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Research article

Existence of solutions to Schrödinger systems on locally finite graphs

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Abstract: We consider the existence of solutions for some Schrödinger systems on locally finite graphs. Using variationial method and Nehari manifold method, the existence and multiplicity of nontrivial solutions are proved. More explicitly, we first give a functional setting of problem and prove the compactness of Sobolev embedding. Then for the Schrödinger system with quadratical nonlinear terms, the existence of nontrivial solutions is proved by using the Nehari manifold method. For the Schrödinger system with cubic type nonlinearity, we first prove the existence of ground state solution and then prove the multiplicity of solutions by combining the Nehari manifold method as well as the Lusternik-Schnirelmann theory.

Keywords: Schrödinger system; locally finite graphs; Lusternik-Schnirelmann theory; Nehari manifold; variational method

1. Introduction

Partial or ordinary differential equations defined on Euclidean space and smooth manifolds have a long and rich history, the differential equations on the graph as their discrete version, have attracted increasing attention in the last decade. In the series seminal works of Grigor'yan, Lin, and Yang [1–3], the authors studied the Yamabe equations, Schrödinger equations and Kazdan-Warner equations on graphs. In these papers, the variational framework was systematically established and critical point theory was applied to prove the existence of solution to the above mentioned equation. An important ingredient in finding solutions is the precompactness of Sobolev embedding. If the graph has finite vertices, Sobolev spaces are finite dimensional and hence the Sobolev embedding is precompact (see [1,2]). If the graph is locally finite and has a positive measure with positive lower bound, it was proved that the Sobolev embedding is also pre-compact in [3]. In [4], Lin and Yang proved existence and positivity results for several equations including Schrödinger equations, the Yamabe equation and the mean field equation on locally finite graphs using a novel local-to-global variational scheme.

We assume that $G = (V, E)$ is a connected locally finite graph, where V is the vertex set and E is the edge set. We also assume the following four conditions on $G = (V, E)$ are satisfied.

(G_1) (Locally finite property) For all $x \in V$, there exists at most finitely many vertices $y \in V$ such that the edge $xy \in E$.

(G_2) (Connected property) For all $x, y \in V$, there exists a path with at most finitely many edges connecting the vertex x and the vertex y.

(G_3) (Symmetric weight property) For any two different vertices $x, y \in V$, there exists a positive symmetric weight function $w : V \times V \to \mathbb{R}$, that is, $w_{xy} > 0$ and $w_{xy} = w_{yx}$.

(G_4) (Positive finite measure) There exists a finite positive measure μ on G.

For a function $u : V \to \mathbb{R}$, define the μ-Laplacian (or Laplacian) of u as follows:

$$\Delta u(x) := \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} (u(y) - u(x)), \tag{1.1}$$

where $y \sim x$ means the edge xy belongs to E. The corresponding gradient form of two functions u and v can be read as

$$\Gamma(u, v)(x) := \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy} (u(y) - u(x)) (v(y) - v(x)).$$

If $u = v$, we denote $\Gamma(u) = \Gamma(u, u)$. The length of the gradient for u can be represented as

$$|\nabla u(x)| := \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy} (u(y) - u(x))^2 \right)^{\frac{1}{2}}. \tag{1.2}$$

Let us denote the integral of a function f on the graph G by

$$\int_V f \mathrm{d}\mu := \sum_{x \in V} \mu(x) f(x).$$

For all $0 < p < +\infty$, $L^p(V)$ on G can be defined as

$$L^p(V) := \left\{ u : V \to \mathbb{R} \mid \|u\|_{L^p(V)} < +\infty \right\},$$

and the norm of $u \in L^p(V)$ can be defined by

$$\|u\|_{L^p(V)} := \left(\int_V |u|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} = \left(\sum_{x \in V} \mu(x) |u(x)|^p \right)^{\frac{1}{p}}.$$

Moreover, $L^\infty(V)$ is the normed space of functions $u : V \to \mathbb{R}$ with

$$\|u\|_{L^\infty(V)} := \sup_{x \in V} |u(x)| < +\infty.$$

For any two vertices $x, y \in V$, because we assume the graph G is connected, there is a shortest path γ connecting the vertex x and the vertex y. Then the distance $d(x, y)$ between the vertex x and the vertex y can be defined as the number of edges of the shortest path γ. More specifically, if $xy \in E$, then we have $d(x, y) = 1$. If $xy \notin E$, then there is a shortest path $\gamma = \{x_1, x_2, \ldots, x_{k+1}\}$ connecting the vertex x and the vertex y, and the distance of x and y is $d(x, y) = k$. In this paper, for a fixed vertex $\theta \in V$, we will denote the distance between a vertex x and θ as

$$d(x) = d(x, \theta).$$

Recall that the Sobolev space can be defined as

$$W^{1,2}(V) := \{u \in L^2(V) \mid \int_V |\nabla u|^2 d\mu < +\infty\},$$

and its norm defined by

$$\|u\|_{W^{1,2}(V)} := \left(\int_V (|\nabla u|^2 + u^2) d\mu\right)^{\frac{1}{2}}. \tag{1.3}$$

Define the set of functions with compact support as $C_c(V) = \{u : V \to \mathbb{R} \mid \text{supp } u \subset V$ contains only finitely many vertices$\}$, and $W_0^{1,2}(V)$ is the completion of $C_c(V)$ with norm defined in (1.3). Then both $W^{1,2}(V)$ and $W_0^{1,2}(V)$ are Hilbert spaces with inner product $\langle u, v\rangle = \int_V (\nabla u \nabla v + uv)\, d\mu$. Let $a(x)$ be a function defined on V, satisfying $a(x) \geq a_0 > 0$ for all $x \in V$. Define the following space of functions

$$\mathcal{H} := \left\{u \in W_0^{1,2}(V) \mid \int_V (|\nabla u|^2 + a(x)u^2) d\mu < +\infty\right\} \tag{1.4}$$

and its norm

$$\|u\|_{\mathcal{H}} := \left(\int_V (|\nabla u|^2 + a(x)u^2) d\mu\right)^{\frac{1}{2}}. \tag{1.5}$$

Clearly, $\mathcal{H}$ is a Hilbert space with the following inner product:

$$\langle u, v\rangle_{\mathcal{H}} := \int_V (\nabla u \nabla v + a(x)uv) d\mu, \quad \forall\, u, v \in \mathcal{H}.$$

For locally finite graph V with infinitely many vertices, both $W^{1,2}(V)$ and $\mathcal{H}$ are infinite dimensional vector spaces.

Based on the above notations, let us discuss some more related works. Chang and Zhang [5] relaxed the conditions and extended the discussion in [2] to the locally finite graph, proving that a class of the p-th nonlinear equation has a strictly positive global solution under appropriate conditions. Liu [6, 7] considered the multiplicity of solutions to a perturbed Yamabe equation and the nonlinear Dirichlet boundary condition problem on graphs. Han and Shao [8, 9] studied the existence and multiplicity of solutions to nonlinear p-Laplacian equations on a locally finite graph. Yang and Zhang [10] proposed and analyzed the existence and multiplicity of nontrivial solutions of the (p, q)-Laplacian coupled system with two parameters on a locally finite graph. For other papers on elliptic type equations and variational problem on locally finite graph, we refer interested readers to the papers [11–13] and their references.

For the Schrödinger equation, Qiu and Liu [14] extended the local-to-global variationial scheme of [4] to the exponential-nonlinearity Schrödinger equation $-\Delta u + hu = fe^u$, establishing the existence of strictly negative solutions in three distinct regimes for weight function f. Zhang and Zhao [15] considered the Schrödinger equation $-\Delta u + (\lambda(a(x) + 1)u = |u|^{p-1}u$ with polynomial nonlinearity on a locally finite graph and proved that for all $\lambda > 1$, the equation has a ground state solution if the weight function $a(x)$ satisfies suitable conditions using the Nehari manifold method. Chang et al. [16] studied the logarithmic Schrödinger equation $-\Delta u + a(x)u = u \log u^2$ on a locally finite graph; the existence of ground state solutions was obtained by using the Nehari manifold method and the mountain pass theorem. Chang et al. [17] proved the existence and respective limit behavior of sign-changing solutions to the logarithmic Schrödinger equation. Yang and Zhao [18] concerned themselves with the nonlinear equation $-\Delta u + \lambda u = f(u)$ on a locally finite graph, and proved that the equation has a normalized solution by employing variational methods. For other studies of the Schrödinger equation, we refer interested readers to the papers [19–21] and their references.

While these works focused on single-field problems on graphs, many physical and applied settings such as coupled interaction models, nonlinear optics in multi-component media, and binary Bose-Einstein condensates require the study of coupled fields. Xu and Zhao [22] studied the following nonlinear system

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u = \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in} \quad V, \\ -\Delta v + (\lambda b(x) + 1)v = \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in} \quad V, \end{cases}$$

on a locally finite graph $G = (V, E)$ and proved that this system has a nontrivial ground state solution depending on λ under suitable assumptions on the potential functions $a(x)$ and $b(x)$ using the mountain pass theorem.

In this paper we first consider the two-field coupled Schrödinger system on a locally finite graph as follows,

$$\begin{cases} -\Delta u + a(x)u = u^3 - \beta uv & \text{in} \quad V, \\ -\Delta v + b(x)v = \frac{1}{2}|v|v - \frac{\beta}{2}u^2 & \text{in} \quad V, \end{cases} \tag{1.6}$$

where Δ is the Laplacian given as in (1.1). To study problem (1.6), it is natural to consider the vector space

$$H := \left\{(u, v) \in W^{1,2}(V) \times W^{1,2}(V) \mid \int_V a(x)u^2 + b(x)v^2 d\mu < +\infty\right\}, \tag{1.7}$$

with norm

$$\|(u, v)\|_H := \left(\int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 \mathrm{d}\mu\right)^{\frac{1}{2}}.$$

The space H is also a Hilbert space with the inner product defined by

$$\langle (u, v), (\varphi, \phi)\rangle_H := \int_V \nabla u \nabla \varphi + a(x)u\varphi + \nabla v \nabla \phi + b(x)v\phi \mathrm{d}\mu, \quad \forall (u, v), (\varphi, \phi) \in H.$$

We assume that the weight functions $a(x)$ and $b(x)$ satisfy the following assumptions:

(A_1) there exist two positive constants a_0 and b_0 satisfying $a(x) \geq a_0 > 0$, $b(x) \geq b_0 > 0$.
(A_2) As $d(x, \theta) \to +\infty$, it holds that $a(x) \to +\infty$, $b(x) \to +\infty$.

The first result of this paper can be stated as follows.

Theorem 1.1. *Let $G = (V, E)$ be a locally finite graph satisfying assumptions $(G_1) - (G_4)$. For all $x \in V$, $\mu(x) \geq \mu_0 > 0$, if the weight functions $a(x)$, $b(x)$ satisfy the assumptions (A_1) and (A_2), then the system (1.6) has a nontrivial solution (u, v).*

The second system we concern is the following system with cubic nonlinearity,

$$\begin{cases} -\varepsilon^2 \Delta u + a(x)u = \mu_1 u^3 + \beta v^2 u & \text{in} \quad V, \\ -\varepsilon^2 \Delta v + b(x)v = \mu_2 v^3 + \beta u^2 v & \text{in} \quad V, \end{cases} \tag{1.8}$$

where Δ is the Laplacian given as in (1.1) and $\mu_1, \mu_2 > 0$ and $\beta \neq 0$ are constants. The energy functional $J_\varepsilon : H_\varepsilon \to \mathbb{R}$ corresponding to system (1.8) can be defined as

$$\begin{aligned} J_\varepsilon(u, v) :=& \frac{1}{2} \int_V \varepsilon^2 |\nabla u|^2 + a(x)u^2 + \varepsilon^2 |\nabla v|^2 + b(x)v^2 d\mu - \frac{1}{4} \int_V \left(\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 \right) d\mu \\ =& \frac{1}{2} \|(u, v)\|_{H_\varepsilon}^2 - \frac{1}{4} \int_V \left(\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 \right) d\mu. \end{aligned} \tag{1.9}$$

Here we define H_ε as

$$H_\varepsilon := \left\{ (u, v) \in W^{1,2}(V) \times W^{1,2}(V) \mid \int_V \varepsilon^2 |\nabla u|^2 + a(x)u^2 + \varepsilon^2 |\nabla v|^2 + b(x)v^2 d\mu < +\infty \right\}$$

with norm

$$\|(u, v)\|_{H_\varepsilon} := \left(\int_V \varepsilon^2 |\nabla u|^2 + a(x)u^2 + \varepsilon^2 |\nabla v|^2 + b(x)v^2 d\mu \right)^{\frac{1}{2}}.$$

The Nehari manifold corresponding to problem (1.8) can be defined as

$$\mathcal{N}_\varepsilon := \{(u, v) \in H_\varepsilon \setminus \{(0, 0)\} \mid \langle J_\varepsilon'(u, v), (u, v) \rangle = 0\}.$$

Namely,

$$\mathcal{N}_\varepsilon = \left\{ (u, v) \in H_\varepsilon \setminus \{(0, 0)\} \mid \|(u, v)\|_{H_\varepsilon}^2 = \int_V \left(\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 \right) d\mu \right\}. \tag{1.10}$$

Let m be the constant

$$m := \inf_{(u,v) \in \mathcal{N}_\varepsilon} J_\varepsilon(u, v).$$

If m is achieved by some vectors $(u, v) \in \mathcal{N}_\varepsilon$, then (u, v) will have least energy among functions in the Nehari manifold $\mathcal{N}_\varepsilon$, and it is indeed a critical point of the energy functional J_ε and hence a weak solution of (1.8) (u, v) will be called as a ground state solution to the system (1.8). Using the Nehari manifold method, we have the following result.

Theorem 1.2. *Let $G = (V, E)$ be a locally finite graph satisfying assumptions $(G_1) - (G_4)$. For all $x \in V$, $\mu(x) \geq \mu_0 > 0$, the weight functions $a(x)$, $b(x)$ satisfies the assumptions (A_1) and (A_2). For every $\varepsilon > 0$ and for all $|\beta| < \sqrt{\mu_1\mu_2}$, system (1.8) has a ground state solution $(u_\varepsilon, v_\varepsilon)$.*

Finally, let us consider the multiplicity of solutions to the following system:

$$\begin{cases} -\Delta u + a(x)u = u^3 + \beta v^2 u & \text{in} \quad V, \\ -\Delta v + a(x)v = v^3 + \beta u^2 v & \text{in} \quad V, \end{cases} \tag{1.11}$$

we have

Theorem 1.3. *Let $G = (V, E)$ be a locally finite graph satisfying assumptions $(G_1) - (G_4)$. For all $x \in V$, $\mu(x) \geq \mu_0 > 0$, the weight function $a(x)$ satisfies the assumptions (A_1) and (A_2), $\beta < 0$. Then*

(i) If $\beta \leq -1$, then system (1.11) has a solution sequence (u_k, v_k) satisfying $\|(u_k, v_k)\|_H \to \infty$;
(ii) For any positive integer k, there exists a real number $\beta_k > -1$; when $\beta < \beta_k$, the system (1.11) has at least k pairs of solutions (u, v), (v, u).

The continuous counterpart of (1.6), the so-called Schrödinger-Korteweg-de Vries (Schrödinger-KdV) system, appears in models for interaction phenomena between short waves and long waves, such as resonant interaction between short and long capillary-gravity water waves. Interested readers are refered to [23] and their references for a comprehensive introduction to this system. The proof of Theorem 1.1 is inspired by the seminal papers [24–26]. To prove it, we need the Sobolev-type inequality for two-component vectors on locally finite graphs. Because the locally finite graphs are generally unbounded, we use assumptions (A_1) and (A_2) on the weight functions $a(x)$, $b(x)$ to control the L^2 integral near infinity; see (2.1). Another difficulty is that the system (1.6) has a semi-trivial solution, namely, a solution of form $(0, v)$ with v satisfying the equation $-\Delta v + b(x)v = \frac{1}{2}|v|v$; see Section 3.2. Therefore, we distinguish the nontrivial solution from the semi-trivial one by comparing their energies; see Section 3.3.

The continuous counterpart of (1.8) and (1.11) arises from Bose-Einstein condensates and nonlinear optics theory, which has been extensively studied in the past two decades. The proof of Theorems 1.2 and 1.3 are inspired by [27–29]. To prove Theorem 1.2, we study the minimizing problem $\inf_{(u,v)\in\mathcal{N}_\varepsilon} J_\varepsilon(u, v)$ and prove that the minimum can be achieved based on the structure of the Nehari manifold $\mathcal{N}_\varepsilon$, it is indeed a ground state solution of system (1.8) (see Section 4). To prove Theorem 1.3, we need to choose a suitable Nahari manifold which is invariant under the convolution map $\sigma(u, v) = (v, u)$, see (5.1). Then we can apply the genus theory to obtain the existence of infinitely many nontrivial solutions (see Section 5).

For a metric graph, Schrödinger equations and systems study of propagation of optical pulses in nonlinear optics, or of matter waves (in the theory of Bose-Einstein condensates) in ramified structures such as T-junctions or X-junctions. For example, ground states of nonlinear Schrödinger equations (NLS) were studied in [30–36] on noncompact metric graphs, the NLS on compact metric graphs were considered in [37–39]; [40] investigated the existence of ground states for the NLS and their dynamics on star graphs; [41] concerned the existence of ground states for the NLS on noncompact quantum graphs; [42] proved the existence and stability of the standing waves on noncompact quantum graphs; [43] studied the quintic NLS on tadpole graphs; Bose-Einstein condensation in Josephson junctions star graph arrays was observed in [44]. We refer the interested readers to [45] for an introduction to the NLS on star graphs and [46] for an introduction to quantum graphs.

2. The Sobolev embedding theorem for two-component system on locally finite graphs

Based on references [3] and [22], one has the following Sobolev embedding theorem.

Proposition 2.1. *Let $G = (V, E)$ be a locally finite graph satisfying assumptions $(G_1) - (G_4)$. For all $x \in V$, $\mu(x) \geq \mu_0 > 0$, the weight functions $a(x)$, $b(x)$ satisfying conditions (A_1) and (A_2), then, for all $q \in [2, \infty]$, the Hilbert space H defined in (1.7) can be embedded continuously into $L^q(V, \mathbb{R}^2)$. That is to say, for all $(u, v) \in H$, there exists a positive constant $C > 0$ which depends on q such that the following inequality holds:*

$$\|(u, v)\|_{L^q(V,\mathbb{R}^2)} \leq C\|(u, v)\|_H.$$

Furthermore, for all $q \in [2, \infty]$, the embedding map from H into $L^q(V, \mathbb{R}^2)$ is a compact map, which means that for any bounded sequence $\{(u_k, v_k)\}_{k=1}^{\infty} \subset H$, there exists an element $(u, v) \subset H$ satisfying (up to a subsequence)

$$(u_k, v_k) \rightharpoonup (u, v) \text{ weakly in } H,$$

$$(u_k, v_k) \to (u, v) \text{ for all } x \in V,$$

$$(u_k, v_k) \to (u, v) \text{ in } L^q(V, \mathbb{R}^2).$$

Proof. The proof is similar to that of [22]; we give it here for reader's convenience. For a fixed vertex $x_0 \in V$, we have the estimate

$$\begin{aligned}
\|(u, v)\|_H^2 &= \int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 d\mu \\
&\geq \int_V |\nabla u|^2 + a_0 u^2 + |\nabla v|^2 + b_0 v^2 d\mu \\
&\geq a_0 \int_V u^2 d\mu \\
&= a_0 \sum_{x\in V} \mu(x)u^2(x) \\
&\geq a_0\mu_0 u^2(x_0),
\end{aligned}$$

which yields $u(x_0) \leq \sqrt{\frac{1}{a_0\mu_0}}\|(u, v)\|_H$. Similarly, one has $v(x_0) \leq \sqrt{\frac{1}{b_0\mu_0}}\|(u, v)\|_H$. Thus it holds that

$$\|(u, v)\|_{L^\infty(V)} = \sup_{x\in V} |u(x)| + \sup_{x\in V} |v(x)| \leq \left(\sqrt{\frac{1}{a_0\mu_0}} + \sqrt{\frac{1}{b_0\mu_0}}\right)\|(u, v)\|_H,$$

which tells us the embedding $H \hookrightarrow L^\infty(V, \mathbb{R}^2)$ is continuous. It follows that the embedding map $H \hookrightarrow L^2(V, \mathbb{R}^2)$ is also continuous. Then, the interpolation inequality gives us that $H \hookrightarrow L^q(V, \mathbb{R}^2)$ continuously for any $q \in [2, \infty]$.

Now we prove that the embedding map $H \hookrightarrow L^q(V, \mathbb{R}^2)$ is compact for $q \in [2, \infty]$.

For a sequence $\{(u_k, v_k)\}_{k=1}^{\infty} \subset H$ which is bounded, there exists weak limit $(u, v) \in H$ such that (up to a subsequence)

$$(u_k, v_k) \rightharpoonup (u, v) \text{ weakly in } H,$$
$$(u_k, v_k) \rightharpoonup (u, v) \text{ in } L^2(V, \mathbb{R}^2).$$

Then

$$\begin{aligned}&\lim_{k\to\infty} \int_V [(u_k - u)(x)\xi(x) + (v_k - v)(x)\eta(x)]d\mu \\ &= \lim_{k\to\infty} \sum_{x\in V} [\mu(x)(u_k - u)(x)\xi(x) + \mu(x)(v_k - v)(x)\eta(x)] \\ &= 0\end{aligned}$$

for all $(\xi, \eta) \in L^2(V, \mathbb{R}^2)$. For all $x_0 \in V$, let $(\xi_1, \eta_1) = (\delta_{x_0}, 0)$, $(\xi_2, \eta_2) = (0, \delta_{x_0})$ respectively (here, $\delta_{x_0}(x_0) = 1$ and $\delta_{x_0}(x) = 0$ for $x \neq x_0$); thus

$$\lim_{k\to\infty} \mu(x_0)(u_k - u)(x_0) = 0,$$
$$\lim_{k\to\infty} \mu(x_0)(v_k - v)(x_0) = 0.$$

Because $\mu(x) \geq \mu_0 > 0$, it holds that $(u_k, v_k)(x) \to (u, v)(x)$ for all $x \in V$ as $k \to \infty$.

Now we will prove $(u_k, v_k) \to (u, v)$ in the $L^q(V, \mathbb{R}^2)$ sense. Without loss of generality, assume that $(u, v) = (0, 0)$. Because the sequence $\{(u_k, v_k)\}_{k=1}^{\infty} \subset H$ is bounded, $\|(u_k, v_k)\|_H^2 \leq C$ for some constant $C > 0$. From the assumptions on the functions $a(x)$ and $b(x)$, for any $\varepsilon > 0$, there is a constant $R > 0$ satisfying $a(x) \geq \frac{2C}{\varepsilon}$ and $b(x) \geq \frac{2C}{\varepsilon}$ for $x \in V$ satisfying $d(x, \theta) > R$. It then holds that

$$\begin{aligned}&\int_{d(x,\theta)>R} (|u_k|^2 + |v_k|^2)d\mu \\ &\leq \frac{\varepsilon}{2C} \int_{d(x,\theta)>R} [a(x)|u_k|^2 + b(x)|v_k|^2]d\mu \\ &\leq \frac{\varepsilon}{2C} \|(u_k, v_k)\|_H^2 \leq \frac{\varepsilon}{2}.\end{aligned} \tag{2.1}$$

Note that the set $\{x \in V \mid d(x, \theta) \leq R\}$ is finite, and as $k \to \infty$ $u_k(x) \to 0$, $v_k(x) \to 0$ for all vertex $x \in V$, there is a positive constant $k_0 > 0$ satisfying $\int_{d(x,\theta)\leq R}(|u_k|^2 + |v_k|^2)d\mu \leq \frac{\varepsilon}{2}$ when $k > k_0$. Hence $\int_V(|u_k|^2 + |v_k|^2)d\mu \leq \varepsilon$ when k is large enough. This gives us that $\lim_{k\to\infty} \|(u_k, v_k)\|_{L^2(V,\mathbb{R}^2)} = 0$.

For all $x \in V$, it holds that

$$\|(u_k, v_k)\|^2_{L^2(V,\mathbb{R}^2)} \geq \mu_0 u_k^2(x), \quad \|(u_k, v_k)\|^2_{L^2(V,\mathbb{R}^2)} \geq \mu_0 v_k^2(x) \tag{2.2}$$

and hence

$$\begin{aligned}\|(u_k, v_k)\|_{L^\infty(V,\mathbb{R}^2)} &= \sup_{x\in V} |u_k(x)| + \sup_{x\in V} |v_k(x)| \\ &\leq 2\sqrt{\frac{1}{\mu_0}}\|(u_k, v_k)\|_{L^2} \to 0,\end{aligned}$$

as $k \to \infty$. For any given $q \in [2, \infty]$, there holds

$$\begin{aligned}
&\|(u_k, v_k)\|^q_{L^q(V,\mathbb{R}^2)} = \int_V |u_k|^q + |v_k|^q d\mu \\
&\le \left[\sup_{x\in V} |u_k(x)|\right]^{q-2} \int_V u_k^2 d\mu + \left[\sup_{x\in V} |v_k(x)|\right]^{q-2} \int_V v_k^2 d\mu \\
&\le \|(u_k, v_k)\|^{q-2}_{L^\infty(V,\mathbb{R}^2)} \int_V \left(u_k^2 + v_k^2\right) d\mu \to 0,
\end{aligned}$$

as $k \to \infty$. This concludes the proof.

Remark 2.1.1. *In the $L^\infty(V, \mathbb{R}^2)$ estimate, we use the point-wise estimate (2.2), which does not hold in the Euclidean case, and therefore the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ is not true (see [47]).*

3. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1.

3.1. The framework

We define the energy functional $E : H \to \mathbb{R}$ corresponding to the system (1.6) on H as follows:

$$\begin{aligned}
E(u, v) :=& \frac{1}{2} \int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 d\mu \\
&- \frac{1}{4} \int_V u^4 d\mu - \frac{1}{6} \int_V |v|^3 d\mu + \frac{\beta}{2} \int_V u^2 v d\mu, \quad \forall (u, v) \in H,
\end{aligned} \tag{3.1}$$

and set $\Lambda = \inf_{(u,v)\in H} E(u, v)$. Clearly we have

$$\Lambda \le J(0, 0) = 0. \tag{3.2}$$

Now define the corresponding Nehari manifold by

$$\mathcal{M} := \{(u, v) \in H \setminus \{(0, 0)\} \mid \langle E'(u, v), (u, v)\rangle = 0\}.$$

Lemma 3.1. *Given any element $(u, v) \in H \setminus \{(0, 0)\}$, there is a unique positive number $\bar{t}$ depending on (u, v) satisfying $(\bar{t}u, \bar{t}v) \in \mathcal{M}$. The maxima of the function defined by $\tilde{g}(t) = E(tu, tv)$ for $t \ge 0$ can be achieved at the point $\bar{t}$.*

Proof. Let us define the following functional $\hat{P} : H \to \mathbb{R}$ by

$$\begin{aligned}
\hat{P}(u, v) :=& \int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 d\mu \\
&- \int_V u^4 d\mu - \frac{1}{2} \int_V |v|^3 d\mu + \frac{3\beta}{2} \int_V u^2 v d\mu.
\end{aligned} \tag{3.3}$$

Then $(u, v) \in H \setminus \{(0, 0)\}$ is in the Nehari manifold $\mathcal{M}$ if and only if the condition $\hat{P}(u, v) = 0$ holds. Assuming that there is a positive number $\bar{t}$ satisfying $(\bar{t}u, \bar{t}v) \in \mathcal{M}$, we have

$$\hat{P}(\bar{t}u, \bar{t}v) = \bar{t}^2 \int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 d\mu \\ - \bar{t}^4 \int_V u^4 d\mu - \frac{1}{2}\bar{t}^3 \int_V |v|^3 d\mu + \frac{3\beta}{2}\bar{t}^3 \int_V u^2 v d\mu. \tag{3.4}$$

Hence, we can obtain that

$$\begin{aligned}\hat{\rho}(\bar{t}) &= \int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 d\mu - \bar{t}^2 \int_V u^4 d\mu - \frac{1}{2}\bar{t} \int_V |v|^3 d\mu + \frac{3\beta}{2}\bar{t} \int_V u^2 v d\mu \\ &= -\bar{t}^2 \int_V u^4 d\mu + \left(\frac{3\beta}{2} \int_V u^2 v d\mu - \frac{1}{2} \int_V |v|^3 d\mu\right)\bar{t} + \int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 d\mu \\ &= 0.\end{aligned}$$

If $\int_V u^4 d\mu \neq 0$, from the fact that $\hat{\rho}(0) = \int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 d\mu > 0$, there exists a unique positive real number $\bar{t}$ satisfying $\hat{\rho}(\bar{t}) = 0$, which means $(\bar{t}u, \bar{t}v) \in \mathcal{M}$. If $\int_V u^4 d\mu = 0$. We then have $v \neq 0$, and there is a unique positive real number $\bar{t} = \bar{t}(u, v)$ depending on (u, v) and satisfying $(\bar{t}u, \bar{t}v) \in \mathcal{M}$.

Fix $(\tilde{u}, \tilde{v}) \in \mathcal{M}$, which yields

$$\langle E'(\tilde{u}, \tilde{v}), (\tilde{u}, \tilde{v})\rangle = \int_V |\nabla \tilde{u}|^2 + a(x)\tilde{u}^2 + |\nabla \tilde{v}|^2 + b(x)\tilde{v}^2 d\mu \\ - \int_V \tilde{u}^4 d\mu - \frac{1}{2} \int_V |\tilde{v}|^3 d\mu + \frac{3\beta}{2} \int_V \tilde{u}^2 \tilde{v} d\mu = 0. \tag{3.5}$$

Thus

$$E(\tilde{u}, \tilde{v}) = \frac{1}{6} \int_V |\nabla \tilde{u}|^2 + a(x)\tilde{u}^2 + |\nabla \tilde{v}|^2 + b(x)\tilde{v}^2 d\mu + \frac{1}{12} \int_V \tilde{u}^4 d\mu,$$

and

$$\int_V |\nabla \tilde{u}|^2 + a(x)\tilde{u}^2 + |\nabla \tilde{v}|^2 + b(x)\tilde{v}^2 d\mu = \int_V \tilde{u}^4 d\mu + \frac{1}{2} \int_V |\tilde{v}|^3 d\mu - \frac{3\beta}{2} \int_V \tilde{u}^2 \tilde{v} d\mu \\ \leq C_1 \|(\tilde{u}, \tilde{v})\|_H^4 + C_2 \|(\tilde{u}, \tilde{v})\|_H^3. \tag{3.6}$$

Therefore, $\|(\tilde{u}, \tilde{v})\|_H \geq \rho$ for some positive real number $\rho > 0$ and $E(\tilde{u}, \tilde{v}) \geq \frac{1}{6}\rho^2$.

Observe that the function

$$\tilde{g}(t) = \frac{t^2}{2} \int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 d\mu \\ - \frac{t^4}{4} \int_V u^4 d\mu - \frac{t^3}{6} \int_V |v|^3 d\mu + \frac{\beta t^3}{2} \int_V u^2 v d\mu \tag{3.7}$$

satisfies $\tilde{g}(t) \to 0$ as $t \to 0$ and $\tilde{g}(t) \to -\infty$ as $t \to +\infty$, $\tilde{g}(\bar{t}) \geq \frac{1}{6}\rho^2$; its maximum must be achieved at the interior of interval $[0, +\infty)$. Supposing we have $\tilde{g}(\tilde{t}) = \max \tilde{g}(t)$, then it holds that $\tilde{g}'(\tilde{t}) = 0$ and $\langle E'(\tilde{t}u, \tilde{t}v), (\tilde{t}u, \tilde{t}v)\rangle = 0$ and therefore $\tilde{t} = \bar{t}$.

For all $(u, v) \in \mathcal{M}$,

$$\langle \hat{P}'(u, v), (u, v)\rangle_H = -\int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 d\mu - \int_V u^4 d\mu \le -\frac{\rho^2}{6}.$$

From the implicit function theorem, one can deduce that the Nehari manifold $\mathcal{M}$ is an infinite dimensional C^1-manifold.

Lemma 3.2. *An element $(u, v) \in H$ is a critical point to the energy functional E if and only if it is a constraint critical point of E on the Nehari manifold $\mathcal{M}$.*

Proof. If for $(u, v) \in \mathcal{M}$, $E|'_{\mathcal{M}}(u, v) = 0$, then $E'(u, v) - \omega\hat{P}'(u, v) = 0$ in H. From the definition of the Nehari manifold $\mathcal{M}$, one has $\langle E|'_{\mathcal{M}}(u, v), (u, v)\rangle = -\omega\langle \hat{P}'(u, v), (u, v)\rangle = 0$. Hence $\omega = 0$ and $\langle E|'_{\mathcal{M}}(u, v), (u, v)\rangle = 0$. Conversely, if (u, v) is a non-trivial critical point of the energy functional E, clearly it is a critical point of $E|_{\mathcal{M}}$.

Lemma 3.3. *The energy functional E satisfies the PS condition on the Nehari manifold $\mathcal{M}$.*

Proof. Assume $\{(u_n, v_n)\}_{n=1}^{\infty} \subset \mathcal{M}$ is a Palais-Smale (PS) sequence of the energy functional E, then $E(u_n, v_n) \to c$ for some real constant $c \in \mathbb{R}$ and $E|'_{\mathcal{M}}(u_n, v_n) \to 0$. Then the sequence $\{(u_n, v_n)\}_{n=1}^{\infty}$ is bounded in H, and up to subsequence one can assume $(u_n, v_n) \rightharpoonup (u, v) \in H$. From the compactness of Sobolev embedding in Proposition 2.1, for $p \in [2, \infty]$, $u_n \to u$, $v_n \to v$ in the $L^p(V)$ sense. Furthermore,

$$\int_V u_n^4 d\mu + \frac{1}{2}\int_V |v_n|^3 d\mu - \frac{3\beta}{2}\int_V u_n^2 v_n d\mu \to \int_V u^4 d\mu + \frac{1}{2}\int_V |v|^3 d\mu - \frac{3\beta}{2}\int_V u^2 v d\mu$$

and

$$\int_V u^4 d\mu + \frac{1}{2}\int_V |v|^3 d\mu - \frac{3\beta}{2}\int_V u^2 v d\mu \ge \rho^2.$$

It also holds that

$$E|'_{\mathcal{M}}(u_n, v_n) = E'(u_n, v_n) - \omega_n\hat{P}'(u_n, v_n) \to 0$$

for a sequence $\{\omega_n\} \subset \mathbb{R}$. Then we have

$$\langle E|'_{\mathcal{M}}(u_n, v_n), (u_n, v_n)\rangle = \langle E'(u_n, v_n), (u_n, v_n)\rangle - \omega_n\langle \hat{P}'(u_n, v_n), (u_n, v_n)\rangle \to 0.$$

Because $\langle E'(u_n, v_n), (u_n, v_n)\rangle_H = 0$, one has $\omega_n\langle \hat{P}'(u_n, v_n), (u_n, v_n)\rangle \to 0$ and $\omega_n \to 0$. Denote

$$\begin{aligned} E(u, v) &= \frac{1}{2}\int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 d\mu - \hat{F}(u, v) - \hat{G}(u, v) + \beta\hat{T}(u, v) \\ &= \frac{1}{2}\langle (u_n, v_n), (u_n, v_n)\rangle_H - \hat{F}(u, v) - \hat{G}(u, v) + \beta\hat{T}(u, v), \end{aligned} \tag{3.8}$$

here $\hat{F}(u, v) = \frac{1}{4}\int_V u^4 d\mu$, $\hat{G}(u, v) = \frac{1}{6}\int_V |v|^3 d\mu$, $\hat{T}(u, v) = \frac{1}{2}\int_V u^2 v d\mu$, and we have

$$E'(u_n, v_n) = (u_n, v_n) - \hat{F}'(u_n, v_n) - \hat{G}'(u_n, v_n) + \beta\hat{T}'(u_n, v_n)$$

and

$$\hat{P}'(u_n, v_n) = 2(u_n, v_n) - 4\hat{F}'(u_n, v_n) - 3\hat{G}'(u_n, v_n) + 3\beta\hat{T}'(u_n, v_n).$$

Hence we deduce that

$$(1-2\omega_n)(u_n,v_n)=(1-4\omega_n)\hat{F}'(u_n,v_n)+(1-3\omega_n)\hat{G}'(u_n,v_n)+(3\omega_n-1)\beta\hat{T}'(u_n,v_n)+o(1).$$

We now claim that the operators $\hat{F}'$, $\hat{G}'$, $\hat{T}'$ are compact mappings. Indeed, taking any $h=(h_1,h_2)\in H$, it holds that

$$\begin{aligned}\left|\langle \hat{T}'_u(u_n,v_n)-\hat{T}'_u(u,v),h\rangle\right| &= \left|\int_V (u_nv_n-uv)h_1d\mu\right| \le \int_V |u_nv_n-uv||h_1|d\mu\\ &\le \int_V |u_nv_n-u_nv||h_1|d\mu+\int_V |u_nv-uv||h_1|d\mu\\ &\le \int_V |v_n-v||u_n||h_1|d\mu+\int_V |u_n-u||v||h_1|d\mu\\ &\le \|v_n-v\|_{L^3(V)}\|u_n\|_{L^3(V)}\|h_1\|_{L^3(V)}+\|u_n-u\|_{L^3(V)}\|v\|_{L^3(V)}\|h_1\|_{L^3(V)}\to 0,\end{aligned}$$

$$\begin{aligned}\left|\langle \hat{T}'_v(u_n,v_n)-\hat{T}'_v(u,v),h\rangle\right| &= \left|\frac{1}{2}\int_V (u_n^2-u^2)h_2d\mu\right|\\ &\le \frac{1}{2}\left(\int_V |u_n^2-v_n^2|^2d\mu\right)^{\frac{1}{2}}\left(\int_V |h_2|^2d\mu\right)^{\frac{1}{2}}.\end{aligned}$$

Because $u_n^4+u^4-|u_n^4-u^4|\ge 0$, from Fatou's lemma, one has

$$\int_V \liminf_{n\to+\infty}\left(u_n^4+u^4-|u_n^4-u^4|\right)d\mu \le \liminf_{n\to+\infty}\int_V \left(u_n^4+u^4-|u_n^4-u^4|\right)d\mu;$$

furthermore, it holds that $\liminf_{n\to+\infty}\int_V |u_n^4-u^4|d\mu\to 0$ and $\int_V |u_n^2-u^2|^2d\mu\le\int_V |u_n^4-u^4|d\mu\to 0$, thus $\langle \hat{T}'_v(u_n,v_n)-\hat{T}'_v(u,v),h\rangle\to 0$. Therefore the operator $\hat{T}'$ is compact. Similarly, we deduce that

$$\begin{aligned}\left|\langle \hat{F}'_u(u_n,v_n)-\hat{F}'_u(u,v),h\rangle\right| &= \left|\int_V (u_n^3-u^3)h_1d\mu\right| \le \int_V |u_n^3-u^3||h_1|d\mu\\ &\le \left(\int_V |u_n^3-u^3|^{\frac{4}{3}}d\mu\right)^{\frac{3}{4}}\|h_1\|_{L^4(V)}\\ &\le \left(\int_V |u_n^4-u^4|d\mu\right)^{\frac{3}{4}}\|h_1\|_{L^4(V)}\to 0,\end{aligned}$$

$$\begin{aligned}\left|\langle \hat{G}'_v(u_n,v_n)-\hat{G}'_u(u,v),h\rangle\right| &= \left|\frac{1}{2}\int_V (|v_n|v_n-|v|v)h_2d\mu\right|\\ &\le \frac{1}{2}\left(\int_V ||v_n|v_n-|v|v|^2d\mu\right)^{\frac{1}{2}}\left(\int_V |h_2|^2d\mu\right)^{\frac{1}{2}}\\ &\le \frac{1}{2}\left(\int_V |v_n^4-v^4|d\mu\right)^{\frac{1}{2}}\left(\int_V |h_2|^2d\mu\right)^{\frac{1}{2}}\to 0.\end{aligned}$$

Therefore the operators $\hat{F}'$ and $\hat{G}'$ are compact mappings.

Therefore we have $(u_n,v_n)\to\hat{F}'(u,v)+\hat{G}'(u,v)-\beta\hat{T}'(u,v)$ and $(u,v)=\hat{F}'(u,v)+\hat{G}'(u,v)-\beta\hat{T}'(u,v)$. Thus $\int_V |\nabla u|^2+a(x)u^2+|\nabla v|^2+b(x)v^2d\mu=4\hat{F}(u,v)+3\hat{G}(u,v)-3\beta\hat{T}(u,v)$, which implies that $(u,v)\in\mathcal{M}$.

3.2. On the equation $-\Delta v + b(x)v = \frac{1}{2}|v|v$

To proceed, we need some properties of the single equation

$$-\Delta v + b(x)v = \frac{1}{2}|v|v. \tag{3.9}$$

Let us define the corresponding energy functional $E_1 : H_1 \to \mathbb{R}$ by

$$E_1(v) := \frac{1}{2}\int_V |\nabla v|^2 + b(x)v^2 d\mu - \frac{1}{6}\int_V |v|^3 d\mu, \quad \forall v \in H_1,$$

and

$$H_1 := \left\{ u \in W^{1,2}(V) \mid \int_V b(x)u^2 d\mu < +\infty \right\}.$$

We define the corresponding Nehari manifold as

$$\mathcal{M}_1 := \{v \in H_1 \setminus \{0\} \mid \langle E_1'(v), v\rangle = 0\}.$$

Then the following results hold.

Lemma 3.4. *$\mathcal{M}_1$ is a nonempty C^1 smooth manifold. For all $v \in H_1 \setminus \{0\}$, there exists a unique positive real number $\bar{t}$ depending on v satisfying $\bar{t}v \in \mathcal{M}_1$. The maxima of the function $\hat{g}(t) = E_1(tv)$ of $t \geq 0$ can be achieved at the point $\bar{t} > 0$.*

Proof. Take any $v \in H_1 \setminus \{0\}$. Let us consider the following relation on $t > 0$:

$$\tilde{P}_1(tv) = t^2 \int_V |\nabla v|^2 + b(x)v^2 d\mu - \frac{t^3}{2}\int_V |v|^3 d\mu = 0.$$

Because $\int_V |v|^3 d\mu \neq 0$, there exists a unique real number $\bar{t} = \bar{t}(v) > 0$ such that $\bar{t}v \in \mathcal{M}_1$.

For any $\tilde{v} \in \mathcal{M}_1$, one has

$$\tilde{P}_1(\tilde{v}) = \int_V |\nabla \tilde{v}|^2 + b(x)\tilde{v}^2 d\mu - \frac{1}{2}\int_V |\tilde{v}|^3 d\mu = 0.$$

Then

$$\int_V |\nabla \tilde{v}|^2 + b(x)\tilde{v}^2 d\mu = \frac{1}{2}\int_V |\tilde{v}|^3 d\mu \leq C\left(\int_V |\nabla \tilde{v}|^2 + b(x)\tilde{v}^2 d\mu\right)^{\frac{3}{2}}.$$

Thus it holds that $\int_V |\nabla \tilde{v}|^2 + b(x)\tilde{v}^2 d\mu \geq \rho^2$ for some $\rho > 0$ and $E_1(\tilde{v}) = \frac{1}{6}\int_V |\nabla \tilde{v}|^2 + b(x)\tilde{v}^2 d\mu \geq \frac{1}{6}\rho^2$. Moreover, we have

$$\langle \tilde{P}_1'(\tilde{v}), \tilde{v}\rangle = 2\int_V |\nabla \tilde{v}|^2 + b(x)\tilde{v}^2 d\mu - \frac{3}{2}\int_V |\tilde{v}|^3 d\mu = -\int_V |\nabla \tilde{v}|^2 + b(x)\tilde{v}^2 d\mu < -\rho^2.$$

Therefore, the Nehari manifold $\mathcal{M}_1$ is a C^1 manifold.

Because for the function

$$\hat{g}(t) = \frac{t^2}{2}\int_V |\nabla v|^2 + b(x)v^2 d\mu - \frac{t^3}{6}\int_V |v|^3 d\mu,$$

it holds that $\hat{g}(t) \to 0$ as $t \to 0$, $\hat{g}(t) \to -\infty$ as $t \to +\infty$, $\hat{g}(\bar{t}) \geq \frac{1}{6}\rho^2$, the maxima of $\hat{g}(t)$ must be achieved at the interior of the interval $[0, +\infty]$. Suppose we have $\hat{g}(\tilde{t}) = \max_{t\in[0,+\infty]} \hat{g}(t)$, we have $\hat{g}'(\tilde{t}) = 0$, hence $\tilde{P}_1(\tilde{v}) = \langle E_1'(\tilde{t}v), \tilde{t}v\rangle = 0$, therefore $\tilde{t} = \bar{t}$.

Lemma 3.5. *Suppose $v_0 \in H_1$ is a minimizer to the energy functional E_1 constraining on the Nehari manifold $\mathcal{M}_1$, i.e., $E_1(v_0) = \inf_{w\in\mathcal{M}_1} E_1(w)$. Then v is a weak solution of (3.9).*

Lemma 3.6. *The energy functional E_1 constraining on the Nehari manifold $\mathcal{M}_1$ admits a minimizer v_0; therefore, it is a weak solution of (3.9).*

The proof of Lemmas 3.5 and 3.6 are standard arguments as in [48]; therefore, we omit them here.

3.3. Proof of Theorem 1.1

We will prove that (1.6) possesses a nontrivial solution which is different from the trivial solutions $\mathbf{u}_0$ and $\mathbf{u}_1$, here $\mathbf{u}_0 := (0, v_0)$ and $\mathbf{u}_1 = (0, -v_0)$; v_0 is the solution obtained in Lemma 3.6. Let us define the following constant

$$\gamma := \inf_{\varphi=(\varphi_1,\varphi_2)\in H\setminus\{(0,0)\}} \frac{\|\varphi\|_H^2}{\int_V v_0\varphi_2^2 d\mu}.$$

From the Sobolev embedding theorem, we can deduce that the constant γ is positive. Furthermore we have the following result.

Proposition 3.7. *(1) If the parameter β satisfies $\beta < -\gamma$, then $\mathbf{u}_0$ is a saddle point to the energy functional E on the Nehari manifold $\mathcal{M}$, and it holds that* $\inf_{\mathcal{M}} E < E(\mathbf{u}_0)$.
(2) If the parameter β satisfies $\beta > \gamma$, then $\mathbf{u}_1$ is a saddle point to the energy functional E on the Nehari manifold $\mathcal{M}$, and it holds that $\inf_{\mathcal{M}} E < E(\mathbf{u}_1)$.

Define $D^2E|_{\mathcal{M}}(\mathbf{u}_0)$ and $D^2E|_{\mathcal{M}}(\mathbf{u}_1)$ as the second derivative of E constrained on the Nehari manifold $\mathcal{M}$. Because $E'(\mathbf{u}_0) = 0$ and $E'(\mathbf{u}_1) = 0$, we then have

$$D^2E|_{\mathcal{M}}(\mathbf{u}_0)[\mathbf{h},\mathbf{h}] = E''(\mathbf{u}_0)[\mathbf{h},\mathbf{h}], \quad \forall \mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_0}\mathcal{M},$$

$$D^2E|_{\mathcal{M}}(\mathbf{u}_1)[\mathbf{h},\mathbf{h}] = E''(\mathbf{u}_1)[\mathbf{h},\mathbf{h}], \quad \forall \mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_1}\mathcal{M}.$$

One also has

$$D^2E_1|_{\mathcal{M}_1}(v_0)[h,h] = E_1''(v_0)[h,h], \quad \forall h \in T_{v_0}\mathcal{M}_1,$$

$$D^2E_1|_{\mathcal{M}_1}(-v_0)[h,h] = E_1''(-v_0)[h,h], \quad \forall h \in T_{-v_0}\mathcal{M}_1.$$

It then follows that

Lemma 3.8.

$$\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_0}\mathcal{M} \text{ if and only if } h_2 \in T_{v_0}\mathcal{M}_1,$$

$$\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_1}\mathcal{M} \text{ if and only if } h_2 \in T_{-v_0}\mathcal{M}_1.$$

Proof of Proposition 3.7: For any $(u, v) \in H$ and $\mathbf{h} = (h_1, h_2) \in H$, then

$$E''(u,v)[\mathbf{h},\mathbf{h}] = \int_V |\nabla h_1|^2 + a(x)h_1^2 + |\nabla h_2|^2 + b(x)h_2^2 d\mu - 3\int_V u^2h_1^2 d\mu - \int_V vh_2^2 d\mu$$
$$+\beta\int_V vh_1^2 d\mu + \beta\int_V uh_1h_2 d\mu.$$

If $(u, v) = \mathbf{u}_0 = (0, v_0)$, we then have

$$E''(\mathbf{u}_0)[\mathbf{h}, \mathbf{h}] = \int_V |\nabla h_1|^2 + a(x)h_1^2 + |\nabla h_2|^2 + b(x)h_2^2 d\mu - \int_V v_0 h_2^2 d\mu + \beta \int_V v_0 h_1^2 d\mu.$$

By taking $\mathbf{h} = (h_1, 0) \in T_{\mathbf{u}_0}\mathcal{M}$, then

$$E''(\mathbf{u}_0)[(h_1, 0), (h_1, 0)] = \int_V (|\nabla h_1|^2 + a(x)h_1^2) d\mu + \beta \int_V v_0 h_1^2 d\mu.$$

From the definition of the constant γ when the parameter β satisfies $\beta < -\gamma$, there is $\tilde{h}_1 \in H_1 \setminus \{0\}$, satisfying

$$\gamma \leq \frac{\int_V (|\nabla \tilde{h}_1|^2 + b(x)\tilde{h}_1^2) d\mu}{\int_V v_0 \tilde{h}_1^2 d\mu} < -\beta.$$

Therefore,

$$E''(\mathbf{u}_0)[(\tilde{h}_1, 0), (\tilde{h}_1, 0)] = \int_V (|\nabla \tilde{h}_1|^2 + \tilde{h}_1^2) d\mu + \beta \int_V v_0 \tilde{h}_1^2 d\mu < 0.$$

Hence $\mathbf{u}_0$ is a saddle point to the energy functional E restricting on $\mathcal{M}$.

If we take $(u, v) = \mathbf{u}_1 = (0, -v_0)$, then

$$E''(\mathbf{u}_1)[\mathbf{h}, \mathbf{h}] = \int_V |\nabla h_1|^2 + a(x)h_1^2 + |\nabla h_2|^2 + b(x)h_2^2 d\mu + \int_V v_0 h_2^2 d\mu - \beta \int_V v_0 h_1^2 d\mu.$$

By taking $\mathbf{h} = (h_1, 0) \in T_{\mathbf{u}_0}\mathcal{M}$, then we obtain

$$E''(\mathbf{u}_0)[(h_1, 0), (h_1, 0)] = \int_V |\nabla h_1|^2 + a(x)h_1^2 d\mu - \beta \int_V v_0 h_1^2 d\mu.$$

Using the definition of the constant γ when $\beta > \gamma$, there exists $\tilde{h}_2 \in H_1 \setminus \{0\}$, satisfying

$$\gamma \leq \frac{\int_V |\nabla \tilde{h}_2|^2 + b(x)\tilde{h}_2^2 d\mu}{\int_V v_0 \tilde{h}_2^2 d\mu} < \beta.$$

Therefore,

$$E''(\mathbf{u}_1)[(\tilde{h}_2, 0), (\tilde{h}_2, 0)] = \int_V (|\nabla \tilde{h}_2|^2 + b(x)\tilde{h}_2^2) d\mu - \beta \int_V v_0 \tilde{h}_2^2 d\mu < 0.$$

Thus $\mathbf{u}_1$ is a saddle point of the energy functional E restricting on $\mathcal{M}$.

From the Ekeland variational principle, we know that $\inf_{\mathcal{M}} E$ can be achieved at some point $(\tilde{u}, \tilde{v}) \neq 0$, which is a weak solution of (1.6). If $\beta < -\gamma$, $E(\tilde{u}, \tilde{v}) < E(\mathbf{u}_0)$. If $\beta > \gamma$, $E(\tilde{u}, \tilde{v}) < E(\mathbf{u}_1)$. Observe that $E(\mathbf{u}_0) = E(\mathbf{u}_1)$. Now we show that $(\tilde{u}, \tilde{v})$ is nontrivial, that is to say, $\tilde{u} \neq 0$ and $\tilde{v} \neq 0$.

If $\tilde{v} = 0$, then $\tilde{u} = 0$, contradicting the fact that $(\tilde{u}, \tilde{v}) \neq 0$. If $\tilde{u} = 0$ and $\tilde{v} \neq 0$, then $E_1(\tilde{v}) = E(\tilde{u}, \tilde{v}) \leq E(0, u) = E_1(u)$ for any $u \in \mathcal{M}_1$; therefore, $\tilde{v}$ is a minimum of E_1 on $\mathcal{M}_1$. Hence $E(\tilde{u}, \tilde{v}) = E(\mathbf{u}_0) = E(\mathbf{u}_1)$, which is also a contradiction.

4. Proof of Theorem 1.2

First, we prove several properties of the Nehari manifold $\mathcal{N}_\varepsilon$.

Lemma 4.1. *The Nehari manifold $\mathcal{N}_\varepsilon$ is non-empty.*

Proof. $\forall (u, v) \in H_\varepsilon \setminus \{(0, 0)\}$, let us define

$$f(t) := J_\varepsilon(tu, tv) = \frac{t^2}{2}\|(u, v)\|^2_{H_\varepsilon} - \frac{t^4}{4}\int_V \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 d\mu, \quad t > 0.$$

Then

$$f'(t) = t\|(u, v)\|^2_{H_\varepsilon} - t^3\int_V \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 d\mu.$$

If $f'(t) = 0$, then

$$t = \left[\frac{\|(u, v)\|^2_{H_\varepsilon}}{\int_V \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 d\mu}\right]^{\frac{1}{2}}.$$

Denote $\tilde{F}(u, v) := \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2$, then

$$\tilde{F}(u, v) = \begin{pmatrix} u^2 & v^2 \end{pmatrix}\begin{pmatrix} \mu_1 & \beta \\ \beta & \mu_2 \end{pmatrix}\begin{pmatrix} u^2 \\ v^2 \end{pmatrix}$$

is quadratic about u^2, v^2. The $\tilde{F}$ is positive is equivalent to $\begin{pmatrix} \mu_1 & \beta \\ \beta & \mu_2 \end{pmatrix}$ being a positive definite matrix, and it is also equivalent to $\mu_1 > 0$, $\mu_1\mu_2 - \beta^2 > 0$, which gives $|\beta| < \sqrt{\mu_1\mu_2}$, then $\int_V \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 d\mu > 0$. Hence there exists a positive constant t satisfying $f'(t) = 0$, that is, $\langle J'_\varepsilon(tu, tv), (tu, tv)\rangle = 0$, which implies that $(tu, tv) \in \mathcal{N}_\varepsilon$.

Lemma 4.2. *The infimum of the problem $m_\varepsilon = \inf_{(u,v)\in\mathcal{N}_\varepsilon} J_\varepsilon(u, v)$ is positive.*

Proof. If $(u, v) \in \mathcal{N}_\varepsilon$, we have $\|(u, v)\|^2_{H_\varepsilon} = \int_V \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 d\mu$, so

$$J_\varepsilon(u, v) = \frac{1}{4}\|(u, v)\|^2_{H_\varepsilon}. \tag{4.1}$$

By Proposition 2.1, we have

$$\int_V \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2 d\mu \le C\|(u, v)\|^4_{H_\varepsilon},$$

where C is a positive constant of the embedding $H_\varepsilon \hookrightarrow L^4(V, \mathbb{R}^2)$. We then get

$$\|(u, v)\|^2_{H_\varepsilon} \ge \frac{1}{C} > 0.$$

Therefore,

$$m_\varepsilon = \inf_{(u,v)\in\mathcal{N}_\varepsilon} J_\varepsilon(u, v) = \frac{1}{4}\inf_{(u,v)\in\mathcal{N}_\varepsilon}\|(u, v)\|^2_{H_\varepsilon} \ge \frac{1}{4C} > 0.$$

This completes the proof.

Lemma 4.3. *There exists some* $(u_\varepsilon, v_\varepsilon) \in \mathcal{N}_\varepsilon$ *such that* $J_\varepsilon(u_\varepsilon, v_\varepsilon) = m_\varepsilon$.

Proof. Consider a sequence of vectors $\{(u_k, v_k)\}_k^\infty \in \mathcal{N}_\varepsilon$ satisfying $\lim_{k\to\infty} J_\varepsilon(u_k, v_k) = m_\varepsilon$. From (4.1) we get $\|(u_k, v_k)\|_{H_\varepsilon}^2 = 4J(u_k, v_k) \to 4m_\varepsilon$ as $k \to \infty$; therefore, $\{(u_k, v_k)\}_k^\infty$ is a bounded sequence in H_ε. From Proposition 2.1, there exists some vector $(u_\varepsilon, v_\varepsilon) \in H_\varepsilon$ such that, as $k \to \infty$,

$$\begin{cases} (u_k, v_k) \rightharpoonup (u_\varepsilon, v_\varepsilon) & in \quad H_\varepsilon, \\ (u_k, v_k) \to (u_\varepsilon, v_\varepsilon) & \forall \quad x \in V, \\ (u_k, v_k) \to (u_\varepsilon, v_\varepsilon) & in \quad L^4(V, \mathbb{R}^2). \end{cases} \tag{4.2}$$

From (4.2) we get

$$\lim_{k\to\infty} \int_V \mu_1 u_k^4 + \mu_2 v_k^4 + 2\beta u_k^2 v_k^2 d\mu = \int_V \mu_1 u_\varepsilon^4 + \mu_2 v_\varepsilon^4 + 2\beta u_\varepsilon^2 v_\varepsilon^2 d\mu. \tag{4.3}$$

Because the norm of the Hilbert space H_ε is weak lower semi-continuity, we have

$$\|(u_\varepsilon, v_\varepsilon)\|_{H_\varepsilon}^2 \leq \liminf_{k\to\infty} \|(u_k, v_k)\|_{H_\varepsilon}^2. \tag{4.4}$$

From (4.3) and (4.4) we get

$$\begin{aligned} J_\varepsilon(u_\varepsilon, v_\varepsilon) &= \frac{1}{2}\|(u_\varepsilon, v_\varepsilon)\|_{H_\varepsilon}^2 - \frac{1}{4}\int_V \mu_1 u_\varepsilon^4 + \mu_2 v_\varepsilon^4 + 2\beta u_\varepsilon^2 v_\varepsilon^2 d\mu \\ &\leq \liminf_{k\to\infty} \left[\frac{1}{2}\|(u_k, v_k)\|_{H_\varepsilon}^2 - \frac{1}{4}\int_V \mu_1 u_k^4 + \mu_2 v_k^4 + 2\beta u_k^2 v_k^2 d\mu\right] \\ &= \liminf_{k\to\infty} J_\varepsilon(u_k, v_k) \\ &= m_\varepsilon. \end{aligned} \tag{4.5}$$

By taking a subsequence, we can assume $\|(u_k, v_k)\|_{H_\varepsilon}^2 \to C$ for suitable positive constant C as $k \to \infty$. Because $(u_k, v_k) \in \mathcal{N}_\varepsilon$, we have

$$\|(u_k, v_k)\|_{H_\varepsilon}^2 = \int_V \mu_1 u_k^4 + \mu_2 v_k^4 + 2\beta u_k^2 v_k^2 d\mu = C \quad as \quad k \to \infty.$$

Combining (4.3) and (4.4) we get

$$\begin{aligned} \|(u_\varepsilon, v_\varepsilon)\|_{H_\varepsilon}^2 &\leq \liminf_{k\to\infty} \|(u_k, v_k)\|_{H_\varepsilon}^2 \\ &= \liminf_{k\to\infty} \int_V \mu_1 u_k^4 + \mu_2 v_k^4 + 2\beta u_k^2 v_k^2 d\mu \\ &= \int_V \mu_1 u_\varepsilon^4 + \mu_2 v_\varepsilon^4 + 2\beta u_\varepsilon^2 v_\varepsilon^2 d\mu. \end{aligned}$$

If $\|(u_\varepsilon, v_\varepsilon)\|_{H_\varepsilon}^2 < \int_V \mu_1 u_\varepsilon^4 + \mu_2 v_\varepsilon^4 + 2\beta u_\varepsilon^2 v_\varepsilon^2 d\mu$, similar arguments as in Lemma 4.1 show that there exists some constant

$$t = \left[\frac{\|(u_\varepsilon, v_\varepsilon)\|_{H_\varepsilon}^2}{\int_V \mu_1 u_\varepsilon^4 + \mu_2 v_\varepsilon^4 + 2\beta u_\varepsilon^2 v_\varepsilon^2 d\mu}\right]^{\frac{1}{2}} \in (0, 1)$$

such that $(tu_\varepsilon, tv_\varepsilon) \in \mathcal{N}_\varepsilon$, combined with (4.4), we get

$$\begin{aligned} 0 < m_\varepsilon \le J_\varepsilon(tu_\varepsilon, tv_\varepsilon) &= \frac{1}{4}\|(tu_\varepsilon, tv_\varepsilon)\|^2_{H_\varepsilon} = \frac{t^2}{4}\|(u_\varepsilon, v_\varepsilon)\|^2_{H_\varepsilon} \\ &\le t^2 \liminf_{k\to\infty} \frac{1}{4}\|(u_k, v_k)\|^2_{H_\varepsilon} \\ &= t^2 \liminf_{k\to\infty} J_\varepsilon(u_k, v_k) \\ &= t^2 m_\varepsilon \\ &< m_\varepsilon. \end{aligned}$$

This result is contradictory, so we have $\|(u_\varepsilon, v_\varepsilon)\|^2_{H_\varepsilon} = \int_V \mu_1 u_\varepsilon^4 + \mu_2 v_\varepsilon^4 + 2\beta u_\varepsilon^2 v_\varepsilon^2 d\mu$, which means that $(u_\varepsilon, v_\varepsilon) \in \mathcal{N}_\varepsilon$. It is easy to get $J_\varepsilon(u_\varepsilon, v_\varepsilon) \ge m_\varepsilon$, and combining with (4.5), we get $J_\varepsilon(u_\varepsilon, v_\varepsilon) = m_\varepsilon$. Thus we finish the proof.

Lemma 4.4. *The minimizer $(u_\varepsilon, v_\varepsilon) \in \mathcal{N}_\varepsilon$ obtained in Lemma 4.3 is a ground state solution to the system (1.8).*

Proof. We are going to prove that for all $(\varphi, \phi) \in C_c(V)$, it holds that $\langle J'_\varepsilon(u_\varepsilon, v_\varepsilon), (\varphi, \phi)\rangle = 0$. We choose $\varepsilon_0 > 0$ satisfying $(u_s, v_s) := (u_\varepsilon + s\varphi, v_\varepsilon + s\phi) \not\equiv (0, 0)$ if $s \in (-\varepsilon_0, \varepsilon_0)$. For $s \in (-\varepsilon_0, \varepsilon_0)$, there exists a positive constant $t(s)$ depending on s satisfying $(t(s)u_s, t(s)v_s) \in \mathcal{N}_\varepsilon$. We can take

$$t(s) = \left[\frac{\|(u_\varepsilon + s\varphi, v_\varepsilon + s\phi)\|^2_{H_\varepsilon}}{\int_V \mu_1(u_\varepsilon + s\varphi)^4 + \mu_2(v_\varepsilon + s\phi)^4 + 2\beta(u_\varepsilon + s\varphi)^2(v_\varepsilon + s\phi)^2 d\mu}\right]^{\frac{1}{2}} > 0.$$

Particularly, one has $t(0) = 1$. Let us define a function $\gamma(s) : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ by

$$\gamma(s) := J_\varepsilon(t(s)u_s, t(s)v_s).$$

Because $(t(s)u_s, t(s)v_s) \in \mathcal{N}_\varepsilon$ and $J_\varepsilon(u_\varepsilon, v_\varepsilon) = \inf_{(u,v)\in\mathcal{N}_\varepsilon} J_\varepsilon(u, v)$, we find that $\gamma(s)$ has a minimum at the point $s = 0$. Therefore we have

$$\begin{aligned} 0 = \gamma'(0) &= \langle J'_\varepsilon(t(0)u_\varepsilon, t(0)v_\varepsilon), t'(0)(u_\varepsilon, v_\varepsilon) + t(0)(\varphi, \phi)\rangle \\ &= \langle J'_\varepsilon(u_\varepsilon, v_\varepsilon), t'(0)(u_\varepsilon, v_\varepsilon) + (\varphi, \phi)\rangle \\ &= \langle J'_\varepsilon(u_\varepsilon, v_\varepsilon), (\varphi, \phi)\rangle. \end{aligned}$$

In the last equality, we use the fact $(u_\varepsilon, v_\varepsilon) \in \mathcal{N}_\varepsilon$ and $\langle J'_\varepsilon(u_\varepsilon, v_\varepsilon), (u_\varepsilon, v_\varepsilon)\rangle = 0$, which concludes the proof.

Now we prove Theorem 1.2.

Proof. By the arguments above, for any $|\beta| < \sqrt{\mu_1\mu_2}$, $(u_\varepsilon, v_\varepsilon)$ obtained in Lemma 4.3 is a critical point of the energy functional $J_\varepsilon(u, v)$ when $\varepsilon > 0$; thus $(u_\varepsilon, v_\varepsilon)$ is a ground state solution of system (1.8).

5. Proof of Theorem 1.3

Let us consider the corresponding energy functional $J \in C^2(H, \mathbb{R})$ for problem (1.11)

$$J(u,v) := \frac{1}{2}\int_V |\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + a(x)v^2 d\mu \\ - \frac{1}{4}\int_V (u^+)^4 + (v^+)^4 + 2\beta u^2 v^2 d\mu.$$

Here, $u^+ = \max\{u, 0\}$ is the positive part of u and $u^- = -\min\{u, 0\}$ is the negative part of u. We are interested in nontrivial critical points of the energy functional J, which means that $u \neq 0$ and $v \neq 0$, which is opposed to the semi-trivial critical points with form $(u, 0)$ or $(0, v)$. We have

Lemma 5.1. *Every nontrivial critical point $(u, v) \in H$ of the energy functional J is a classical solution to the problem (1.11).*

Proof. Recall that a critical point $(u, v) \in H$ is a weak solution to the following system:

$$\begin{cases} -\Delta u + (a(x) - \beta v^2)u = (u^+)^3 \text{ on } V, \\ -\Delta v + (a(x) - \beta u^2)v = (v^+)^3 \text{ on } V. \end{cases}$$

Testing the above system with negative parts u^- and v^-, respectively, and taking integration, we get

$$\int_V |\nabla u^-|^2 d\mu + \int_V (a(x) - \beta v^2)|u^-|^2 d\mu = 0 = \int_V |\nabla v^-|^2 d\mu + \int_V (a(x) - \beta u^2)|v^-|^2 d\mu.$$

Because $\beta < 0$, we conclude that $u \geq 0$ and $v \geq 0$ on V. Because $(u, v) \in H$ is nontrivial, we conclude that $u > 0$ and $v > 0$. Indeed, if $u(x_0) = 0$ for some x_0, then we have $u(x) = 0$ for all x satisfying $x \sim x_0$. Because x_0 is arbitrary, $u \equiv 0$, which is a contradiction.

We define the modified Nehari manifold for the problem (1.11) as follows:

$$\mathcal{N} := \{(u,v) \in H \mid u \neq 0, v \neq 0, \int_V |\nabla u|^2 d\mu + \int_V a(x)u^2 d\mu - \beta \int_V u^2 v^2 d\mu = \int_V |u^+|^4 d\mu, \\ \int_V |\nabla v|^2 d\mu + \int_V a(x)v^2 d\mu - \beta \int_V u^2 v^2 d\mu = \int_V |v^+|^4 d\mu\} \tag{5.1}$$

Clearly, nontrivial critical points of the energy functional J are contained in the modified Nehari manifold $\mathcal{N}$. Observe that in the case of $a(x) = b(x)$, the space H can be decomposed as $H = \mathcal{H} \times \mathcal{H}$, with $\mathcal{H}$ defined in (1.4) and its norm in (1.5). We will denote $\|\cdot\|_{\mathcal{H}}$ as $\|\cdot\|$ for brevity. We need some properties about the modified Nehari manifold $\mathcal{N}$ summarized as follows.

Lemma 5.2. *(1) The modified Nehari manifold $\mathcal{N}$ is a C^2-submanifold of the Hilbert space H with co-dimension two.*

(2) If (u, v) is a critical point of $J_{\mathcal{N}}$, which is the restricted functional of J to the modified Nehari manifold $\mathcal{N}$, then it is a nontrivial critical point of the energy functional J.

(3) For any $(u, v) \in \mathcal{N}$, we have $J(u, v) = \frac{1}{4}(\|u\|^2 + \|v\|^2)$.

(4) The restricted functional $J_{\mathcal{N}} : \mathcal{N} \to \mathbb{R}$ satisfies the PS condition.

Proof. (1) From the Sobolev embedding relation $H \hookrightarrow L^4(V, \mathbb{R}^2)$, we have

$$C\|u\|^4 \geq \|u\|^4_{L^4(V)} \geq \|u\|^2, \quad C\|v\|^4 \geq \|v\|^4_{L^4(V)} \geq \|v\|^2$$

with a universal constant $C > 0$ for $(u, v) \in \mathcal{N}$; hence we have

$$\|u\| \geq C^{-1/2} \text{ and } \|v\| \geq C^{-1/2} \text{ for } (u, v) \in \mathcal{N}. \tag{5.2}$$

Furthermore, $\mathcal{N} = \{(u, v) \in H : u \neq 0, v \neq 0, \mathcal{F}(u, v) = (0, 0)^T\}$, where $\mathcal{F} \in C^2(H, \mathbb{R}^2)$ is defined as

$$\mathcal{F}(u, v) = \begin{pmatrix} \mathcal{F}_1(u, v) \\ \mathcal{F}_2(u, v) \end{pmatrix} = \begin{pmatrix} \int_V |\nabla u|^2 d\mu + \int_V a(x)u^2 d\mu - \beta \int_V u^2v^2 d\mu - \int_V |u^+|^4 d\mu \\ \int_V |\nabla v|^2 d\mu + \int_V a(x)v^2 d\mu - \beta \int_V u^2v^2 d\mu - \int_V |v^+|^4 d\mu \end{pmatrix}.$$

Observe that if $(u, v) \in \mathcal{N}$, it holds that

$$\partial_u \mathcal{F}_1(u, v)u = -2 \int_V |u^+|^4 d\mu \neq 0$$

and

$$\partial_v \mathcal{F}_2(u, v)v = -2 \int_V |v^+|^4 d\mu \neq 0,$$

where as $\partial_u \mathcal{F}_2(u, v)u = \partial_v \mathcal{F}_1(u, v)v = -2\beta \int_V u^2v^2 d\mu$. Consequently,

$$T_{u,v} := \begin{pmatrix} \partial_u \mathcal{F}_1(u, v)u & \partial_u \mathcal{F}_2(u, v)u \\ \partial_v \mathcal{F}_1(u, v)v & \partial_v \mathcal{F}_2(u, v)v \end{pmatrix} = \begin{pmatrix} -2 \int_V |u^+|^4 d\mu & -2\beta \int_V u^2v^2 d\mu \\ -2\beta \int_V u^2v^2 d\mu & -2 \int_V |v^+|^4 d\mu \end{pmatrix}.$$

Because $(u, v) \in \mathcal{N}$, $\int_V |u^+|^4 d\mu > -\beta \int_V u^2v^2 d\mu \geq 0$ and $\int_V |v^+|^4 d\mu > -\beta \int_V u^2v^2 d\mu \geq 0$, from which we know that the matrix $T_{u,v}$ is negative definite. Therefore $\mathcal{F}'(u, v)(u, 0)$ and $\mathcal{F}'(u, v)(0, v)$ are linearly independent vectors in $\mathbb{R}^2$; thus $\mathcal{F}'(u, v) : H \to \mathbb{R}^2$ is an onto map. From implicit function theorem we know that $\mathcal{N}$ is a co-dimension two C^2-submanifold in the Hilbert space H.

(2) Supposing $(u, v) \in \mathcal{N}$ is a critical point of the restricted functional $J_{\mathcal{N}}$, then there exist two Lagrangian multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfying

$$\lambda_1 \mathcal{F}_1'(u, v) + \lambda_2 \mathcal{F}_2'(u, v) = J'(u, v).$$

Testing this equation against $(u, 0)$, $(0, v)$, respectively, gives

$$T_{u,v} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because $T_{u,v}$ is a negative definite matrix, we have $\lambda_1 = \lambda_2 = 0$, so $J'(u, v) = 0$.

(3) Take $(u, v) \in \mathcal{N}$, which holds that

$$
\begin{aligned}
J(u,v) &= \frac{1}{2}\int_V |\nabla u|^2 d\mu + \frac{1}{2}\int_V a(x)u^2 d\mu \\
&\quad - \frac{1}{4}\left(\int_V |\nabla u|^2 d\mu + \int_V a(x)u^2 d\mu - \beta\int_V u^2v^2 d\mu\right) \\
&\quad + \frac{1}{2}\int_V |\nabla v|^2 d\mu + \frac{1}{2}\int_V a(x)v^2 d\mu \\
&\quad - \frac{1}{4}\left(\int_V |\nabla v|^2 d\mu + \int_V a(x)v^2 d\mu - \beta\int_V u^2v^2 d\mu\right) \\
&\quad - \frac{\beta}{2}\int_V u^2v^2 d\mu \\
&= \frac{1}{4}\left(\|u\|^2 + \|v\|^2\right).
\end{aligned}
$$

(4) Let us take a Palais-Smale (PS) sequence $\{(u_k, v_k)\}_{k=1}^{\infty} \subset \mathcal{N}$ to the restricted energy functional $J_{\mathcal{N}}$. Then the sequence $\{(u_k, v_k)\}_{k=1}^{\infty}$ is bounded in the Hilbert space H. Therefore up to a subsequence, one can assume $(u_k, v_k) \rightharpoonup (u, v) \in H$, $u_k \to u$, $v_k \to v$ in the $L^4(V)$ space. We then have

$$u^+ \neq 0, \quad v^+ \neq 0.$$

Indeed, if $u^+ = 0$, then

$$\lim_{k\to\infty} \|u_k^+\|_{L^4(V)} \to 0 \quad \text{and} \quad \limsup_{k\to\infty} \beta\int_V u_k^2 v_k^2 d\mu \leq 0,$$

and thus $\|u_k\| \to 0$ because $(u_k, v_k) \in \mathcal{N}$, which gives a contradiction to the estimates in (5.2). Similar arguments shows that $v^+ \neq 0$.

Observe that

$$o(1) = J'_{\mathcal{N}}(u_k, v_k) = J'(u_k, v_k) - \lambda_1^k \mathcal{F}_1'(u_k, v_k) - \lambda_2^k \mathcal{F}_2'(u_k, v_k) \quad \text{when} \quad k \to \infty$$

for appropriate sequences $\{\lambda_1^k\}_k, \{\lambda_2^k\}_k \subset \mathbb{R}$, the functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined in step (2). Then we have

$$
\begin{aligned}
o(1) &= \begin{pmatrix} J'(u_k, v_k)(u_k, 0) - [\lambda_1^k \mathcal{F}_1'(u_k, v_k) + \lambda_2^k \mathcal{F}_2'(u_k, v_k)](u_k, 0) \\ J'(u_k, v_k)(0, v_k) - [\lambda_1^k \mathcal{F}_1'(u_k, v_k) + \lambda_2^k \mathcal{F}_2'(u_k, v_k)](0, v_k) \end{pmatrix} \\
&= -\begin{pmatrix} [\lambda_1^k \mathcal{F}_1'(u_k, v_k) + \lambda_2^k \mathcal{F}_2'(u_k, v_k)](u_k, 0) \\ [\lambda_1^k \mathcal{F}_1'(u_k, v_k) + \lambda_2^k \mathcal{F}_2'(u_k, v_k)](0, v_k) \end{pmatrix} = -T_{u_k, v_k} \begin{pmatrix} \lambda_1^k \\ \lambda_2^k \end{pmatrix} \\
&= (-T_{u,v} + o(1)) \begin{pmatrix} \lambda_1^k \\ \lambda_2^k \end{pmatrix}.
\end{aligned}
$$

Because we have $(u_k, v_k) \in \mathcal{N}$, the weak convergence implies

$$\|u\|^2 - \beta\int_V u^2v^2 d\mu \leq \int_V |u^+|^4 d\mu \quad \text{and} \quad \|v\|^2 - \beta\int_V u^2v^2 d\mu \leq \int_V |v^+|^4 d\mu.$$

Therefore $T_{u,v}$ is a negative definite matrix, so we have $\lambda_1^k, \lambda_2^k \to 0$. Because the functionals $\mathcal{F}_1'(u_k, v_k)$ and $\mathcal{F}_2'(u_k, v_k)$ are still bounded, we then have $J'(u_k, v_k) \to 0$ strongly. Thus (u, v) is weak solution to the system

$$\begin{cases} -\Delta u + a(x)u = (u^+)^3 + \beta v^2 u \quad \text{on} \quad V, \\ -\Delta v + a(x)v = (v^+)^3 + \beta u^2 v \quad \text{on} \quad V. \end{cases}$$

Then we have

$$\|u\|^2 = |u^+|_4^4 + \beta \int_M v^2 u^2 d\mu_g = \lim_{k\to\infty} \left(|u_k^+|_4^4 + \beta \int_V u_k^2 v_k^2 d\mu \right) = \lim_{k\to\infty} \|u_k\|^2.$$

From this we know that $u_k \to u$ strongly in the Hilbert space $\mathcal{H}$. Similar arguments show that $v_k \to v$ strongly in the Hilbert space $\mathcal{H}$, hence (u_k, v_k) converges strongly to (u, v) in the Hilbert space H.

To prove Theorem 1.3, for any $c \in \mathbb{R}$, let us consider the sub-level sets defined by $\mathcal{N}^c := \{(u, v) \in \mathcal{N} : J(u, v) \le c\}$ and level sets defined by

$$K_c := \{(u, v) \in \mathcal{N} \mid J(u, v) = c, J'(u, v) = 0\} = \{(u, v) \in \mathcal{N} : J_{\mathcal{N}}(u, v) = c, J_{\mathcal{N}}'(u, v) = 0\}.$$

Observe that the functional J, the modified Nehari manifold $\mathcal{N}$ and the sets $\mathcal{N}^c$, K_c are invariant under the involution map

$$\sigma : H \to H, \quad (u, v) \to \sigma(u, v) = (v, u).$$

Define

$$c(\beta) := \inf\{J(u, v) \mid (u, v) \in \mathcal{N} \text{ is a fixed point of the involution map } \sigma\}.$$

Then we have

Lemma 5.3. *When $\beta \le -1$, we have $c(\beta) = \infty$; furthermore, it holds that* $\lim_{\beta\to-1,\beta>-1} c(\beta) = \infty$.

Proof. When $\beta < -1$, by the definition of the modified Nehari manifold $\mathcal{N}$, we know that the involution map σ does not possess fixed points; therefore, $c(\beta) = \infty$. If $-1 < \beta < 0$, $(u, u) \in \mathcal{N}$ with $u \in \mathcal{H}$, we have

$$\|u\|^2 = \|u^+\|_{L^4(V)}^4 + \beta \|u\|_{L^4(V)}^4 \le (1 + \beta)\|u\|_{L^4(V)}^4 \le C(1 + \beta)\|u\|^4,$$

where $C > 0$ is independent of β since we have the Sobolev embedding $\mathcal{H} \hookrightarrow L^4(V)$. Thus $\|u\|^2 \ge \frac{1}{C(1+\beta)}$ and $J(u, u) \ge \frac{1}{2C(1+\beta)}$. Because $\lim_{\beta\to-1} \frac{1}{2C(1+\beta)} = \infty$, we have $\lim_{\beta\to-1,\beta>-1} c(\beta) = \infty$.

Because the restricted functional $J_{\mathcal{N}} : \mathcal{N} \to \mathbb{R}$ satisfies the PS condition, standard arguments give us the following equivariant version of the deformation lemma as in [49], so we omit the proof here.

Lemma 5.4. *Suppose $c \in \mathbb{R}$, $N \subset \mathcal{N}$ is a σ-invariant relative open neighborhood of K_c. There exist $\varepsilon > 0$ and a C^1 map $\eta : [0, 1] \times \mathcal{N}^{c+\varepsilon} \setminus N \to \mathcal{N}^{c+\varepsilon}$ such that, for any $s \in [0, 1]$ and $(u, v) \in \mathcal{N}^{c+\varepsilon} \setminus N$, one has*

$$\eta(0, (u, v)) = (u, v), \quad \eta(1, (u, v)) \in \mathcal{N}^{c-\varepsilon} \textit{ as well as } \sigma[\eta(s, (u, v))] = \eta(s, \sigma(u, v)).$$

For a σ-invariant closed subset $A \subset \mathcal{N}$, let us define the genus $\gamma(A)$ to be the least $n \in \mathbb{N} \cup \{0\}$ such that there is a continuous map $h : A \to \mathbb{R}^N \setminus \{0\}$ satisfying $h(\sigma(u, v)) = -h(u, v)$ for any $(u, v) \in A$. If such map h does not exist, then we define $\gamma(A) = \infty$. Particularly, if the set A has a fixed point to the involution map σ, then $\gamma(A) = \infty$. It also holds that $\gamma(\emptyset) = 0$.

Lemma 5.5.

(1) If σ-invariant closed subsets A and B satisfy $A \subset B$, then it hold that $\gamma(A) \leq \gamma(B)$.

(2) For σ-invariant closed subsets A and B, it holds that $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

(3) For a continuous and σ-equivalent map $h : A \to \mathcal{N}$, it holds that $\gamma(A) \leq \gamma(\overline{h(A)})$.
For a σ-invariant closed set A which does not contain fixed points of the involution map σ, one has the following:

(4) If the genus of A satisfies $\gamma(A) > 1$, then the cardinality $\sharp A = \infty$.

(5) If the set A is compact, then the genus satisfies $\gamma(A) < \infty$; furthermore, there is a σ-invariant relatively open neighborhood N of A in $\mathcal{N}$ satisfying $\gamma(A) = \gamma(\bar{N})$.

(6) If S be the boundary of a symmetrical bounded neighborhood of zero in a normed vector space with dimension k, $\psi : S \to \mathcal{N}$ is a continuous map satisfying the condition $\psi(-u) = \sigma(\psi(u))$, then it holds that $\gamma(\psi(S)) \geq k$.

Proof. We prove (4): for a σ-invariant finite subset $A \subset \mathcal{N}$ which does not contain fixed point of the involution map σ, we can write it as follows:

$$A = \{(u_1, v_1), \cdots, (u_n, v_n), \sigma(u_1, v_1), \cdots, \sigma(u_n, v_n)\}.$$

Define a continuous map $h : A \to \mathbb{R} \setminus \{0\}$ by

$$h(u_i, v_i) = -1, \quad h(\sigma(u_i, v_i)) = -1, \quad i = 1, \cdots, n,$$

which shows that the genus $\gamma(A)$ satisfies $\gamma(A) \leq 1$.

To prove (6), assume by contradiction that there is a continuous map $h : \psi(S) \to \mathbb{R}^{k-1} \setminus \{0\}$ satisfying $h(\sigma(u, v)) = -h(u, v)$. From this we know that $h \circ \psi : S \to \mathbb{R}^{k-1} \setminus \{0\}$ is a continuous and odd map, which gives a contradiction to the Borsuk-Ulam theorem. The other properties are clear.

Lemma 5.6. *If the constant c satisfies $c < c(\beta)$, then we have $\gamma(K_c) < \infty$; moreover, there is $\varepsilon > 0$ satisfying*

$$\gamma(\mathcal{N}^{c+\varepsilon}) \leq \gamma(\mathcal{N}^{c-\varepsilon}) + \gamma(K_c).$$

Proof. Because the restricted energy functional $J_{\mathcal{N}}$ satisfies the PS condition, K_c is a compact set. From the definition of the constant $c(\beta)$, we deduce that K_c does not contain any fixed point of the involution map σ. Therefore $\gamma(K_c) < \infty$, and there is a σ-invariant relative open neighborhood $N \subset \mathcal{N}$ of K_c in the modified Nehari manifold $\mathcal{N}$ satisfying $\gamma(\bar{N}) = \gamma(K_c)$. Let $\varepsilon > 0$ and $\eta : [0, 1] \times \mathcal{N}^{c+\varepsilon} \setminus N \to \mathcal{N}^{c+\varepsilon}$ be the deformation map given by Lemma 5.4. Set $\eta_1 := \eta(1, \cdot) : \mathcal{N}^{c+\varepsilon} \setminus N \to \mathcal{N}^{c-\varepsilon}$. Because η_1 is σ-equivalent, then we have $\gamma(\mathcal{N}^{c+\varepsilon} \setminus N) \leq \gamma(\mathcal{N}^{c-\varepsilon})$ and therefore

$$\gamma(\mathcal{N}^{c+\varepsilon}) \leq \gamma(\mathcal{N}^{c+\varepsilon} \setminus N) + \gamma(\bar{N}) \leq \gamma(\mathcal{N}^{c-\varepsilon}) + \gamma(K_c).$$

We define $c_k := \inf\{c \in \mathbb{R} : \gamma(\mathcal{N}^c) \geq k\}$, $k \in \mathbb{N}$, which are the Lusternik-Schnirelmann-type level sets corresponding to the genus γ. Clearly, $\{c_k\}_{k=1}^{\infty}$ is a non-decreasing sequence. Then the following holds.

Lemma 5.7. *(1) The sequence $\{c_k\}_{k=1}^{\infty}$ is uniformly bounded with an upper bound independently of the constant β.*

(2) As $k \to \infty$, we have $c_k \to \bar{c}$, and the limit $\bar{c}$ satisfies the estimate $c(\beta) \leq \bar{c} \leq \infty$.

(3) If for some $l \geq k$ we have $c := c_k = c_{k+1} = \cdots = c_l < c(\beta)$, it then holds that $\gamma(K_c) \geq l - k + 1$.

(4) For $c_k < c(\beta)$, $K_{c_k} \neq \emptyset$; furthermore, $\mathcal{N}^{c_k}$ has at leaset k pairs of critical points (u, v), (v, u) to the energy functional J.

Proof. (1) We take $W \subset H$ to be a k-dimensional subspace of H which consists of functions $u \in H$ satisfying the property $\int_V u d\mu = 0$. If we define $S := \{u \in W : \|u\| = 1\}$ to be the unit sphere of W, then for every $u \in S$ it holds that $u^+ \neq 0$ and $u^- \neq 0$. Let us consider the map $\psi : S \to \mathcal{N}$ defined as

$$\psi(u) := \left(\left(\frac{\|u^+\|^2}{\|u^+\|^4} \right)^{1/2} u^+, \left(\frac{\|u^-\|^2}{\|u^-\|^4} \right)^{1/2} u^- \right).$$

Clearly for every $u \in S$, $\psi(-u) = \sigma(\psi(u))$ and it is a continuous map. Therefore we have $\gamma(\psi(S)) \geq k$ and $c_k \leq \sup_{u \in S} J(\psi(u)) < \infty$. $\sup_{u \in S} J(\psi(u))$ is independent of the parameter β, so the claim follows.

(2) By contradiction we assume that as $k \to \infty$ it holds that $c_k \to \bar{c} < c(\beta)$. For the constant $c = \bar{c}$, let $\varepsilon > 0$ be the constant given in Lemma 5.6. We find that $\bar{c} - \varepsilon < c_k$ when k is large enough, hence $\gamma(\mathcal{N}^{\bar{c}-\varepsilon})$ is a finite number. Therefore we have the estimates $\gamma(\mathcal{N}^{\bar{c}+\varepsilon}) \leq \gamma(\mathcal{N}^{\bar{c}-\varepsilon}) + \gamma(K_{\bar{c}}) < \infty$. This gives a contradiction to the fact $\bar{c} \geq c_k$ for any k.

(3) By assumption and the definition of the Lusternik-Schnirelmann values c_k, for every $\varepsilon > 0$ it holds that $\gamma(\mathcal{N}^{c-\varepsilon}) \leq k - 1$, $\gamma(\mathcal{N}^{c+\varepsilon}) \geq l$; therefore we have the estimate $\gamma(K_c) \geq l - k + 1$.

(4) For $c_k < c(\beta)$, property (3) implies that $\gamma(K_{c_k}) \geq 1$; hence the σ-invariant set K_{c_k} is nonempty. For $c_1 < c_2 < \cdots < c_k$, we conclude that there are at least k pairs of critical points to the energy functional J in $\mathcal{N}^{c_k}$. If for some $i < k$ and $j > i$, $c_i = c_j$, then by property (3) we have $\gamma(K_{c_i}) > 1$; hence the set K_{c_i} contains infinitely many elements, corresponding to the fact that $\mathcal{N}^{c_k}$ has infinitely many pairs of critical points to the energy functional J.

Proof of Theorem 1.3. (i) For every k, let us choose $(u_k, v_k) \in K_{c_k}$, which gives a sequence of nontrivial critical points to the energy functional J with $J(u_k, v_k) \to \infty$. We therefore have $\|u_k\|^2 + \|v_k\|^2 \to \infty$.

(ii) Given a positive integer k, there is $\beta_k > -1$ satisfying $c_k < c(\beta)$ for $\beta < \beta_k$. Hence there exist at least k pairs of nontrivial critical points to the energy functional J, and therefore (1.11) has at least k pairs of solutions (u, v), (v, u).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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