Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(12): 7385–7427.
DOI: 10.3934/era.2025326
Received: 13 October 2025
Revised: 18 November 2025
Accepted: 24 November 2025
Published: 08 December 2025

Research article

Asymptotic profiles of a diffusive SEIR epidemic model with underlying disease and mass action infection mechanism

Chengxia Lei, Bo Li*and Xinyi Li

School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, China

* **Correspondence:** Email: libo5181923@163.com.

Abstract: We study a susceptible–exposed–infected–recovered (SEIR) reaction–diffusion epidemic model that includes susceptible individuals with underlying diseases, focusing on how these comorbidities, diffusion coefficients, and spatial heterogeneity affect disease's spread. The basic reproduction number R_0 is central to understanding and controlling infectious diseases' spread. We define R_0, analyze its behavior under low diffusion rates, and investigate the persistence of infection in relation to R_0. Our results show that underlying health conditions increase the value of R_0, enhancing the disease's transmission potential and persistence. In a homogeneous environment, if $R_0 > 1$, the system admits a constant endemic equilibrium that is globally asymptotically stable; if $R_0 < 1$, the disease-free equilibrium is globally attractive, implying eventual disease eradication. Furthermore, we analyze the asymptotic behavior of the endemic equilibrium as the diffusion rates approach zero. Our results indicate that limiting the mobility of susceptible, exposed, and infectious individuals alone is insufficient to eliminate the disease. By examining the influence of diffusion coefficients on the spatial dynamics and disease persistence, we conclude that effective control strategies must extend beyond diffusion control and incorporate interventions targeting additional transmission factors.

Keywords: SEIR reaction–diffusion epidemic model; spatially heterogeneous environment; basic reproduction number; asymptotic profiles

1. Introduction

Infectious diseases have posed significant challenges to human health throughout history. To improve our ability to prevent and control these diseases, a scientific understanding of their mechanisms and transmission dynamics is essential. The development of dynamic models of infectious disease serves as a powerful tool for understanding and managing these diseases. Such epidemic models are vital for capturing the complex interactions among pathogens, hosts, and the environment. In 1927, Kermack and McKendrick proposed the well-known susceptible–infected–recovered (SIR) compartment model [1] to describe the spread of the plague epidemic in Bombay. Later, they developed an susceptible–infected–susceptible (SIS) epidemic model to describe infectious diseases that do not confer immunity after recovery [2]. If an individual experiences an exposure period before becoming infectious (e.g., malaria, West Nile virus, etc.), the disease dynamics can be described by the susceptible–exposed–infected–recovered (SEIR) compartment model (see, for example, [3–7]. When recovered individuals lose immunity and return to the susceptible class, the SEIR–susceptible (SEIRS) model is applicable (see, for example, [8, 9]). Various compartmental models are employed to make dynamic models of infectious disease more accurate and applicable to real-world scenarios [10–18] and the references therein. Each of these models incorporates different aspects of disease transmission and progression to more accurately reflect the complexities of specific infectious diseases. In particular, the SEIR model is especially effective for diseases with a well-defined incubation period, such as COVID-19, as it provides a comprehensive framework for understanding and predicting disease progression.

In [3], the authors consider the following SEIR ordinary differential equation (ODE) model:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta IS - dS, & t > 0, \\ \frac{dE}{dt} = \beta IS - (\sigma + d)E, & t > 0, \\ \frac{dI}{dt} = \sigma E - (\gamma + d)I, & t > 0, \\ \frac{dR}{dt} = \gamma I - dR, & t > 0. \end{cases} \tag{1.1}$$

Here, $S = S(t)$, $E = E(t)$, $I = I(t)$, and $R = R(t)$ represent the densities of susceptible, exposed, infectious, and recovered individuals at time t, respectively. The parameters Λ, d, β, σ, and γ are all positive constants, representing the recruitment rate (including births and immigration), the natural death rate, the transmission rate due to effective contact between susceptible and infectious individuals, the rate at which exposed individuals become infectious, and the recovery rate of infectious individuals, respectively.

The term βIS in the SEIR epidemic model (1.1) corresponds to the incidence of mass–action (also known as the density-dependent transmission mechanism). This form of incidence arises naturally from the assumption of homogeneous mixing and was central to the foundational work of Kermack and McKendrick in their 1927 trilogy on the mathematical theory of epidemics [1]. Their frame work laid the groundwork for modern compartmental modeling and continues to influence epidemiological theory and public health practice (see, for example, [11, 12, 17, 19–26], and the references therein). A more general nonlinear incidence rate of the form $\beta I^p S^q$ $(p, q > 0)$ was investigated in [27] and further analyzed in [28], providing a flexible framework for capturing complex transmission dynamics beyond standard mass-action assumptions. It has since been widely adopted in epidemic modeling, with particular attention to the global stability of equilibria in systems incorporating such nonlinearities [4, 13]. Another widely used transmission mechanism is the standard incidence, given by $\frac{\beta SI}{N}$ with $N = S + E + I + R$, also referred to as frequency-dependent transmission [29]. This form has attracted significant attention in mathematical epidemiology (see, e.g., [12, 14, 16, 30, 31] and references therein). McCallum et al. [32] compared models employing standard incidence and mass-action mechanisms, concluding that the appropriateness of each depends on the specific mode of disease transmission, with both having distinct advantages in different biological contexts.

A survey conducted in [33] indicates that a significant proportion of COVID-19 patients who died had pre-existing chronic diseases, highlighting the critical role of underlying health conditions in influencing disease severity. Specifically, individuals with chronic conditions are more susceptible to COVID-19 [34]. To better understand the dynamics of the disease, it is essential to consider the impact of underlying health conditions on susceptibility. The corresponding transmission mechanism and model structure are illustrated in [34]. This leads to the following extended model:

$$\begin{cases} \frac{dS_1}{dt} = \Lambda - \beta_1 S_1 I - \theta S_1 - dS_1, & t > 0, \\ \frac{dS_2}{dt} = \theta S_1 - \beta_2 S_2 I - dS_2, & t > 0, \\ \frac{dE}{dt} = \beta_1 S_1 I + \beta_2 S_2 I - (d + \sigma)E, & t > 0, \\ \frac{dI}{dt} = \sigma E - (\gamma + d + \alpha)I, & t > 0, \\ \frac{dR}{dt} = \gamma I - dR, & t > 0, \end{cases} \tag{1.2}$$

where $S_1 = S_1(t)$, and $S_2 = S_2(t)$ denote the densities of individuals who are susceptible without underlying conditions and those with underlying conditions, respectively, at time t. The roles of the positive constants Λ, d, σ, and γ are the same as those in (1.1). The parameter $\beta_1 > 0$ denotes the transmission rate for susceptible individuals without underlying conditions after effective contact with infectious individuals. The positive constant $\beta_2 (> \beta_1)$ represents the transmission rate after effective contact between susceptible individuals with underlying conditions and infectious individuals. The parameter $\alpha > 0$ is the disease-induced mortality rate, and the positive constant θ denotes the rate at which healthy individuals develop underlying conditions. Subsequently, Yang et al. [35] incorporated time delays into the system (1.2) to account for the effects of delays and underlying health conditions on disease transmission.

Recently, Allen et al. [36] proposed a frequency-dependent SIS epidemic patch model and investigated the effects of spatial heterogeneity, habitat connectivity, and movement rates on the persistence and extinction of infectious diseases. Later on, they studied this epidemic model in a continuous-time and continuous-space SIS model [30]. They focus on the existence, uniqueness, and particularly the asymptotic profile of the steady states in a spatially heterogeneous environment. Their study suggests that controlling the mobility of susceptible individuals may be more effective in limiting disease's spread than restricting the mobility of infectious individuals. Spatial heterogeneity plays a crucial role in disease transmission, making its understanding essential for the development of effective public health strategies. To more accurately capture the spatial characteristics and enhance the realism of mathematical models, researchers have extended traditional ODE models by incorporating the Laplace operator to represent population movement. This approach corresponds to Brownian motion and is commonly referred to as local diffusion. For further details and related studies, see [9, 15, 16, 37–43] and the references therein. In addition to classical diffusion, nonlocal diffusion models incorporating integral operators to account for long–range spatial correlations, have been introduced to describe population movement more realistically [44]. Alternatively, in [45], the authors model population mobility using a graph Laplacian operator to represent discrete movement networks.

In this paper, we propose a reaction–diffusion model that incorporates spatial heterogeneity, underlying health conditions, and population mobility to capture the complex dynamics of infectious disease transmission. The model is an extension of the system (1.2), given by:

$$\begin{cases} \frac{\partial S_1}{\partial t} = d_1 \Delta S_1 + \Lambda(x) - \beta_1(x) S_1 I - \theta(x) S_1 - d(x) S_1, & x \in \Omega, \ t > 0, \\ \frac{\partial S_2}{\partial t} = d_2 \Delta S_2 - \beta_2(x) S_2 I + \theta(x) S_1 - d(x) S_2, & x \in \Omega, \ t > 0, \\ \frac{\partial E}{\partial t} = d_E \Delta E + \beta_1(x) S_1 I + \beta_2(x) S_2 I - [d(x) + \sigma(x)]E, & x \in \Omega, \ t > 0, \\ \frac{\partial I}{\partial t} = d_I \Delta I + \sigma(x) E - [\gamma(x) + d(x) + \alpha(x)]I, & x \in \Omega, \ t > 0, \\ \frac{\partial R}{\partial t} = d_R \Delta R + \gamma(x) I - d(x) R, & x \in \Omega, \ t > 0, \\ \frac{\partial S_1}{\partial n} = \frac{\partial S_2}{\partial n} = \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n} = 0, & x \in \partial\Omega, \ t > 0. \end{cases} \tag{1.3}$$

Here, the spatial domain $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded region with a smooth boundary $\partial\Omega$. The positive diffusion coefficients d_1, d_2, d_E, and d_I, represent the movement rates of susceptible individuals without underlying conditions, susceptible individuals with underlying conditions, exposed individuals, and infectious individuals, respectively. The positive functions $\Lambda(x)$, $d(x)$, $\alpha(x)$, $\beta_1(x)$, $\beta_2(x)$, $\sigma(x)$, $\theta(x)$, and $\gamma(x)$ are all *Hölder* continuous on $\bar{\Omega}$ and retain the same meanings as in the system (1.2). The initial data are assumed to be nonnegative and sufficiently smooth, satisfying

$$\begin{cases} S_1(x,0), S_2(x,0), E(x,0), I(x,0), R(x,0) \geq 0 \text{ for any } x \in \bar{\Omega}, \\ \int_\Omega I(x,0)\,\mathrm{d}x > 0 \text{ or } \int_\Omega E(x,0)\,\mathrm{d}x > 0. \end{cases} \tag{1.4}$$

By the standard regularity theory for parabolic equations (see [46]) and under the assumption of (1.4), we can establish the existence and uniqueness of a classical solution

$$(S_1, S_2, E, I, R) \in \left[C^{2,1}(\bar{\Omega} \times (0,\infty))\right]^5$$

to the system (1.3). Furthermore, by the strong maximum principle for parabolic equations (see [47]), it follows that

$$S_1 > 0, S_2 > 0, E > 0, I > 0 \text{ and } R > 0$$

for any $x \in \Omega$, $t > 0$.

Since S_1, S_2, E, and I are independent of R in the system (1.3), and thus R does not appear in the equations governing the dynamics of the other compartments, and it does not influence the evolution of S_1, S_2, E, and I. Therefore, we simplify the model by removing the equation for R and focus solely on the dynamics of the system

$$\begin{cases} \frac{\partial S_1}{\partial t} = d_1\Delta S_1 + \Lambda(x) - \beta_1(x)S_1 I - \theta(x)S_1 - d(x)S_1, & x \in \Omega, t > 0, \\ \frac{\partial S_2}{\partial t} = d_2\Delta S_2 - \beta_2(x)S_2 I + \theta(x)S_1 - d(x)S_2, & x \in \Omega,\ t > 0, \\ \frac{\partial E}{\partial t} = d_E\Delta E + \beta_1(x)S_1 I + \beta_2(x)S_2 I - [d(x) + \sigma(x)]E, & x \in \Omega,\ t > 0, \\ \frac{\partial I}{\partial t} = d_I\Delta I + \sigma(x)E - [\gamma(x) + d(x) + \alpha(x)]I, & x \in \Omega,\ t > 0, \\ \frac{\partial S_1}{\partial n} = \frac{\partial S_2}{\partial n} = \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ S_1(x,0) = S_{1,0}(x), S_2(x,0) = S_{2,0}(x), & x \in \Omega, \\ E(x,0) = E_0(x), I(x,0) = I_0(x), & x \in \Omega. \end{cases} \tag{1.5}$$

If susceptible individuals with underlying conditions are not considered, then the model (1.5) reduces to

$$\begin{cases} \frac{\partial S_1}{\partial t} = d_{S_1}\Delta S_1 + \Lambda(x) - \beta_1(x)S_1 I - d(x)S_1, & x \in \Omega,\ t > 0, \\ \frac{\partial E}{\partial t} = d_E\Delta E + \beta_1(x)S_1 I - [d(x) + \sigma(x)]E, & x \in \Omega,\ t > 0, \\ \frac{\partial I}{\partial t} = d_I\Delta I + \sigma(x)E - [\gamma(x) + d(x) + \alpha(x)]I, & x \in \Omega,\ t > 0, \\ \frac{\partial S_1}{\partial n} = \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.6}$$

This model was previously studied by the first author and collaborators in [15], where the effects of spatial heterogeneity and individual movements on disease transmission were investigated.

In the present work, we analyze the role of susceptible individuals with underlying health conditions in shaping the dynamics of infectious diseases, with particular emphasis on the mechanisms of disease's persistence and extinction. Additionally, we investigate the asymptotic behavior of positive steady states in spatially heterogeneous environments. The corresponding steady-state problem associated with the reaction–diffusion system (1.5) is given by the following elliptic system:

$$\begin{cases} d_1\Delta\tilde{S}_1 + \Lambda(x) - \beta_1(x)\tilde{S}_1\tilde{I} - \theta(x)\tilde{S}_1 - d(x)\tilde{S}_1 = 0, & x \in \Omega, \\ d_2\Delta\tilde{S}_2 - \beta_2(x)\tilde{S}_2\tilde{I} + \theta(x)\tilde{S}_1 - d(x)\tilde{S}_2 = 0, & x \in \Omega, \\ d_E\Delta\tilde{E} + \beta_1(x)\tilde{S}_1\tilde{I} + \beta_2(x)\tilde{S}_2\tilde{I} - [d(x) + \sigma(x)]\tilde{E} = 0, & x \in \Omega, \\ d_I\Delta\tilde{I} + \sigma(x)\tilde{E} - [\gamma(x) + d(x) + \alpha(x)]\tilde{I} = 0, & x \in \Omega, \\ \frac{\partial\tilde{S}_1}{\partial n} = \frac{\partial\tilde{S}_2}{\partial n} = \frac{\partial\tilde{E}}{\partial n} = \frac{\partial\tilde{I}}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.7}$$

The steady-state solution of the system (1.5), denoted by $(\tilde{S}_1, \tilde{S}_2, \tilde{E}, \tilde{I})$, satisfies the elliptic system (1.7). For any $x \in \Omega$, if $\tilde{I}(x) \equiv 0$, then the solution reduces to the form $(\hat{S}_1(x), \hat{S}_2(x), 0, 0)$, which is referred to as the disease-free equilibrium (DFE). Here, $(\hat{S}_1(x), \hat{S}_2(x))$ is the unique positive solution to the following system:

$$\begin{cases} -d_1\Delta S_1 = \Lambda(x) - (\theta(x) + d(x))S_1, & x \in \Omega, \\ -d_2\Delta S_2 = \theta(x)S_1 - d(x)S_2, & x \in \Omega, \\ \frac{\partial S_1}{\partial n} = \frac{\partial S_2}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.8}$$

If $\tilde{I}(x) > 0$ for some $x \in \Omega$, then the solution $(\tilde{S}_1, \tilde{S}_2, \tilde{E}, \tilde{I})$ of the system (1.7) is referred to as the endemic equilibrium (EE). In this case, the strong maximum principle for elliptic equations implies that

$$\tilde{S}_1(x) > 0,\ \tilde{S}_2(x) > 0,\ \tilde{E}(x) > 0,\ \tilde{I}(x) > 0,$$

for any $x \in \bar{\Omega}$.

The remainder of this paper is organized as follows. In Section 2, we define the basic reproduction number R_0 for the model described by the system (1.5) and analyze the influence of the diffusion coefficients on R_0, highlighting the role of spatial movements in disease transmission. In Section 3, we establish the uniform boundedness and uniform persistence of solutions to the system (1.5). In Section 4, we construct a Lyapunov function to prove the global attractivity of both the DFE and the EE in the spatially homogeneous case. In Section 5, we study the asymptotic behavior of the EE as the diffusion coefficients d_1, d_2, d_E, and d_I approach zero, revealing how reduced mobility affects disease dynamics. Section 6 concludes with a discussion of the results and their implications.

2. The basic reproduction number

Inspired by [9] and [48], this section defines the basic reproduction number R_0 for the system (1.5) and analyzes its dependence on the diffusion coefficients.

We linearize the system (1.5) about the DFE $E_0(\hat{S}_1, \hat{S}_2, 0, 0)$. The resulting linearized system is governed by

$$\begin{cases} \frac{\partial\bar{S}_1}{\partial t} = d_1\Delta\bar{S}_1 - \beta_1(x)\hat{S}_1\bar{I} - \theta(x)\bar{S}_1 - d(x)\bar{S}_1, & x \in \Omega,\ t > 0, \\ \frac{\partial\bar{S}_2}{\partial t} = d_2\Delta\bar{S}_2 - \beta_2(x)\hat{S}_2\bar{I} + \theta(x)\bar{S}_1 - d(x)\bar{S}_2, & x \in \Omega,\ t > 0, \\ \frac{\partial\bar{E}}{\partial t} = d_E\Delta\bar{E} + \beta_1(x)\hat{S}_1\bar{I} + \beta_2(x)\hat{S}_2\bar{I} - [d(x) + \sigma(x)]\bar{E}, & x \in \Omega,\ t > 0, \\ \frac{\partial\bar{I}}{\partial t} = d_I\Delta\bar{I} + \sigma(x)\bar{E} - [\gamma(x) + d(x) + \alpha(x)]\bar{I}, & x \in \Omega,\ t > 0, \\ \frac{\partial\bar{S}_1}{\partial n} = \frac{\partial\bar{S}_2}{\partial n} = \frac{\partial\bar{E}}{\partial n} = \frac{\partial\bar{I}}{\partial n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.1}$$

Here, $\bar{S}_1(x,t) = S_1(x,t) - \hat{S}_1(x)$, $\bar{S}_2(x,t) = S_2(x,t) - \hat{S}_2(x)$, $\bar{E}(x,t) = E(x,t)$, and $\bar{I}(x,t) = I(x,t)$. Following the framework in [48], we define the operators L, and the matrices $F(x)$, $V(x)$ as follows:

$$L = \operatorname{diag}\left(-d_E\Delta, -d_I\Delta\right),$$
$$F(x) = \begin{pmatrix} 0 & \beta_1(x)\hat{S}_1(x) + \beta_2(x)\hat{S}_2(x) \\ 0 & 0 \end{pmatrix},$$
$$V(x) = \begin{pmatrix} d(x) + \sigma(x) & 0 \\ -\sigma(x) & \gamma(x) + d(x) + \alpha(x) \end{pmatrix}.$$

We apply the theoretical framework developed in [49] and [48] to define the basic reproduction number R_0 for the system (1.5). This leads to the following result.

Lemma 2.1. *The eigenvalue problem*

$$\begin{cases} -d_E\Delta\varphi_E + [d(x)+\sigma(x)]\varphi_E = \omega_0(\beta_1(x)\hat{S}_1(x) + \beta_2(x)\hat{S}_2(x))\varphi_I, & x \in \Omega, \\ -d_I\Delta\varphi_I - \sigma(x)\varphi_E + [\gamma(x) + d(x) + \alpha(x)]\varphi_I = 0, & x \in \Omega, \\ \frac{\partial\varphi_E}{\partial n} = \frac{\partial\varphi_I}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{2.2}$$

admits a unique positive eigenvalue, denoted by ω_0, with a corresponding pair of positive eigenfunctions (φ_E, φ_I). Moreover, the basic reproduction number R_0 satisfies

$$R_0 = \frac{1}{\omega_0}.$$

The proof of Lemma 2.1 is identical to those in [9, 15] and is therefore omitted.

We now consider the eigenvalue problem

$$\begin{cases} -d_E\Delta\phi_E + [d(x)+\sigma(x)]\phi_E - (\beta_1(x)\hat{S}_1(x) + \beta_2(x)\hat{S}_2(x))\phi_I = \lambda\phi_E, & x \in \Omega, \\ -d_I\Delta\phi_I - \sigma(x)\phi_E + [\gamma(x) + d(x) + \alpha(x)]\phi_I = \lambda\phi_I, & x \in \Omega, \\ \frac{\partial\phi_E}{\partial n} = \frac{\partial\phi_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

It follows from the Krein–Rutman theorem [50] that the principal eigenvalue λ_1 is real and, algebraically simple, and possesses a strictly positive eigenfunction. Specifically, the eigenfunction $\phi = (\phi_E, \phi_I)^T$, can be chosen such that $\phi_E(x) > 0$ and $\phi_I(x) > 0$ for all $x \in \bar{\Omega}$. Similar to [9, 15], we can then derive the relationship between the basic reproduction number and the principal eigenvalue.

Lemma 2.2. $(1 - R_0)$ *has the same sign as λ_1, where λ_1 denotes the principal eigenvalue of the eigenvalue problem* (2.3).

Moreover, it follows that $\frac{1}{R_0}$ is the principal eigenvalue of the adjoint eigenvalue problem associated with (2.2), namely

$$\begin{cases} -d_E\Delta\varphi_E^* + [d(x)+\sigma(x)]\varphi_E^* = \sigma(x)\varphi_I^*, & x \in \Omega, \\ -d_I\Delta\varphi_I^* + [\gamma(x) + d(x) + \alpha(x)]\varphi_I^* = \frac{1}{R_0}(\beta_1(x)\hat{S}_1(x) + \beta_2(x)\hat{S}_2(x))\varphi_E^*, & x \in \Omega, \\ \frac{\partial\varphi_E^*}{\partial n} = \frac{\partial\varphi_I^*}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

Let $\varphi^* = (\varphi_E^*, \varphi_I^*)^T$ denote the strictly positive eigenfunction corresponding to the principal eigenvalue $\frac{1}{R_0}$. We now analyze the asymptotic behavior of the basic reproduction number R_0 as the diffusion coefficients d_1, d_2, d_E, and d_I approach zero or infinity.

Theorem 2.3. *The following statements about $R_0 := R_0(d_1, d_2, d_E, d_I)$ hold:*

(i) *For a fixed d_2, d_E, and $d_I > 0$, we have $R_0 \to \tilde{R}_0$ as $d_1 \to 0$, where $\tilde{R}_0$ is the principal eigenvalue of the problem*

$$\begin{cases} -d_E\Delta\varphi_E + [d(x) + \sigma(x)]\varphi_E = \frac{1}{\tilde{R}_0}\left(\frac{\beta_1(x)\Lambda(x)}{\theta(x)+d(x)} + \beta_2(x)\tilde{S}_2(x)\right)\varphi_I, & x \in \Omega, \\ -d_I\Delta\varphi_I - \sigma(x)\varphi_E + [\gamma(x) + d(x) + \alpha(x)]\varphi_I = 0, & x \in \Omega, \\ \frac{\partial\varphi_E}{\partial n} = \frac{\partial\varphi_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

Here, $\tilde{S}_2$ is the unique positive solution of

$$\begin{cases} d_2\Delta S_2 + \frac{\theta(x)\Lambda(x)}{\theta(x)+d(x)} - d(x)S_2 = 0, & x \in \Omega, \\ \frac{\partial S_2}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.6}$$

(ii) *For a fixed d_1, d_E, and $d_I > 0$, we have $R_0 \to \hat{R}_0$ as $d_2 \to 0$, where $\hat{R}_0$ is the principal eigenvalue of the problem*

$$\begin{cases} -d_E\Delta\varphi_E + [d(x) + \sigma(x)]\varphi_E = \frac{1}{\hat{R}_0}\left(\beta_1(x) + \frac{\beta_2(x)\theta(x)}{d(x)}\right)\hat{S}_1\varphi_I, & x \in \Omega, \\ -d_I\Delta\varphi_I - \sigma(x)\varphi_E + [\gamma(x) + d(x) + \alpha(x)]\varphi_I = 0, & x \in \Omega, \\ \frac{\partial\varphi_E}{\partial n} = \frac{\partial\varphi_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Here, $\hat{S}_1$ is the unique positive solution of

$$\begin{cases} d_1\Delta S_1 + \Lambda(x) - (\theta(x) + d(x))S_1 = 0, & x \in \Omega, \\ \frac{\partial S_1}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.7}$$

(iii) *For a fixed d_1, d_2, and $d_I > 0$, we have $R_0 \to \bar{R}_0 := \frac{1}{\omega_1}$ as $d_E \to 0$, where ω_1 is the principal eigenvalue of the problem*

$$\begin{cases} -d_I\Delta\bar{\varphi}_I + [\gamma(x) + d(x) + \alpha(x)]\bar{\varphi}_I = \frac{\omega\sigma(x)\left(\beta_1(x)\hat{S}_1(x)+\beta_2(x)\hat{S}_2(x)\right)}{d(x)+\sigma(x)}\bar{\varphi}_I, & x \in \Omega, \\ \frac{\partial\bar{\varphi}_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.8}$$

Furthermore, $\bar{R}_0$ is a monotone decreasing function of d_I and satisfies

$$\bar{R}_0 \to \max\left\{\frac{\sigma(x)\left(\beta_1(x)\hat{S}_1(x) + \beta_2(x)\hat{S}_2(x)\right)}{(d(x) + \sigma(x))(\gamma(x) + d(x) + \alpha(x))}\middle| x \in \overline{\Omega}\right\} \text{ as } d_I \to 0 \tag{2.9}$$

and

$$\bar{R}_0 \to \frac{\int_\Omega \frac{\sigma(x)\left(\beta_1(x)\hat{S}_1(x)+\beta_2(x)\hat{S}_2(x)\right)}{d(x)+\sigma(x)}\mathrm{d}x}{\int_\Omega \gamma(x) + d(x) + \alpha(x)\mathrm{d}x} \text{ as } d_I \to \infty. \tag{2.10}$$

Here, $(\hat{S}_1, \hat{S}_2)$ is the solution of (1.8).

(iv) *For a fixed d_1, d_2, and $d_E > 0$, we have $R_0 \to R_0^* := \frac{1}{\omega_2}$ as $d_I \to 0$, where ω_2 is the principal eigenvalue of the problem*

$$\begin{cases} -d_E\Delta\bar{\varphi}_E + [d(x) + \sigma(x)]\bar{\varphi}_E = \frac{\omega\sigma(x)\left(\beta_1(x)\hat{S}_1(x)+\beta_2(x)\hat{S}_2(x)\right)}{\gamma(x)+d(x)+\alpha(x)}\bar{\varphi}_E, & x \in \Omega, \\ \frac{\partial\bar{\varphi}_E}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Furthermore, R_0^* is a monotone decreasing function of d_E and satisfies

$$R_0^* \to \max\left\{ \frac{\sigma(x)\left(\beta_1(x)\hat{S}_1(x) + \beta_2(x)\hat{S}_2(x)\right)}{(d(x)+\sigma(x))(\gamma(x)+d(x)+\alpha(x))} \middle| x \in \overline{\Omega} \right\} \text{ as } d_E \to 0$$

and

$$R_0^* \to \frac{\int_\Omega \frac{\sigma(x)\left(\beta_1(x)\hat{S}_1(x)+\beta_2(x)\hat{S}_2(x)\right)}{\gamma(x)+d(x)+\alpha(x)}\mathrm{d}x}{\int_\Omega d(x)+\sigma(x)\mathrm{d}x} \text{ as } d_E \to \infty.$$

Here, $(\hat{S}_1, \hat{S}_2)$ is the solution of (1.8).

Proof. Since the solution $(\hat{S}_1(x), \hat{S}_2(x))$ of (1.8) depends on d_1, we denote it by $(\hat{S}_{1,d_1}(x), \hat{S}_{2,d_1}(x))$. Then it follows from (1.8) and [59, Lemma 2.4] that as $d_1 \to 0$, we have

$$\hat{S}_{1,d_1}(x) \to \frac{\Lambda(x)}{\theta(x)+d(x)} \text{ uniformly on } \bar{\Omega}.$$

Moreover, by applying the L^p theory and the Sobolev embedding theorem, we deduce that

$$\hat{S}_{2,d_1}(x) \to \tilde{S}_2(x) \text{ uniformly on } \bar{\Omega} \text{ as } d_1 \to 0.$$

For any $\epsilon_0 > 0$, a sufficiently small $\sigma_0 > 0$ exists such that, for all $0 < d_1 < \sigma_0$, the solution $(\hat{S}_{1,d_1}(x), \hat{S}_{2,d_1}(x))$ of (1.8) satisfies

$$0 < (1-\epsilon_0)\frac{\Lambda(x)}{\theta(x)+d(x)} < \hat{S}_{1,d_1}(x) < (1+\epsilon_0)\frac{\Lambda(x)}{\theta(x)+d(x)} \tag{2.11}$$

and

$$0 < (1-\epsilon_0)\tilde{S}_2(x) < \hat{S}_{2,d_1}(x) < (1+\epsilon_0)\tilde{S}_2(x) \tag{2.12}$$

on $\bar{\Omega}$.

As in the proof of Lemma 2.1, we deduce that (2.5) admits a principal eigenvalue $\tilde{\omega}_0 = \frac{1}{\tilde{R}_0}$. Let $\tilde{\varphi} := (\tilde{\varphi}_E, \tilde{\varphi}_I)^T$ denote the corresponding eigenfunction. Then, by (2.11) and (2.12), we obtain

$$\begin{cases} -d_E\Delta\tilde{\varphi}_E + [d(x)+\sigma(x)]\tilde{\varphi}_E \geq \frac{\tilde{\omega}_0}{1+\epsilon_0}\left(\beta_1(x)\hat{S}_{1,d_1}(x) + \beta_2(x)\hat{S}_{2,d_1}(x)\right)\tilde{\varphi}_I, & x \in \Omega, \\ -d_I\Delta\tilde{\varphi}_I - \sigma(x)\tilde{\varphi}_E + [\gamma(x)+d(x)+\alpha(x)]\tilde{\varphi}_I = 0, & x \in \Omega, \\ \frac{\partial\tilde{\varphi}_E}{\partial n} = \frac{\partial\tilde{\varphi}_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.13}$$

Equation (2.13) can be rewritten as

$$L\tilde{\varphi} + V\tilde{\varphi} \geq \frac{\tilde{\omega}_0}{1+\epsilon_0}F\tilde{\varphi}, \ x \in \Omega, \quad \frac{\partial\tilde{\varphi}}{\partial n}\Big|_{\partial\Omega} = 0. \tag{2.14}$$

The equation in (2.4) can be rewritten as

$$L\varphi^* + V^T\varphi^* = \frac{1}{R_0}F^T\varphi^*, \ x \in \Omega, \quad \frac{\partial\varphi^*}{\partial n}\Big|_{\partial\Omega} = 0. \tag{2.15}$$

We then multiply the inequality in (2.14) by $(\varphi^*)^T$ and the equation in (2.15) by $(\tilde{\varphi})^T$, subtract the resulting expressions, and integrate over Ω to obtain

$$\left(\frac{1}{R_0} - \frac{\tilde{\omega}_0}{1+\epsilon_0}\right)\int_\Omega \left(\beta_1(x)\hat{S}_{1,d_1}(x) + \beta_2(x)\hat{S}_{2,d_1}(x)\right)\tilde{\varphi}_I \varphi_E^* \mathrm{d}x \geq 0,$$

which implies that $R_0 \leq \frac{1+\epsilon_0}{\tilde{\omega}_0}$. Similarly, by interchanging the roles of (2.14) and (2.15), we obtain the lower bound $R_0 \geq \frac{1-\epsilon_0}{\tilde{\omega}_0}$. Letting $\epsilon_0 \to 0$, it follows that $R_0 \to \frac{1}{\tilde{\omega}_0} = \tilde{R}_0$. This completes the proof of (i).

Next, based on [59, Lemma 2.4], we have

$$\hat{S}_2(x) \to \frac{\theta(x)}{d(x)}\hat{S}_1(x) \text{ uniformly on } \bar{\Omega} \text{ as } d_2 \to 0,$$

where $\hat{S}_1$ is determined by (2.7). By applying an argument similar to that used in the proof of Part (i), we can conclude that the assertion in Part (ii) holds.

Now, we proceed to show that the assertions (iii) and (iv) hold by employing an argument analogous to that used in the proof of [9]. Since $A = \left\{u \in C^2(\bar{\Omega}) \mid \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}$ is dense in $C(\bar{\Omega})$, for any $\varepsilon \in (0,1)$, we can choose the functions $\beta_i^*(x),\ \beta_i^{**}(x) \in A$ for $i = 1,2$, such that

$$\frac{\beta_i(x)}{(1+\varepsilon)(d(x)+\sigma(x))} < \beta_i^{**}(x) < \frac{\beta_i(x)}{d(x)+\sigma(x)}$$

and

$$\frac{\beta_i(x)}{d(x)+\sigma(x)} < \beta_i^*(x) < \frac{\beta_i(x)}{(1-\varepsilon)(d(x)+\sigma(x))}.$$

Let $\bar{\varphi}_I$ denote the eigenfunction corresponding to the principal eigenvalue ω_1 of (2.8). Set

$$(\hat{\varphi}_E, \hat{\varphi}_I) = \left(\omega_1(\beta_1^{**}(x)\hat{S}_1(x) + \beta_2^{**}(x)\hat{S}_2(x))\bar{\varphi}_I, \bar{\varphi}_I\right),$$

$$(\check{\varphi}_E, \check{\varphi}_I) = \left(\omega_1(\beta_1^{*}(x)\hat{S}_1(x) + \beta_2^{*}(x)\hat{S}_2(x))\bar{\varphi}_I, \bar{\varphi}_I\right).$$

For the given $\varepsilon \in (0,1)$, $\delta > 0$ exists such that, for $0 < d_E < \delta$,

$$\begin{cases} -d_E\Delta\hat{\varphi}_E \geq -(d(x)+\sigma(x))\left(1 - \frac{f_1(x)}{1+\varepsilon}\right)\hat{\varphi}_E, & x \in \Omega, \\ \frac{\partial\hat{\varphi}_E}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{2.16}$$

and

$$\begin{cases} -d_E\Delta\check{\varphi}_E \leq -(d(x)+\sigma(x))\left(1 - \frac{f_2(x)}{1-\varepsilon}\right)\check{\varphi}_E, & x \in \Omega, \\ \frac{\partial\check{\varphi}_E}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where

$$f_1(x) := \frac{\beta_1(x)\hat{S}_1(x) + \beta_2(x)\hat{S}_2(x)}{(d(x)+\sigma(x))(\beta_1^{**}(x)\hat{S}_1(x) + \beta_2^{**}(x)\hat{S}_2(x))},$$

and

$$f_2(x) := \frac{\beta_1(x)\hat{S}_1(x) + \beta_2(x)\hat{S}_2(x)}{(d(x)+\sigma(x))(\beta_1^{*}(x)\hat{S}_1(x) + \beta_2^{*}(x)\hat{S}_2(x))}.$$

By virtue of (2.16), we have

$$-d_E\Delta\hat{\varphi}_E + [d(x)+\sigma(x)]\hat{\varphi}_E \geq \frac{\beta_1(x)\hat{S}_1(x)+\beta_2(x)\hat{S}_2(x)}{(1+\varepsilon)(\beta_1^{**}(x)\hat{S}_1(x)+\beta_2^{**}(x)\hat{S}_2(x))}\hat{\varphi}_E$$

$$= \frac{\omega_1}{1+\varepsilon}(\beta_1(x)\hat{S}_1(x)+\beta_2(x)\hat{S}_2(x))\hat{\varphi}_I,\ x\in\Omega.$$

For $x\in\Omega$, using (2.8), we have

$$-d_I\Delta\hat{\varphi}_I + [\gamma(x)+d(x)+\alpha(x)]\hat{\varphi}_I = \frac{\omega_1\sigma(x)\left(\beta_1(x)\hat{S}_1(x)+\beta_2(x)\hat{S}_2(x)\right)}{\omega_1\left(\beta_1^{**}(x)\hat{S}_1(x)+\beta_2^{**}(x)\hat{S}_2(x)\right)(d(x)+\sigma(x))}\hat{\varphi}_E.$$

Since

$$\frac{\beta_1(x)\hat{S}_1(x)+\beta_2(x)\hat{S}_2(x)}{d(x)+\sigma(x)} > \beta_1^{**}(x)\hat{S}_1(x)+\beta_2^{**}(x)\hat{S}_2(x),$$

and noting that $\omega=\omega_1$, it follows that

$$-d_I\Delta\hat{\varphi}_I + [\gamma(x)+d(x)+\alpha(x)]\hat{\varphi}_I - \sigma(x)\hat{\varphi}_E \geq 0,\ x\in\Omega.$$

Therefore, the following inequality system holds:

$$\begin{cases} -d_E\Delta\hat{\varphi}_E + [d(x)+\sigma(x)]\hat{\varphi}_E \geq \frac{\omega_1}{1+\varepsilon}\left(\beta_1(x)\hat{S}_1(x)+\beta_2(x)\hat{S}_2(x)\right)\hat{\varphi}_I, & x\in\Omega,\\ -d_I\Delta\hat{\varphi}_I + [\gamma(x)+d(x)+\alpha(x)]\hat{\varphi}_I - \sigma(x)\hat{\varphi}_E \geq 0, & x\in\Omega,\\ \frac{\partial\hat{\varphi}_E}{\partial n} = \frac{\partial\hat{\varphi}_I}{\partial n} = 0, & x\in\partial\Omega. \end{cases} \tag{2.17}$$

As in the proof of (i), we have

$$\frac{1-\varepsilon}{\omega_1} \leq R_0 \leq \frac{1+\varepsilon}{\omega_1}.$$

Letting $\varepsilon\to 0$, we have $R_0\to\bar{R}_0$. Consequently, (2.9) and (2.10) follow from [30, Lemma 2.1].

For the proof of (iv), we can use a similar argument as in (iii), and thus we omit the details here. □

By the proof of Theorem 2.3 (i), for a fixed $d_E, d_I > 0$, the principal eigenvalue $\tilde{R}_0$ of (2.5) approaches $\tilde{R}_0^*$ as $d_2\to 0$, where $\tilde{R}_0^*$ is the principal eigenvalue of

$$\begin{cases} -d_E\Delta\varphi_E + (d(x)+\sigma)\varphi_E = \frac{1}{\tilde{R}_0^*}\left(\frac{\beta_1(x)\Lambda(x)}{\theta(x)+d(x)} + \frac{\beta_2(x)\theta(x)\Lambda(x)}{d(x)(\theta(x)+d(x))}\right)\varphi_I, & x\in\Omega,\\ -d_I\Delta\varphi_I - \sigma\varphi_E + [\gamma(x)+d(x)+\alpha(x)]\varphi_I = 0, & x\in\Omega,\\ \frac{\partial\varphi_E}{\partial n} = \frac{\partial\varphi_I}{\partial n} = 0, & x\in\partial\Omega. \end{cases} \tag{2.18}$$

For the system (1.6), the basic reproduction number $\mathcal{R}_0$ is characterized by the following eigenvalue problem:

$$\begin{cases} -d_E\Delta\psi_E + [d(x)+\sigma(x)]\psi_E = \frac{1}{\mathcal{R}_0}\beta_1(x)\hat{S}(x)\psi_I, & x\in\Omega,\\ -d_I\Delta\psi_I - \sigma(x)\psi_E + [\gamma(x)+d(x)+\alpha(x)]\psi_I = 0, & x\in\Omega,\\ \frac{\partial\psi_E}{\partial n} = \frac{\partial\psi_I}{\partial n} = 0, & x\in\partial\Omega, \end{cases}$$

where $\hat{S}(x)$ is the unique positive solution to

$$\begin{cases} -d\Delta S = \Lambda(x) - d(x)S, & x\in\Omega,\\ \frac{\partial S}{\partial n} = 0, & x\in\partial\Omega. \end{cases}$$

As the diffusion rate d of the susceptible population tends to zero, it follows from [15, Theorem 2.3] that $\mathcal{R}_0 \to \tilde{\mathcal{R}}_0$, where $\tilde{\mathcal{R}}_0$ is the principal eigenvalue of the problem.

$$\begin{cases} -d_E\Delta\psi_E + (d(x)+\sigma)\psi_E = \frac{1}{\tilde{\mathcal{R}}_0}\beta_1(x)\frac{\Lambda(x)}{d(x)}\psi_I, & x \in \Omega, \\ -d_I\Delta\psi_I - \sigma\psi_E + [\gamma(x)+d(x)+\alpha(x)]\psi_I = 0, & x \in \Omega, \\ \frac{\partial\psi_E}{\partial n} = \frac{\partial\psi_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.19}$$

Lemma 2.4. *If $\frac{\beta_1\Lambda}{d}$ and σ are positive constants, then $\tilde{R}_0^* > \tilde{\mathcal{R}}_0$.*

Proof. Let $\frac{\beta_1(x)\Lambda(x)}{d(x)} := C > 0$ be a constant. Multiplying the first equation of (2.18) by ψ_E and the first equation of (2.19) by φ_E, subtracting the resulting equations and integrating over Ω, we obtain

$$\frac{C}{\tilde{\mathcal{R}}_0}\int_\Omega \psi_I(x)\varphi_E(x)\mathrm{d}x = \frac{1}{\tilde{R}_0^*}\int_\Omega \left(\frac{\beta_1(x)\Lambda(x)}{d(x)+\theta(x)} + \frac{\beta_2(x)\theta(x)\Lambda(x)}{d(x)(\theta(x)+d(x))}\right)\varphi_I(x)\psi_E(x)\mathrm{d}x. \tag{2.20}$$

Next, multiplying the second equation in (2.18) by ψ_I and the second equation in (2.19) by φ_I, subtracting the resulting equations and integrating over Ω, we obtain

$$\int_\Omega \psi_I(x)\varphi_E(x)\mathrm{d}x = \int_\Omega \varphi_I(x)\psi_E(x)\mathrm{d}x.$$

Noting that $\beta_2(x) > \beta_1(x)$ and using (2.20), we conclude that

$$\left(\frac{1}{\tilde{\mathcal{R}}_0} - \frac{1}{\tilde{R}_0^*}\right)\int_\Omega \varphi_I(x)\psi_E(x)\mathrm{d}x > 0.$$

Since $\psi_I(x)$ and $\varphi_E(x)$ are both positive, this implies that $\tilde{R}_0^* > \tilde{\mathcal{R}}_0$. Therefore, the inclusion of susceptible individuals with underlying conditions can enlarge the basic reproduction number. □

Remark 2.5. The inequality $\tilde{R}_0^* > \tilde{\mathcal{R}}_0$ reflects that accounting for underlying health conditions increases the basic reproduction number. This arises because individuals with pre-existing comorbidities are generally more susceptible to infection and may experience prolonged or more severe infectious periods, thereby amplifying the overall transmission potential of the disease in the population.

3. Uniform bounds and the persistence of solutions to (1.5)

In this section, we establish uniform bounds and demonstrate the uniform persistence property of the solutions to the system (1.5). We begin by deriving the uniform bounds for these solutions as follows.

Lemma 3.1. *There is a positive constant C, independent of the initial data, and a time $T > 0$, such that the solution $(S_1, S_2, E, I) \in \left[C^{2,1}(\bar{\Omega}\times(0,\infty))\right]^4$ to (1.5) satisfies*

$$\|S_1(\cdot,t)\|_{L^\infty(\Omega)} + \|S_2(\cdot,t)\|_{L^\infty(\Omega)} + \|E(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} \le C \text{ for } t \ge T. \tag{3.1}$$

Proof. From the first equation of the system (1.5), we have

$$\begin{cases} \frac{\partial S_1}{\partial t} - d_1\Delta S_1 \le \max_{x\in\bar{\Omega}}\Lambda(x) - \min_{x\in\bar{\Omega}}(d(x)+\theta(x))S_1, & x \in \Omega,\ t > 0, \\ \frac{\partial S_1}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ S_1(x,0) = S_{1,0}(x) \ge 0, & x \in \Omega. \end{cases}$$

By the parabolic comparison principle, it follows that

$$S_1(x,t) \le u(x,t) \text{ for } x \in \bar{\Omega},\ t > 0,$$

where u is the unique solution to

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = \max\limits_{x\in\bar{\Omega}} \Lambda(x) - u \min\limits_{x\in\bar{\Omega}}(d(x) + \theta(x)), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \max\limits_{x\in\bar{\Omega}} S_{1,0}(x) \ge 0, & x \in \Omega. \end{cases} \tag{3.2}$$

It is clear that the positive constant

$$C_1 = \max \left\{ \frac{\max\limits_{x\in\bar{\Omega}} \Lambda(x)}{\min\limits_{x\in\bar{\Omega}}(d(x) + \theta(x))}, \max_{x\in\bar{\Omega}} S_{1,0}(x) \right\}$$

serves as an upper solution of (3.2).

Next, we consider the following parabolic initial value problem:

$$\begin{cases} \frac{\partial v}{\partial t} - d_2 \Delta v = C_1 \max\limits_{x\in\bar{\Omega}} \theta(x) - v \min\limits_{x\in\bar{\Omega}} d(x), & x \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ v(x,0) = \max\limits_{x\in\bar{\Omega}} S_{2,0}(x) \ge 0, & x \in \Omega. \end{cases} \tag{3.3}$$

It is evident that the constants

$$C_2 = \max \left\{ \frac{C_1 \max\limits_{x\in\bar{\Omega}} \theta(x)}{\min\limits_{x\in\bar{\Omega}} d(x)}, \max_{x\in\bar{\Omega}} S_{2,0}(x) \right\}$$

and 0 are the upper and lower solutions of (3.3), respectively. Thus, by the theory of parabolic equations, the system (3.3) admits a unique positive solution, denoted by v. Now, consider

$$\begin{cases} \frac{\partial S_2}{\partial t} - d_2 \Delta S_2 \le \theta(x) C_1 - d(x) S_2, & x \in \Omega,\ t > 0, \\ \frac{\partial S_2}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ S_2(x,0) = S_{2,0}(x) \ge 0, & x \in \Omega, \end{cases}$$

and apply the parabolic comparison principle to obtain

$$S_2(x,t) \le v(x,t),\ x \in \bar{\Omega}, t > 0.$$

Therefore, it follows that

$$S_1(x,t) \le u(x,t) \le C_1,\ \ S_2(x,t) \le v(x,t) \le C_2,\ \ x \in \bar{\Omega},\ \ t \ge 0.$$

Since

$$U(t) = \frac{\max\limits_{x\in\bar{\Omega}} \Lambda(x)}{\min\limits_{x\in\bar{\Omega}}(d(x) + \theta(x))} - \frac{\max\limits_{x\in\bar{\Omega}} \Lambda(x) - \max\limits_{x\in\bar{\Omega}} S_{1,0}(x) \min\limits_{x\in\bar{\Omega}}(d(x) + \theta(x))}{\min\limits_{x\in\bar{\Omega}}(d(x) + \theta(x))} e^{-t \min\limits_{x\in\bar{\Omega}}(d(x)+\theta(x))}$$

is an upper solution of (3.2), we have

$$\lim_{t\to\infty} u(x,t) \le \frac{\max_{x\in\bar{\Omega}} \Lambda(x)}{\min_{x\in\bar{\Omega}}(d(x)+\theta(x))}$$

uniformly on $\bar{\Omega}$. Therefore, we obtain the following uniform bounds:

$$\limsup_{t\to\infty} S_1(x,t) \le \frac{\max_{x\in\bar{\Omega}} \Lambda(x)}{\min_{x\in\bar{\Omega}}(d(x)+\theta(x))} \text{ uniformly on } \bar{\Omega}. \tag{3.4}$$

Similarly, we deduce

$$\limsup_{t\to\infty} S_2(x,t) \le \frac{C_1 \max_{x\in\bar{\Omega}} \theta(x)}{\min_{x\in\bar{\Omega}} d(x)} \text{ uniformly on } \bar{\Omega}. \tag{3.5}$$

We apply a similar approach as in the proof of [15] to find that

$$\|S_1(\cdot,t)\|_{L^1(\Omega)}, \|S_2(\cdot,t)\|_{L^1(\Omega)}, \|E(\cdot,t)\|_{L^1(\Omega)}, \|I(\cdot,t)\|_{L^1(\Omega)}$$

are uniformly bounded for $t > T$. According to [51, Lemma 2.1], by choosing $p_0 = 1$ and combining this with the bounds established in (3.4) and (3.5), we conclude that the uniform bound (3.1) holds. This completes the proof. □

We now recall the Agmon–Douglas–Nirenberg theorem from [52] to present the Schauder theory for second-order elliptic boundary value problems of the form

$$\begin{cases} \mathcal{L}u = f(x) & x \in \Omega \subset \mathbb{R}, \\ a(x)\frac{\partial u}{\partial n} + b(x)u = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{3.6}$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L}u = -\sum_{i,j}^{n} a_{i,j}(x)D_{i,j}u + \sum_{i}^{n} b_i(x)D_i u + c(x)u.$$

We propose the following hypothesis:

(H): $\mathcal{L}$ is elliptic in Ω with the coefficients $a_{i,j}$, b_i, $c \in C^{\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. Moreover, the constants $\Lambda \ge \lambda > 0$ and $\Lambda_\alpha > 0$ exist such that

$$\lambda|\xi|^2 \le a_{i,j}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \text{ for all } x \in \bar{\Omega},\ \xi \in \mathbb{R}^n$$

and

$$\frac{1}{\lambda}\left(|a_{i,j}|_\alpha + |b_i|_\alpha + |c|_\alpha\right) \le \Lambda_\alpha.$$

Theorem 3.2. (Agmon–Douglas–Nirenberg [52] or [53, Theorem A.5])*Assume that Condition* (H) *holds, $\partial\Omega \in C^{2+\alpha}$, $b \in C^{1+\alpha}(\partial\Omega)$, and $c \ge 0$, with at least one of c or b being not identical to zero.*

Furthermore, suppose that $\varphi \in C^{2+\alpha}(\partial\Omega)$ *when* $a = 0$, *and* $\varphi \in C^{1+\alpha}(\partial\Omega)$ *when* $a = 1$. *Then the boundary value problem* (3.6) *admits a unique classical solution* $u \in C^{2+\alpha}(\bar{\Omega})$, *satisfying the the estimates*

$$|u|_{2+\alpha,\bar{\Omega}} \leq C\left(\frac{1}{\lambda}|f|_{\alpha,\bar{\Omega}} + |\varphi|_{2+\alpha,\partial\Omega}\right) \text{ when } a = 0,$$

$$|u|_{2+\alpha,\bar{\Omega}} \leq C^*\left(\frac{1}{\lambda}|f|_{\alpha,\bar{\Omega}} + |\varphi|_{1+\alpha,\partial\Omega}\right) \text{ when } a = 1,$$

where the positive constants C *and* C^* *depend on* Ω, α, λ, *and* Λ_α; *moreover,* C^* *also depends on* $|b|_{1+\alpha,\partial\Omega}$.

We now utilize the uniform bounds established in Lemma 3.1 and the Agmon–Douglas–Nirenberg theorem to prove the uniform persistence of solutions to the system (1.5).

Theorem 3.3. *The following two statements hold:*
(i) *If* $R_0 < 1$, *then the DFE* $(\hat{S}_1, \hat{S}_2, 0, 0)$ *is linearly stable.*
(ii) *If* $R_0 > 1$, *then the system* (1.5) *is uniformly persistent: There is a constant* $\varepsilon_0 > 0$, *independent of the initial data* $(S_{1,0}, S_{2,0}, E_0, I_0)$, *such that*

$$\liminf_{t\to\infty} \left\| (S_1(\cdot,t), S_2(\cdot,t), E(\cdot,t), I(\cdot,t)) - \left(\hat{S}_1, \hat{S}_2, 0, 0\right) \right\|_{L^\infty(\Omega)} > \varepsilon_0. \tag{3.7}$$

Furthermore, system (1.5) *admits at least one EE. Here,* $(\hat{S}_1, \hat{S}_2)$ *is given by* (1.8).

Proof. (i) By substituting $(\bar{S}_1, \bar{S}_2, \bar{E}, \bar{I}) = (e^{-\lambda t}\varphi_{S_1}(x), e^{-\lambda t}\varphi_{S_2}(x), e^{-\lambda t}\varphi_E(x), e^{-\lambda t}\varphi_I(x))$ into the linearized system (2.1) and dividing through by $e^{-\lambda t}$, we obtain the following eigenvalue problem:

$$\begin{cases} d_1\Delta\varphi_{S_1} - (\theta + d)\varphi_{S_1} - \beta_1\hat{S}_1\varphi_I + \lambda\varphi_{S_1} = 0, & x \in \Omega, \\ d_2\Delta\varphi_{S_2} + \theta\varphi_{S_1} - d\varphi_{S_2} - \beta_2\hat{S}_2\varphi_I + \lambda\varphi_{S_2} = 0, & x \in \Omega, \\ d_E\Delta\varphi_E + \beta_1\hat{S}_1\varphi_I + \beta_2\hat{S}_2\varphi_I - (d + \sigma)\varphi_E + \lambda\varphi_E = 0, & x \in \Omega, \\ d_I\Delta\varphi_I + \sigma\varphi_E - (\gamma + d + \alpha)\varphi_I + \lambda\varphi_I = 0, & x \in \Omega, \\ \frac{\partial\varphi_{S_1}}{\partial n} = \frac{\partial\varphi_{S_2}}{\partial n} = \frac{\partial\varphi_E}{\partial n} = \frac{\partial\varphi_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.8}$$

Assume that $R_0 < 1$. We will show that the DFE is linearly stable. That is, for any solution

$$(\lambda, \varphi_{S_1}, \varphi_{S_2}, \varphi_E, \varphi_I)$$

of the eigenvalue problem (3.8), if at least one of the components $\varphi_{S_1}, \varphi_{S_2}, \varphi_E, \varphi_I$ is not identically zero, then the real part of the eigenvalue must satisfy $Re(\lambda) > 0$. Suppose, for contradiction, that $(\lambda, \varphi_{S_1}, \varphi_{S_2}, \varphi_E, \varphi_I)$ is a solution of (3.8), with at least one of $\varphi_{S_1}, \varphi_{S_2}, \varphi_E, \varphi_I$ being non-zero, and assume that $Re(\lambda) \leq 0$.

We first show that $\varphi_I \not\equiv 0$ in Ω. Otherwise, $\varphi_I \equiv 0$ in Ω. Then, by the fourth equation of (3.8), we immediately have $\varphi_E \equiv 0$ in Ω. It follows from the first two equations of (3.8) that φ_{S_2} is not identical to zero. Therefore, we consider two possible cases: (a) $\varphi_{S_2} \not\equiv 0, \varphi_{S_1} \equiv 0$; (b) $\varphi_{S_1} \not\equiv 0, \varphi_{S_2} \not\equiv 0$.

In Case (*a*), the eigenvalue problem (3.8) reduces to

$$\begin{cases} d_2\Delta\varphi_{S_2} - d\varphi_{S_2} + \lambda\varphi_{S_2} = 0, & x \in \Omega, \\ \frac{\partial\varphi_{S_2}}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.9}$$

Let λ_2^* denote the principal eigenvalue of the problem (3.9). By the eigenvalue comparison principle and the fact that $d(x) > 0$ on $\bar{\Omega}$, we have $\lambda_2^* \geq \min_{\bar{\Omega}} d(x) > 0$. Moreover, since the operator $d_2\Delta - d$ is self-adjoint under Neumann boundary conditions, all eigenvalues are real. From the assumption $Re(\lambda) \leq 0$, we have $\lambda_2^* \leq \lambda \leq 0$. This contradiction implies that Case (a) cannot occur.

For Case (b), we consider the equation for φ_{S_1}, which reduces to the following eigenvalue problem:

$$\begin{cases} d_1\Delta\varphi_{S_1} - (\theta + d)\varphi_{S_1} + \lambda\varphi_{S_1} = 0, & x \in \Omega, \\ \frac{\partial\varphi_{S_1}}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.10}$$

An argument analogous to that in the proof of Case (a) shows that this case cannot occur.

The contradictions derived above imply that $\varphi_I \not\equiv 0$. Therefore, $(\lambda, \varphi_I, \varphi_E)$ corresponds to a nontrivial solution of the system (2.3). Since $\varphi_I(x) > 0$ and $\varphi_E(x) > 0$ in Ω by the maximum principle, $\lambda = \lambda_1$ is the principal eigenvalue, and (φ_I, φ_E) is the corresponding positive eigenfunction of (2.3). Next, we consider the following two elliptic boundary value problems:

$$\begin{cases} d_1\Delta\varphi_{S_1} - (\theta + d)\varphi_{S_1} + \lambda_1\varphi_{S_1} = \beta_1\hat{S}_1\varphi_I, & x \in \Omega, \\ \frac{\partial\varphi_{S_1}}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{3.11}$$

and

$$\begin{cases} d_2\Delta\varphi_{S_2} - d\varphi_{S_2} + \lambda_1\varphi_{S_2} = \beta_2\hat{S}_2\varphi_I - \theta\varphi_{S_1}, & x \in \Omega, \\ \frac{\partial\varphi_{S_2}}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.12}$$

Since $\beta_1\hat{S}_1\varphi_I \in C^\alpha$, the Agmon–Douglas–Nirenberg theorem (Theorem 3.2) guarantees the existence and uniqueness of a classical solution φ_{S_1} to (3.11). Similarly, the right-hand side of (3.12) also belongs to C^α, so there is a unique solution φ_{S_2} to (3.12). Therefore, we conclude that $(\lambda_1, \varphi_{S_1}, \varphi_{S_2}, \varphi_E, \varphi_I)$ solves the full eigenvalue problem (3.8). Since λ_1 is the principal eigenvalue of (2.3), it is real and satisfies $\lambda_1 \leq \lambda \leq 0$. However, by Lemma 2.2, we know that $R_0 \geq 1$ if and only if $\lambda_1 \leq 0$. This contradicts the assumption $R_0 < 1$. Thus, the DFE is linearly stable.

(ii) We now establish the uniform persistence of the system (1.5) by applying the theory of abstract dynamic systems developed in [54, 55]. Let $X = C(\bar{\Omega}, \mathbb{R}_+^4)$. Define

$$W_0 := \{(S_{1,0}, S_{2,0}, E_0, I_0) \mid E_0 \not\equiv 0\ ,\ I_0 \not\equiv 0\}$$

and

$$\partial W_0 := X \setminus W_0 = \{(S_{1,0}, S_{2,0}, E_0, I_0) \mid E_0 = 0 \text{ or } I_0 = 0\}.$$

It is easy to verify that W_0 and ∂W_0 are relatively open and closed subsets of X, respectively, and that W_0 is convex. By the regularity theory for parabolic equations, for each initial condition $(S_{1,0}, S_{2,0}, E_0, I_0) \in X$, the system (1.5) generates a semiflow $\Phi_t : X \longmapsto X$:

$$\Phi_t\,(S_{1,0}, S_{2,0}, E_0, I_0) = (S_1(\cdot, t), S_2(\cdot, t), E(\cdot, t), I(\cdot, t)),$$

where $(S_1(\cdot, t), S_2(\cdot, t), E(\cdot, t), I(\cdot, t)) \in X$ is the unique classical solution of the system (1.5) corresponding to the given initial data. Furthermore, the standard L^p theory for parabolic equations, together with Sobolev embedding theorems, ensures that for each fixed $t > 0$, the map Φ_t is compact. In addition, by Lemma 3.1, the semiflow Φ_t is point-dissipative.

As in the proof of [15, Theorem 3.2], we deduce that $\Phi_t(W_0) \subset W_0$ for all $t > 0$, and that the maximal positively invariant set of Φ_t in ∂W_0, i.e.,

$$\begin{aligned} A_\partial : &= \{(S_{1,0}, S_{2,0}, E_0, I_0) \in \partial W_0 \mid \Phi_t(S_{1,0}, S_{2,0}, E_0, I_0) \in \partial W_0, t \geq 0\} \\ &= \{(S_{1,0}, S_{2,0}, E_0, I_0) \in X \mid E_0 = 0, I_0 = 0\}. \end{aligned}$$

Furthermore, we conclude that the DFE $(\hat{S}_1, \hat{S}_2, 0, 0)$ is a compact and isolated invariant set for the semiflow Φ_t restricted to A_∂.

Denote the stable set of $(\hat{S}_1, \hat{S}_2, 0, 0)$ by $W^s((\hat{S}_1, \hat{S}_2, 0, 0))$. According to [54], it remains to verify that $W^s((\hat{S}_1, \hat{S}_2, 0, 0))$ does not intersect W_0, i.e., $W^s((\hat{S}_1, \hat{S}_2, 0, 0)) \cap W_0 = \emptyset$. Suppose, for the sake of contradiction, that there is a point $(S_{1,0}, S_{2,0}, E_0, I_0) \in W_0$ lying in the stable set of $(\hat{S}_1, \hat{S}_2, 0, 0)$. Then the unique solution (S_1, S_2, E, I) satisfies

$$\lim_{t\to\infty} S_1(x,t) = \hat{S}_1(x), \quad \lim_{t\to\infty} S_2(x,t) = \hat{S}_2(x) \text{ uniformly on } \bar{\Omega}.$$

For any small $0 < \varepsilon < 1$, a time $T_1 > 0$, exists such that

$$0 < \hat{S}_1(x) - \varepsilon < S_1(t,x) \text{ and } 0 < \hat{S}_2(x) - \varepsilon < S_2(t,x) \text{ for } (x,t) \in \overline{\Omega} \times [T_1, \infty).$$

Since $R_0 > 1$, Lemma 2.2 implies that $\lambda_1 < 0$, where λ_1 is the principal eigenvalue of (2.3). Therefore, a sufficiently small ε exists such that the principal eigenvalue

$$\begin{cases} -d_E \Delta \varphi_E + [d(x) + \sigma(x)]\varphi_E - [\beta_1(x)(\hat{S}_1(x) - \varepsilon) + \beta_2(x)(\hat{S}_2(x) - \varepsilon)]\varphi_I = \lambda_1(\varepsilon)\varphi_E, & x \in \Omega, \\ -d_I \Delta \varphi_I - \sigma(x)\varphi_E + [\gamma(x) + d(x) + \alpha(x)]\varphi_I = \lambda_1(\varepsilon)\varphi_I, & x \in \Omega, \\ \frac{\partial \varphi_E}{\partial n} = \frac{\partial \varphi_I}{\partial n} = 0, & x \in \partial\Omega \end{cases}$$

remains negative. Let $(\varphi_E^\varepsilon, \varphi_I^\varepsilon)$ be the corresponding positive eigenfunction associated with $\lambda_1(\varepsilon)$.

Next, we enlarge T_1 if necessary and consider the parabolic system

$$\begin{cases} \frac{\partial \omega(x,t)}{\partial t} - d_E \Delta \omega(x,t) = [\beta_1(x)(\hat{S}_1(x) - \varepsilon) + \beta_2(x)(\hat{S}_2(x) - \varepsilon)]\nu(x,t) & \\ \qquad\qquad - [d(x) + \sigma(x)]\omega(x,t), & x \in \Omega,\ t > T_1, \\ \frac{\partial \nu(x,t)}{\partial t} - d_I \Delta \nu(x,t) = \sigma(x)\omega(x,t) - [\gamma(x) + d(x) + \alpha(x)]\nu(x,t), & x \in \Omega,\ t > T_1, \\ \frac{\partial \omega(x,t)}{\partial n} = \frac{\partial \nu(x,t)}{\partial n} = 0, & x \in \partial\Omega,\ t > T_1, \\ \omega(x, T_1) = E(x, T_1) > 0, \nu(x, T_1) = I(x, T_1) > 0, & x \in \Omega. \end{cases} \tag{3.13}$$

We now choose a sufficiently small constant $\varrho > 0$ such that

$$E(x, T_1) \geq \varrho e^{-\lambda(\varepsilon)T_1}\varphi_E^\varepsilon(x), I(x, T_1) \geq \varrho e^{-\lambda(\varepsilon)T_1}\varphi_I^\varepsilon(x) \text{ on } \bar{\Omega}.$$

By the comparison principle, the pair $(\varrho e^{-\lambda(\varepsilon)t}\varphi_E^\varepsilon(x), \varrho e^{-\lambda(\varepsilon)t}\varphi_I^\varepsilon(x))$ is a subsolution of the system (3.13) for $t \geq T_1$. It is clear that $(E(x,t), I(x,t))$ is a supersolution of (3.13). Hence, we obtain

$$E(x,t) \geq \varrho e^{-\lambda(\varepsilon)t}\varphi_E^\varepsilon(x) \text{ and } I(x,t) \geq \varrho e^{-\lambda(\varepsilon)t}\varphi_I^\varepsilon(x),$$

for all $x \in \bar{\Omega}$, $t \geq T_1$. Since $\lambda_1(\varepsilon) < 0$, we deduce that

$$\varrho e^{-\lambda(\varepsilon)t}\varphi_E^\varepsilon(x) \to \infty \text{ and } \varrho e^{-\lambda(\varepsilon)t}\varphi_I^\varepsilon(x) \to \infty,$$

uniformly on $\bar{\Omega}$, as $t \to \infty$. Therefore,

$$E(x,t),\ I(x,t) \to \infty \text{ uniformly on } \bar{\Omega} \text{ as } t \to \infty,$$

which contradicts the uniform boundedness of the solutions established in Lemma 3.1. This contradiction implies that E_0 is isolated in X, and that $W^s(E_0) \cap W_0 = \emptyset$.

In summary, it follows from [54, Theorem 4.5] or [55, Theorem 1.3.1] that (3.7) holds. Furthermore, by [54, Theorem 4.7] or [55, Theorem 1.3.7], uniform persistence implies the existence of an EE. This completes the proof. □

4. Global stability of the DFE and EE

In this section, we assume that the parameters $\Lambda, \beta_1, \beta_2, \gamma, d, \alpha, \sigma$, and θ are all positive constants. Under this assumption, the basic reproduction number R_0 can be expressed as

$$\begin{aligned} R_0 &= \frac{\sigma\Lambda(\beta_1 d + \beta_2\theta)}{d(d+\sigma)(d+\theta)(\gamma+d+\alpha)} \\ &= \frac{\sigma\Lambda\beta_1}{(d+\sigma)(d+\theta)(\gamma+d+\alpha)} + \frac{\sigma\Lambda\theta\beta_2}{d(d+\sigma)(d+\theta)(\gamma+d+\alpha)}. \end{aligned} \tag{4.1}$$

If $R_0 \le 1$, there is a unique constant DFE

$$E_0 = (\frac{\Lambda}{\theta+d}, \frac{\theta\Lambda}{d(\theta+d)}, 0, 0).$$

If $R_0 > 1$, the unique constant EE $\tilde{E}(S_1^*, S_2^*, E^*, I^*)$ satisfies the following system of equations:

$$\begin{cases} \Lambda - \beta_1 S_1 I - \theta S_1 - dS_1 = 0, \\ \theta S_1 - \beta_2 S_2 I - dS_2 = 0, \\ \beta_1 S_1 I + \beta_2 S_2 I - (d+\sigma)E = 0, \\ \sigma E - (\gamma+d+\alpha)I = 0. \end{cases} \tag{4.2}$$

By direct calculation from (4.2), the unique positive solution is given by

$$I^* = \frac{b + \sqrt{b^2 + 4(d+\sigma)(\gamma+d+\alpha)\beta_1\beta_2[d(d+\theta)(d+\sigma)(\gamma+d+\alpha)(R_0-1)]}}{2(d+\sigma)(\gamma+d+\alpha)\beta_1\beta_2},$$

where $b = \sigma\Lambda\beta_1\beta_2 - (d+\sigma)(\gamma+d+\alpha)(\beta_1 d + \beta_2\theta + \beta_2 d)$. It is straightforward to verify that

$$S_1^* = \frac{\Lambda}{\beta_1 I^* + \theta + d},\ S_2^* = \frac{\theta\Lambda}{(\beta_1 I^* + \theta + d)(\beta_2 I^* + d)},\ E^* = \frac{\gamma+d+\alpha}{\sigma} I^*.$$

To investigate the global stability of the DFE and the EE, we construct an appropriate Lyapunov function.

Theorem 4.1. *Assume that $R_0 < 1$. Then the DFE $(\hat{S}_1, \hat{S}_2, 0, 0)$ is the global attractor of the system* (1.5).

Proof. Define the Lyapunov functional

$$W(t) = \int_\Omega [L(S_1(x,t), S_2(x,t), E(x,t), I(x,t))]\,\mathrm{d}x \text{ for } t > 0,$$

where

$$L(S_1, S_2, E, I) = \frac{1}{2}\left(S_1 - \hat{S}_1\right)^2 + \frac{1}{2}a\left(S_2 - \hat{S}_2\right)^2 + bE + cI,$$

and a, b, c are positive constants to be determined.

For simplicity, we define the following functions:

$$f_1(S_1, S_2, E, I) = \Lambda - \beta_1 S_1 I - \theta S_1 - dS_1,$$

$$f_2(S_1, S_2, E, I) = -\beta_2 S_2 I + \theta S_1 - dS_2,$$

$$f_3(S_1, S_2, E, I) = \beta_1 S_1 I + \beta_2 S_2 I - (d + \sigma)E,$$

$$f_4(S_1, S_2, E, I) = \sigma E - (\gamma + d + \alpha)I.$$

By direct computation, we obtain

$$\begin{aligned}
\frac{\mathrm{d}W}{\mathrm{dt}} &= \int_\Omega \left[L_{S_1}\frac{\partial S_1}{\partial t} + L_{S_2}\frac{\partial S_2}{\partial t} + L_E\frac{\partial E}{\partial t} + L_I\frac{\partial I}{\partial t}\right]\mathrm{d}x \\
&= \int_\Omega (d_1 L_{S_1}\Delta S_1 + d_2 L_{S_2}\Delta S_2 + d_E L_E \Delta E + d_I L_I \Delta I)\,\mathrm{d}x \\
&\quad + \int_\Omega (L_{S_1} f_1 + L_{S_2} f_2 + L_E f_3 + L_I f_4)\,\mathrm{d}x \\
&= H_1 + H_2,
\end{aligned}$$

where

$$\begin{aligned}
H_1 &= \int_\Omega (d_1 L_{S_1}\Delta S_1 + d_2 L_{S_2}\Delta S_2 + d_E L_E \Delta E + d_I L_I \Delta I)\,\mathrm{d}x \\
&= d_1 \int_\Omega \left(S_1 - \hat{S_1}\right)\Delta S_1 \mathrm{d}x + a d_2 \int_\Omega \left(S_2 - \hat{S_2}\right)\Delta S_2 \mathrm{d}x \\
&\quad + b d_E \int_\Omega \Delta E \mathrm{d}x + c d_I \int_\Omega \Delta I \mathrm{d}x \\
&= -d_1 \int_\Omega |\nabla S_1|^2 \mathrm{d}x - d_2 a \int_\Omega |\nabla S_2|^2 \mathrm{d}x \\
&\leq 0
\end{aligned}$$

and

$$\begin{aligned}
H_2 &= \int_\Omega (L_{S_1} f_1 + L_{S_2} f_2 + L_E f_3 + L_I f_4)\,\mathrm{d}x \\
&= -\int_\Omega (\theta + d + \beta_1 I)\left(S_1 - \hat{S}_1\right)^2 \mathrm{d}x - \int_\Omega a(d + \beta_2 I)\left(S_2 - \hat{S}_2\right)^2 \mathrm{d}x \\
&\quad + \int_\Omega a\theta\left(S_1 - \hat{S}_1\right)\left(S_2 - \hat{S}_2\right)\mathrm{d}x + \int_\Omega \beta_1\left(b - \hat{S}_1\right)\left(S_1 - \hat{S}_1\right) I \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \beta_2\left(b-a\hat{S}_2\right)\left(S_2-\hat{S}_2\right)I\mathrm{d}x+\int_\Omega b\left(\beta_1\hat{S}_1+\beta_2\hat{S}_2\right)I\mathrm{d}x\\
&-\int_\Omega c(d+\gamma+\alpha)I\mathrm{d}x+\int_\Omega[\sigma c-(d+\sigma)b]E\mathrm{d}x\\
\le&-\int_\Omega(\theta+d)\left(S_1-\hat{S}_1\right)^2\mathrm{d}x-\int_\Omega ad\left(S_2-\hat{S}_2\right)^2\mathrm{d}x+\int_\Omega a\theta\left(S_1-\hat{S}_1\right)\left(S_2-\hat{S}_2\right)\mathrm{d}x\\
&+\int_\Omega \beta_1\left(b-\hat{S}_1\right)\left(S_1-\hat{S}_1\right)I\mathrm{d}x+\int_\Omega \beta_2\left(b-a\hat{S}_2\right)\left(S_2-\hat{S}_2\right)I\mathrm{d}x\\
&+\int_\Omega\left[b\left(\beta_1\hat{S}_1+\beta_2\hat{S}_2\right)-c(d+\gamma+\alpha)\right]I\mathrm{d}x+\int_\Omega[\sigma c-(d+\sigma)b]E\mathrm{d}x.
\end{aligned}
$$

Set

$$a=\frac{\hat{S}_1}{\hat{S}_2}=\frac{d}{\theta},b=\hat{S}_1.$$

Therefore

$$
\begin{aligned}
H_2\le&-\int_\Omega(\theta+d)\left(S_1-\hat{S}_1\right)^2\mathrm{d}x-\int_\Omega \frac{d^2}{\theta}\left(S_2-\hat{S}_2\right)^2\mathrm{d}x+\int_\Omega d\left(S_1-\hat{S}_1\right)\left(S_2-\hat{S}_2\right)\mathrm{d}x\\
&+\int_\Omega\left[b\left(\beta_1\hat{S}_1+\beta_2\hat{S}_2\right)-c(d+\gamma+\alpha)\right]I\mathrm{d}x+\int_\Omega[\sigma c-(d+\sigma)b]E\mathrm{d}x.
\end{aligned}
$$

Let

$$
\begin{aligned}
F(S_1,S_2,E,I)\triangleq&-(\theta+d)\left(S_1-\hat{S}_1\right)^2-\frac{d^2}{\theta}\left(S_2-\hat{S}_2\right)^2+d\left(S_1-\hat{S}_1\right)\left(S_2-\hat{S}_2\right)\\
&+\left[b\left(\beta_1\hat{S}_1+\beta_2\hat{S}_2\right)-c(d+\gamma+\alpha)\right]I+[c\sigma-b(d+\sigma)]E.
\end{aligned}
$$

Consider the quadratic form in the variables $S_1-\hat{S}_1$ and $S_2-\hat{S}_2$ given by

$$-(\theta+d)\left(S_1-\hat{S}_1\right)^2-\frac{d^2}{\theta}\left(S_2-\hat{S}_2\right)^2+d\left(S_1-\hat{S}_1\right)\left(S_2-\hat{S}_2\right).\tag{4.3}$$

The discriminant of this quadratic form is $\Delta=\frac{-3\theta d^2-4d^3}{\theta}<0$. Therefore, the quadratic form (4.3) is negative definite. Recall that $R_0<1$ (i.e., $R_0=\frac{\sigma(\beta_1\hat{S}_1+\beta_2\hat{S}_2)}{(d+\sigma)(d+\gamma+\alpha)}<1$), it then follows that $\frac{\beta_1\hat{S}_1+\beta_2\hat{S}_2}{d+\gamma+\alpha}<\frac{d+\sigma}{\sigma}$. If c satisfies

$$\frac{\hat{S}_1(\beta_1\hat{S}_1+\beta_2\hat{S}_2)}{d+\gamma+\alpha}<c<\frac{\hat{S}_1(d+\sigma)}{\sigma},$$

then we can see that $H_2\le 0$. Therefore, $\frac{\mathrm{dW}}{\mathrm{dt}}\le 0$. It is evident that $\frac{\mathrm{dW}}{\mathrm{dt}}=0$ if and only if $(S_1,S_2,E,I)=(\hat{S}_1,\hat{S}_2,0,0)$. Thus, W is a Lyapunov functional of the system (1.5). Furthermore,

$$(S_1(x,t),S_2(x,t),E(x,t),I(x,t))\to(\hat{S}_1,\hat{S}_2,0,0),$$

uniformly in $[L^\infty(\Omega)]^4$ as $t\to\infty$. This implies that $(\hat{S}_1,\hat{S}_2,0,0)$ is the global attractor of (1.5). □

Theorem 4.2. *If $R_0>1$, then the endemic equilibrium $\tilde{E}(S_1^*,S_2^*,E^*,I^*)$ is the global attractor of* (1.5).

Proof. We define

$$W(t) = \int_\Omega [L(S_1(x,t), S_2(x,t), E(x,t), I(x,t))]\,\mathrm{d}x, \quad \forall t > 0,$$

where

$$\begin{aligned} L(S_1, S_2, E, I) =& \left(S_1 - S_1^* - S_1^* \ln \frac{S_1}{S_1^*}\right) + \left(S_2 - S_2^* - S_2^* \ln \frac{S_2}{S_2^*}\right) \\ &+ \left(E - E^* - E^* \ln \frac{E}{E^*}\right) + \frac{d+\sigma}{\sigma}\left(I - I^* - I^* \ln \frac{I}{I^*}\right). \end{aligned}$$

By some calculations, we have

$$\begin{aligned} \frac{\mathrm{d}W(t)}{\mathrm{d}t} =& \int_\Omega \left[L_{S_1}\frac{\partial S_1}{\partial t} + L_{S_2}\frac{\partial S_2}{\partial t} + L_E\frac{\partial E}{\partial t} + L_I\frac{\partial I}{\partial t}\right]\mathrm{d}x \\ =& \int_\Omega (d_1 L_{S_1}\Delta S_1 + d_2 L_{S_2}\Delta S_2 + d_E L_E \Delta E + d_I L_I \Delta I)\,\mathrm{d}x \\ &+ \int_\Omega (L_{S_1} f_1 + L_{S_2} f_2 + L_E f_3 + L_I f_4)\,\mathrm{d}x \\ =& V_1 + V_2, \end{aligned}$$

where

$$f_1(S_1, S_2, E, I),\ f_2(S_1, S_2, E, I),\ f_3(S_1, S_2, E, I) \text{ and } f_4(S_1, S_2, E, I)$$

is given by Theorem 4.1. By direct calculation, we obtain

$$\begin{aligned} V_1 =& \int_\Omega (d_1 L_{S_1}\Delta S_1 + d_2 L_{S_2}\Delta S_2 + d_E L_E \Delta E + d_I L_I \Delta I)\,\mathrm{d}x \\ =& d_1 \int_\Omega \left(1 - \frac{S_1^*}{S_1}\right)\Delta S_1 \mathrm{d}x + d_2 \int_\Omega \left(1 - \frac{S_2^*}{S_2}\right)\Delta S_2 \mathrm{d}x \\ &+ d_E \int_\Omega \left(1 - \frac{E^*}{E}\right)\Delta E \mathrm{d}x + d_I \frac{d+\sigma}{\sigma}\int_\Omega \left(1 - \frac{I^*}{I}\right)\Delta I \mathrm{d}x \\ =& - d_1 S_1^* \int_\Omega \left(\frac{|\nabla S_1|^2}{S_1^2}\right)\mathrm{d}x - d_2 S_2^* \int_\Omega \left(\frac{|\nabla S_2|^2}{S_2^2}\right)\mathrm{d}x \\ &- d_E E^* \int_\Omega \left(\frac{|\nabla E|^2}{E^2}\right)\mathrm{d}x - d_I I^* \frac{d+\sigma}{\sigma}\int_\Omega \left(\frac{|\nabla I|^2}{I^2}\right)\mathrm{d}x \\ \le& 0, \end{aligned}$$

and

$$\begin{aligned} V_2 =& \int_\Omega (L_{S_1} f_1 + L_{S_2} f_2 + L_E f_3 + L_I f_4)\,\mathrm{d}x \\ =& \int_\Omega (S_1 - S_1^*)\left(\frac{\Lambda}{S_1} - \beta_1 I - \theta - d\right)\mathrm{d}x + \frac{d+\sigma}{\sigma}\int_\Omega (I - I^*)\left[\frac{\sigma E}{I} - (\gamma + d + \alpha)\right]\mathrm{d}x \\ &+ \int_\Omega (S_2 - S_2^*)\left(\frac{\theta S_1}{S_2} - \beta_2 I - d\right)\mathrm{d}x + \int_\Omega (E - E^*)\left[\frac{\beta_1 S_1 I + \beta_2 S_2 I}{E} - (d+\sigma)\right]\mathrm{d}x. \end{aligned}$$

Given that (S_1^*, S_2^*, E^*, I^*) is a solution to (4.2), we obtain

$$\begin{aligned}V_2 =& \int_\Omega (S_1 - S_1^*)\left[\Lambda\left(\frac{1}{S_1} - \frac{1}{S_1^*}\right) - \beta_1 (I - I^*)\right]\mathrm{d}x + \int_\Omega (S_2 - S_2^*)\left[\theta\left(\frac{S_1}{S_2} - \frac{S_1^*}{S_2^*}\right) - \beta_2 (I - I^*)\right]\mathrm{d}x \\ &+ \int_\Omega (E - E^*)\left[\beta_1\left(\frac{S_1 I}{E} - \frac{S_1^* I^*}{E^*}\right) + \beta_2\left(\frac{S_2 I}{E} - \frac{S_2^* I^*}{E^*}\right)\right]\mathrm{d}x + (d+\sigma)\int_\Omega (I - I^*)\left[\frac{E}{I} - \frac{E^*}{I^*}\right]\mathrm{d}x \\ =& \int_\Omega \beta_1 S_1^* I^*\left(3 - \frac{S_1^*}{S_1} - \frac{S_1 E^* I}{S_1^* E I^*} - \frac{E I^*}{E^* I}\right)\mathrm{d}x + \int_\Omega d S_2^*\left(3 - \frac{S_1^*}{S_1} - \frac{S_2}{S_2^*} - \frac{S_1 S_2^*}{S_1^* S_2}\right)\mathrm{d}x \\ &+ \int_\Omega d S_1^*\left(2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*}\right)\mathrm{d}x + \int_\Omega \beta_2 S_2^* I^*\left(4 - \frac{S_1^*}{S_1} - \frac{S_1 S_2^*}{S_1^* S_2} - \frac{S_2 E^* I}{S_2^* E I^*} - \frac{E I^*}{E^* I}\right)\mathrm{d}x.\end{aligned}$$

For any S_1, S_2, E, and $I > 0$, by the inequality between the geometric mean and the arithmetic mean, we have

$$2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*} \le 0,\ 3 - \frac{S_1^*}{S_1} - \frac{S_2}{S_2^*} - \frac{S_1 S_2^*}{S_1^* S_2} \le 0,$$
$$3 - \frac{S_1^*}{S_1} - \frac{S_1 E^* I}{S_1^* E I^*} - \frac{E I^*}{E^* I} \le 0,\ 4 - \frac{S_1^*}{S_1} - \frac{S_1 S_2^*}{S_1^* S_2} - \frac{S_2 E^* I}{S_2^* E I^*} - \frac{E I^*}{E^* I} \le 0.$$

Therefore, $V_2 \le 0$. It follows that $\frac{\mathrm{dW}}{\mathrm{dt}} \le 0$, and equality holds if and only if

$$(S_1, S_2, E, I) = (S_1^*, S_2^*, E^*, I^*).$$

Hence, W is a Lyapunov functional for the system (1.5). Furthermore

$$(S_1(x,t), S_2(x,t), E(x,t), I(x,t)) \to (S_1^*, S_2^*, E^*, I^*)$$

uniformly in $[L^\infty(\Omega)]^4$ as $t \to \infty$, which implies that (S_1^*, S_2^*, E^*, I^*) is the global attractor of (1.5).

□

5. Asymptotic profiles of the EE

To simplify the notation, we write

$$f^* = \max_{x\in\bar{\Omega}} f(x) \ \text{ and } \ f_* = \min_{x\in\bar{\Omega}} f(x),$$

where $f = \Lambda, \beta_1, \beta_2, \sigma, d, \theta, \alpha, \gamma$. We now present a useful lemma that will be employed in this section.

Lemma 5.1. ([56] or [57, Lemma 3.1]) *Assume that $\omega \in C^2(\overline{\Omega})$ and satisfies $\frac{\partial\omega}{\partial n} = 0, x \in \partial\Omega$. Then the following properties hold:*

(i) *If w has a local maximum at $x_0 \in \bar{\Omega}$, then $\nabla\omega(x_0) = 0$ and $\Delta\omega(x_0) \le 0$.*

(ii) *If w has a local minimum at $y_0 \in \bar{\Omega}$, then $\nabla\omega(y_0) = 0$ and $\Delta\omega(y_0) \ge 0$.*

Recall that the system defined by (1.5) admits at least one positive equilibrium when $R_0 > 1$, as established in Theorem 3.3(ii). We now investigate the behavior of the EE in the limit as the diffusion parameters tend to zero. The results presented in this section may offer some insight into how spatial heterogeneity influences the disease's dynamics and could inform the development of control strategies under certain conditions.

5.1. The case of $d_1 \to 0$

We first examine the asymptotic behavior of the EE to in the system (1.5) as $d_1 \to 0$, while $d_2 > 0$, $d_E > 0$, and $d_I > 0$ are fixed. By Theorem 2.3(i) and Theorem 3.3(ii), if $\tilde{R}_0 > 1$, then the system (1.5) admits at least one EE. The main result in this case is stated below.

Theorem 5.2. *Assume that $\tilde{R}_0 > 1$. For fixed $d_2 > 0$, $d_E > 0$, and $d_I > 0$, and let $d_1 \to 0$. Then every positive solution (S_1, S_2, E, I) of (1.7) (up to a subsequence as $d_1 \to 0$) satisfies*

$$(S_1, S_2, E, I) \to (\tilde{S}_1, \tilde{S}_2, \tilde{E}, \tilde{I}) \text{ uniformly on } \bar{\Omega}$$

where

$$\tilde{S}_1(x) = \frac{\Lambda(x)}{\theta(x) + d(x) + \beta_1(x)\tilde{I}(x)},$$

and $(\tilde{S}_2, \tilde{E}, \tilde{I})$ is a positive solution of the following problem:

$$\begin{cases} -d_2\Delta\tilde{S}_2 = \theta(x)\tilde{S}_1 - \beta_2(x)\tilde{S}_2\tilde{I} - d(x)\tilde{S}_2, & x \in \Omega, \\ -d_E\Delta\tilde{E} = \beta_1(x)\tilde{S}_1\tilde{I} + \beta_2(x)\tilde{S}_2\tilde{I} - [d(x) + \sigma(x)]\tilde{E}, & x \in \Omega, \\ -d_I\Delta\tilde{I} = \sigma(x)\tilde{E} - [\gamma(x) + d(x) + \alpha(x)]\tilde{I}, & x \in \Omega, \\ \frac{\partial\tilde{S}_2}{\partial n} = \frac{\partial\tilde{E}}{\partial n} = \frac{\partial\tilde{I}}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

Proof. **Step 1.** Estimates of the upper and lower bounds.

Assume that $S_1(x_0) = \max_{x\in\bar{\Omega}} S_1(x)$, $x_0 \in \bar{\Omega}$. Applying Lemma 5.1 to the first equation of (1.7), we obtain

$$\Lambda(x_0) \geq [\theta(x_0) + d(x_0)]S_1(x_0) + \beta_1(x_0)S_1(x_0)I(x_0) \geq [\theta(x_0) + d(x_0)]S_1(x_0),$$

which implies that

$$S_1(x) \leq S_1(x_0) \leq \frac{\Lambda^*}{\theta_* + d_*} \text{ for all } x \in \bar{\Omega}.$$

Let $S_2(x_1) = \max_{x\in\bar{\Omega}} S_2(x)$ for some $x_1 \in \bar{\Omega}$. It follows from the second equation of (1.7) and Lemma 5.1 that

$$\theta(x_1)S_1(x_1) \geq \beta_2(x_1)S_2(x_1)I(x_1) + d(x_1)S_2(x_1) \geq d(x_1)S_2(x_1).$$

Combining this with (5.1), we obtain

$$S_2(x) \leq S_2(x_1) \leq \frac{\theta^*}{d_*}S_1(x_1) \leq \frac{\theta^*\Lambda^*}{d_*(\theta_* + d_*)} \text{ for all } x \in \bar{\Omega}.$$

We now set $V(x) = d_1S_1 + d_2S_2 + d_EE + d_II$. Adding the equations in (1.7) yields

$$\begin{aligned} &-\Delta(d_1S_1 + d_2S_2 + d_EE + d_II) \\ =&\Lambda(x) - d(x)S_1 - d(x)S_2 - d(x)E - (\gamma(x) + d(x) + \alpha(x))I. \end{aligned}$$

Assume that $V(x_2) = \max_{x\in\bar{\Omega}} V(x)$, $x_2 \in \bar{\Omega}$. By Lemma 5.1, we have

$$\Lambda(x_2) \geq d(x_2)S_1(x_2) + d(x_2)S_2(x_2) + d(x_2)E(x_2) + (\gamma(x_2) + d(x_2) + \alpha(x_2))I(x_2).$$

This implies that

$$S_1(x_2) + S_2(x_2) + E(x_2) + I(x_2) \le \frac{\Lambda(x_2)}{d(x_2)} \le \frac{\Lambda^*}{d_*}.$$

Without loss of generality, assume that $0 < d_1 < 1$. In this case,

$$\max_{x\in\bar\Omega}(d_1S_1(x) + d_2S_2(x) + d_EE(x) + d_II(x)) \le \max_{x\in\bar\Omega} V(x) = V(x_2) \le M\frac{\Lambda^*}{d_*},$$

where $M = \max\{1, d_2, d_E, d_I\}$. Therefore, for all $x \in \bar\Omega$, we obtain

$$E(x) \le \frac{1}{d_E}\max_{x\in\bar\Omega} V(x) \le \frac{M}{d_E}\frac{\Lambda^*}{d_*},$$

and

$$I(x) \le \frac{1}{d_I}\max_{x\in\bar\Omega} V(x) \le \frac{M}{d_I}\frac{\Lambda^*}{d_*}.$$

Then there is a positive constant C, independent of d_1, such that

$$E(x), I(x) \le C \text{ for all } x \in \bar\Omega.$$

Assume that $S_1(x_3) = \min_{x\in\bar\Omega} S_1(x)$, $x_3 \in \bar\Omega$. Applying Lemma 5.1 to the first equation of (1.7), we obtain

$$S_1(x) \ge S_1(x_3) \ge \frac{\Lambda_*}{\beta_1^*C + \theta^* + d^*} \text{ for all } x \in \bar\Omega.$$

We can then find a positive constant C, independent of d_1, such that

$$S_1(x) \ge \min_{x\in\bar\Omega} S_1(x) = S_1(x_3) \ge C > 0 \text{ for all } x \in \bar\Omega.$$

Next, we estimate the lower bound of S_2. Let $S_2(x_4) = \min_{x\in\bar\Omega} S_2(x)$ for some $x_4 \in \bar\Omega$. We apply Lemma 5.1 to the second equation of (1.7) to conclude that

$$S_2(x) \ge S_2(x_4) \ge \frac{\theta(x_4)S_1(x_4)}{\beta_2(x_4)I(x_4) + d(x_4)}.$$

Since $S_1(x)$ has a positive lower bound and $I(x)$ has a positive upper bound, it follows that $\frac{\theta(x_4)S_1(x_4)}{\beta_2(x_4)I(x_4)+d(x_4)}$ is also bounded below by a positive constant. Therefore, there is a constant $C > 0$, independent of d_1, such that

$$S_2(x) \ge \min_{x\in\bar\Omega} S_2(x) = S_2(x_4) \ge C > 0 \text{ for all } x \in \bar\Omega.$$

From the analysis above, a constant $C > 0$ independent of d_1 exists such that

$$\frac{1}{C} \le S_1(x),\ S_2(x) \le C \text{ and } I(x),\ E(x) \le C \text{ for all } x \in \bar\Omega. \tag{5.2}$$

Next, we claim that E also has a positive lower bound. We argue by contradiction. Suppose, to the contrary, that no such lower bound exists. Then there is a sequence $d_n := d_{1,n} \to 0$ as $n \to \infty$, and a corresponding positive solution

$$(S_{1,n}, S_{2,n}, E_n, I_n) = (S_{1,d_n}, S_{2,d_n}, E_{d_n}, I_{d_n}),$$

of system (1.7), such that $\min_{x\in\bar{\Omega}} E_n \to 0$ as $n \to \infty$. From the third equation of (1.7), we observe that

$$-d_E \Delta E_n + [d(x) + \sigma(x)]E_n \geq 0, \; x \in \Omega; \quad \frac{\partial E_n}{\partial n} = 0, \; x \in \partial\Omega.$$

Therefore, by applying [58, Lemma 2.1] with $p = 1$, we obtain

$$\|E_n\|_{L^1(\Omega)} \leq C \inf_{\Omega} E_n.$$

This implies

$$\|E_n\|_{L^1(\Omega)} \to 0 \text{ as } n \to \infty.$$

We now integrate the third equation of (1.7) over Ω, using (5.2) to obtain

$$\frac{\beta_{1,*} + \beta_{2,*}}{C} \int_\Omega I_n dx \leq \int_\Omega (\beta_1(x) S_{1,n} + \beta_2(x) S_{2,n}) I_n dx = \int_\Omega [d(x) + \sigma(x)] E_n dx.$$

It follows that $\|I_n\|_{L^1(\Omega)} \to 0$ as $n \to \infty$.

We examine the following equation satisfied by I_n:

$$\begin{cases} -d_I \Delta I_n + [\gamma(x) + d(x) + \alpha(x)] I_n = \sigma(x) E_n, & x \in \Omega, \\ \frac{\partial I_n}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{5.3}$$

From (5.2), we know that

$$\|\sigma(x) E_n\|_{L^p(\Omega)} \leq C \text{ for } p \geq 1.$$

By standard elliptic L^p theory and the Sobolev embedding theorem, we deduce that

$$\|I_n\|_{C^{1+\alpha}(\bar{\Omega})} \leq C \|I_n\|_{W^{2,p}(\Omega)} \leq C \|\sigma(x) E_n\|_{L^p(\Omega)} \leq C$$

for some $\alpha \in (0, 1)$, where C is independent of n. Therefore, the sequence $\{I_n\}_{0<d_{1,n}\leq 1}$ is precompact in $C^1(\bar{\Omega})$. Thus, there is a subsequence of $d_{1,n} \to 0$, still denoted by d_n, and a corresponding sequence of positive solutions $(S_{1,n}, S_{2,n}, E_n, I_n)$ to (1.7), such that

$$I_n \to \bar{I} \text{ in } C^1(\bar{\Omega}) \text{ as } n \to \infty.$$

Consequently, $\bar{I} \equiv 0$. Indeed, if $\bar{I} \not\equiv 0$, then $\int_\Omega I_n \mathrm{d}x \to \int_\Omega \bar{I} \mathrm{d}x > 0$, which contradicts the fact that $\|I_n\|_{L^1(\Omega)} \to 0$ as $n \to \infty$. Therefore, we conclude that

$$I_n \to 0 \;\text{ in }\; C^1(\bar{\Omega}), \;\text{as } n \to \infty. \tag{5.4}$$

In the following, we focus on the first equation of the system (1.7) as follows:

$$-d_n \Delta S_{1,n} = \Lambda(x) - \beta_1(x) S_{1,n} I_n - \theta(x) S_{1,n} - d(x) S_{1,n}, \; x \in \Omega; \quad \frac{\partial S_{1,n}}{\partial n} = 0, \; x \in \partial\Omega.$$

In view of (5.4), for any $\varepsilon > 0$, a positive integer N, exists such that, for all $n > N$,

$$0 < I_n < \varepsilon, \; x \in \bar{\Omega}. \tag{5.5}$$

Consequently, for all $n \geq N$, it follows that

$$\Lambda(x) - \beta_1(x)S_{1,n}I_n - \theta(x)S_{1,n} - d(x)S_{1,n} \leq \Lambda(x) - \theta(x)S_{1,n} - d(x)S_{1,n},$$

and

$$\Lambda(x) - \beta_1(x)S_{1,n}I_n - \theta(x)S_{1,n} - d(x)S_{1,n} \geq \Lambda(x) - \theta(x)S_{1,n} - d(x)S_{1,n} - \varepsilon\beta_1(x)S_{1,n}.$$

By the comparison principle, we know that $S_{1,n}$ is an upper solution of the problem

$$-d_n\Delta\underline{Z} = \Lambda(x) - (\varepsilon\beta_1(x) + \theta(x) + d(x))\underline{Z},\ x \in \Omega; \quad \frac{\partial\underline{Z}}{\partial n} = 0,\ x \in \partial\Omega. \tag{5.6}$$

Here, $\underline{Z}_n$ denotes the unique positive solution of (5.6). Using an argument analogous to the proof of [59, Lemma 2.4], we deduce that

$$\underline{Z}_n \to \frac{\Lambda(x)}{\varepsilon\beta_1(x) + \theta(x) + d(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty.$$

Therefore, $S_{1,n}$ satisfies

$$\liminf_{n\to\infty} S_{1,n} \geq \lim_{n\to\infty} \underline{Z}_n = \frac{\Lambda(x)}{\varepsilon\beta_1(x) + \theta(x) + d(x)} \text{ on } \bar{\Omega}. \tag{5.7}$$

Similarly, we consider the problem

$$-d_n\Delta\overline{Z} = \Lambda(x) - (\theta(x) + d(x))\overline{Z},\ x \in \Omega; \quad \frac{\partial\overline{Z}}{\partial n} = 0,\ x \in \partial\Omega,$$

which admits a unique positive solution, denoted by $\overline{Z}_n$. In fact, by the comparison principle, $S_{1,n}$ serves as a lower solution to this elliptic problem. Furthermore, we use [59, Lemma 2.4] to conclude that

$$\limsup_{n\to\infty} S_{1,n} \leq \lim_{n\to\infty} \overline{Z}_n = \frac{\Lambda(x)}{\theta(x) + d(x)}. \tag{5.8}$$

By the arbitrariness of ε, together with (5.7) and (5.8), we obtain

$$S_{1,n}(x) \to \hat{S}_1(x) := \frac{\Lambda(x)}{\theta(x) + d(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty. \tag{5.9}$$

Next, we focus on the following equation satisfied by $S_{2,n}$:

$$-d_2\Delta S_{2,n} = \theta(x)S_{1,n} - \beta_2(x)S_{2,n}I_n - d(x)S_{2,n},\ x \in \Omega; \quad \frac{\partial S_{2,n}}{\partial n} = 0,\ x \in \partial\Omega.$$

Thanks to (5.4) and (5.9), the method of upper and lower solutions can be applied to show that

$$S_{2,n} \to \tilde{S}_2 \text{ uniformly on } \bar{\Omega}, \text{ as } n \to \infty,$$

where $\tilde{S}_2$ is determined by (2.6).

Define

$$\|E_n\|_{L^\infty} + \|I_n\|_{L^\infty} = N,\ \ \hat{E}_n = \frac{E_n}{N},\ \ \hat{I}_n = \frac{I_n}{N}. \tag{5.10}$$

It then follows that

$$\|\hat{E}_n\|_{L^\infty} + \|\hat{I}_n\|_{L^\infty} = 1, \quad \hat{E}_n,\ \hat{I}_n > 0 \tag{5.11}$$

and $(\hat{E}_n, \hat{I}_n)$ satisfies the system

$$\begin{cases} -d_E\Delta\hat{E}_n = \beta_1(x)S_{1,n}\hat{I}_n + \beta_2(x)S_{2,n}\hat{I}_n - [\sigma(x)+d(x)]\hat{E}_n, & x\in\Omega, \\ -d_I\Delta\hat{I}_n = \sigma(x)\hat{E}_n - [\gamma(x)+d(x)+\alpha(x)]\hat{I}_n, & x\in\Omega, \\ \frac{\partial\hat{E}_n}{\partial n} = \frac{\partial\hat{I}_n}{\partial n} = 0, & x\in\partial\Omega. \end{cases} \tag{5.12}$$

By a standard compactness argument for elliptic equations, and after passing to a subsequence if necessary, we obtain

$$\hat{E}_n \to \hat{E} \text{ and } \hat{I}_n \to \hat{I} \text{ as } d_n \to 0 \ \text{ in } C^1(\bar{\Omega}),$$

where $(\hat{E}, \hat{I})$ satisfies

$$\begin{cases} -d_E\Delta\hat{E} = \beta_1(x)\hat{S}_1\hat{I} + \beta_2(x)\tilde{S}_2\hat{I} - [\theta(x)+d(x)]\hat{E}, & x\in\Omega, \\ -d_I\Delta\hat{I} = \sigma(x)\hat{E} - [\gamma(x)+d(x)+\alpha(x)]\hat{I}, & x\in\Omega, \\ \frac{\partial\hat{E}}{\partial n} = \frac{\partial\hat{I}}{\partial n} = 0, & x\in\partial\Omega. \end{cases}$$

Applying the strong maximum principle, we conclude that either

$$\hat{E}(x) > 0, \hat{I}(x) > 0 \text{ or } \hat{E}(x) \equiv 0, \hat{I}(x) \equiv 0 \text{ on } \overline{\Omega}.$$

However, it follows from (5.11) that $\hat{E}(x), \hat{I}(x) > 0$ for all $x \in \overline{\Omega}$, which implies that $\tilde{R}_0 = 1$, contradicting our assumption. Hence, there is a positive constant C, independent of d_n, such that

$$E(x) \geq C \ \text{ for all } x \in \overline{\Omega}. \tag{5.13}$$

Let $I(x_1) = \min_{x\in\bar{\Omega}} I(x)$, $x_1 \in \bar{\Omega}$. From the third equation in (1.7) and by applying [57, Lemma 3.1], we obtain

$$[\gamma(x_1) + d(x_1) + \alpha(x_1)]I(x_1) \geq \sigma(x_1)E(x_1).$$

In view of (5.13), it follows that

$$I(x) \geq I(x_1) \geq \frac{C\sigma_*}{\gamma^* + d^* + \alpha^*} \text{ for all } x \in \bar{\Omega}. \tag{5.14}$$

Hence, by (5.2), (5.13), and (5.14), one can conclude that there is a positive constant C, independent of $0 < d_1 < 1$, such that

$$\frac{1}{C} < S_1(x), S_2(x), E(x), I(x) < C \text{ for all } x \in \bar{\Omega}. \tag{5.15}$$

Step 2. Convergence of S_2, E, and I.

We now consider the following problem:

$$\begin{cases} -d_E\Delta E_n + [d(x)+\sigma(x)]E_n = \beta_1(x)S_{1,n}I_n + \beta_2(x)S_{2,n}I_n, & x\in\Omega, \\ \frac{\partial E_n}{\partial n} = 0, & x\in\partial\Omega. \end{cases} \tag{5.16}$$

By virtue of (5.15), we can get

$$\|(\beta_1(x)S_{1,n}+\beta_2(x)S_{2,n})I_n\|_{L^p(\Omega)}\le C \text{ for all } p\ge 1.$$

Applying standard elliptic L^p theory and the Sobolev embedding theorem, we obtain

$$\|E_n\|_{C^{1+\alpha}(\bar\Omega)}\le C\|E_n\|_{W^{2,p}(\Omega)}\le C,\ \ 0<\alpha<1.$$

Therefore, the sequence $\{E_n\}_{0<d_1\le 1}$ is precompact in $C^1(\bar\Omega)$. Consequently, there is a subsequence of d_n, still denoted by $d_n := d_{1,n}$ with $d_n\to 0$ as $n\to\infty$, and a corresponding positive solution $(S_{1,n}, S_{2,n}, E_n, I_n)$ of (1.7), such that

$$E_n\to\tilde E \text{ in } C^1(\bar\Omega) \text{ as } n\to\infty.$$

Since (5.13) holds, we deduce that $\tilde E>0$.

Next, recalling that I satisfies (5.3), and $S_{2,n}$

$$\begin{cases} -d_2\Delta S_{2,n}+\beta_2(x)S_{2,n}I_n+d(x)S_{2,n}=\theta(x)S_{1,n}, & x\in\Omega,\\ \frac{\partial S_{2,n}}{\partial n}=0, & x\in\partial\Omega.\end{cases} \tag{5.17}$$

By (5.15), we have

$$\|\sigma(x)E_n\|_{L^p(\Omega)}\le C,\ \|\theta(x)S_{1,n}\|_{L^p(\Omega)}\le C \text{ for all } p\ge 1.$$

We then apply a similar argument as above to conclude that

$$\|I_n\|_{C^{1+\alpha}(\bar\Omega)}\le C,\ \left\|S_{2,n}\right\|_{C^{1+\alpha}(\bar\Omega)}\le C \text{ for some } 0<\alpha<1.$$

Thus, the sequences $\{S_{2,n}\}_{0<d_{1,n}\le 1}$ $\{I_n\}_{0<d_{1,n}\le 1}$ are precompact in $C^1(\bar\Omega)$. Hence, possibly after passing to a further subsequence (still denoted by $d_n := d_{1,n}$), a corresponding positive solution $(S_{1,n}, S_{2,n}, E_n, I_n)$ of (1.7) exists such that

$$S_{2,n}\to\tilde S_2,\ I_n\to\tilde I>0 \text{ in } C^1(\bar\Omega) \text{ as } n\to\infty, \tag{5.18}$$

thanks to the lower bound provided in (5.15).

Step 3. The convergence of S_1.

For each $n\ge 1$, $S_{1,n}$ satisfies the following problem:

$$\begin{cases} -d_n\Delta S_{1,n}=\Lambda(x)-\beta_1(x)S_{1,n}I_n-\theta(x)S_{1,n}-d(x)S_{1,n}, & x\in\Omega,\\ \frac{\partial S_{1,n}}{\partial n}=0, & x\in\partial\Omega.\end{cases} \tag{5.19}$$

In view of (5.18), for any $0<\varepsilon<\min_{\bar\Omega}\tilde I$, a constant N, exists such that, for all $n>N$, we have

$$0<\tilde I-\varepsilon\le I_n(x)\le\tilde I+\varepsilon \text{ for all } x\in\bar\Omega.$$

Thus, for a sufficiently large n, it follows that

$$\Lambda(x)-\beta_1(x)S_{1,n}I_n-\theta(x)S_{1,n}-d(x)S_{1,n}\ge\Lambda(x)-\theta(x)S_{1,n}-d(x)S_{1,n}-\beta_1(x)S_{1,n}(\tilde I+\varepsilon)$$

and

$$\Lambda(x) - \beta_1(x)S_{1,n}I_n - \theta(x)S_{1,n} - d(x)S_{1,n} \le \Lambda(x) - \theta(x)S_{1,n} - d(x)S_{1,n} - \beta_1(x)S_{1,n}(\tilde{I} - \varepsilon).$$

Now, consider the following auxiliary problem for a fixed large n:

$$\begin{cases} -d_n\Delta \underline{W} = \Lambda(x) - (\beta_1(x)(\tilde{I} + \varepsilon) + \theta(x) + d(x))\underline{W}, & x \in \Omega, \\ \frac{\partial \underline{W}}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{5.20}$$

which admits a unique positive solution, denoted by $\underline{W}_n$. Furthermore, by [59, Lemma 2.4], one can show that

$$\underline{W}_n \to \frac{\Lambda(x)}{\beta_1(x)(\tilde{I} + \varepsilon) + \theta(x) + d(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty.$$

One can easily verify that $S_{1,n}$ satisfies the conditions of an upper solution for (5.20) in the sense of the maximum principle. Therefore

$$\liminf_{n\to\infty} S_{1,n} \ge \lim_{n\to\infty} \underline{W}_n = \frac{\Lambda(x)}{\beta_1(x)(\tilde{I} + \varepsilon) + \theta(x) + d(x)} \text{ on } \bar{\Omega}. \tag{5.21}$$

Let $\overline{W}_n$ be the unique positive solution to the following problem:

$$\begin{cases} -d_n\Delta \overline{W} = \Lambda(x) - (\beta_1(x)(\tilde{I} - \varepsilon) + \theta(x) + d(x))\overline{W}, & x \in \Omega, \\ \frac{\partial \overline{W}}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

which is an upper solution to (5.19) by the maximum principle. Applying [59, Lemma 2.4] again yields

$$\limsup_{n\to\infty} S_{1,n}(x) \le \lim_{n\to\infty} \overline{W}_n = \frac{\Lambda(x)}{\beta_1(x)(\tilde{I} - \varepsilon) + \theta(x) + d(x)} \text{ on } \bar{\Omega}. \tag{5.22}$$

By the arbitrariness of $\varepsilon > 0$ and combining (5.21) and (5.22), we conclude that

$$S_{1,n}(x) \to \tilde{S}_1(x) := \frac{\Lambda(x)}{\beta_1(x)\tilde{I} + \theta(x) + d(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty. \tag{5.23}$$

It is evident that $(\tilde{S}_2, \tilde{E}, \tilde{I})$ satisfies (5.1). This completes the proof. □

5.2. *The case of $d_2 \to 0$*

In this subsection, we investigate the asymptotic behavior of the EE as $d_2 \to 0$, while keeping $d_1 > 0$, $d_E > 0$, and $d_I > 0$ fixed. By Theorem 2.3(ii) and Theorem 3.3(ii), if $\hat{R}_0 > 1$, then the system (1.7) admits at least one EE. The corresponding result in this limiting regime is stated below.

Theorem 5.3. *Suppose that $\hat{R}_0 > 1$ and fix $d_1 > 0$, $d_E > 0$, and $d_I > 0$. Then, for any positive solution (S_1, S_2, E, I) of (1.7), a subsequence (still denoted by (S_1, S_2, E, I) for simplicity) exists such that, as $d_2 \to 0$, we have*

$$(S_1, S_2, E, I) \to (\tilde{S}_1, \tilde{S}_2, \tilde{E}, \tilde{I}) \text{ uniformly on } \bar{\Omega},$$

where

$$\tilde{S}_2(x) = \frac{\theta(x)\tilde{S}_1(x)}{d(x) + \beta_2(x)\tilde{I}(x)},$$

and $(\tilde{S}_1, \tilde{E}, \tilde{I})$ is a positive solution of the following problem:

$$\begin{cases} -d_1\Delta\tilde{S}_1 = \Lambda(x) - \beta_1(x)\tilde{S}_1\tilde{I} - \theta(x)\tilde{S}_1 - d(x)\tilde{S}_1, & x \in \Omega, \\ -d_E\Delta\tilde{E} = \beta_1(x)\tilde{S}_1\tilde{I} + \beta_2(x)\tilde{S}_2\tilde{I} - [d(x) + \sigma(x)]\tilde{E}, & x \in \Omega, \\ -d_I\Delta\tilde{I} = \sigma(x)\tilde{E} - [\gamma(x) + d(x) + \alpha(x)]\tilde{I}, & x \in \Omega, \\ \frac{\partial\tilde{S}_1}{\partial n} = \frac{\partial\tilde{E}}{\partial n} = \frac{\partial\tilde{I}}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{5.24}$$

Proof. **Step 1.** Estimates of the upper and lower bounds.

The upper and lower bound estimates for S_1, and S_2, and the upper bounds for E and I established in Theorem 5.2 still hold. Therefore, there is a positive constant C, independent of d_2, such that

$$E(x), I(x) \le C \text{ and } \frac{1}{C} \le S_1(x), S_2(x) \le C \text{ for } x \in \overline{\Omega}.$$

Next, we estimate the lower bound of E. Arguing by contradiction, suppose that E does not admit a positive lower bound. Then a sequence $d_n := d_{2,n} \to 0$ as $n \to \infty$, exists along which a corresponding positive solution $(S_{1,n}, S_{2,n}, E_n, I_n) = \left(S_{1,d_{2,n}}, S_{2,d_{2,n}}, E_{d_{2,n}}, I_{d_{2,n}}\right)$ of the system (1.7) satisfies $\min_{x\in\bar{\Omega}} E_n \to 0$. Using similar arguments as in the first step of the proof of Theorem 5.2, we conclude that (5.4) holds. We now consider the elliptic equation

$$\begin{cases} -d_1\Delta S_{1,n} + (\beta_1(x)I_n + \theta + d)S_{1,n} = \Lambda(x), & x \in \Omega, \\ \frac{\partial S_{1,n}}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{5.25}$$

It follows from the method of the upper and lower solutions to deduce that

$$S_{1,n} \to \hat{S}_1 \text{ uniformly on } \bar{\Omega} \text{ as } d_2 \to 0. \tag{5.26}$$

Here, $\hat{S}_1$ is determined by (2.7).

Given any $\varepsilon > 0$, it follows from (5.4) and (5.26) that $N > 0$ exists such that for all $n \ge N$,

$$0 < \hat{S}_1(x) - \varepsilon \le S_{1,n}(x) \le \hat{S}_1(x) + \varepsilon, \;\; 0 \le I_n(x) \le \varepsilon \text{ in } \Omega.$$

For all $n \ge N$, it then follows that

$$\theta(x)S_{1,n} - \beta_2(x)S_{2,n}I_n - d(x)S_{2,n} \ge \theta(x)(\hat{S}_1(x) - \varepsilon) - [\varepsilon\beta_2(x) + d(x)]S_{2,n}$$

and

$$\theta(x)S_{1,n} - \beta_2(x)S_{2,n}I_n - d(x)S_{2,n} \le \theta(x)(\hat{S}_1(x) + \varepsilon) - d(x)S_{2,n}.$$

For fixed a sufficiently large n, $S_{2,n}$ serves as an upper solution to

$$\begin{cases} -d_n\Delta\underline{U} = \theta(x)(\hat{S}_1(x) - \varepsilon) - [\varepsilon\beta_2(x) + d(x)]\underline{U}, & x \in \Omega, \\ \frac{\partial\underline{U}}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{5.27}$$

and a lower solution to

$$\begin{cases} -d_n\Delta\overline{U} = \theta(x)(\hat{S}_1(x) + \varepsilon) - d(x)\overline{U}, & x \in \Omega, \\ \frac{\partial\overline{U}}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{5.28}$$

respectively. It is known that both (5.27) and (5.28) admit unique positive solutions, denoted $\underline{U}_n$ and $\overline{U}_n$, respectively. Moreover, by a similar argument as in [59, Lemma 2.4], we have

$$\underline{U}_n(x) \to \frac{\theta(x)(\hat{S}_1(x) - \varepsilon)}{\varepsilon\beta_2(x) + d(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty,$$

and

$$\overline{U}_n \to \frac{\theta(x)(\hat{S}_1(x) + \varepsilon)}{d(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty.$$

Therefore, we conclude that

$$\frac{\theta(x)(\hat{S}_1(x) - \varepsilon)}{\varepsilon\beta_2(x) + d(x)} \leq \liminf_{n\to\infty} S_{2,n}(x) \leq \limsup_{n\to\infty} S_{2,n}(x) \leq \frac{\theta(x)(\hat{S}_1(x) + \varepsilon)}{d(x)}. \tag{5.29}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (5.29) that

$$S_{2,n}(x) \to \hat{S}_2(x) := \frac{\theta(x)\hat{S}_1(x)}{d(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty.$$

By the definition of (5.10), we know that $(\hat{E}_n, \hat{I}_n)$ satisfies (5.12). By a standard compactness argument for elliptic equations, after passing to a subsequence if necessary, we obtain

$$\hat{E}_n \to \hat{E} \text{ and } \hat{I}_n \to \hat{I} \text{ in } C^1(\overline{\Omega}) \text{ as } d_n \to 0,$$

where $(\hat{E}, \hat{I})$ satisfies the following elliptic system:

$$\begin{cases} -d_E\Delta\hat{E} = \beta_1(x)\hat{S}_1\hat{I} + \beta_2(x)\hat{S}_2\hat{I} - [\theta(x) + d(x)]\hat{E}, & x \in \Omega, \\ -d_I\Delta\hat{I} = \sigma(x)\hat{E} - [\gamma(x) + d(x) + \alpha(x)]\hat{I}, & x \in \Omega, \\ \frac{\partial\hat{E}}{\partial n} = \frac{\partial\hat{I}}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Applying the strong maximum principle, we conclude that either

$$\hat{E}(x) > 0, \hat{I}(x) > 0 \text{ or } \hat{E}(x) \equiv 0, \hat{I}(x) \equiv 0 \text{ in } \Omega.$$

It follows from (5.11) that $\hat{E}(x), \hat{I}(x) > 0$, which implies $\hat{R}_0 = 1$, leading to a contradiction. Hence, a positive constant C independent of d_2 exists such that

$$E(x) \geq C \text{ for all } x \in \overline{\Omega}. \tag{5.30}$$

Let $I(x_2) = \min_{x\in\bar{\Omega}} I(x)$ for some $x_2 \in \bar{\Omega}$. By the fourth equation of (1.7), we obtain

$$[\gamma(x_2) + d(x_2) + \alpha(x_2)]I(x_2) \geq \sigma(x_2)E(x_2).$$

In view of (5.30), this yields

$$I(x) \geq I(x_2) \geq \frac{C\sigma_*}{\gamma^* + d^* + \alpha^*} \text{ for all } x \in \bar{\Omega}.$$

Therefore, we can find a positive constant C, independent of $0 < d_2 < 1$, such that

$$\frac{1}{C} < S_1(x), S_2(x), E(x), I(x) < C \text{ for all } x \in \bar{\Omega}. \tag{5.31}$$

Step 2. Convergence of S_1, S_2, E, and I.

Since $||\Lambda(x)||_{L^p(\Omega)} \leq C$ for any $p \geq 1$, we apply the L^p theory and the Sobolev embedding theorem to (5.25) to deduce that $||S_{1,n}||_{C^{1+\alpha}(\bar{\Omega})} \leq C$, for some $0 < \alpha < 1$. Hence, $S_{1,n}$ is precompact in $C^1(\bar{\Omega})$, so a subsequence $d_n := d_{2,n} \to 0$ as $n \to \infty$ exists, and a corresponding positive solution $(S_{1,n}, S_{2,n}, E_n, I_n)$ of (1.7), such that

$$S_{1,n} \to \tilde{S}_1 \text{ in } C^1(\bar{\Omega}) \text{ as } n \to \infty. \tag{5.32}$$

By virtue of (5.31), we find that $\tilde{S}_1 > 0$ in $C^1(\bar{\Omega})$.

In light of (5.31), we apply a standard compactness argument to the elliptic equations (5.3) and (5.16); after passing to a subsequence if necessary, it follows that

$$||E_n||_{C^{1+\alpha}(\bar{\Omega})} \leq C, \; ||I_n||_{C^{1+\alpha}(\bar{\Omega})} \leq C, \;\; 0 < \alpha < 1.$$

Thus, $\{E_n\}_{0<d_n\leq 1}$ and $\{I_n\}_{0<d_{2,n}\leq 1}$ are precompact in $C^1(\bar{\Omega})$. Therefore, passing to a subsequence if necessary (still denoted by $d_n := d_{2,n} \to 0$ as $n \to \infty$), a corresponding sequence of positive solutions $(S_{1,n}, S_{2,n}, E_n, I_n)$ to (1.7) exists, such that

$$E_n \to \tilde{E} > 0, \; I_n \to \tilde{I} > 0 \text{ in } C^1(\bar{\Omega}), \text{ as } n \to \infty, \tag{5.33}$$

and by (5.31).

For any $\varepsilon > 0$, by (5.32) and (5.33), $N > 0$ exists such that for all $n \geq N$,

$$0 < \tilde{S}_1 - \varepsilon \leq S_{1,n} \leq \tilde{S}_1 + \varepsilon \text{ and } \; 0 < \tilde{I} - \varepsilon \leq I_n \leq \tilde{I} + \varepsilon.$$

For all $n \geq N$, it then follows that

$$\theta(x)S_{1,n} - \beta_2(x)S_{2,n}I_n - d(x)S_{2,n} \geq \theta(x)(\tilde{S}_1 - \varepsilon) - [\beta_2(x)(\tilde{I} + \varepsilon) + d(x)]S_{2,n},$$

and

$$\theta(x)S_{1,n} - \beta_2(x)S_{2,n}I_n - d(x)S_{2,n} \leq \theta(x)(\tilde{S}_1 + \varepsilon) - [\beta_2(x)(\tilde{I} - \varepsilon) + d(x)]S_{2,n}.$$

Let $\underline{Q}_n$ and $\overline{Q}_n$ be the unique positive solutions of

$$\begin{cases} -d_n\Delta\underline{Q} = \theta(x)(\tilde{S}_1 - \varepsilon) - [\beta_2(x)(\tilde{I} + \varepsilon) + d(x)]\underline{Q}, & x \in \Omega, \\ \frac{\partial \underline{Q}}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{5.34}$$

and

$$\begin{cases} -d_n\Delta\overline{Q} = \theta(x)(\tilde{S}_1 + \varepsilon) - [\beta_2(x)(\tilde{I} - \varepsilon) + d(x)]\overline{Q}, & x \in \Omega, \\ \frac{\partial \overline{Q}}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{5.35}$$

respectively. For a sufficiently large n, it follows from the comparison principle that

$$\underline{Q}_n(x) \leq S_{2,n}(x) \leq \overline{Q}_n(x) \text{ for all } x \in \Omega.$$

Moreover, by [59, Lemma 2.4], we obtain

$$\underline{Q}_n(x) \to \frac{\theta(x)(\tilde{S}_1(x) - \varepsilon)}{\beta_2(x)(\tilde{I}(x) + \varepsilon) + d(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty$$

and

$$\overline{Q}_n(x) \to \frac{\theta(x)(\tilde{S}_1(x) + \varepsilon)}{\beta_2(x)(\tilde{I}(x) - \varepsilon) + d(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty.$$

Hence

$$\frac{\theta(x)(\tilde{S}_1(x) - \varepsilon)}{\beta_2(x)(\tilde{I}(x) + \varepsilon) + d(x)} \le \liminf_{n\to\infty} S_{2,n}(x) \le \limsup_{n\to\infty} S_{2,n}(x) \le \frac{\theta(x)(\tilde{S}_1(x) + \varepsilon)}{\beta_2(x)(\tilde{I}(x) - \varepsilon) + d(x)}.$$

By the arbitrariness of $\varepsilon > 0$, we conclude that

$$S_{2,n}(x) \to \tilde{S}_2(x) := \frac{\theta(x)\tilde{S}_1(x)}{\beta_2(x)\tilde{I}(x) + d(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty.$$

It is now clear that $(\tilde{S}_1, \tilde{E}, \tilde{I})$ satisfies (5.24). The proof is complete. □

5.3. The case of $d_E \to 0$

Assume that

$$\int_\Omega \frac{\sigma(x)(\beta_1(x)\hat{S}_1(x) + \beta_2(x)\hat{S}_2(x))}{d(x) + \sigma(x)} \mathrm{d}x > \int_\Omega (\gamma(x) + d(x) + \alpha(x))\mathrm{d}x, \tag{5.36}$$

where, $(\hat{S}_1, \hat{S}_2)$ is uniquely determined by (1.8). By Theorem 2.3(iii), we then have $\bar{R}_0 > 1$ for any fixed d_1, d_2, and $d_I > 0$ as $d_E \to 0$. Consequently, Theorem 3.3 guarantees the existence of an EE for the system (1.7).

In this subsection, we investigate the asymptotic behavior of this EE as the diffusion rate d_E tends to zero. Our main result is stated below.

Theorem 5.4. *Suppose that* (5.36) *holds and fix $d_1 > 0$, $d_2 > 0$, and $d_I > 0$. Then for any positive solution (S_1, S_2, E, I) of* (1.7), *there is a subsequence* (*still denoted by (S_1, S_2, E, I) for simplicity*) *such that, as $d_E \to 0$, we have*

$$(S_1, S_2, E, I) \to (\tilde{S}_1, \tilde{S}_2, \tilde{E}, \tilde{I}) \text{ uniformly on } \bar{\Omega},$$

where

$$\tilde{E}(x) = \frac{\beta_1(x)\tilde{S}_1(x)\tilde{I}(x) + \beta_2(x)\tilde{S}_2(x)\tilde{I}(x)}{d(x) + \sigma(x)},$$

and $(\tilde{S}_1, \tilde{S}_2, \tilde{I})$ is a positive solution of the following problem:

$$\begin{cases} -d_1\Delta\tilde{S}_1 = \Lambda(x) - \beta_1(x)\tilde{S}_1\tilde{I} - \theta(x)\tilde{S}_1 - d(x)\tilde{S}_1, & x \in \Omega, \\ -d_2\Delta\tilde{S}_2 = \theta(x)\tilde{S}_1 - \beta_2(x)\tilde{S}_2\tilde{I} - d(x)\tilde{S}_2, & x \in \Omega, \\ -d_I\Delta\tilde{I} = \sigma(x)\tilde{E} - [\gamma(x) + d(x) + \alpha(x)]\tilde{I}, & x \in \Omega, \\ \frac{\partial\tilde{S}_1}{\partial n} = \frac{\partial\tilde{S}_2}{\partial n} = \frac{\partial\tilde{I}}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{5.37}$$

Proof. In the following, C is a positive constant independent of $d_E > 0$.

Step 1. Estimates of the upper and lower bounds.

Recalling the estimates of S_1, S_2, and I in Theorem 5.2, we can find a constant $C > 0$, independent of d_E, such that

$$I(x) \leq C,\ \frac{1}{C} \leq S_1(x), S_2(x) \leq C \text{ for all } x \in \bar{\Omega}. \tag{5.38}$$

Let $E(x_1) = \max_{\bar{\Omega}} E(x)$ for some $x_1 \in \bar{\Omega}$. Applying Lemma 5.1 to the third equation in the system (1.7), we obtain

$$(d(x_1) + \sigma(x_1))E(x_1) \leq [\beta_1(x_1)S_1(x_1) + \beta_2(x_1)S_2(x_1)]I(x_1) \leq C.$$

Thus, we derive the upper bound

$$E(x) \leq E(x_1) \leq \frac{C}{d_* + \sigma_*} \text{ for all } x \in \bar{\Omega}. \tag{5.39}$$

We now proceed to prove the lower bound of I. Assume, for the sake of contradiction, that I does not admit a positive lower bound. Then there is a sequence $d_n := d_{E,n} \to 0$ as $n \to \infty$, and a corresponding sequence of positive solutions

$$(S_{1,n}, S_{2,n}, E_n, I_n) = (S_{1,d_n}, S_{2,d_n}, E_{d_n}, I_{d_n})$$

to the system (1.7), such that $\min_{x \in \bar{\Omega}} I_n \to 0$. From the fourth equation in (1.7), we have

$$-d_I \Delta I_n + [\gamma(x) + d(x) + \alpha(x)]I_n \geq 0, x \in \Omega; \quad \frac{\partial I_n}{\partial n} = 0, x \in \partial\Omega.$$

By [58, Lemma 2.1] with $p = 1$, it follows that

$$\|I_n\|_{L^1(\Omega)} \leq C \inf_{\Omega} I_n.$$

Consequently, $\|I_n\|_{L^1(\Omega)} \to 0$ as $n \to \infty$. Applying standard elliptic regularity theory to (5.3), we deduce that (5.4) holds, following arguments analogous to those in the proof of Theorem 5.2. Recalling that $S_{1,n}$ satisfies (5.25), and $S_{2,n}$ satisfies (5.17), we apply the method of upper and lower solutions to conclude that

$$S_{1,n} \to \hat{S}_1 \text{ and } S_{2,n} \to \hat{S}_2 \text{ uniformly on } \bar{\Omega} \tag{5.40}$$

as $d_n \to 0$. Here, $(\hat{S}_1, \hat{S}_2)$ is the unique solution determined by (1.8).

In view of the definition in (5.10) and the uniqueness of the non-negative solution $\hat{I}_n$ to (5.3) with E_n replaced by $\hat{E}_n$, standard compactness arguments for elliptic equations show, after passing to a further subsequence if necessary, that

$$\hat{I}_n \to \hat{I} \text{ in } C^1(\overline{\Omega}),\ \text{ as } d_E \to 0, \tag{5.41}$$

where $\hat{I}$ satisfies

$$\begin{cases} -d_I \Delta \hat{I} + (\gamma(x) + d(x) + \alpha(x))\hat{I} \geq 0, & x \in \Omega, \\ \frac{\partial \hat{I}}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

By the strong maximum principle, we conclude that either $\hat{I} \equiv 0$ or $\hat{I} > 0$ in $\bar{\Omega}$. In view of (5.11), we obtain $\hat{I} > 0$.

We now consider the following equation satisfied by $\hat{E}_n$:

$$\begin{cases} -d_n \Delta \hat{E}_n = (\beta_1(x) S_{1,n} + \beta_2(x) S_{2,n}) \hat{I}_n - [d(x) + \sigma(x)] \hat{E}_n, & x \in \Omega, \\ \frac{\partial \hat{E}_n}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

In view of (5.40) and (5.41), we apply the method from [59, Lemma 2.4] to deduce that

$$\hat{E}_n \to \frac{(\beta_1 \hat{S}_1 + \beta_2 \hat{S}_2)\hat{I}}{d(x) + \sigma(x)} := \hat{E} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty.$$

Consequently, $\hat{I}$ satisfies

$$-d_I \Delta \hat{I} + [\gamma(x) + d(x) + \alpha(x)]\hat{I} = \sigma(x) \frac{(\beta_1(x)\hat{S}_1(x) + \beta_2(x)\hat{S}_2(x))\hat{I}}{d(x) + \sigma(x)}, \ x \in \Omega; \ \frac{\partial \hat{I}}{\partial n} = 0, \ x \in \partial\Omega.$$

This implies that the principal eigenvalue $\omega_1 = 1$ of (2.8) satisfies $\omega_1 = 1$, with $\hat{I}$ being the corresponding positive eigenfunction.

However, by Theorem 2.3(iii), we have $R_0^n \to \bar{R}_0 = 1$ as $d_n \to 0$, where R_0^n is the basic reproduction number associated with (2.2) when the diffusion coefficient is d_n. On the other hand, the assumption (5.36), together with Theorem 2.3(iii), implies that $\bar{R}_0 > 1$. This contradicts the previous conclusion that $\bar{R}_0 = 1$. Hence, our initial assumption must be false. Consequently, a positive constant C, independent of $d_E > 0$ exists, such that

$$I(x) \geq C \text{ for all } x \in \Omega. \tag{5.42}$$

Let $E(x_2) = \min_{x \in \bar{\Omega}} E(x)$ with $x_2 \in \bar{\Omega}$. From the third equation of (1.7) and Lemma 5.1, it follows that

$$E(x_2) \geq \frac{\beta_1(x_2) S_1(x_2) + \beta_2(x_2) S_2(x_2)}{d(x_2) + \sigma(x_2)} I(x_2).$$

Therefore, using (5.38), and (5.42), we deduce

$$E(x) \geq E(x_2) \geq \frac{C(\beta_{1,*} + \beta_{2,*})}{d^* + \sigma^*} \text{ for all } x \in \Omega. \tag{5.43}$$

Step 2. Convergence of S_1, S_2, and I.

First, observe that

$$\|\beta_1(x) I_n + \theta(x) - d(x)\|_{L^\infty(\Omega)}, \ \|\Lambda(x)\|_{L^p(\Omega)} \leq C, \text{ for all } p \geq 1.$$

Applying the L^p theory for elliptic equations together with the Sobolev embedding theorem to (5.25) yields

$$\left\| S_{1,n} \right\|_{C^{1+\alpha}(\bar{\Omega})} \leq C, \ \ 0 < \alpha < 1.$$

Hence, $S_{1,n}$ is precompact in $C^1(\bar{\Omega})$. Consequently, a subsequence $d_n := d_{E,n} \to 0$ as $n \to \infty$ exists, and a corresponding positive solution $(S_{1,n}, S_{2,n}, E_n, I_n)$ of (1.7) such that

$$S_{1,n} \to \tilde{S}_1 \text{ in } C^1(\bar{\Omega}), \text{ as } n \to \infty. \tag{5.44}$$

Because of (5.38), we have $\tilde{S}_1 > 0$ on $C^1(\bar{\Omega})$.

Next, consider Eq (5.17). By the standard L^p elliptic regularity theory and the Sobolev embedding theorem, the sequence $\{S_{2,n}\}$ is precompact in $C^1(\bar{\Omega})$. Hence, upon passing to a further subsequence if necessary (still denoted by $S_{2,n}$ for simplicity), we can assume that

$$S_{2,n} \to \tilde{S}_2 > 0 \ \text{ in } C^1(\bar{\Omega}), \ \text{as } n \to \infty, \tag{5.45}$$

where the positivity follows from (5.38). Since I_n satisfies (5.3) and, by (5.39), we have

$$\|\sigma(x)E_n\|_{L^p(\Omega)} \le C \text{ for } p \ge 1,$$

the L^p estimates and the Sobolev embedding theorem yield

$$\|I_n\|_{C^{1+\alpha}(\bar{\Omega})} \le C\,\|I_n\|_{W^{2,p}(\Omega)} \le C \text{ for } 0 < \alpha < 1.$$

Thus, along the subsequence

$$I_n \to \tilde{I} \ \text{ in } C^1(\bar{\Omega}), \ \text{as } n \to \infty \tag{5.46}$$

and $\tilde{I} > 0$ by (5.42).

Step 3. The convergence of E.

In view of (5.44)–(5.46), for any $\varepsilon > 0$, a positive constant N exists, such that for all $n > N$, we have

$$0 < \tilde{I}(x) - \varepsilon \le I_n(x) \le \tilde{I}(x) + \varepsilon, \ x \in \bar{\Omega},$$

$$0 < \tilde{S}_1(x) - \varepsilon \le S_{1,n}(x) \le \tilde{S}_1(x) + \varepsilon, \ \ x \in \bar{\Omega},$$
$$0 < \tilde{S}_2(x) - \varepsilon \le S_{2,n}(x) \le \tilde{S}_2(x) + \varepsilon, \ \ x \in \bar{\Omega}.$$

Therefore, for a sufficiently large n, we obtain the following inequalities:

$$(\beta_1 S_{1,n} + \beta_2 S_{2,n})I_n - [d(x) + \sigma(x)]E_n \le [\beta_1(\tilde{S}_1 + \varepsilon) + \beta_2(\tilde{S}_2 + \varepsilon)](\tilde{I} + \varepsilon) - [d(x) + \sigma(x)]E_n,$$

$$(\beta_1 S_{1,n} + \beta_2 S_{2,n})I_n - [d(x) + \sigma(x)]E_n \ge [\beta_1(\tilde{S}_1 - \varepsilon) + \beta_2(\tilde{S}_2 - \varepsilon)](\tilde{I} - \varepsilon) - [d(x) + \sigma(x)]E_n.$$

By applying the perturbation argument [59, Lemma 2.4] as in the proof of [15, Theorem 5.4], we then conclude that

$$\frac{[\beta_1(x)(\tilde{S}_1(x) - \varepsilon) + \beta_2(x)(\tilde{S}_2(x) - \varepsilon)](\tilde{I}(x) - \varepsilon)}{d(x) + \sigma(x)} \le \liminf_{n\to\infty} E_n(x) \le$$

$$\limsup_{n\to\infty} E_n(x) \le \frac{[\beta_1(x)(\tilde{S}_1(x) + \varepsilon) + \beta_2(x)(\tilde{S}_2(x) + \varepsilon)](\tilde{I}(x) + \varepsilon)}{d(x) + \sigma(x)}.$$

Since ε is arbitrary, we obtain

$$E_n(x) \to \frac{(\beta_1(x)\tilde{S}_1(x) + \beta_2(x)\tilde{S}_2(x))\tilde{I}(x)}{d(x) + \sigma(x)} \ \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty.$$

It is evident that the triple $(\tilde{S}_1, \tilde{S}_2, \tilde{I})$ forms a positive solution to (5.37). □

5.4. The case of $d_I \to 0$

Assume that
$$\int_\Omega \frac{\sigma(x)(\beta_1(x)\hat{S}_1(x)+\beta_2(x)\hat{S}_2(x))}{\gamma(x)+d(x)+\alpha(x)}\mathrm{d}x > \int_\Omega (d(x)+\sigma(x))\mathrm{d}x, \tag{5.47}$$
where $(\hat{S}_1, \hat{S}_2)$ is the unique solution of (1.8). By Theorem 2.3(iv) and Theorem 3.3, the system (1.7) admits a positive steady-state solution—that is, an EE.

In this subsection, we investigate the asymptotic behavior of this EE as the diffusion rate d_I tends to zero. The main result is stated as follows.

Theorem 5.5. *Suppose that the condition* (5.47) *holds and fix $d_1 > 0$, $d_2 > 0$, and $d_E > 0$. Then, for any positive solution (S_1, S_2, E, I) of* (1.7)*, a subsequence* (*still denoted by (S_1, S_2, E, I) for simplicity*) *exists such that, as $d_I \to 0$*
$$(S_1, S_2, E, I) \to (\tilde{S}_1, \tilde{S}_2, \tilde{E}, \tilde{I}) \text{ uniformly on } \bar{\Omega},$$
where
$$\tilde{I}(x) = \frac{\sigma(x)\tilde{E}(x)}{\gamma(x)+d(x)+\alpha(x)}$$
and $(\tilde{S}_1, \tilde{S}_2, \tilde{E})$ is a positive solution to the following problem
$$\begin{cases} -d_1\Delta\tilde{S}_1 = \Lambda(x) - \beta_1(x)\tilde{S}_1\tilde{I} - \theta(x)\tilde{S}_1 - d(x)\tilde{S}_1, & x \in \Omega, \\ -d_2\Delta\tilde{S}_2 = \theta(x)\tilde{S}_1 - \beta_2(x)\tilde{S}_2\tilde{I} - d(x)\tilde{S}_2, & x \in \Omega, \\ -d_E\Delta\tilde{E} = \beta_1(x)\tilde{S}_1\tilde{I} + \beta_2(x)\tilde{S}_2\tilde{I} - [d(x)+\sigma(x)]\tilde{E}, & x \in \Omega, \\ \frac{\partial \tilde{S}_1}{\partial n} = \frac{\partial \tilde{S}_2}{\partial n} = \frac{\partial \tilde{E}}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{5.48}$$

Proof. In the proof, C denotes a positive constant independent of d_I. We begin by establishing the upper and lower bounds. By the estimates for S_1, S_2, and E in Theorem 5.2, a constant $C > 0$, independent of d_I exists, such that
$$E(x),\ S_1(x),\ S_2(x) \le C \text{ for all } x \in \bar{\Omega}.$$

Let $I(x_1) = \max_{\bar{\Omega}} I(x)$ for some $x_1 \in \bar{\Omega}$. Applying Lemma 5.1 to the fourth equation of (1.7), we obtain
$$\sigma(x_1)E(x_1) \ge [\gamma(x_1) + d(x_1) + \alpha(x_1)]I(x_1).$$
Since $E(x)$ is uniformly bounded above, we conclude that
$$I(x) \le I(x_1) \le \frac{\sigma^* C}{\gamma_* + d_* + \alpha_*} \text{ for all } x \in \bar{\Omega}.$$

Assume that $S_1(x_2) = \min_{x\in\bar{\Omega}} S_1(x)$, $x_2 \in \bar{\Omega}$, applying Lemma 5.1 to the first equation of (1.7), we have
$$S_1(x) \ge S_1(x_2) \ge \frac{\Lambda_*}{\beta_1^* C + \theta^* + d^*} \text{ for all } x \in \bar{\Omega}.$$

This implies that a positive constant C, independent of d_I exists, such that
$$S_1(x) \ge \min_{x\in\bar{\Omega}} S_1(x) = S_1(x_2) \ge C > 0 \text{ for all } x \in \bar{\Omega}. \tag{5.49}$$

Now, we estimate the lower bound of S_2. Let $S_2(x_3) = \min_{x\in\bar{\Omega}} S_2(x),\ x_3 \in \bar{\Omega}$. We use Lemma 5.1 to the second equation of (1.7) to derive

$$S_2(x) \geq S_2(x_3) \geq \frac{\theta(x_3)S_1(x_3)}{\beta_2(x_3)I(x_3) + d(x_3)} \text{ for all } x \in \bar{\Omega}.$$

Since $S_1(x)$ has a positive lower bound and $I(x)$ has a positive upper bound, there is a positive constant C_0 independent of d_I such that

$$S_2(x) \geq \min_{x\in\bar{\Omega}} S_2(x) = S_2(x_3) \geq C_0 > 0 \text{ for all } x \in \bar{\Omega}. \tag{5.50}$$

We employ an argument similar to that in the proof of [15, Theorem 5.5] to show that there is a positive constant C, independent of $d_I > 0$, such that

$$E(x) \geq C \quad \text{for all } \ x \in \bar{\Omega}.$$

Set $I(x_4) = \min_{x\in\bar{\Omega}} I(x)$ for some $x_4 \in \bar{\Omega}$. Applying Lemma 5.1([57, Lemma 3.1]), we obtain

$$[\gamma(x_4) + d(x_4) + \alpha(x_4)]I(x_4) \geq \sigma(x_4)E(x_4).$$

It follows that

$$I(x) \geq I(x_4) \geq \frac{C\sigma_*}{\gamma^* + d^* + \alpha^*} \text{ for all } x \in \bar{\Omega}.$$

Observe that S_1, S_2, and E satisfy the following system:

$$\begin{cases} -d_1\Delta S_1 + (\theta(x) + d(x))S_1 + \beta_1(x)S_1 I = \Lambda(x), & x \in \Omega, \\ -d_2\Delta S_2 + d(x)S_2 + \beta_2(x)S_2 I = \theta(x)S_1, & x \in \Omega, \\ -d_E\Delta E + [d(x) + \sigma(x)]E = \beta_1(x)S_1 I + \beta_2(x)S_2 I, & x \in \Omega, \\ \frac{\partial S_1}{\partial n} = \frac{\partial S_2}{\partial n} = \frac{\partial E}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Following arguments analogous to those in the proofs of Theorems 5.2–5.4, and considering a sequence $\{d_{I,n}\}$ with $d_{I,n} \to 0$ as $n \to \infty$, we obtain

$$S_{1,n} \to \tilde{S}_1,\ S_{2,n} \to \tilde{S}_2,\ E_n \to \tilde{E} \text{ in } C^1(\bar{\Omega}),\ \text{as } n \to \infty. \tag{5.51}$$

Moreover, since S_1, S_2, and E are uniformly bounded below by positive constants (independent of n), it follows that $\tilde{S}_1, \tilde{S}_2, \tilde{E} > 0$ in $\mathrm{C}^1(\bar{\Omega})$.

Now consider the following equation satisfied by I_n:

$$\begin{cases} -d_n\Delta I_n = \sigma(x)E_n - [\gamma(x) + d(x) + \alpha(x)]I_n, & x \in \Omega, \\ \frac{\partial I_n}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Applying the perturbation argument from [59, Lemma 2.4], we deduce that

$$I_n(x) \to \frac{\sigma(x)\tilde{E}(x)}{\gamma(x) + d(x) + \alpha(x)} \text{ uniformly on } \bar{\Omega} \text{ as } n \to \infty.$$

It is now clear that $(\tilde{S}_1, \tilde{S}_2, \tilde{E})$ is a positive solution of the system (5.48). This completes the proof. □

6. Conclusions and discussion

In this paper, we study an SEIR reaction–diffusion epidemic model (1.3) that incorporates susceptible individuals with underlying health conditions. We investigate the effects of spatial heterogeneity, individual movements, and underlying diseases on the persistence and extinction of infectious diseases. Similar problems have been addressed in our previous study [15] on the classical SEIR model without underlying diseases (1.6), and many of the results obtained here are consistent with those obtained in [15]. However, our results indicate that the presence of susceptible individuals with underlying health conditions can elevate the risk of disease transmission.

In what follows, we compare and discuss how factors such as environmental heterogeneity, diffusion rates, and the presence of underlying diseases affect the basic reproduction number, and also examine how these factors influence the disease's persistence or extinction, as well as the asymptotic profiles of the steady states.

- **The effects of spatial heterogeneity and the mobility**

  In a heterogeneous environment, the basic reproduction number R_0 is defined by the eigenvalue problem (2.2), whereas in the homogeneous case, R_0 is given explicitly by Eq (4.1). Compared with (4.1), the expression (2.2) indicates that R_0 depends not only on the spatial heterogeneity, such as resource distribution, population density, and contact rates, but also on the diffusion rates of various compartments, including susceptible individuals without underlying diseases (d_1), those with underlying diseases (d_2), the exposed (d_E), and the infectious (d_I). The asymptotic behavior of R_0 with respect to these diffusion rates is analyzed in Theorem 2.3, revealing that the inclusion of susceptible individuals with underlying diseases introduces additional complexity beyond that of the classical model (1.6).

  Both spatial heterogeneity and mobility play critical roles in shaping R_0 by altering the transmission dynamics and disease spread patterns. The persistence results for the infectious population (see Theorem 3.3) offer important insights into the conditions under which the disease can remain endemic, emphasizing the central importance of controlling R_0 for effective epidemic management. A primary objective of this study is to examine how spatial heterogeneity and diffusion rates influence the asymptotic profiles of EEs; see Theorems 5.2–5.5. Analogous results for the model (1.6) have been established in [15, Theorems 5.3–5.5]. These findings highlight the complexity of infectious disease dynamics in spatially structured populations and demonstrate the robustness of disease persistence even as the mobility rates of various compartments tend to zero.

- **The effects of underlying diseases**

  The basic reproduction number plays a fundamental role in determining how a disease spreads within a population, and is also affected by the proportion of susceptible individuals (see [15, Theorem 2.3]). However, in our model (1.3), the basic reproduction number depends not only on the density of susceptible individuals without underlying diseases, denoted by $\hat{S}_1(x)$, but also on that of those with underlying diseases, $\hat{S}_2(x)$. Furthermore, the transmission coefficients differ between these two groups: β_1 for individuals without underlying diseases and $\beta_2(> \beta_1)$ for those with underlying diseases. This dual contribution highlights the added complexity of the basic reproduction number introduced by underlying health conditions. According to Lemma 2.4, such conditions can, in certain scenarios, lead to an increase in the basic reproduction number.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

C. Lei was partially supported by National Natural Science Foundation of China (No. 11801232, 12271486, and 12471168), the Natural Science Foundation of Jiangsu Province (No. BK20180999), and the Foundation of Jiangsu Normal University (No. 17XLR008).

We thank the handling editor and the anonymous referees for their careful reading and valuable suggestions. C. Lei would like to thank the China Scholarship Council (202310090060) for financial support during the period of the overseas study and to express her gratitude to the School of Science and Technology, University of New England, for its kind hospitality.

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. W. Kermack, À. G. McKendrick, A contribution to the mathematical theory of epidemics, *Proc. R. Soc. London, Ser. A* , **115** (1927), 700–721. https://doi.org/10.1098/rspa.1927.0118
2. W. Kermack, À. G. McKendrick, Contributions to the mathematical theory of epidemics: V. Analysis of experimental epidemics of mouse-typhoid; a bacterial disease conferring incomplete immunity, *J. Hyg.*, **39** (1939), 271–288. https://doi.org/10.1017/S0022172400011918
3. S. Ansumali, S. Kaushal, A. Kumar, M. Prakash, M. Vidyasagar, Modelling a pandemic with asymptomatic patients, impact of lockdown and herd immunity, with applications to SARS-CoV-2, *Annu. Rev. Control*, **50** (2020), 432–447. https://doi.org/10.1016/j.arcontrol.2020.10.003
4. S. Han, C. Lei, Global stability of equilibria of a diffusive SEIR epidemic model with nonlinear incidence, *Appl. Math. Lett.*, **98** (2019), 114–120. https://doi.org/10.1016/j.aml.2019.05.045
5. M. Y. Li, H. L. Smith, L. Wang, Global dynamics of an SEIR epidemic model with vertical transmission, *SIAM J. Appl. Math.*, **62** (2001), 58–69. https://doi.org/10.1137/S0036139999359860
6. I. B. Schwartz, H. L. Smith, Infinite subharmonic bifurcation in an SEIR epidemic model, *J. Math. Biol.*, **18** (1983), 233–253. https://doi.org/10.1007/BF00276090
7. P. Wintachai, K. Prathom, Stability analysis of SEIR model related to efficiency of vaccines for COVID-19 situation, *Heliyon*, **7** (2021), e06812. https://doi.org/10.1016/j.heliyon.2021.e06812
8. G. Lu, Z. Lu, Geometric approach to global asymptotic stability for the SEIRS models in epidemiology, *Nonlinear Anal. Real World Appl.*, **36** (2017), 20–43. https://doi.org/10.1016/j.nonrwa.2016.12.005
9. P. Song, Y. Lou, Y. Xiao, A spatial SEIRS reaction-diffusion model in heterogeneous environment, *J. Differ. Equations*, **267** (2019), 5084–5114. https://doi.org/10.1016/j.jde.2019.05.022

10. F. Brauer, C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Springer-Verlag, 2001. https://doi.org/10.1007/978-1-4614-1686-9

11. R. S. Cantrell, C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, Wiley and Sons, Chichester West Sussex, 2003. https://doi.org/10.1002/0470871296

12. R. Cui, K. Y. Lam, Y. Lou, Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments, *J. Differ. Equations*, **263** (2017), 2343–2373. https://doi.org/10.1016/j.jde.2017.03.045

13. L. Dong, S. Hou, C. Lei, Global attractivity of the equilibria of the diffusive SIR and SEIR epidemic models with multiple parallel infectious stages and nonlinear incidence mechanism, *Appl. Math. Lett.*, **134** (2022), 108352. https://doi.org/10.1016/j.aml.2022.108352

14. S. Han, C. Lei, X. Zhang, Qualitative analysis on a diffusive SIRS epidemic model with standard incidence infection mechanism, *Z. Angew. Math. Phys.*, **71** (2020). https://doi.org/10.1007/s00033-020-01418-1

15. C. Lei, H. Li, Y. Zhao, Dynamical behavior of a reaction-diffusion SEIR epidemic model with mass action infection mechanism in a heterogeneous environment, *Discrete Contin. Dyn. Syst. Ser. B*, **29** (2024), 3163–3198. https://doi.org/10.3934/dcdsb.2023216

16. C. Lei, X. Zhou, Concentration phenomenon of the endemic equilibrium of a reaction-diffusion-advection SIS epidemic model with spontaneous infection, *Discrete Contin. Dyn. Syst. Ser. B*, **27** (2022), 3077–3100. https://doi.org/10.3934/dcdsb.2021174

17. X. Wang, S. Liu, Global properties of a delayed SIR epidemic model with multiple parallel infectious stages, *Math. Biosci. Eng.*, **9** (2012), 685–695. https://doi.org/10.3934/mbe.2012.9.685

18. J. Zhou, Y. Zhao, Y. Ye, Complex dynamics and control strategies of SEIR heterogeneous network model with saturated treatment, *Physica A*, **608** (2022), 128287. https://doi.org/10.1016/j.physa.2022.128287

19. R. M. Anderson, R. M. May, Population biology of infectious diseases Part I, *Nature*, **280** (1979), 361–367. https://doi.org/10.1038/280361a0

20. Y. Cai, X. Lian, Z. Peng, W. Wang, Spatiotemporal transmission dynamics for influenza disease in a heterogenous environment, *Nonlinear Anal. Real World Appl.*, **46** (2019), 178–194. https://doi.org/10.1016/j.nonrwa.2018.09.006

21. R. Cui, Asymptotic profiles of the endemic equilibrium of a reaction-diffusion-advection SIS epidemic model with saturated incidence rate, *Discrete Contin. Dyn. Syst. Ser. B*, **26** (2021), 2997–3022. https://doi.org/10.3934/dcdsb.2020217

22. R. Cui, H. Li, R. Peng, M. Zhou, Concentration behavior of endemic equilibrium for a reaction-diffusion-advection SIS epidemic model with mass action infection mechanism, *Calc. Var. Partial Differ. Equations*, **60** (2021), 184. https://doi.org/10.1007/s00526-021-01992-w

23. O. Diekmann, J. A. P. Heesterbeek, *Mathematical Epidemiology of Infectious Diseases in Model Building, Analysis and Interpretation*, John Wiley & Sons, 2000.

24. H. W. Hethcote, The mathematics of infectious diseases, *SIAM Rev.*, **42** (2000), 599–653. https://doi.org/10.1137/S0036144500371907

25. Z. Lu, X. Chi, L. Chen, The effect of constant and pulse vaccination on SIR epidemic model with horizontal and vertical transmission, *Math. Comput. Modell.*, **36** (2002), 1039–1057. https://doi.org/10.1016/S0895-7177(02)00257-1

26. Y. Yang, Y. R. Yang, X. J. Jiao, Traveling waves for a nonlocal dispersal SIR model equipped delay and generalized incidence, *Electron. Res. Arch.*, **28** (2020), 1–13. https://doi.org/10.3934/era.2020001

27. W. M. Liu, H. W. Hethcote, S. A. Levin, Dynamical behavior of epidemiological models with nonlinear incidence rates, *J. Math. Biol.*, **25** (1987), 359–380. https://doi.org/10.1007/BF00277162

28. A. Korobeinikov, P. K. Maini, A Lyapunov function and global properties for SIR and SEIR epidemiological models with nonlinear incidence, *Math. Biosci. Eng.*, **1** (2004), 57–60. https://doi.org/10.3934/mbe.2004.1.57

29. M. C. M. de Jong, O. Diekmann, H. Heesterbeek, How does transmission of infection depend on population size? in Epidemic Models: Their Structure and Relation to Data, Cambridge University Press, (1996), 84–94.

30. L. J. S. Allen, B. M. Bolker, Y. Lou, A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, *Discrete Contin. Dyn. Syst., Ser. A*, **21** (2008), 1–20. https://doi.org/10.3934/dcds.2008.21.1

31. H. Li, R. Peng, T. Xiang, Dynamics and asymptotic profiles of endemic equilibrium for two frequency-dependent SIS epidemic models with cross-diffusion, *Eur. J. Appl. Math.*, **31** (2020), 26–56. https://doi.org/10.1017/s0956792518000463

32. H. McCallum, N. Barlow, J. Hone, How should pathogen transmission be modelled?, *Trends Ecol. Evol.*, **16** (2001), 295–300.

33. C. Russell, N. Lone, J. Baillie, Comorbidities, multimorbidity and COVID-19, *Nat. Med.*, **29** (2023), 334–343. https://doi.org/10.1038/s41591-022-02156-9

34. Y. Zhai, X. Lin, J. Li, W. Liang, Research on the spread of COVID-19 based on the SEIR model for susceptible populations with basic medical history, *Appl. Math. Mech.*, **42** (2021), 413–421. https://doi.org/10.21656/1000-0887.410313

35. H. Yang, X. L. Lin, J. Wu, Qualitative analysis of a time-delay transmission model for COVID-19 based on susceptible populations with basic medical history, *Qeios*, **2023** (2023). https://doi.org/10.32388/8a5osc.2

36. L. J. S. Allen, B. M. Bolker, Y. Lou, A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic patch model, *SIAM J. Appl. Math.*, **67** (2007), 1283–1309. https://doi.org/10.1137/060672522

37. D. Gao, C. Lei, R. Peng, B. Zhang, A diffusive SIS epidemic model with saturated incidence function in a heterogeneous environment, *Nonlinearity*, **37** (2024), 025002. https://doi.org/10.1088/1361-6544/ad1495

38. C. Lei, J. Xiong, X. Zhou, Qualitative analysis on an SIS epidemic reaction-diffusion model with mass action infection mechanism and spontaneous infection in a heterogeneous environment, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 81–98. https://doi.org/10.3934/dcdsb.2019173

39. H. Li, R. Peng, F. Wang, Varying total population enhances disease persistence: qualitative analysis on a diffusive SIS epidemic model, *J. Differ. Equations*, **262** (2017), 885–913. https://doi.org/10.1016/j.jde.2016.09.044

40. H. Nie, Y. Shi, J. Wu, The effect of diffusion on the dynamics of a predator-prey chemostat model, *SIAM J. Appl. Math.*, **82** (2022), 821–848. https://doi.org/10.1137/21M1432090

41. R. Peng, X. Q. Zhao, A reaction-diffusion SIS epidemic model in a time-periodic environment, *Nonlinearity*, **25** (2012), 1451–1471. https://doi.org/10.1088/0951-7715/25/5/1451

42. N. Sun, C. Lei, Long-time behavior of a reaction-diffusion model with strong Allee effect and free boundary: effect of protection zone, *J. Dyn. Differ. Equations*, **35** (2023), 737–770. https://doi.org/10.1007/s10884-021-10027-z

43. X. Yan, H. Nie, P. Zhou, On a competition-diffusion-advection system from river ecology: mathematical analysis and numerical study, *SIAM J. Appl. Dyn. Syst.*, **21** (2022), 438–469. https://doi.org/10.1137/20m1387924

44. Y. X. Feng, W. T. Li, S. Ruan, F. Y. Yang, Dynamics and asymptotic profiles of a nonlocal dispersal SIS epidemic model with bilinear incidence and Neumann boundary conditions, *J. Differ. Equations*, **335** (2022), 294–346. https://doi.org/10.1016/j.jde.2022.07.003

45. R. Li, Y. Song, H. Wang, G. P. Jiang, M. Xiao, Reactive–diffusion epidemic model on human mobility networks: Analysis and applications to COVID-19 in China, *Physica A*, **609** (2023), 128337. https://doi.org/10.1016/j.physa.2022.128337

46. D. Henry, *Geometric theory of semilinear parabolic equations*, in *Lecture Notes in Mathematics*, Springer-Verlag, 1981.

47. M. H. Protter, H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, 1984.

48. W. Wang, X. Q. Zhao, Basic reproduction numbers for reaction-diffusion epidemic models, *SIAM J. Appl. Dyn. Syst.*, **11** (2012), 1652–1673. https://doi.org/10.1137/120872942

49. H. R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, *SIAM J. Appl. Math.*, **70** (2009), 188–211. https://doi.org/10.1137/080732870

50. M. G. Krein, M. A. Rutman, Linear operators leaving invariant a cone in a Banach space, *Usp. Mat. Nauk*, **3** (1948), 3–95.

51. Z. Du, R. Peng, A priori L^∞ estimates for solutions of a class of reaction-diffusion systems, *J. Math. Biol.*, **72** (2016), 1429–1439. https://doi.org/10.1007/s00285-015-0914-z

52. S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for the solutions of elliptic differential equations satisfying general boundary values I, *Commun. Pure Appl. Math.*, **12** (1959), 623–727.

53. M. Wang, *Lecture of Reaction Diffusion Equations*, Shanxi University Press, 2025.

54. P. Magal, X. Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems, *SIAM J. Math. Anal.*, **37** (2005), 251–275. https://doi.org/10.1137/s0036141003439173

55. X. Q. Zhao, *Dynamical Systems in Population Biology*, Springer-Verlag, 2003.

56. Y. Lou, W. M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differ. Equations*, **131** (1996), 79–131. https://doi.org/10.1006/jdeq.1996.0157

57. R. Peng, Qualitative analysis on a diffusive and ratio-dependent predator-prey model, *IMA J. Appl. Math.*, **78** (2013), 566–586. https://doi.org/10.1093/imamat/hxr066

58. R. Peng, J. Shi, M. Wang, On stationary patterns of a reaction-diffusion model with autocatalysis and saturation law, *Nonlinearity*, **21** (2008), 1471–1488. https://doi.org/10.1088/0951-7715/21/7/006

59. Y. Du, R. Peng, M. Wang, Effect of a protection zone in the diffusive Leslie predator-prey model, *J. Differ. Equations*, **246** (2009), 3932–3956. https://doi.org/10.1016/j.jde.2008.11.007