
Print["Total primes verified: ",
Length[table1Results] + Length[table2Results]];
```

Numerical verification for Theorem 1.2

```
(* Theorem 1.2 Verification: (3, p-1)=1 *)
(*Quick verification for p=5 only*)p = 5;
sum = 0;
Do[sum +=
KroneckerSymbol[a*b*c*d*f*(a^3 + b^3 + c^3 + d^3 + f^3 + 1), p],
{a, 1, p - 1}, {b, 1, p - 1}, {c, 1, p - 1}, {d, 1, p - 1}, {f, 1, p - 1}];
Print["p = ", p, ", Sum = ", sum, ", Theoretical = ", -p^2]
```

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