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Research article

Turing instability and bifurcation in a predator-prey model with nonlocal fear and nonlinear cross-diffusion

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Abstract: This paper has studied a predator-prey model that incorporates a nonlocal fear effect and nonlinear cross-diffusion under homogeneous Neumann boundary conditions. We have derived necessary and sufficient conditions for Turing instability in the presence of both nonlocal fear and nonlinear cross-diffusion by means of linear stability analysis. Moreover, we have investigated steady-state bifurcations induced by the nonlocal fear effect using the Lyapunov-Schmidt reduction.

Keywords: predator-prey model; nonlinear cross-diffusion; nonlocal fear effect; Turing instability; steady-state bifurcation

1. Introduction

Predator-prey interactions play a crucial role in maintaining ecosystem equilibrium. Although it is generally believed that predators predominantly affect prey populations through direct hunting, a 2011 study by Zanette et al. [1] on song sparrows showed that even without direct predation, the perceived predation risk could lead to a 40% reduction in their offspring. To quantify the cost of the risk, in 2016, Wang et al. [2] proposed a predator-prey model incorporating a fear factor to quantify the defense costs induced by fear perception. The model is presented as follows.

$$\begin{cases} \frac{du}{dt} = ruf(k,v) - du - au^2 - puv, \\ \frac{dv}{dt} = cpuv - mv. \end{cases} \tag{1.1}$$

Here $f(k,v) = \frac{1}{1+kv}$, and k stands for the level of fear that incites the prey to exhibit anti-predator behaviors. The results showed that the high fear levels stabilize the system by excluding periodic solutions, while low levels induce multiple limit cycles via subcritical Hopf bifurcations.

Fear of predators increases the survival probability of prey but leads to a cost of prey reproduction. Wang and Zou proposed a predator-prey model with the cost of fear and adaptive avoidance of predators. Mathematical analyses have shown that the fear effect can interplay with maturation delay between juvenile prey and adult prey in determining the long-term population dynamics [3]. And then, some experts and scholars focused on the effects of fear on some predator-prey models, such as the three-species food chain model [4], the predator-prey model incorporating the prey refuge [5], the prey-predator model of the crowding effect in predators [6], and so on. In addition, some experts and scholars have also discussed the effects of white noise and fear effects on predator-prey models [7].

Many experts and scholars also considered the influence of the spatial factor and fear on the dynamic behaviors of some prey-predator models, such as the nonexistence of nonconstant positive steady states [8] and high co-dimensional bifurcation (Hopf-Hopf bifurcation [9] and Turing-Hopf bifurcation [10]). Han et al. [11] investigated a cross-diffusive Leslie-Gower predator-prey model with the fear effect. They revealed that high fear leads to stripe patterns, low fear leads to spot patterns, and intermediate fear leads to a mix of both. In addition, pattern formations of a spatial fractional diffusive predator-prey system with refuge and fear [12] were investigated, and high-codimension bifurcation of some predator-prey systems with chemotaxis and fear effect [13, 14] was studied.

In 1989, Furter and Grinfeld [15] hypothesized that the presence of a predator at a given spatial location is determined not only by its local characteristics but also by the density of predators in adjacent regions. This consideration led to the incorporation of nonlocal interactions into the single-species population dynamics. And then they studied the local bifurcation structure by the Lyapunov-Schmidt reduction. Since then, there have been many works concerned with the dynamical behaviors of some reaction-diffusion systems with nonlocal effects. Based on the hypothesis in [15], Dong and Niu [16] introduced the nonlocal fear effect $f(k, \bar{v})$ into a predator-prey model, where $\bar{v}$ is the nonlocal term. In addition, in 2023, Sun [17] investigated the implications of the nonlocal fear effect within a diffusive predator-prey model. The results showed that the fear effect in the system can alter the stability of the constant steady state and lead to spatially nonhomogeneous steady states. Moreover, high-level fear can stabilize the model by excluding periodic solutions. Hence, the model (1.1) with the nonlocal fear effect and spatial diffusion reads as follows.

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u + \frac{ru}{1+k\bar{v}} - du - au^2 - puv, x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_2\Delta u + cpuv - m_1 v - m_2 v^2, x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, \frac{\partial v}{\partial \mathbf{n}} = 0, x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), x \in \Omega, \end{cases} \tag{1.2}$$

where $\bar{v} = \frac{1}{|\Omega|}\int_\Omega v(x, t)dx$, $u(x, t)$, and $v(x, t)$ represent the density of the prey and the predator at time t and location $x \in \Omega$, respectively. The $d_1, d_2 > 0$ are the diffusion rates. Other parameters are positive values; for their specific biological interpretations, please refer to [16]. $\Omega \in \mathbb{R}^n$ is a bounded domain with a smooth boundary and $\mathbf{n}$ being the outward unit normal vector over $\partial\Omega$. The initial data u_0, v_0 are continuous functions.

Taking into account the effects of the nonlocal fear effect, the prey species tend to avoid regions with high-predator density. This pattern can be mathematically represented by the term $\alpha\Delta(uv)$, where $\alpha < 0$ quantifies the avoidance intensity. Furthermore, the population pressure of predator species may weaken in the high density location of prey species, which can be modeled by $\Delta\frac{v}{1+\beta u}$, where β is a saturation coefficient [18]. Furthermore, recent research efforts have extended to analyzing systems with general incidence rate [19]. For instance, Li et al. [20] established criteria for exponential stability in a multi-stage epidemic system featuring a discontinuous incidence rate. Hence, as analyzed above, we study the following system.

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta[(d_1 + \alpha v)u] + \frac{ru}{1+k\bar{v}} - du - au^2 - puv, x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = \Delta[(d_2 + \frac{1}{1+\beta u})v] + cpuv - m_1 v - m_2 v^2, x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, \frac{\partial v}{\partial \mathbf{n}} = 0, x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), x \in \Omega. \end{cases} \tag{1.3}$$

Turing's pioneering work was that a system of coupled reaction-diffusion equations could be used to describe spatiotemporal patterns in biological systems in 1952 [21]. From this theory, the diffusion could destabilize an otherwise stable equilibrium point and cause spatial patterns. Such instability is often referred to as diffusion-driven instability or Turing instability. Recently, there have been many works focused on Turing instability of reaction-diffusion models by taking into account the effect of cross-diffusion and nonlocal terms [22, 23]. For example, Liu et al. [24] introduced the super cross-diffusion terms in a predator-prey system and studied the Turing instability and pattern formation in such a system with Michaelis-Menten-type predator harvesting. Liu and Guo [25] introduced the nonlinear cross-diffusion terms in a Lotka-Volterra competition model and mainly studied the nonexistence of nonconstant solutions and sufficient conditions ensuring the existence of nonconstant solutions by using Leray-Schauder degree theory. Fu et al. [26] introduced the cross-diffusion and nonlocal terms in an activator-inhibitor (depletion) model and studied the Turing instability and the existence of bifurcating solutions by using bifurcation theory. For more related works, please refer to [27] and the references therein.

The structure of this paper follows this organizational framework. Section 2 analyzes the stability conditions of positive constant steady-state solutions, examining how nonlinear diffusion influences their stability properties and triggers Turing instability. Section 3 employs the Lyapunov-Schmidt reduction to investigate steady-state bifurcation near the trivial steady-state E_2. Numerical simulations and illustrative cases are presented in Section 4 to validate the theoretical results.

2. Turing instability of the positive stationary solution

In this section, one studies the stability of constant steady-state solutions of system (1.3). Clearly, the system (1.3) has three constant steady states: (i) $E_0 = (0,0)$; (ii) $E_1 = (\frac{r_0-d}{a}, 0)(r_0 > d)$; (iii) an interior positive constant steady state $E_2 = (u^*, v^*)$ if and only if $r > d + \frac{am_1}{cp}$, where $v^* = \frac{-(kcpd+akm_1+cp^2+am_2)+\sqrt{\Delta}}{2k(cp^2+am_2)}, u^* = \frac{m_1+m_2v^*}{cp}$, and $\mathbf{\Delta} = (kcpd + akm_1 + cp^2 + am_2)^2 - 4k(cp^2 + am_2)(cpd + am_1 - r_0cp)$.

Based on the analysis of [16], one has the following lemma.

Lemma 2.1. *[16] (i) E_0 is always unstable.*
(ii) If $\frac{cp(r-d)}{a} < m_1$, then the positive constant steady-state E_1 is locally asymptotically stable. Conversely, it is unstable.
(iii) If $r > d + \frac{am_1}{cp}$, then the positive constant steady-state E_2 exists and is locally asymptotically stable.

Linearizing the system (1.3) at the constant equilibrium E_2 gives

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = D\Delta\begin{bmatrix} u \\ v \end{bmatrix} + J_U\begin{bmatrix} u \\ v \end{bmatrix} + J_{\bar{U}}\begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix}, \tag{2.1}$$

where

$$D = \begin{bmatrix} d_1 + \alpha e_2 & \alpha e_1 \\ -\frac{\beta e_2}{(1+\beta e_1)^2} & d_2 + \frac{1}{1+\beta e_1} \end{bmatrix} J_U = \begin{bmatrix} -ae_1 & -pe_1 \\ cpe_2 & -m_2 e_2 \end{bmatrix}, J_{\bar{U}} = \begin{bmatrix} 0 & -\frac{kre_1^2}{(1+ke_2)^2} \\ 0 & 0 \end{bmatrix}, (e_1, e_2) \in \mathbb{X}^2.$$

Substitute $\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} a_i \\ b_i \end{pmatrix} \varphi_i(x)$ into (2.1), where φ_i are eigenfunctions of the following eigenvalue problem

$$-\Delta u = \lambda u, x \in (0, l\pi), u'(0) = u'(l\pi) = 0.$$

The eigenvalues are $\lambda_i = \frac{i^2}{l^2}$ with corresponding eigenfunctions $\varphi_i(x) = cos(\frac{ix}{l}), i \in \mathbb{N}_0$.

The operator is defined as follows.

$$L(\alpha, i) = \begin{pmatrix} -d_{11}\lambda_i + f_u & -d_{12}\lambda_i + \bar{f}_v \\ -d_{21}\lambda_i + g_u & -d_{22}\lambda_i + g_v \end{pmatrix},$$

where

$$\bar{f}_u = \begin{cases} f_u + f_{\bar{u}}, & i = 0, \\ f_u, & i \neq 0. \end{cases}$$

Hence, when $i = 0$, one gets

$$L(\alpha, 0) \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \lambda \begin{pmatrix} a_0 \\ b_0 \end{pmatrix},$$

when $i \neq 0$, one gets

$$L(\alpha, i) \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \lambda \begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$

Taking $i = 0$, thus λ satisfies the equation

$$\lambda^2 - \lambda trL(\alpha, 0)_{(u^*,v^*)} + detL(\alpha, 0)_{(u^*,v^*)} = 0,$$

where

$$trL(\alpha, 0)_{(u^*,v^*)} = -au^* - m_2 v^*,$$
$$detL(\alpha, 0)_{(u^*,v^*)} = am_2 u^* v^* + cpu^* \left(\frac{kr(u^*)^2}{(1 + kv^*)^2} + pu^* \right).$$

Taking $i \neq 0$, thus λ satisfies the equation

$$\lambda^2 - \lambda trL(\alpha, i)_{(u^*,v^*)} + detL(\alpha, i)_{(u^*,v^*)} = 0, \tag{2.2}$$

where $trL(\alpha, i)_{(u^*,v^*)} = -(d_1 + d_2 + \alpha v^* + \frac{1}{1+\beta u^*})\lambda_i - au^* - m_2 v^*$, $detL(\alpha, i)_{(u^*,v^*)} = A\lambda_i^2 + B\lambda_i + C$, here, $A = (d_1 + \alpha v^*)(d_2 + \frac{1}{1+\beta u^*}) + \frac{\alpha\beta u^* v^*}{(1+\beta u^*)^2}$, $B = au^*(d_2 + \frac{1}{1+\beta u^*}) + m_2 v^*(d_1 + \alpha v^*) + \alpha cpu^* v^* + pu^* \frac{\beta v^*}{(1+\beta u^*)^2}$, $C = (am_2 + cp)u^* v^*$.

Remark 2.2. *If* $\alpha > 0$, $trL(\alpha, i)_{(u^*,v^*)} < 0$ *and* $detL(\alpha, i)_{(u^*,v^*)} > 0$ *always hold. Then, the characteristic equation (2.2) has two negative roots, so the constant steady state* E_2 *is always asymptotically stable. When* $\alpha < 0$, $trL(\alpha, i)_{(u^*,v^*)} < 0$ *and* $detL(\alpha, i)_{(u^*,v^*)} < 0$ *may hold, the characteristic equation (2.2) has one positive root, and then the instability may occur.*

Theorem 2.3. *Assume that* $r > d + \frac{am_1}{cp}$ *holds, if*

$$\alpha < \alpha_T \ \ and \ \ \lambda_i > \lambda_{iT}$$

hold, then the nonlinear diffusion may induce the Turing instability to the positive constant steady-state E_2 *of the system (1.3), where* $\alpha_T = -\frac{d_1[d_2(1+\beta u^*)+1](1+\beta u^*)}{v^*(1+\beta u^*)[d_2(1+\beta u^*)+1]+\beta u^* v^*}$, $\lambda_{iT} = max\{\lambda_i | A\lambda_i^2 + B\lambda_i + C = 0\}$.

Proof. Under $r > d + \frac{am_1}{cp}$, by Lemma 2.1, the unique interior equilibrium point E_2 is stable without the nonlinear diffusion.

In addition, after a simple calculation, $A = (d_1 + \alpha u^*)(d_2 + \frac{1}{1+\beta u^*}) + \frac{\alpha\beta u^* v^*}{(1+\beta u^*)^2} < 0$, then $\alpha < \alpha_T$. Here, $\alpha_T = -\frac{d_1[d_2(1+\beta u^*)+1](1+\beta u^*)}{v^*(1+\beta u^*)[d_2(1+\beta u^*)+1]+\beta u^* v^*}$.

$$\lim_{i\to\infty} detL(\alpha, i)_{(u^*,v^*)} = \lim_{i\to\infty}(A\lambda_i^2 + B\lambda_i + C) = -\infty,$$

with $A < 0, C > 0$, then $detL(\alpha, i)_{(u^*,v^*)} = 0$ has at least one positive root. Let $\lambda_{iT} = \max\{\lambda_i | A\lambda_i^2 + B\lambda_i + C = 0\}$, one can deduce that $detL(\alpha, i)_{(u^*,v^*)} < 0$ when $\lambda_i > \lambda_{iT}$ for a fixed l, which implies that the characteristic equation (2.2) has a positive root. i.e., as α decreases, the nonlinear diffusion can induce Turing instability to the trivial steady-state E_2.

Theorem 2.4. *If* $\alpha_c \le \alpha \le \bar{\alpha}_T, \bar{\alpha}_T := \min\{\alpha_T, \alpha_t\}$*, then system (1.3) undergoes Turing instability at the positive constant steady-state* E_2.

Proof. By Theorem 2.3, suppose that $A < 0$ always holds ($\alpha < \alpha_T$), then $detL(\alpha, i)_{(u^*,v^*)} = 0$ only has a positive root, which is $\lambda_i = -\frac{B+\sqrt{B^2-4AC}}{2A}$. Hence, in order to get the Turing instability, one must have $\lambda_i \ge 1$, i.e.,

$$-\frac{B + \sqrt{B^2 - 4AC}}{2A} \ge 1.$$

If $-2A \ge B$, i.e., $\alpha \le \alpha_t =: -\frac{(2d_1+au^*)(d_2+\frac{1}{1+\beta * u^*})+d_1 m_2 v^* + \frac{p\beta u^* v^*}{(1+\beta * u^*)^2}}{2v^*(d_2+\frac{1}{1+\beta * u^*})+m_2(v^*)^2+\frac{2\beta u^* v^*}{(1+\beta * u^*)^2}}$, then $A + B + C \ge 0$. Through simple calculation, one obtains that

$$\alpha \ge \alpha_c := -\frac{(d_1 + au^*)(d_2 + \frac{1}{1+\beta u^*}) + d_1 m_2 v^* + \frac{p\beta u^* v^*}{(1+\beta * u^*)^2} + (am_2 + cp)u^* v^*}{v^*(d_2 + \frac{1}{1+\beta u^*}) + \frac{\beta u^* v^*}{(1+\beta u^*)^2} + m_2 v^* v^* + cpu^* v^*}.$$

Hence, if $\alpha_c \le \alpha \le \bar{\alpha}_T$ holds, Turing instability may occur at the trivial steady-state E_2, where $\bar{\alpha}_T = \min\{\alpha_T, \alpha_t\}$.

3. Existence of non-constant steady-state solutions

In this section, we mainly aim to investigate the existence of nonconstant steady-state solutions near the positive constant steady-state E_2 under the condition $r > d + \frac{am_1}{cp}$.

$$\begin{cases} 0 = \Delta[(d_1 + \alpha v)u] + \frac{ru}{1+k\bar{v}} - du - au^2 - puv, \\ 0 = \Delta[(d_2 + \frac{1}{1+\beta u})v] + cpuv - m_1 v - m_2 v^2, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, \frac{\partial v}{\partial \mathbf{n}} = 0, x \in \partial\Omega. \end{cases} \tag{3.1}$$

The first Fréchet derivative of system (3.1) at E_2 can be given as follows:

$$R_\alpha := \begin{bmatrix} d_{11}\Delta - a_{11} & d_{12}\Delta - a_{12} \\ d_{21}\Delta + a_{21} & d_{22}\Delta - a_{22} \end{bmatrix}. \tag{3.2}$$

Moreover, the adjoint operator R_α^* of R_α reads as follows:

$$R_\alpha^* := \begin{bmatrix} d_{11}\Delta - a_{11} & d_{21}\Delta + a_{21} \\ d_{12}\Delta - a_{12} & d_{22}\Delta - a_{22} \end{bmatrix},$$

where $d_{11} = d_1 + \alpha v^*, d_{12} = \alpha u^*, d_{21} = -\frac{\beta v^*}{(1+\beta u^*)^2}, d_{22} = d_2 + \frac{1}{1+\beta u^*}, a_{11} = au^*, a_{12} = pu^* + \frac{kru^*}{(1+kv^*)^2} J_{\bar{U}} *, a_{21} = cpv^*, a_{22} = m_2 v^*, J_{\bar{U}} * u = \int_\Omega u(x,t)dx.$

To investigate the existence of nonconstant steady-state solutions near trivial steady-state E_2, one chooses α as the bifurcation parameter and gives the following hypothesis.

$$(H_{i_2}) : \alpha < \alpha_T, detL(\lambda_{i_2}, \alpha^0) = 0, detL(\lambda_s, \alpha^0) \neq 0, \forall s \in N_0 \backslash \{i_2\}.$$

From the condition (H_{i_2}) and (3.2), we have

$$\alpha^0 = -\frac{d_1 d_{22}\lambda_{i_2}^2 + (a_{12}d_{21} - a_{11}d_{22} - d_1 a_{22})\lambda_{i_2} + C}{(d_{22}v^* + d_{21}u^*)\lambda_{i_2}^2 + (a_{21}u^* - a_{22}v^*)\lambda_{i_2}}.$$

Let $K = KerR_{\alpha^0}, K^* = KerR^*_{\alpha^0}, S = span\{q_{i_2}\phi_{i_2}\}, S^* = span\{p_{i_2}\phi_{i_2}\}$

$$q_{i_2} = \begin{pmatrix} 1 \\ \frac{-d_{11}\lambda_{i_2} + a_{11}}{a_{12} - d_{12}\lambda_{i_2}} \end{pmatrix}, p_{i_2} = \begin{pmatrix} d_{22}\lambda_{i_2} - a_{22} \\ a_{12} - d_{12}\lambda_{i_2} \end{pmatrix}.$$

In addition, $q_{i_2} \cdot p_{i_2} = a_{11} + a_{22} - (d_{11} + d_{22})\lambda_{i_2}$ and

$$\kappa_0 \triangleq q_{i_2} \cdot R_\alpha p_{i_2} = det(\lambda_{i_2}, \alpha) = A\lambda_{i_2}^2 + B\lambda_{i_2} + C.$$

To find the bifurcating solutions near the trivial steady-state E_2 of system (1.3), define $G = (G_1, G_2)^T : \mathbb{X}^2 \times \mathbb{R} \to \mathbb{Y}^2$ by

$$G_1(u,v) = \Delta[(d_1 + \alpha v)u] + \frac{r_0 u}{1 + k\bar{v}} - du - au^2 - puv,$$
$$G_2(u,v) = \Delta[(d_2 + \frac{1}{1 + \beta u})v] + cpuv - m_1 v - m_2 v^2,$$

for $E = (u,v)^T \in \mathbb{X}^2$ and $\alpha \in \mathbb{R}$. Then, one objective is to find other solutions for the equation $G(E,\alpha) = 0$ when $E \to E_2 \in \mathbb{X}^2$ and $\alpha \to \alpha_0 \in \mathbb{R}$. For any $\varsigma = (\varsigma_1,\varsigma_2)^T, \xi = (\xi_1,\xi_2)^T$ and $\zeta = (\zeta_1,\zeta_2)^T \in K$. The Fréchet derivative forms of $d^2F_z(\varsigma,\xi)$ and $d^3F_z(\varsigma,\xi,\zeta)$, respectively.

$$\begin{aligned} d^2F_z(\varsigma,\xi) &= \begin{pmatrix} A_1\varsigma_1\xi_1 + A_2(\Delta)(\varsigma_1\xi_2+\varsigma_2\xi_1) \\ +A_3(\varsigma_1\bar{\xi}_2+\bar{\varsigma}_2\xi_1) + A_4\bar{\varsigma}_2\bar{\xi}_2 \\ A_5(\Delta)\varsigma_1\xi_1 + A_6(\Delta)(\varsigma_1\xi_2+\varsigma_2\xi_1) + A_7\varsigma_2\xi_2 \end{pmatrix}, \\ d^3F_z(\varsigma,\xi,\zeta) &= \begin{pmatrix} B_1(\varsigma_1\bar{\xi}_2\bar{\zeta}_2 + \hat{\varsigma}_2\bar{\xi}_2\zeta_1) + B_2\bar{\varsigma}_2\bar{\xi}_2\bar{\zeta}_2 \\ B_3(\Delta)(\varsigma_1\xi_2\zeta_1 + \varsigma_2\xi_1\zeta_1 + \varsigma_1\xi_1\zeta_2) + B_4(\Delta)\varsigma_1\xi_1\zeta_1 \end{pmatrix}. \end{aligned}$$

Here,

$$\begin{gathered} A_1 = -2, A_2(\Delta) = (\alpha\Delta - p), A_3 = -\tfrac{rk}{(1+kv^*)^2}, A_4 = \tfrac{2rk^2u^*}{(1+kv^*)^3}, \\ A_5(\Delta) = 2\tfrac{\beta^2v^*}{(1+\beta u^*)^3}\Delta, A_6(\Delta) = (cp - \tfrac{\beta}{(1+\beta u^*)^2}\Delta), A_7 = -2m_2, \\ B_1 = \tfrac{2rk^2}{(1+kv^*)^3}, B_2 = -6\tfrac{rk^3u^*}{(1+kv^*)^4}, B_3(\Delta) = 2\tfrac{\beta^2}{(1+\beta u^*)^3}\Delta, B_4(\Delta) = -6\tfrac{\beta^3v^*}{(1+\beta u^*)^3}\Delta. \end{gathered}$$

One has the following decompositions:

$$\mathbb{X}^2 = S \bigoplus \mathbb{X}_{i_2}, \mathbb{Y}^2 = S^* \bigoplus \mathbb{Y}_{i_2}$$

and

$$\begin{aligned} \mathbb{X}_{i_2} &= \{\psi \in \mathbb{X}^2 | \langle \phi_{i_2}, p_{i_2}\cdot\psi \rangle\}, \\ \mathbb{Y}_{i_2} &= \{\psi \in \mathbb{Y}^2 | \langle \phi_{i_2}, q_{i_2}\cdot\psi \rangle\}. \end{aligned}$$

Based on the analysis of [28], one knows that $R_\alpha : \mathbb{X}^2 \to \mathbb{Y}^2$ is a Fredholm operator with zero index, and $R_\alpha|_{\mathbb{X}_{i_2}} : \mathbb{X}_{i_2} \to \mathbb{Y}_{i_2}$ is invertible and has a bounded inverse. Suppose M and $I - M$ are the projection operators from $\mathbb{Y}^2$ to $\mathbb{Y}_{i_2}$ and S^*, respectively. Then for every $u(x) \in \mathbb{Y}^2$,

$$ME = E(x) - \frac{\langle \phi_{i_2}, q_{i_2}\cdot E\rangle}{q_{i_2}\cdot p_{i_2}}\phi_{i_2}p_{i_2}$$

for all $E \in \mathbb{X}^2$.

Hence, $G(E,\alpha) = 0$ is equivalent to the system reading as

$$MG(E,\alpha) = \mathbf{0}, (I - M)G(E,\alpha) = \mathbf{0}. \tag{3.3}$$

For every $E \in \mathbb{X}^2$, there is a unique decomposition.

$$E = E_2 + z\phi_{i_2}p_{i_2} + w,$$

where $z \in \mathbb{R}$ and $w \in \mathbb{X}_{i_2}$.

Hence, the first equation of (3.3) can be rewritten as

$$MG(E_2 + z\phi_{i_2}p_{i_2} + w, \alpha) = \mathbf{0}.$$

A map $MF : K \times X_{i_2} \times R \to Y_{i_2}$, which satisfies the conditions of the implicit function theorem: $MG(E_2, \alpha) = \mathbf{0}, MG_w(E_2, \alpha) = MR_\alpha = R_\alpha$. Therefore, one obtains a unique continuously differentiable map $w = (w_1, w_2)^T$ in two open neighborhoods δ of 0 in $\mathbb{R}^2$ and ε of α^0 in $\mathbb{R}$ respectively, which satisfies

$$for\ all\ (z, \alpha) \in \delta \times \varepsilon, w(0, \alpha) = 0.$$

Substituting $u = u^* + zq_{i_2} + w(z, \alpha)$ into the first equation of (3.3), then

$$\forall (z, \alpha) \in \delta \times \varepsilon, MG(u^* + z\phi_{i_2} p_{i_2} + w(z, \alpha), \alpha) = \mathbf{0}. \tag{3.4}$$

In addition, $\langle \phi_{i_2}, p_{i_2} \cdot w(z, \alpha) \rangle \equiv 0$.

Therefore, $\forall (z, \alpha) \in \delta \times \varepsilon$,

$$G(u^* + z\phi_{i_2} p_{i_2} + w(z, \alpha), \alpha) \in K^*. \tag{3.5}$$

Substituting $w = w(z, \alpha)$ into the second equation of (3.3), then

$$\mathcal{G}(z, \alpha) \triangleq (I - M)G(u^* + z\phi_{i_2} p_{i_2} + w(z, \alpha), \alpha) = 0.$$

From (3.4), then

$$\begin{array}{lcl} w(z, \alpha) & = & \frac{1}{2} w_2 z^2 + \frac{1}{6} w_3 z^3 + \cdots, \\ w_2 & = & -(R_{\alpha^0})^{-1} d^2 F_z(\phi_{i_2} q_{i_2}, \phi_{i_2} q_{i_2}), \end{array}$$

where

$$(R_{\alpha^0})^{-1} = \frac{1}{detR_{\alpha^0}} \begin{bmatrix} d_{22}\Delta + a_{22} & -d_{12}\Delta - a_{12} \\ -d_{21}\Delta - a_{21} & d_{11}\Delta + a_{11} \end{bmatrix}.$$

Calculating the inner product of (3.5) with $q_{i_2}\phi_{i_2}$, derive that

$$\mathcal{G}^*(z, \alpha) = z(\kappa_0 + \frac{1}{2}\kappa_1 z + \frac{1}{6}\kappa_2 z^2) + o(|\alpha - \alpha^0|, |z|^3),$$

where

$$\begin{aligned} \kappa_1 &= \left\langle \phi_{i_2}, q_{i_2} \cdot d^2 F_z(\phi_{i_2} p_{i_2}, \phi_{i_2} p_{i_2}) \right\rangle, \\ \kappa_2 &= \left\langle \phi_{i_2}, q_{i_2} \cdot d^3 F_z(\phi_{i_2} p_{i_2}, \phi_{i_2} p_{i_2}, \phi_{i_2} p_{i_2}) \right\rangle + 3 \left\langle \phi_{i_2}, q_{i_2} \cdot d^2 F_z(\phi_{i_2} p_{i_2}, \phi_{i_2} p_{i_2}, w_{zz}(0, \alpha^0)) \right\rangle. \end{aligned}$$

If $\kappa_1 \neq 0$, by the implicit function theorem, one obtains a unique continuously differentiable map $\alpha \to z(\alpha)$ in two open neighborhoods, δ of 0 in $\mathbb{R}$, ε of α^0 in $\mathbb{R}$, which satisfies $\mathcal{G}^*(z, \alpha) = 0$ and

$$z(\alpha) = -\frac{2\kappa_0}{\kappa_1} + o(|\alpha - \alpha^0|).$$

i.e., the system (3.1) has a non-constant steady-state $E_\alpha = E_2 + z(\alpha) p_{i_2} \phi_{i_2} + w(z(\alpha), \alpha)$.

For simplicity, for any $\delta > 0$, one defines the following intervals, read as

$$I_1(\alpha, \delta) = \{(\alpha^0 - \delta, \alpha^0 + \delta) : \kappa_0 > 0\},$$

$$I_2(\alpha,\delta)=\{(\alpha^0-\delta,\alpha^0+\delta):\kappa_0<0\}.$$

If $\kappa_1 = 0, \kappa_2 < 0$ (respectively > 0), then there exists $\delta > 0$ and two continuously differentiable mappings $z^{\pm}$ from $I_1(\alpha,\delta)$ (respectively, $I_2(\alpha,\delta)$) to $\mathbb{R}$ such that system (3.1) has two nontrivial solutions $E^{\pm}_{\alpha} \in \mathbb{X}^2$, where

$$E^{\pm}_{\alpha} = E_2 + z^{\pm}(\alpha)p_{i_2}\phi_{i_2} + w(z^{\pm}(\alpha),\alpha),$$

when $\alpha \to \alpha^0, E^{\pm}_{\alpha} \to E_2$.

Theorem 3.1. *If the assumption (H_{i_2}) holds.*
(i) If $\kappa_1 \neq 0$, there exists a continuously differentiable map $\alpha \to z_\alpha$ in $(0-\delta, 0+\delta) \in \mathbb{R}, (\alpha^0-\varepsilon, \alpha^0+\varepsilon) \in \mathbb{R}$, satisfing that system (3.1) near E_2; there exists has only one nonconstant steady-state solution $E_\alpha = E_2 + z(\alpha)p_{i_2}\phi_{i_2} + w(z(\alpha),\alpha)$. In addition, $\lim_{\alpha\to\alpha^0} E_\alpha = E_2$.
(ii) If $\kappa_1 = 0, \kappa_2 < 0$ (respectively > 0), then there exist $\delta > 0$ and two continuously differentiable mapping $z^{\pm}$ from $I_1(\alpha,\delta)$ (respectively, $I_2(\alpha,\delta)$) to $\mathbb{R}$ such that (3.1) has two nontrivial solutions $E^{\pm}_{\alpha} \in \mathbb{X}^2$, where

$$E^{\pm}_{\alpha} = E_2 + z^{\pm}(\alpha)p_{i_2}\phi_{i_2} + w(z^{\pm}(\alpha),\alpha),$$

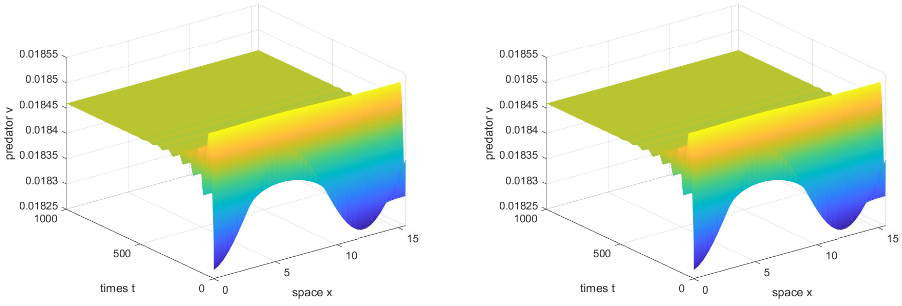
when $\alpha \to \alpha^0, E^{\pm}_{\alpha} \to E_2$.

4. Numerical simulation

In this section, numerical simulations are carried out to demonstrate the Turing instability and the existence of nonconstant steady-states for system (1.3) in the one-dimensional spatial domain $(0, l\pi)$ based on the theoretical results in Sections 2 and 3.

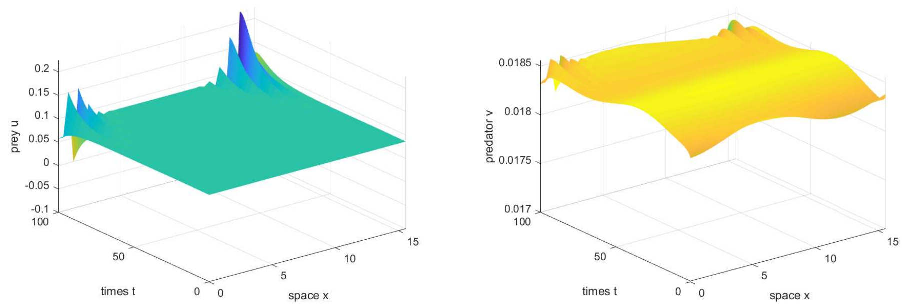
Taking the parameters as follows: $d = 1.1, r = 1.2, p = 1.1, m_1 = 0.1, m_2 = 0.2, c = 1.1$, and $\beta = 0.1$. By direct calculation, $\frac{cp(r-d)}{a} = m_1$; according to Lemma 2.1, the E_2 exists provided $a < 1.21$.

Let $d_1 = 0.1, d_2 = 0.2, a = 0.2, l = 5$; then the constant steady-state $E_2 = (0.0857, 0.0184)$. By Theorems 2.3 and 2.4, one gets $\alpha_c = -7.5811, \bar{\alpha}_T = -5.6305$. Hence, when $\alpha = -1$, E_2 is locally asymptotically stable. As shown in Figure 1. When $\alpha_c < \alpha = -5.7 < \bar{\alpha}_T$, E_2 is unstable, i.e., Turing instability occurs. As shown in Figure 2. However, when choosing $k = 10$, the other parameters remain unchanged; the critical parameters of Turing instability are $\alpha_c = -20.4872, \bar{\alpha}_T = -15.3135$. When $\alpha = -5.5$, by Theorems 2.3 and 2.4, the constant steady-state E_2 is also locally asymptotically stable. The results show that a high level of the fear effect can eliminate the Turing instability when the other parameters are fixed. As shown in Figure 3.



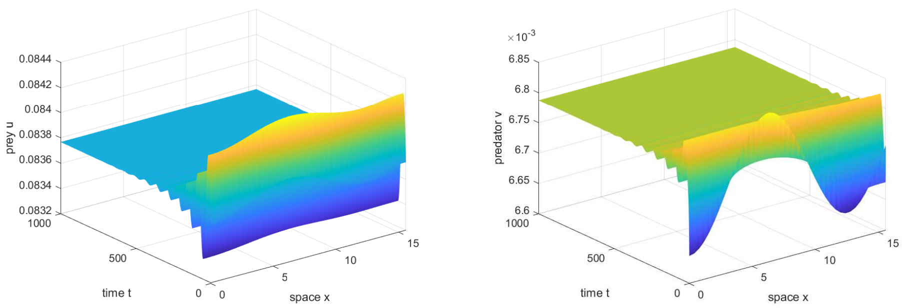
(a) (b)

Figure 1. The initial conditions $(u_0, v_0) = (0.0857 - 0.0001cos\frac{x}{5}, 0.0184 - 0.0001cos\frac{x}{5})$. When $k = 3, \alpha = -1$, and $l = 5$, E_2 of system (1.3) is stable.



(a) (b)

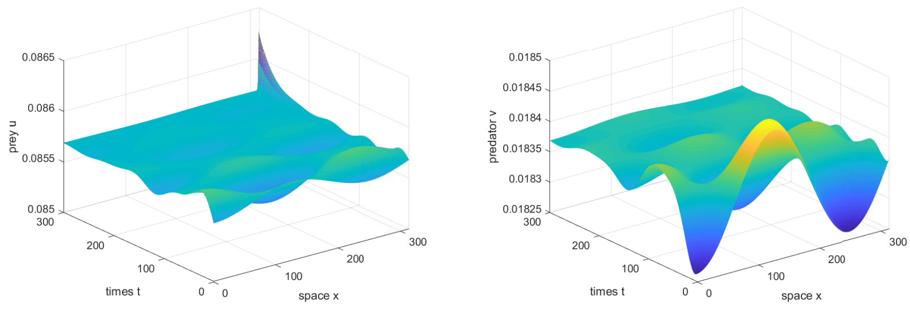
Figure 2. The initial conditions $(u_0, v_0) = (0.0857 - 0.0001cos\frac{x}{5}, 0.0184 - 0.0001cos\frac{x}{5})$. When $k = 3, \alpha = -5.7$, and $l = 5$, E_2 of system (1.3) is unstable, Turing instability occurs.



(a) (b)

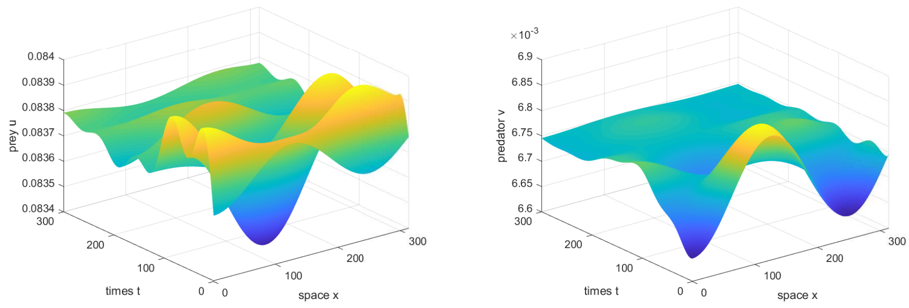
Figure 3. The initial conditions $(u_0, v_0) = (0.0838 - 0.0001cos\frac{x}{5}, 0.0067 - 0.0001cos\frac{x}{5})$. When $k = 10, \alpha = -5.5$, and $l = 5$, E_2 of system (1.3) is stable.

Let $d_1 = 0.1, d_2 = 0.2, a = 0.2$, and $l = 100$. By Theorem 3.1, there exists a λ_i such that the hypothesis $H_{\lambda_{i_2}}$ holds, and $\kappa_1 \neq 0$, then system (1.3) has one nonconstant steady-state solution, as shown in Figure 4.



(a) (b)

Figure 4. The initial conditions $(u_0, v_0) = (0.0857 - 0.0001cos\frac{x}{100}, 0.0184 - 0.0001cos\frac{x}{100})$, When $k = 3, \alpha = -5.5$, and $l = 100$, system (1.3) has a nonconstant steady-state solution.



(a) (b)

Figure 5. The initial conditions $(u_0, v_0) = (0.0838 - 0.0001cos\frac{x}{100}, 0.0067 - 0.0001cos\frac{x}{100})$, When $k = 10, \alpha = -5.5$, and $l = 100$, system (1.3) has a nonconstant steady-state solution.

5. Discussion and conclusions

Turing instability leading to spatial pattern formation occurs exclusively when the prey avoidance coefficient satisfies $\alpha < 0$ within the critical range $\alpha_c \leq \alpha \leq \bar{\alpha}_T$. This instability arises from nonlinear cross-diffusion destabilizing the homogeneous equilibrium E_2, provided the spatial wavenumber λ_i exceeds a threshold λ_{iT}. Crucially, the nonlocal fear effect (k) exhibits dual roles: while low fear levels permit pattern emergence under suitable α, high fear levels suppress Turing instability by elevating the critical threshold $|\alpha_c|$, thereby stabilizing E_2. Numerical simulations validate this antagonistic mechanism (Figures 1–3), showing that increased k eliminates instability even at fixed α. Furthermore, steady-state bifurcation analysis via Lyapunov-Schmidt reduction confirms the existence of spatially heterogeneous solutions near E_2 (Figures 4 and 5), extending pattern formation beyond linear instability. Ecologically, $\alpha < 0$ models prey aggregation induced by predator-avoidance behavior, whereas high k reduces prey fitness, mediating stability-perturbation trade-offs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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