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Research article

Chains of Cantor subspaces of Mahler T-Numbers and the middle-third Cantor set

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Abstract: We investigate Mahler's real T-numbers through Cantor-type constructions inside $\mathcal{T} \cap \mathbb{G}$, where $\mathbb{G}$ is the middle-third Cantor set and $\mathcal{T}$ is the set of real Mahler T-numbers. We build explicit families $\mathcal{T}^{(t)}$ which are homeomorphic to Cantor space and use them to analyze structural, combinatorial, and additive properties of $\mathcal{T}$. Our results include the existence of descending chains of Cantor subsets of $\mathcal{T}$ of length $\mathfrak{c}$; a characterization of ternary expansions in $\mathcal{T}^{(t)}$, showing non-normality but maximal block complexity under sparse forcing, linked to the Adamczewski–Bugeaud criterion; and a sumset theorem proving that for suitable parameters t_1, t_2 one has the interval identity $\mathcal{T}^{(t_1)} + \mathcal{T}^{(t_2)} = [0, 1]$, which yields the global corollary $\mathcal{T} + \mathcal{T} = \mathbb{R}$ (Erdős property) by integer translation invariance. We also discuss implications for cardinal invariants and entropy of the shift map, highlighting the interplay between thin Diophantine sets and large additive structure. To address a natural concern about existence, we include a non-emptiness lemma which shows that our scheduled deletion-and-witness procedure always leaves a non-empty perfect set $\mathcal{T}^{(t)}$.

Keywords: Mahler T-numbers; Cantor set; descending chain; cardinality; Erdős property

1. Introduction

Mahler's classification of transcendental numbers divides $\mathbb{R}$ into S-, T-, and U-numbers according to their approximation properties by algebraic numbers [1]. While T-numbers occupy a delicate middle ground–being transcendentals with exceptionally large irrationality exponent–they remain poorly understood as a class.

In this paper we present a systematic study of Cantor-type subsets of the set $\mathcal{T}$ of T-numbers with the goal of elucidating both their fine combinatorial properties and their large-scale arithmetic structure.

We construct a family of Cantor sets $\{\mathcal{T}^{(t)} : t \in \mathcal{P}\}$, each homeomorphic to the Cantor space $2^{\mathbb{N}}$, lying inside $\mathcal{T} \cap \mathbb{G}$, where $\mathbb{G}$ is the classical middle-third Cantor set. These sets provide a flexible framework in which to explore several natural questions:

1) *Poset structure.* We show the existence of descending chains of Cantor subsets of $\mathcal{T}$ of length $\mathfrak{c}$ and discuss the possibilities for antichains and order types.

2) *Fractal geometry.* We analyze the size of $\mathcal{T} \cap \mathbb{G}$ in terms of Hausdorff dimension.

3) *Digital complexity.* We prove that elements of $\mathcal{T}^{(t)}$ are never normal in base 3 but under sparse forcing exhibit maximal block complexity and relative normality on the alphabet $\{0, 2\}$, thus linking to the Adamczewski–Bugeaud criterion for T-numbers.

4) *Sumsets.* Our main additive result shows that for suitable t_1, t_2, $\mathcal{T}^{(t_1)} + \mathcal{T}^{(t_2)} = [0, 1]$, which implies the global identity $\mathcal{T} + \mathcal{T} = \mathbb{R}$ (Erdős property) by integer translations [2].

5) *Cardinal invariants.* We investigate cofinality and maximal elements in the poset of Cantor subsets of $\mathcal{T}$, showing the existence of families indexed by ω_1 under CH.

6) *Dynamics.* Finally, we study the shift map on the ternary expansions of $\mathcal{T}^{(t)}$, establishing entropy bounds and discussing possible invariant measures.

The combination of these results highlights an unexpected phenomenon: although T-numbers are defined by extreme Diophantine properties and form a set of Lebesgue measure zero, their Cantor subsets behave much like classical fractals, and their sumsets can fill the entire unit interval. This interplay between thin Diophantine sets, combinatorial complexity, and large additive structure forms the central theme of the paper.

Throughout, we write $\mathcal{T}$ for the set of Mahler T-numbers and reserve plain T for parameters (e.g., $T(t)$) only when explicitly indicated. To dispel any doubt about existence, we prove a *Non-emptiness Lemma* (Lemma 4) ensuring that the scheduled deletions and witness insertions always leave a non-empty perfect set $\mathcal{T}^{(t)}$.

2. Mahler's classification

Let $H(P)$ denote the maximum absolute value of the coefficients of a polynomial $P \in \mathbb{Z}[X]$.

Definition 1 (Mahler [1])**.** *For a real number x and integer $n \geq 1$, define*

$$w_n(x) \;=\; \sup\Big\{ w : \; \exists^\infty P \in \mathbb{Z}[X],\ \deg P \leq n,\ P(x) \neq 0,\ |P(x)| < H(P)^{-w} \Big\}.$$

Then:

- *x is an S-number if $w_n(x) < \infty$ for all n.*
- *x is a U_m-number if $w_m(x) = \infty$ but $w_{m-1}(x) < \infty$.*
- *x is a T-number if $w_n(x) < \infty$ for all n, but*

$$\limsup_{n\to\infty} \frac{w_n(x)}{n} = \infty.$$

Thus T-numbers are precisely those reals that admit infinitely strong polynomial approximations with approximation exponents diverging along a subsequence of degrees.

3. Auxiliary Lemmas

The construction relies on two standard ingredients: (i) the ability to produce small values of binomials of large degree and (ii) an a priori counting bound for algebraic numbers of fixed degree.

Lemma 1 (Binomial small values; MVT version)**.** *Fix $m \geq 2$ and $\tau > 0$. For all sufficiently large integers q there exist $p \in \mathbb{Z}$ and an interval I centered at $\alpha = (p/q)^{1/m} \in (0, 1)$ with*

$$|I| \leq \frac{2}{m} q^{-(\tau+1)}$$

such that for all $x \in I$ we have

$$0 < |\, qx^m - p \,| < q^{-\tau}.$$

Proof. Let $P(x) = qx^m - p$. By the mean value theorem, for x near a real root α,

$$|P(x)| = |P'(\xi)|\,|x - \alpha| \quad \text{for some } \xi \text{ between } x \text{ and } \alpha.$$

Since $x, \xi \in [0, 1]$, we have $|P'(\xi)| = mq\xi^{m-1} \leq mq$. Thus $|P(x)| \leq mq\,|x - \alpha|$. If we impose $|x - \alpha| \leq \frac{1}{m} q^{-(\tau+1)}$, we obtain $|P(x)| \leq q^{-\tau}$, as required; taking the symmetric interval I about α gives $|I| \leq \frac{2}{m} q^{-(\tau+1)}$. □

Lemma 2 (Avoiding rationals; Borel–Cantelli)**.** *Fix $\mu_0 > 1$. There exists Q_0 such that for every $q \geq Q_0$ the union*

$$\bigcup_{p \in \mathbb{Z}} \left(p/q - q^{-(\mu_0+2)},\; p/q + q^{-(\mu_0+2)}\right)$$

has total length $\ll q^{-(\mu_0+1)}$. Consequently, removing these intervals for all $q \geq Q_0$ leaves a closed set $E \subset [0, 1]$ such that

$$w_1(x) \leq \mu_0 + 1 \qquad \forall x \in E.$$

Proof. For each fixed q, at most $q + 1$ integers p meet $[0, 1]$, and the total length is $\leq 2(q + 1)q^{-(\mu_0+2)} \ll q^{-(\mu_0+1)}$. Since $\mu_0 > 1$, the series $\sum_q q^{-(\mu_0+1)}$ converges; thus by Borel–Cantelli almost every x lies in only finitely many of these intervals. Translating back to linear polynomials, $P(X) = qX - p$ shows that $w_1(x) \leq \mu_0 + 1$. □

Lemma 3 (Counting algebraic numbers of fixed degree and height)**.** *For each $m \geq 2$ there exists a constant $C_m > 0$ such that the number of real algebraic numbers $\alpha \in [0, 1]$ with $\deg(\alpha) = m$ and naive height $H(\alpha) \leq H$ is at most $C_m H^{m+1}$.*

Proof sketch and reference. This follows from standard height counts for irreducible integer polynomials of degree m with coefficients bounded by H, together with a uniform bound on the number of real roots in $[0, 1]$ per polynomial. See Baker [3]. □

4. Construction of the nested Cantor families

We build the sets inside $\mathbb{G}$, always working with finite unions of basic ternary cylinders.

4.1. Degree scheduling (witness degrees only)

Choose strictly increasing integers $m_k \uparrow \infty$ and exponents T_k with $T_k/m_k \to \infty$ (e.g. $m_k = 2^k$, $T_k = m_k^2$). At level k we will insert a *witness* for degree m_k.

4.2. Degree-m barriers (to keep $w_m < \infty$ for fixed m)

For each $m \geq 2$, fix $\lambda_m := m + 4$. For $H \geq 2$, let

$$\mathcal{B}_{m,H} := \bigcup_{\alpha} (\alpha - H^{-\lambda_m},\ \alpha + H^{-\lambda_m}),$$

where the union runs over real algebraic $\alpha \in [0, 1]$ with $\deg(\alpha) = m$ and $H(\alpha) \leq H$. By Lemma 3,

$$|\mathcal{B}_{m,H}| \leq 2\,C_m\,H^{m+1-\lambda_m} = 2C_m\,H^{-3},$$

so $\sum_{H\geq 2} |\mathcal{B}_{m,H}| < \infty$. We *schedule* deletions so that for each fixed m the full union $\mathcal{B}_m = \bigcup_{H\geq 2} \mathcal{B}_{m,H}$ is removed by some finite stage, replacing each current ternary cylinder by finitely many subcylinders avoiding $\mathcal{B}_{m,H}$.

4.3. Witness stages (to force large w_{m_k} once per degree)

For definiteness, let the height sequence satisfy $H_k = 3^{k!}$ so that $\log H_{k+1}/\log H_k \to \infty$, and let $T_k = k^2$, which grows faster than linearly. These choices meet all requirements of the subsequent lemmas.

At level k choose large q_k and an integer p_k so that

$$P_k(X) = q_k X^{m_k} - p_k, \qquad H(P_k) = q_k =: H_k,$$

has a real root $\alpha_k = (p_k/q_k)^{1/m_k}$ in $(0, 1)$ (we choose p_k, q_k so that the short intervals from Lemma 1 lie inside basic ternary cylinders).

By Lemma 1, there is a union of short intervals I_k of total length $\ll H_k^{-T_k/m_k}$ centered near α_k on which

$$0 < |P_k(x)| < H_k^{-T_k}.$$

Ensure I_k lies inside the survivors after the barrier deletions at level k; keep all components of I_k as children at level k.

4.4. Parametric freezing scheme and the sets $\mathcal{T}^{(t)}$

Enumerate $\mathbb{Q} \cap (0, 1)$ as $(q_k)_{k\geq 1}$. For $t \in (0, 1)$, define

$$F_t := \{k : q_k \leq t\}, \qquad A_t := \mathbb{N} \setminus F_t.$$

At level k, after applying the scheduled barrier deletions and (if applicable) the witness insertion, we do:

- if $k \in F_t$ (frozen), keep one child subcylinder from each parent;
- if $k \in A_t$ (active), keep two children.

Let $E_k^{(t)}$ be the union of survivors at level k and set

$$\mathcal{T}^{(t)} = \bigcap_{k\geq 1} E_k^{(t)}.$$

Choose H_k so large that $\sum_k 2^k H_k^{-T_k/m_k} < \infty$ (this will give $\dim_H = 0$).

5. Main Theorem

Theorem 1. *For each $t \in (0, 1)$ there exists a perfect, nowhere dense Cantor set $\mathcal{T}^{(t)} \subset \mathbb{G}$ such that every $x \in \mathcal{T}^{(t)}$ is a Mahler T-number. Moreover, if $0 < t < s < 1$, then $\mathcal{T}^{(s)} \subsetneq \mathcal{T}^{(t)}$. In particular, $\{\mathcal{T}^{(t)} : t \in (0, 1)\}$ is a strictly descending chain of Cantor subsets of $\mathbb{G}$, each contained in $\mathcal{T}$; hence also*

$$\bigcap_{t\in(0,1)} \mathcal{T}^{(t)} \subseteq \mathcal{T}.$$

Proof. Cantor structure and containment in $\mathbb{G}$: Each $E_k^{(t)}$ is a finite union of basic ternary cylinders, hence $E_k^{(t)} \subset \mathbb{G}$. Because A_t is infinite, there are infinitely many active levels with two-way branching; thus there are no isolated points. Compactness and total disconnectedness are clear, so $\mathcal{T}^{(t)}$ is Cantor and $\mathcal{T}^{(t)} \subset \mathbb{G}$.

Hausdorff dimension zero: At level k the kept pieces have total length $\ll 2^k H_k^{-T_k/m_k}$. By the choice $\sum_k 2^k H_k^{-T_k/m_k} < \infty$, a standard covering argument yields $\dim_H \mathcal{T}^{(t)} = 0$.

T-membership: Fix $m \geq 2$. Since all $\mathcal{B}_{m,H}$ are deleted by a finite stage, there exists $H_0 = H_0(m)$ such that for every survivor $x \in \mathcal{T}^{(t)}$ and all $H \geq H_0$,

$$\operatorname{dist}(x, \{\alpha : \deg \alpha = m,\ H(\alpha) \leq H\}) \geq H^{-\lambda_m}.$$

This implies $w_m(x) \leq \lambda_m - 1 < \infty$ (otherwise there would be infinitely many degree-m approximants of strength $> \lambda_m$ bringing x into infinitely many $\mathcal{B}_{m,H}$). Hence,

$$w_m(x) < \infty \quad \text{for all fixed } m \geq 2.$$

On the other hand, for each witness level k and every $x \in \mathcal{T}^{(t)} \subset I_k$, we have

$$0 < |P_k(x)| < H_k^{-T_k} \quad \text{with } \deg P_k = m_k,$$

so $w_{m_k}(x) \geq T_k$. Since $T_k/m_k \to \infty$,

$$\limsup_{n\to\infty} \frac{w_n(x)}{n} \geq \lim_{k\to\infty} \frac{T_k}{m_k} = \infty.$$

Combining these two facts gives $x \in T$ for all $x \in \mathcal{T}^{(t)}$.

Strict chain: If $t < s$, let k_0 be the first index with $q_{k_0} \in (t, s]$. Then level k_0 is active for t but frozen for s, hence $E_{k_0}^{(s)} \subsetneq E_{k_0}^{(t)}$ and therefore $\mathcal{T}^{(s)} \subsetneq \mathcal{T}^{(t)}$. The family is strictly descending and has cardinality $\mathfrak{c}$. □

Lemma 4 (Non-emptiness and perfectness under scheduled deletions)**.** *In the construction of $\mathcal{T}^{(t)}$ there exists a scheduling of barrier deletions and witness insertions such that:*

(i) every surviving parent cylinder produces at least one child at each stage; and

(ii) on infinitely many (active) stages, each surviving parent produces at least two children.

Consequently $\mathcal{T}^{(t)} \neq \emptyset$ and is perfect.

Proof. Work level-by-level on the rooted tree C of basic ternary cylinders inside $\mathbb{G}$. At stage k we process finitely many deletions

$$\mathcal{U}_k = \Big(\bigcup_{m \le M_k} \bigcup_{2 \le H \le H(m,k)} \mathcal{B}_{m,H} \Big) \cup (\text{witness exclusions outside } I_k),$$

where $M_k, H(m,k)$ are finite cut-offs. By Lemma 3 and $\lambda_m = m + 4$ we have $|\mathcal{B}_{m,H}| \le 2C_m H^{-3}$ and $\sum_H |\mathcal{B}_{m,H}| < \infty$, so the total deleted length can be bounded by any prescribed ε_k. Choose $\varepsilon_k = 2^{-(k+3)}$ and take the witness height H_k large so that $|I_k| \le 2^{-(k+3)}$. Hence $|\mathcal{U}_k| \le 2^{-(k+2)}$.

Fix a surviving parent cylinder C at the beginning of stage k. Refine C to depth r_k so that at most a $2^{-(k+2)}$-fraction of its 3^{r_k} children meet $\mathcal{U}_k$. Then at least one child avoids $\mathcal{U}_k$; on active stages we ensure at least two such children. This maintains: (i) one surviving child per stage and (ii) two children on infinitely many stages.

By Kőnig's lemma for finitely branching trees, (i) yields an infinite branch, so $\mathcal{T}^{(t)} \neq \emptyset$. Because (ii) holds infinitely often, no point is isolated; the limit set, being closed and totally disconnected in $\mathbb{G}$, is perfect. □

6. Poset constructions inside $\mathcal{T} \cap \mathbb{G}$

Throughout this section we fix a nonempty perfect set $P \subset \mathcal{T} \cap \mathbb{G}$ and a homeomorphism

$$h : 2^{\mathbb{N}} \longrightarrow P.$$

For $A \subseteq \mathbb{N}$ define the clopen cylinder

$$\Sigma_A := \{\sigma \in 2^{\mathbb{N}} : \sigma(i) = 1 \text{ for all } i \in A\}, \qquad C_A = h(\Sigma_A) \subseteq P \subset \mathcal{T} \cap \mathbb{G}.$$

We write $A^c = \mathbb{N} \setminus A$.

Remark 1 (Scheduling philosophy)**.** *We separate roles: (i) for each fixed m we schedule finitely many deletions (the degree-m barriers) so every x that survives has $w_m(x) < \infty$; (ii) along a strictly increasing witness sequence (m_k) we insert one powerful approximation per level, forcing $w_{m_k}(x)$ to be arbitrarily large. This obviates any need for "robustness under arbitrary subsequences."*

The following results produce (i) a strictly descending chain of size $\mathfrak{c}$, (ii) an antichain of size $\mathfrak{c}$, (iii) chains realizing every countable order type, and (iv) an order-embedding of $(\mathcal{P}(\mathbb{N}), \subseteq)$ (with reversed order) into the family of Cantor subsets of $T \cap \mathbb{G}$.

Lemma 5 (Cantor structure and containment)**.** *Let $P \subset \mathcal{T} \cap \mathbb{G}$ be a nonempty perfect set and $h : 2^{\mathbb{N}} \to P$ a homeomorphism. For $A \subseteq \mathbb{N}$ define the clopen cylinder $\Sigma_A = \{\sigma \in 2^{\mathbb{N}} : \sigma(i) = 1\ \forall i \in A\}$ and $C_A = h(\Sigma_A)$. If $A^c = \mathbb{N} \setminus A$ is infinite, then $\Sigma_A \cong 2^{\mathbb{N}}$ and hence C_A is a perfect, compact, totally disconnected set homeomorphic to Cantor space, with $C_A \subset P \subset \mathcal{T} \cap \mathbb{G}$.*

Proof. Free coordinates in A^c give $\Sigma_A \cong 2^{A^c} \cong 2^{\mathbb{N}}$; h transfers Cantor structure and containment to C_A. □

6.1. A continuum descending chain

Fix an enumeration $(q_m)_{m\geq 1}$ of $\mathbb{Q} \cap (0,1)$ without repetitions and a bijection $e : \mathbb{N} \to \mathbb{N}$. For $x \in (0,1)$ set

$$B_x := \{ m \in \mathbb{N} : q_m < x \}, \qquad A_x := e[B_x], \qquad \mathcal{T}^{(x)} := C_{A_x}.$$

Theorem 2 (Chain of cardinality $\mathfrak{c}$)**.** *For $x < y$ in $(0,1)$ one has $\mathcal{T}^{(y)} \subsetneq \mathcal{T}^{(x)}$. Thus $\{\mathcal{T}^{(x)} : x \in (0,1)\}$ is a strictly descending chain of size $\mathfrak{c}$, and each $\mathcal{T}^{(x)}$ is a Cantor subset of $T \cap \mathbb{G}$.*

Proof. If $x < y$, then $B_x \subsetneq B_y$ and hence $A_x \subsetneq A_y$. Therefore $\Sigma_{A_y} \subsetneq \Sigma_{A_x}$, so $C_{A_y} \subsetneq C_{A_x}$ because h is injective. Because $(A_x)^c$ is infinite, Lemma 5 applies. □

6.2. A continuum antichain

Let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection. For $s = (s(1), s(2), \dots) \in 2^{\mathbb{N}}$ define

$$B_s = \{ \langle n, \widehat{s}(n) \rangle : n \geq 1 \}, \quad \text{where} \quad \widehat{s}(n) = \sum_{j=1}^{n} s(j)\, 2^{n-j}.$$

Set $A_s = e[B_s]$, $D_s = C_{A_s}$.

Theorem 3 (Antichain of cardinality $\mathfrak{c}$)**.** *The family $\{D_s : s \in 2^{\mathbb{N}}\}$ is an antichain under inclusion: if $s \neq t$, then neither $D_s \subseteq D_t$ nor $D_t \subseteq D_s$. Each D_s is a Cantor subset of $T \cap \mathbb{G}$.*

Proof. If $s \neq t$, let ℓ be the length of their longest common prefix. Then $\widehat{s}(\ell+1) \neq \widehat{t}(\ell+1)$, hence $\langle \ell+1, \widehat{s}(\ell+1) \rangle \in B_s \setminus B_t$ and vice versa. Thus $A_s \setminus A_t \neq \varnothing$ and $A_t \setminus A_s \neq \varnothing$. Choose $i \in A_s \setminus A_t$. Because $i \notin A_t$, the ith coordinate is free in Σ_{A_t}: choose $\sigma \in \Sigma_{A_t}$ with $\sigma(i) = 0$. Every $\rho \in \Sigma_{A_s}$ has $\rho(i) = 1$, so $h(\sigma) \notin D_s$. Hence $D_t \nsubseteq D_s$. By symmetry, $D_s \nsubseteq D_t$. Finally, $(A_s)^c$ is infinite, so each D_s is Cantor by Lemma 5. □

6.3. Chains of arbitrary countable order type

Theorem 4. *For every countable linear order $(L, <)$ there exists a chain $\{E_\ell : \ell \in L\}$ of Cantor subsets of $T \cap \mathbb{G}$ whose order type is $(L, <)$.*

Proof. Fix an enumeration $L = \{\ell_0, \ell_1, \dots\}$. Recursively assign to each ℓ_j a set $A_{\ell_j} \subseteq \mathbb{N}$ so that if $\ell_i < \ell_j$ then $A_{\ell_j} \supsetneq A_{\ell_i}$ and if ℓ_i and ℓ_j are incomparable in L (which never occurs because L is a linear order), no condition is imposed. This is possible because $\mathbb{N}$ can be partitioned into infinitely many disjoint infinite pieces, and we may reserve one new index to witness each strict extension. Then set $E_\ell = C_{A_\ell}$. By Lemma 5, each E_ℓ is Cantor. The ordering property ensures that $\{E_\ell : \ell \in L\}$ is a chain of order type L. □

6.4. Boolean algebra embedding

Theorem 5. *There exists an order embedding of $(\mathcal{P}(\mathbb{N}), \subseteq)$ with reversed order into the poset of Cantor subsets of $T \cap \mathbb{G}$ under inclusion.*

Proof. For $A \subseteq \mathbb{N}$, define C_A as before. If $A \subseteq B$, then $\Sigma_B \subseteq \Sigma_A$, hence $C_B \subseteq C_A$. Thus $A \mapsto C_A$ is order-preserving for the reversed order. By Lemma 5, C_A is Cantor whenever A^c is infinite. Hence the embedding is valid. □

7. Sumset results

We now turn to additive properties of the Cantor families $\mathcal{T}^{(t)}$. A remarkable phenomenon occurs: although each $\mathcal{T}^{(t)}$ is extremely thin, the sum of two such sets can fill an entire interval.

7.1. Interval filling

Theorem 6. *For each $t \in (0,1)$, let $\mathcal{T}^{(t)} \subset [0,1]$ be the set of reals with base-3 expansion using only digits $\{0,2\}$, and assume there is a set $K_t \subset \mathbb{N}$ (the* forced–2 *positions) such that*

$$x = \sum_{n\geq 1} \frac{a_n}{3^n} \in \mathcal{T}^{(t)} \iff a_n = \begin{cases} 2, & n \in K_t, \\ \text{any of } \{0,2\}, & n \notin K_t. \end{cases}$$

Let $t_1 \neq t_2$ and write $K_i := K_{t_i}$ for $i = 1, 2$. Assume the following two conditions hold:

(A1) $K_1 \cap K_2 = \varnothing$;

(A2) *There exists $N \in \mathbb{N}$ such that $\{N, N+1, N+2, \dots\} \cap (K_1 \cup K_2) = \varnothing$ (Equivalently, beyond some index N both digits are free at every position).*

Then

$$\mathcal{T}^{(t_1)} + \mathcal{T}^{(t_2)} = [0,1].$$

Proof. The inclusion $\mathcal{T}^{(t_1)} + \mathcal{T}^{(t_2)} \subseteq [0,1]$ is immediate since each summand lies in $[0,1]$.

For the reverse inclusion, fix $y \in [0,1]$ and choose its *nonterminating* base-3 expansion

$$y = \sum_{n\geq 1} b_n \, 3^{-n}, \qquad b_n \in \{0,1,2\},$$

i.e., we forbid tails of infinitely many 2's. This expansion is unique.

We construct digits $(a_n^{(1)}, a_n^{(2)}) \in \{0,2\} \times \{0,2\}$ such that the forcing rules for t_1, t_2 are respected and the (base-3) addition with carry holds:

$$a_n^{(1)} + a_n^{(2)} + \kappa_n \equiv b_n \pmod 3, \qquad \kappa_{n+1} = \left\lfloor \frac{a_n^{(1)} + a_n^{(2)} + \kappa_n}{3} \right\rfloor, \qquad \kappa_1 = 0. \tag{7.1}$$

Since $a_n^{(i)} \in \{0,2\}$, the pair-sum $s_n := a_n^{(1)} + a_n^{(2)}$ always lies in $\{0,2,4\}$.

Phase I: indices $1 \leq n < N$. By (A1), at each n at least one of $a_n^{(1)}, a_n^{(2)}$ is free, hence s_n can be chosen from $\{0,2,4\}$ unless one component is forced 2, in which case $s_n \in \{2,4\}$. We proceed inductively on $n = 1, 2, \dots, N-1$ as follows.

Given $\kappa_n \in \{0,1\}$ and $b_n \in \{0,1,2\}$, choose s_n in the admissible set (either $\{0,2,4\}$ or $\{2,4\}$, depending on whether a digit is forced) so that $s_n + \kappa_n \equiv b_n \pmod 3$. This is always possible when the admissible set is $\{0,2,4\}$; if it is $\{2,4\}$ and $b_n \equiv 0 \pmod 3$ with $\kappa_n = 0$, then no choice in $\{2,4\}$ yields residue 0. In that exceptional subcase, necessarily $n < N$ and there exists $m < n$ with both digits free (by (A2) and minimality of N). Modify the prior choice at m by replacing s_m with $s'_m = 4$ (if it was 0 or 2), which toggles κ_{m+1} from 0 to 1 and hence propagates κ_n to 1 (there are no two-forced positions by (A1)). With $\kappa_n = 1$, the residue 0 is achievable at n using $s_n = 2$. This finite adjustment affects only carries between m and n and preserves the already matched digits (all residue conditions are re-satisfied by re-choosing s_j on the finite interval $[m, n]$ in the same way). Thus we can ensure that for every $1 \le n < N$ there exists an admissible s_n satisfying (7.1).

Phase II: indices $n \ge N$. By (A2), both digits are free at every position $n \ge N$, hence $s_n \in \{0, 2, 4\}$ without restriction. We now continue the digitwise induction for $n = N, N+1, N+2, \dots$ choosing, at each step, $s_n \in \{0, 2, 4\}$ so that $s_n + \kappa_n \equiv b_n \pmod 3$. This is always possible because for each fixed $\kappa_n \in \{0, 1\}$ the set $\{0, 2, 4\} + \kappa_n$ hits all residues mod 3:

$$\kappa_n = 0 : \ \{0, 2, 4\} \equiv \{0, 2, 1\} \pmod 3, \qquad \kappa_n = 1 : \ \{1, 0, 2\} \pmod 3.$$

This defines the entire sequences $(a_n^{(1)}), (a_n^{(2)})$ and the carries (κ_n).

Conclusion. Set

$$x := \sum_{n\ge1} \frac{a_n^{(1)}}{3^n} \in \mathcal{T}^{(t_1)}, \qquad z := \sum_{n\ge1} \frac{a_n^{(2)}}{3^n} \in \mathcal{T}^{(t_2)}.$$

By construction, for every $M \ge 1$ we have

$$\sum_{n=1}^{M} \frac{a_n^{(1)} + a_n^{(2)}}{3^n} \equiv \sum_{n=1}^{M} \frac{b_n}{3^n} \pmod{3^{-M}},$$

hence the difference of the partial sums has absolute value at most 3^{-M}. Letting $M \to \infty$ yields $x+z = y$. As $y \in [0, 1]$ was arbitrary, we conclude $\mathcal{T}^{(t_1)} + \mathcal{T}^{(t_2)} = [0, 1]$. □

7.2. *The whole real line*

Theorem 7. *We have*

$$\mathcal{T} + \mathcal{T} = \mathbb{R}.$$

That is, the class of T-numbers has the Erdős property.

Proof. By Theorem 6, $[0, 1] \subseteq \mathcal{T} + \mathcal{T}$. Now, $\mathcal{T}$ is invariant under addition of integers, because Mahler's classification is unaffected by rational translation. Thus $(\mathcal{T} + \mathcal{T}) + m = \mathcal{T} + \mathcal{T}$ for all $m \in \mathbb{Z}$.

Given any $r \in \mathbb{R}$, choose $m \in \mathbb{Z}$ such that $r - m \in [0, 1]$. Then $r = (r - m) + m$ belongs to $\mathcal{T} + \mathcal{T}$. Hence $\mathcal{T} + \mathcal{T} = \mathbb{R}$. □

8. Complexity of expansions in $\mathcal{T}^{(t)}$

A central theme in Diophantine approximation is the connection between Mahler's classification and the complexity of digital expansions. In particular, Adamczewski and Bugeaud [4] proved that T-numbers necessarily give rise to expansions of very high combinatorial complexity. It is therefore natural to analyze the ternary expansions of the Cantor-type families $\mathcal{T}^{(t)} \subset \mathcal{T} \cap \mathbb{G}$ introduced earlier.

By construction, every $x \in \mathcal{T}^{(t)}$ has digits in $\{0, 2\}$ only, with a "certifying layer" K_t of positions forced to be 2 and the complement $R_t = \mathbb{N} \setminus K_t$ free. Hence no element of $\mathcal{T}^{(t)}$ is normal in base 3, yet when K_t is sparse, the sequences still display full block complexity and relative normality over the alphabet $\{0, 2\}$.

8.1. *Non-normality in base 3*

Proposition 1. *No element of $\mathcal{T}^{(t)}$ is normal in base* 3.

Proof. Since $a_n \in \{0, 2\}$ for all n, the digit 1 never appears in the ternary expansion of $x \in \mathcal{T}^{(t)}$. Base-3 normality would require frequency 1/3 for digit 1, a contradiction. □

8.2. *Block complexity under sparse forcing*

For $n \in \mathbb{N}$, define the block complexity of $\mathcal{T}^{(t)}$:

$$p_t(n) \;=\; \#\Big\{ u \in \{0,2\}^n : \;\exists\, x \in \mathcal{T}^{(t)},\; \exists j \geq 1 \; (a_j, \ldots, a_{j+n-1}) = u \Big\}.$$

We say K_t has *arbitrarily long gaps* if for every n there exists j such that $[j, j+n-1] \cap K_t = \varnothing$.

Theorem 8 (Maximal block complexity)**.** *If K_t has arbitrarily long gaps, then $p_t(n) = 2^n$ for all $n \geq 1$.*

Proof. Fix n and choose j with $[j, j+n-1] \cap K_t = \varnothing$. On this window all n digits are free, so every word $u \in \{0,2\}^n$ occurs as $(a_j, \ldots, a_{j+n-1})$ in some $x \in \mathcal{T}^{(t)}$. Thus $p_t(n) = 2^n$. Taking $\lim_{n\to\infty} \frac{1}{n} \log p_t(n)$ shows the shift entropy equals $\log 2$. □

8.3. *Typical frequencies under a natural measure*

Assume K_t has zero upper density:

$$\lim_{N\to\infty} \frac{|K_t \cap [1, N]|}{N} = 0.$$

Define a product measure μ_t on $\mathcal{T}^{(t)}$ by making digits $\{a_n : n \in R_t\}$ i.i.d. with $\mathbb{P}[a_n = 0] = \mathbb{P}[a_n = 2] = \frac{1}{2}$ and fixing $a_n \equiv 2$ on K_t.

Lemma 6. *Fix $k \geq 1$. The proportion of starting positions $1 \leq j \leq N$ for which $[j, j+k-1] \cap K_t \neq \varnothing$ tends to 0 as $N \to \infty$.*

Proof. Each $m \in K_t \cap [1, N+k-1]$ belongs to at most k windows of length k. Thus

$$\#\{1 \leq j \leq N : [j, j+k-1] \cap K_t \neq \varnothing\} \;\leq\; k\,|K_t \cap [1, N+k-1]|.$$

Dividing by N and using $|K_t \cap [1, N+k-1]|/N \to 0$ gives the claim. □

Theorem 9 (Relative normality)**.** *If K_t has zero upper density, then for μ_t-a.e. $x \in \mathcal{T}^{(t)}$:*

1) Digit frequencies exist and

$$\lim_{N\to\infty} \tfrac{1}{N}\, \#\{\, 1 \leq n \leq N : \; a_n = 0 \,\} \;=\; \tfrac{1}{2},$$

$$\lim_{N\to\infty} \tfrac{1}{N}\, \#\{\, 1 \leq n \leq N : \; a_n = 2 \,\} \;=\; \tfrac{1}{2},$$

$$\lim_{N\to\infty} \tfrac{1}{N}\, \#\{\, 1 \leq n \leq N : \; a_n = 1 \,\} \;=\; 0.$$

2) For every block $u \in \{0,2\}^k$,

$$\lim_{N\to\infty} \tfrac{1}{N}\, \#\{\, 1 \leq j \leq N : \; (a_j, \ldots, a_{j+k-1}) = u \,\} \;=\; 2^{-k}.$$

Proof. 1) On R_t, variables $X_n = \mathbf{1}_{\{a_n=2\}}$ are i.i.d. with mean $1/2$. The strong law gives

$$\frac{1}{|R_t \cap [1,N]|} \sum_{n\in R_t\cap[1,N]} X_n \to \tfrac{1}{2} \quad \text{a.s.}$$

Because $|K_t \cap [1,N]| = o(N)$, the forced digits vanish in frequency, yielding the stated limits.

2) By Lemma 6, almost all windows of length k are contained in R_t. On these windows the k-blocks are uniformly distributed in $\{0,2\}^k$. Hence by the ergodic theorem, each occurs with frequency 2^{-k}. Since the bad windows have vanishing density, the unconditional frequency is also 2^{-k}. □

No element of $\mathcal{T}^{(t)}$ is base-3 normal because the digit 1 never appears. However, if the certifying set K_t is very sparse (as in our constructions), then the combinatorial complexity of ternary expansions inside $\mathcal{T}^{(t)}$ is maximal, and almost every element with respect to the natural product measure is "normal relative to the Cantor alphabet $\{0,2\}$," with uniform block frequencies. This perspective connects directly with the work of Adamczewski and Bugeaud [4]. Our Cantor-type families provide concrete examples: although the digits are restricted to $\{0,2\}$ and hence exclude classical normality, they nonetheless achieve maximal block complexity under sparse forcing, illustrating the principle that T-numbers must display strong combinatorial irregularity in their expansions.

Remark 2 (Beyond the sparse case)**.** *If K_t has positive density, then the frequency of digit 2 is biased upwards, and the entropy decreases below* $\log 2$. *In this sense the families $\mathcal{T}^{(t)}$ interpolate between rigid expansions (where many positions are forced) and Bernoulli shifts (where almost all positions are free), providing a concrete laboratory for studying the interaction between digital complexity and Mahler's T-classification [1, 3].*

9. Discussion

The results above show that Mahler's T-numbers admit a remarkably rich internal structure. By constructing Cantor-like families $\mathcal{T}^{(t)} \subset T \cap \mathbb{G}$, we obtained a continuum-long descending chain, continuum antichains, and order embeddings of $\mathcal{P}(\mathbb{N})$. Moreover, the additive structure proved unexpectedly large: the sum of two thin Cantor families fills the entire interval $[0,1]$, and as a corollary one recovers the Erdős property $\mathcal{T} + \mathcal{T} = \mathbb{R}$.

In parallel, we analyzed the ternary expansions of the Cantor families. Although no element is normal in base 3, under mild sparsity assumptions the expansions achieve full block complexity on the restricted alphabet $\{0,2\}$ and relative normality with respect to the natural Bernoulli measure. This situates the construction within the general Adamczewski–Bugeaud framework linking Mahler's classification to expansion complexity.

Taken together, these findings reveal that T-numbers are not only ubiquitous from the viewpoint of Diophantine approximation but also exhibit a striking degree of internal combinatorial and dynamical richness. They support the emerging perspective that the sets $T \cap \mathbb{G}$ and their Cantor subsets are natural laboratories for testing broader conjectures at the interface of transcendence, fractal geometry, and symbolic dynamics.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declare there is no conflict of interest.

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