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Research article

Singular Hopf bifurcation in singularly perturbed system with delay

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Abstract: In this paper, we study the singular Hopf bifurcation in a class of singularly perturbed systems with delay. The key condition for the generation of a singular Hopf bifurcation involves the existence of eigenvalues of the singular pure imaginary part. The necessary and sufficient conditions for the generation of a singular Hopf bifurcation are discussed in a class of general nonlinear $(n, 1)$-fast-slow systems with delay on slow variables $(n \geq 1)$.

Keywords: singularly perturbed system; delay; singular Hopf bifurcation

1. Introduction

The singular Hopf bifurcation is an important phenomenon in singularly perturbed systems. Baer and Erneux [1] studied a class of general nonlinear singularly-perturbed systems with a single fast variable and slow variable, and discussed the existence of a singular Hopf bifurcation. Their two-dimensional setting implies in a trivial way that both timescales are involved in the bifurcation mechanism. Braaksma [2] studied a general nonlinear ordinary differential equation system with m fast variables and n slow variables, and analyzed that the most important condition for generating a singular Hopf bifurcation involves the existence of eigenvalues with singular imaginary parts, which means that when the scale parameter tends to zero, their imaginary parts will grow infinitely. Yang and Zeng [3] further analyzed the stability of the singular Hopf bifurcation. The above studies are based on singularly-perturbed systems of ordinary differential equations, so it is very novel to analyze the singular Hopf bifurcation in singularly perturbed systems of delay differential equations.

The singular Hopf bifurcation in singularly perturbed systems of ordinary differential equations (ODEs) has a wide range of applications in recent years, especially for planar fast-slow systems [4–8]. Furthermore, the singular Hopf bifurcation in high-dimensional fast-slow systems is studied [9–13]. Guckenheimer [14] and Kristiansen [15] discussed the singular Hopf bifurcations in $(1, 2)$-fast-slow systems and $(2, 1)$-fast-slow systems on R^3, respectively. Zhang [16] and Li [17, 18] discussed the relaxation oscillation transformed from the singular Hopf bifurcation. In addition, the singular Hopf bifurcation in the singular perturbation system of partial differential equations (PDEs) is also applied in the chemical reaction system [19]. Delay-differential equations (DDEs) are important mathematical models in many application areas, including optics [20–23], physiology and infectious disease modeling [24–26], mechanics [27–29], neuroscience [30, 31], and others. Xu et al. [32] used the time delay as the bifurcation parameter to analyze the local and global Hopf bifurcation, and proved the boundedness of the global Hopf bifurcation. By using the theory of integral semigroups and the Hopf bifurcation theory of semilinear equations on non-dense domains, Zhang and Liu [33] proved that when the parameters pass through some critical values, there will be a non-trivial periodic oscillation phenomenon through the Hopf bifurcation. The singularly-perturbed system of DDEs has been studied by many scholars in recent years, but the singular Hopf bifurcation in singularly-perturbed systems of DDEs is rarely applied and discussed [34–36], and thus there is a lack of a singular Hopf bifurcation theory in singularly-perturbed systems of DDEs. Therefore, it is of great significance for us to study the singular Hopf bifurcation in the singularly-perturbed systems of general nonlinear DDEs, which provides a theoretical basis for expanding the applicability of the singular Hopf bifurcation.

In this paper, the singular Hopf bifurcation in general nonlinear $(n, 1)$-fast-slow systems with delay on slow variables will be researched. The rest of this paper is organized as follows. In Section 2, the singular Hopf bifurcation theorem in $(1, 1)$-fast-slow systems with delay is given by defining the pure imaginary eigenvalues with the singular imaginary part. In Section 3, the necessary and sufficient conditions for the existence of the singular Hopf bifurcation in $(m, 1)$-fast-slow systems with delay are discussed.

2. $(1, 1)$-fast-slow system with delay

In this section, we develop the necessary and sufficient conditions for the existence of eigenvalues with singular pure imaginary part for the characteristic equation, which is the key condition for the generation of a singular Hopf bifurcation in a $(1, 1)$-fast-slow system. First, we consider a class of general nonlinear $(1, 1)$-fast-slow systems with delay on slow variables as follows:

$$\begin{cases} \dfrac{dx}{dt} = f\left(x, y, y\left(t-\tilde{\tau}\right), \beta\right), \\ \dfrac{dy}{dt} = \varepsilon g\left(x, y, y\left(t-\tilde{\tau}\right), \beta\right), \end{cases} \tag{2.1}$$

where $x(t) \in R$ is a fast variable, $y(t), y(t-\tau) \in R$ are slow variables, $\beta \in R$ is a bifurcation parameter, ε is a time scale parameter, and $0 < \varepsilon \ll 1$. Therefore, system (2.1) is a fast-time system.

Letting $T = \varepsilon t$, we rewrite system (2.1) as

$$\begin{cases} \varepsilon\dfrac{dx}{dT} = f\left(x, y, y\left(T-\tau\right), \beta\right), \\ \dfrac{dy}{dT} = g\left(x, y, y\left(T-\tau\right), \beta\right), \end{cases} \tag{2.2}$$

where $\tau = \varepsilon\tilde{\tau}$, and system (2.2) is called a slow-time system. Let $E^*(x^*, y^*)$ be the equilibrium of system (2.2). Then, we have $f(x^*, y^*, y^*, \beta) = 0$ and $g(x^*, y^*, y^*, \beta) = 0$. Then, the characteristic equation of system (2.1) at the equilibrium $E^*(x^*, y^*)$ is

$$\left|\lambda I - A - Be^{-\lambda\tilde{\tau}}\right| = 0, \tag{2.3}$$

where I is a unit matrix of order 2, and

$$A = \begin{pmatrix} f_x^{'} & f_y^{'} \\ \varepsilon g_x^{'} & \varepsilon g_y^{'} \end{pmatrix} := \begin{pmatrix} a & b \\ \varepsilon c & \varepsilon d \end{pmatrix},\ B = \begin{pmatrix} f_{x(t-\tau)}^{'} & f_{y(t-\tau)}^{'} \\ \varepsilon g_{x(t-\tau)}^{'} & \varepsilon g_{y(t-\tau)}^{'} \end{pmatrix} := \begin{pmatrix} 0 & p \\ 0 & \varepsilon q \end{pmatrix}$$

are the Jacobian matrices of the general term and the delay term in system (2.1), respectively.

Then we give the exact definition of eigenvalues with singular pure imaginary part:

Definition 2.1. Let $0 < \varepsilon \ll \varepsilon_0$, ε_0 be a fixed small positive number. If $\lambda = \lambda(\varepsilon) = \sigma(\varepsilon) + i\varphi(\varepsilon)$ is the eigenvalue of the slow system (2.2) at the equilibrium E^*, and satisfies

$$\sigma = \sigma(\varepsilon) \to 0,\ \varphi = \varphi(\varepsilon) \to \infty\ \ as,\ \varepsilon \to 0,$$

then $\lambda(\varepsilon)$ is called an eigenvalue with a singular pure imaginary part.

If $\lambda = \lambda(\varepsilon) \to O(1)\,(\varepsilon \to 0)$, then $\lambda(\varepsilon)$ is called a regular eigenvalue.

Theorem 2.2. *Consider the characteristic equation family of the slow system (2.2):*

$$P(\lambda) = \left|\lambda I - \varepsilon^{-1}A - \varepsilon^{-1}Be^{-\lambda\tau}\right| = 0,\ i.e., \begin{vmatrix} \lambda - \varepsilon^{-1}a & \varepsilon^{-1}b - \varepsilon^{-1}pe^{-\lambda\tau} \\ -c & \lambda - d - qe^{-\lambda\tau} \end{vmatrix} = 0.$$

The characteristic equation $P(\lambda) = 0$ has two eigenvalues with singular pure imaginary part if and only if

$$(i)\ a = 0;\ (ii)\ \tilde{\tau}cp - d - q = 0;\ (iii)\ c(p+b) < 0.$$

Proof. **Necessity of the conditions.** Let $\lambda_{1,2} = \lambda_{1,2}(\varepsilon) = \sigma(\varepsilon) \pm i\varphi(\varepsilon)\,(\varphi(\varepsilon) > 0)$ be two eigenvalues with singular pure imaginary part, i.e., $\sigma(\varepsilon) \to 0\ (\varepsilon \to 0)$ *and* $\varphi(\varepsilon) \to \infty\ (\varepsilon \to 0)$. Substituting $\lambda_{1,2}$ into the characteristic equation $P(\lambda) = 0$, we can get

$$\lambda^2 - \left(\varepsilon^{-1}a + d\right)\lambda + \varepsilon^{-1}(ad - bc) + \varepsilon^{-1}e^{-\lambda\tau}(aq - cp) - q\lambda e^{-\lambda\tau} = 0. \tag{2.4}$$

Using Euler's formula and separating the real and imaginary parts, we have

$$\begin{cases} \sigma^2 - \varphi^2 - \sigma d - qe^{-\tau\sigma}(\sigma\cos(\varphi\tau) + \varphi\sin(\varphi\tau)) = \varepsilon^{-1}\left(a\sigma - ad - e^{-\tau\sigma}\cos(\varphi\tau)(aq - cp) + bc\right), \\ 2\sigma\varphi - \varphi d + qe^{-\tau\sigma}(\sigma\sin(\varphi\tau) - \varphi\cos(\varphi\tau)) = \varepsilon^{-1}\left(\varphi a + e^{-\tau\sigma}\sin(\varphi\tau)(aq - cp)\right), \end{cases} \tag{2.5}$$

The limit of the left and right sides of (2.5-1) with respect to ε can be obtained as

$$\lim_{\varepsilon\to 0}\left(\varphi(\varepsilon)^2 + q\varphi^2(\varepsilon)\tau(\varepsilon)\right) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(ad + aq - cp - bc). \tag{2.6}$$

According to the above limit and $\varphi(\varepsilon) \to +\infty\,(\varepsilon \to 0)$, we deduce $ad + aq - cp - bc \neq 0$, and further we have $\varphi(\varepsilon) = O\left(\frac{1}{\sqrt{\varepsilon}}\right)(\varepsilon \to 0)$.

We take the limit on the left and right sides of Eq (2.5-2) with respect to ε, and replace them according to the equivalent infinitesimal. We have

$$\lim_{\varepsilon\to 0}\left(\varphi(\varepsilon)\left(\varepsilon^{-1}a + \tilde{\tau}(aq - cp) - 2\sigma(\varepsilon) + d + q\right)\right) = 0. \tag{2.7}$$

According to $\varphi(\varepsilon) \to +\infty$ $(\varepsilon \to 0)$ and $\sigma(\varepsilon) \to 0$ $(\varepsilon \to 0)$, we get $a = o\left(\varepsilon^{\frac{3}{2}}\right)$ $(\varepsilon \to 0)$. Obviously, a has nothing to do with ε, so $a = 0$ is proved. Further, combined with the limit (2.6), it is concluded that $c(p+b) < 0$.

Using $a = 0$, repeating the above limit process, we can get

$$\lim_{\varepsilon\to 0}\left(\varphi(\varepsilon)\left(-\tilde{\tau}cp - 2\sigma(\varepsilon) + d + q\right)\right) = 0. \tag{2.8}$$

Similarly, $\tilde{\tau}cp - d - q = 0$ is proved.

In addition, according to the above two proved conclusions (i) and (ii), we can take the limit again and get

$$\lim_{\varepsilon\to 0}\left(\varphi(\varepsilon)\,\sigma(\varepsilon)\right) = 0.$$

Thus, we have $\sigma(\varepsilon) = O\left(\varepsilon^{\frac{1}{2}}\right)(\varepsilon \to 0)$.

Sufficiency of the conditions. It is known that the conditions $(i) - (iii)$ hold, and the characteristic equation of the slow system (2.2) has two eigenvalues with singular pure imaginary part. With the help of condition (i) $a = 0$, the characteristic equation family $P(\lambda) = 0$ can be simplified as

$$P(\lambda) = \begin{vmatrix} \lambda & -\varepsilon^{-1}b - \varepsilon^{-1}pe^{-\varepsilon\lambda\tilde{\tau}} \\ -c & \lambda - d - qe^{-\varepsilon\lambda\tilde{\tau}} \end{vmatrix} = 0 \text{ i.e., } \lambda^2 - \left(d + qe^{-\varepsilon\lambda\tilde{\tau}}\right)\lambda - \varepsilon^{-1}c\left(b + pe^{-\varepsilon\lambda\tilde{\tau}}\right) = 0.$$

Let $\lambda_{1,2} = \lambda_{1,2}(\varepsilon) = \sigma(\varepsilon) \pm i\varphi(\varepsilon)$ be a pair of eigenvalues of the above characteristic equation family, where $\varphi(\varepsilon) > 0$. Substituting into the above formula, we obtain

$$\begin{aligned} &(\sigma(\varepsilon) + \mathrm{i}\varphi(\varepsilon))^2 - \varepsilon^{-1}c\left(b + pe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\left(\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon)) - \sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\,\mathrm{i}\right)\right) \\ &\quad - \left(d + qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\left(\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon)) - \sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\,\mathrm{i}\right)\right)(\sigma(\varepsilon) + \mathrm{i}\varphi(\varepsilon)) = 0, \end{aligned}$$

and separating the real and imaginary parts of the above equation, we obtained

$$\begin{cases} \sigma(\varepsilon)^2 - \varphi(\varepsilon)^2 - \sigma(\varepsilon)\left(d + qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) \\ -qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\,\varphi(\varepsilon) - \varepsilon^{-1}c\left(b + pe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) = 0, \\ 2\sigma(\varepsilon)\,\varphi(\varepsilon) - \varphi(\varepsilon)\left(d + qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) \\ +qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\,\sigma(\varepsilon) + \varepsilon^{-1}cpe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon)) = 0. \end{cases} \tag{2.9}$$

We consider the reduction to absurdity: supposing $\lim_{\varepsilon\to 0}\varphi(\varepsilon) \neq \infty$, then obviously $\lim_{\varepsilon\to 0}\varphi(\varepsilon) = \varphi_0$, where φ_0 is a nonnegative constant.

Take the limit of Eq (2.9-1) with respect to ε:

$$\lim_{\varepsilon\to 0}\left(\sigma(\varepsilon)^2 - \sigma(\varepsilon)\left(d + qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\right) - qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\varepsilon\tilde{\tau}\varphi(\varepsilon)^2 - \varepsilon^{-1}c\left(b + pe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\right)\right) = \varphi_0^2. \tag{2.10}$$

According to condition (iii), $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = \infty$ can be obtained. The following analyzes the limit distribution of $\sigma(\varepsilon)$:

When $\sigma(\varepsilon) \to +\infty\,(\varepsilon \to 0)$, in order to ensure that the limit (2.10) holds, a necessary condition is $\sigma(\varepsilon) = O\left(\frac{1}{\sqrt{\varepsilon}}\right)(\varepsilon \to 0)$;

When $\sigma(\varepsilon) \to -\infty\,(\varepsilon \to 0)$ and $\varepsilon\sigma(\varepsilon) \to$ *constant* $(\varepsilon \to 0)$, we have $\sigma(\varepsilon) = O\left(\frac{1}{\sqrt{\varepsilon}}\right)(\varepsilon \to 0)$,

otherwise the limit (2.10) does not hold.
When $\sigma(\varepsilon) \to -\infty\,(\varepsilon \to 0)$ and $\varepsilon\sigma(\varepsilon) \to -\infty\,(\varepsilon \to 0)$, according to the limit (2.10), we have $\sigma(\varepsilon) = O\left(\frac{1}{\varepsilon}\right)(\varepsilon \to 0)$, which contradicts $\varepsilon\sigma(\varepsilon) \to -\infty\,(\varepsilon \to 0)$.
Finally, we have $\sigma(\varepsilon) = O\left(\frac{1}{\varepsilon}\right)(\varepsilon \to 0)$.

Taking the limit of Eq (2.9-2) with respect to ε, we get

$$(2 + q\tilde{\tau}) \lim_{\varepsilon\to 0} (\varphi(\varepsilon)\,\sigma(\varepsilon)) = (d + q - cp\tilde{\tau}) \lim_{\varepsilon\to 0} \varphi(\varepsilon) = 0, \tag{2.11}$$

It can be obtained that $\varphi(\varepsilon) = o\left(\sqrt{\varepsilon}\right)(\varepsilon \to 0)$.

The deformation of Eq (2.9-1) is obtained as

$$\begin{aligned}\sigma(\varepsilon)^2 - \varphi(\varepsilon)^2 - \sigma(\varepsilon)\left(d + qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) - qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\,\varphi(\varepsilon)\\ = \varepsilon^{-1}c\left(b + pe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right),\end{aligned}$$

For the left and right sides of the above formula, taking the limit with respect to ε,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\sigma(\varepsilon)^2 - \varphi(\varepsilon)^2 - \sigma(\varepsilon)\left(d + qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) - qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\,\varphi(\varepsilon)\right) = +\infty,\\ &\lim_{\varepsilon\to 0}\left(\varepsilon^{-1}c\left(b + pe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right)\right) = -\infty.\end{aligned}$$

The limit values on the left and right sides of the equation are different, which contradicts the original hypothesis of the reduction to absurdity. Therefore, $\lim_{\varepsilon\to 0}\varphi(\varepsilon) = +\infty$ is proved.

By using reduction to absurdity again, we prove $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = 0$, so we assume that $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = \sigma_0$ or $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = \infty$, where σ_0 is a non-zero constant.

For the case of $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = \sigma_0$, it is obvious that $\varepsilon\varphi(\varepsilon) \to 0\,(\varepsilon \to 0)$, otherwise Eq (2.9-1) does not hold when $\varepsilon \to 0$. Taking the limit of Eq (2.9-2) with respect to ε, we have

$$(2\sigma_0 - d - q + cp\tilde{\tau}) \lim_{\varepsilon\to 0}\varphi(\varepsilon) = 2\sigma_0 \lim_{\varepsilon\to 0}\varphi(\varepsilon) = 0,$$

contrary to $\lim_{\varepsilon\to 0}\varphi(\varepsilon) = +\infty$, so $\lim_{\varepsilon\to 0}\sigma(\varepsilon) \neq \sigma_0$.

For the case of $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = +\infty$, it is not difficult to get $\varepsilon\varphi(\varepsilon) \to 0\,(\varepsilon \to 0)$ and $\varepsilon\sigma(\varepsilon) \to 0\,(\varepsilon \to 0)$, otherwise the Eq (2.9-1) does not hold when $\varepsilon \to 0$. Taking the limit of Eq (2.9-2) with respect to ε, we have

$$2\lim_{\varepsilon\to 0}(\varphi(\varepsilon)\,\sigma(\varepsilon)) = (d + q - cp\tilde{\tau}) \lim_{\varepsilon\to 0}\varphi(\varepsilon) = 0,$$

contrary to $\lim_{\varepsilon\to 0}\varphi(\varepsilon) = +\infty$ and $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = +\infty$, so $\lim_{\varepsilon\to 0}\sigma(\varepsilon) \neq +\infty$.

For the case of $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = -\infty$, we say that $\varepsilon\varphi(\varepsilon) \to 0\,(\varepsilon \to 0)$, otherwise $\varphi(\varepsilon) = O\left(\frac{1}{\varepsilon}\right)(\varepsilon \to 0)$, and for the limit of Eq (2.9-1), we have

$$\lim_{\varepsilon\to 0}\sigma(\varepsilon) = -\lim_{\varepsilon\to 0}\varphi(\varepsilon)\,. \tag{2.12}$$

The two equations of (2.9) are first added and divided by $\varphi(\varepsilon)$, and from then the limit of ε, we can be obtain that

$$\lim_{\varepsilon\to 0}\sigma(\varepsilon) = \frac{1}{2}\left(d + q\lim_{\varepsilon\to 0}\left(e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right)\right) - \frac{1}{2}bc\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\varphi(\varepsilon)}$$

is a finite number, contradictory to the conditions of the case. Thus, $\varepsilon\sigma(\varepsilon) \to 0\,(\varepsilon \to 0)$.

When $\varepsilon\sigma(\varepsilon) \to -\infty\,(\varepsilon \to 0)$, we have $e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)} \to +\infty\,(\varepsilon \to 0)$. In Eq (2.9-1), it is not difficult to see that $q\sigma(\varepsilon)\,e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))$ is the only term that converges fastest to ∞. Therefore, the generalized limit of ε at the left end of Eq (2.9-1) is $\pm\infty$ (depending on the symbol of q, and contrary to the symbol of q), while the right end of Eq (2.9-1) is always 0, which is contradictory. In particular, when $q = 0$, the Eq (2.12) is established and the contradiction arises. Therefore, $\varepsilon\sigma(\varepsilon) \to constant$, and the constant is 0, otherwise it contradicts the following equation in the limit state of ε:

$$\lim_{\varepsilon\to 0}\varepsilon\left(\sigma(\varepsilon)^2 - \varphi(\varepsilon)^2\right) = \lim_{\varepsilon\to 0}\left(\varepsilon\sigma(\varepsilon)\left(d + qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) - \varepsilon qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon)) \right.$$
$$\left. -c\left(b + pe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right)\right).$$

Its left limit is $+\infty$, and the right limit is a constant, so we have $\lim_{\varepsilon\to 0}\sigma(\varepsilon) \neq -\infty$. In the above three cases, $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = 0$ is proved.

Remark 1. We note that the singular Hopf bifurcation theorem in the non-delayed fast-slow system is mainly different from the singular Hopf bifurcation theorem in the delayed fast-slow system under the following condition:

$$(ii)\ \tilde{\tau}cp - d - q = 0 \ \text{ i.e., } \tau cp - \varepsilon d - \varepsilon q = 0.$$

For the case of $\tau = 0$, condition (ii) in Theorem 2.2 naturally holds when $\varepsilon \to 0$. The two have good compatibility. The above shows that (ii) in the singular Hopf bifurcation condition can be approximately regarded as natural under the restriction of small time delay. Under the limitation of large time delay, a singular Hopf bifurcation cannot occur.

We consider the (1,1)-fast-slow system [36]

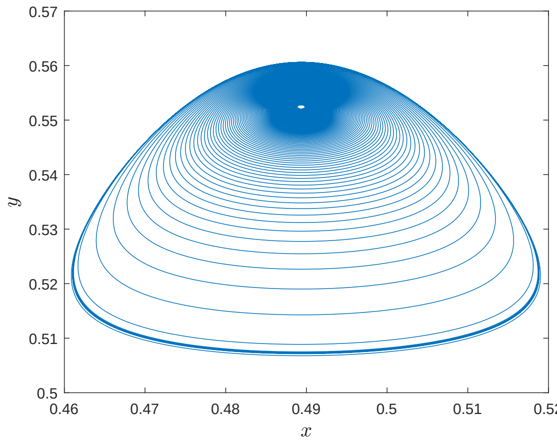
$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x) - \frac{a_1xy}{x+\eta} - h := F(x(t), y(t), \varepsilon),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon y\left(\frac{a_2x}{x+\eta} - cx(t-\tau) - 1\right) := G(x(t), x(t-\tau), y(t), \varepsilon),$$

where a_1, a_2, η, c, and h are the model parameters, ε is the time scale parameter, and τ is the delay parameter.

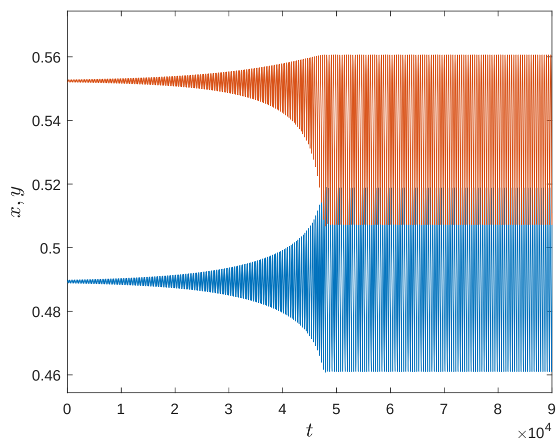
We choose the following parameters:

$$\varepsilon = 0.02,\ a_1 = 0.09,\ a_2 = 3.36,\ \eta = 1.14,\ \tau = 2.71,\ h = 0.12,\ c = 0.49.$$

It satisfies the conditions of Theorem 2.1, and the system produces a singular Hopf bifurcation near the internal equilibrium as shown in Figure 1.



(a)



(b)

Figure 1. The system generates a singular Hopf bifurcation near the internal equilibrium.

3. $(n, 1)$-fast-slow system with delay

On the basis of Section 2, we consider the case of multiple fast variables, and study a class of general nonlinear $(n, 1)$-fast-slow systems with delay on slow variables:

$$\begin{cases} \dfrac{dx}{dt} = f\left(x, y, y\left(t - \tilde{\tau}\right), \beta\right), \\ \dfrac{dy}{dt} = \varepsilon g\left(x, y, y\left(t - \tilde{\tau}\right), \beta\right), \end{cases} \tag{3.1}$$

where the fast variable $x \in R^n$, and the rest is consistent with system (2.1). Letting $T = \varepsilon t$, we get the following slow system:

$$\begin{cases} \varepsilon \dfrac{dx}{dT} = f\left(x, y, y\left(T - \tau\right), \beta\right), \\ \dfrac{dy}{dT} = g\left(x, y, y\left(T - \tau\right), \beta\right). \end{cases} \tag{3.2}$$

Let $E^*(x^*, y^*)$ be the equilibrium of the slow system (3.2). The characteristic equation of system (3.2) at the equilibrium E^* is

$$P\left(\lambda\right) = \left|\lambda I - \varepsilon^{-1}\tilde{A} - \varepsilon^{-1}\tilde{B}e^{-\lambda\tau}\right| = 0, \tag{3.3}$$

where I is the $(n+1)$-order unity matrix, and $\tilde{A}$ and $\tilde{B}$ are the Jacobian matrices of the general term and the delay term in system (3.1), respectively.

$$\tilde{A} = \begin{pmatrix} f_x' & f_y' \\ \varepsilon g_x' & \varepsilon g_y' \end{pmatrix} := \begin{pmatrix} A_{n\times n} & B_{n\times 1} \\ \varepsilon C_{1\times n} & \varepsilon D_{n\times 1} \end{pmatrix}, \; \tilde{B} = \begin{pmatrix} f_{x(t-\tau)}' & f_{y(t-\tau)}' \\ \varepsilon g_{x(t-\tau)}' & \varepsilon g_{y(t-\tau)}' \end{pmatrix} := \begin{pmatrix} 0_{n\times n} & P_{n\times 1} \\ 0_{1\times n} & \varepsilon Q_{1\times 1} \end{pmatrix}.$$

Thus, we get the following form of the characteristic equation:

$$P\left(\lambda\right) = \begin{vmatrix} \lambda I_{n\times n} - \varepsilon^{-1}A_{n\times n} & -\varepsilon^{-1}B_{n\times 1} - \varepsilon^{-1}P_{n\times 1}e^{-\lambda\tau} \\ -C_{1\times n} & \lambda - D_{1\times 1} - Q_{1\times 1}e^{-\lambda\tau} \end{vmatrix} = 0. \tag{3.4}$$

The essence of the singular Hopf bifurcation is the generation of pure imaginary eigenvalues with singular imaginary part. In the singularly perturbed system with multiple fast variables, we have the following theorem.

Theorem 3.1. *Consider the characteristic equation (3.4) on the slow system (3.2). Then, the characteristic equation (3.4) has two eigenvalues with singular pure imaginary part if and only if the following conditions hold:*

$$(i)\ r\begin{pmatrix} -C(B+P)I+(B+P)C \\ A \end{pmatrix} = r(-C(B+P)I+(B+P)C) = r(A) = n-1;$$

$$(ii)\ r\begin{pmatrix} A \\ (D+Q)I-\tilde{\tau}PC \end{pmatrix} = r((D+Q)I-\tilde{\tau}PC) = r(A) = n-1;$$

$$(iii)\ \left(B_{n\times 1}+P_{n\times 1}, C^T_{1\times n}\right) < 0.$$

The inner product mentioned here is the standard inner product in finite-dimensional real vector space.

Proof. **Necessity of the conditions.** In order to further simplify the characteristic equation (3.4), we first prove $\lambda - D - Qe^{-\lambda\tau} \neq 0$. Suppose $\lambda - D - Qe^{-\lambda\tau} = 0$, and let $\lambda(\varepsilon) = \sigma(\varepsilon) + i\varphi(\varepsilon)$, $\lim\limits_{\varepsilon\to 0}\sigma(\varepsilon) = 0$, and $\lim\limits_{\varepsilon\to 0}\varphi(\varepsilon) = \infty$. Separation of the real and imaginary parts can be obtained as

$$\begin{cases} \sigma(\varepsilon) - D - Qe^{-\sigma(\varepsilon)\varepsilon\tilde{\tau}}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon)) = 0, \\ \varphi(\varepsilon) + Qe^{-\sigma(\varepsilon)\varepsilon\tilde{\tau}}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon)) = 0. \end{cases}$$

For the second equation in the above formula, $\lim\limits_{\varepsilon\to 0}\varphi(\varepsilon) = 0$ can be obtained by taking the limit of ε, which is contradictory to the proposition, so that when λ is near the eigenvalue of the singular imaginary part, we get $\lambda - D - Qe^{-\lambda\tau} \neq 0$. Then, the characteristic equation (3.4) is further simplified as

$$P(\lambda) = \left(\lambda - D - Qe^{-\lambda\tau}\right)^{1-n}\left|\left(\lambda - D - Qe^{-\lambda\tau}\right)\left(\lambda I - \varepsilon^{-1}A\right) - \varepsilon^{-1}\left(BC + e^{-\lambda\tau}PC\right)\right| = 0.$$

Let

$$P_1(\lambda) = \left|\left(\lambda - D - Qe^{-\lambda\tau}\right)\cdot\left(\lambda I - \varepsilon^{-1}A\right) - \varepsilon^{-1}\left(BC + e^{-\lambda\tau}PC\right)\right| = 0,$$

$P(\lambda) = 0$, and $P_1(\lambda) = 0$ be co-solutions. Further, we only need to discuss $P_1(\lambda) = 0$. We have

$$P_1(\lambda) = |E + iF| = 0, \tag{3.5}$$

where $E = E(\varepsilon) = -\varphi(\varepsilon)\left(\varphi(\varepsilon) + Qe^{-\tau\sigma(\varepsilon)}\sin(\tau\varphi(\varepsilon))\right)I - \varepsilon^{-1}\left(BC + e^{-\tau\sigma(\varepsilon)}\cos(\tau\varphi(\varepsilon))PC\right) + \left(\sigma(\varepsilon) - D - Qe^{-\tau\sigma(\varepsilon)}\cos(\tau\varphi(\varepsilon))\right)\left(\sigma(\varepsilon)I - \varepsilon^{-1}A\right)$, and $F = F(\varepsilon) = \varepsilon^{-1}e^{-\tau\sigma(\varepsilon)}\sin(\tau\varphi(\varepsilon))PC + \left(\varphi(\varepsilon) + Qe^{-\tau\sigma(\varepsilon)}\sin(\tau\varphi(\varepsilon))\right)\left(\sigma(\varepsilon)I - \varepsilon^{-1}A\right) + \varphi(\varepsilon)\left(\sigma(\varepsilon) - D - Qe^{-\tau\sigma(\varepsilon)}\cos(\tau\varphi(\varepsilon))\right)I$.

According to Eq (3.5), we have the real block matrix

$$\begin{vmatrix} E(\varepsilon)_{n\times n} & -F(\varepsilon)_{n\times n} \\ F(\varepsilon)_{n\times n} & E(\varepsilon)_{n\times n} \end{vmatrix} = 0.$$

Furthermore, there is a nonzero real vector pair (u, v), where u and v are n-order column vectors satisfying $E(\varepsilon) u = F(\varepsilon) v$ and $E(\varepsilon) v = -F(\varepsilon) u$. This is equivalent to the following:

$$\begin{aligned}&\left(\sigma(\varepsilon) - D - Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right)\sigma(\varepsilon) Iu - \varphi(\varepsilon)\left(\varphi(\varepsilon) + Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) Iu\\&-\left(\sigma(\varepsilon) - D - Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right)\varphi(\varepsilon) Iv - \sigma(\varepsilon)\left(\varphi(\varepsilon) + Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) Iv\\&+\varepsilon^{-1}\varphi(\varepsilon) Av = -\varepsilon^{-1}Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon)) Av + \varepsilon^{-1}e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon)) PCv\\&+\varepsilon^{-1}\left(\sigma(\varepsilon) - D - Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon)) Au + \varepsilon^{-1}\left(BC + e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon)) PC\right) u\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\left(\sigma(\varepsilon) - D - Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right)\sigma(\varepsilon) Iv + \varphi(\varepsilon)\left(\sigma(\varepsilon) - D - Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon)) Iu\right.\\&+\left(\varphi(\varepsilon) + Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right)\sigma(\varepsilon) Iu - \varepsilon^{-1}\varphi(\varepsilon) Au - \varphi(\varepsilon)\left(\varphi(\varepsilon) + Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right)\\&Iv - \varepsilon^{-1}\varphi(\varepsilon) Au = \varepsilon^{-1}\left(Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) Au - \varepsilon^{-1}e^{-\varepsilon\tilde{\tau}\sigma(\xi)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon)) PCu\\&+\varepsilon^{-1}\left(\sigma(\varepsilon) - D - Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) Av + \varepsilon^{-1}\left(BC + e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon)) PC\right) v.\end{aligned} \tag{3.7}$$

Next, we look for the same order infinity of $\varphi(\varepsilon)$. Suppose $\varepsilon\varphi(\varepsilon) \to \infty\,(\varepsilon \to 0)$. We multiply Eq (3.6) by ε, and then take the limit of its L^1 norm with respect to ε; the left end of the equation is $+\infty$, and the right end of the equation is constant. Therefore, $\varphi(\varepsilon) \neq o\left(\frac{1}{\varepsilon}\right)(\varepsilon \to 0)$. Further, we have $\varphi(\varepsilon) = O\left(\frac{1}{\varepsilon^a}\right)(\varepsilon \to 0)$, where $0 < a \leq 1$. Taking the limit of its L^1 norm again, we can get that the right end of the equation is still a constant, and the left end of the equation is

$$\lim_{\varepsilon\to 0}\|\varepsilon\varphi(\varepsilon)\left(\varphi(\varepsilon) + Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) Iu + \varphi(\varepsilon) Av - \left(D + Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right)\varepsilon\varphi(\varepsilon) Iv\|_1.$$

In order to make the limit of the above L^1 norm equal, we have the following conclusion:

$$\varphi(\varepsilon) = O\left(\frac{1}{\sqrt{\varepsilon}}\right)(\varepsilon \to 0) \text{ and } Av = 0 \text{ i.e., } |A| = 0.$$

Similarly, we can get $Au = 0$. Further simplifying Eq (3.6) gives

$$\begin{aligned}&\varphi(\varepsilon)\left[\left(\varphi(\varepsilon) + De^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) Iu + 2\sigma(\varepsilon) Iv - \left(D + Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) Iv\right]\\&=\left(\sigma(\varepsilon)^2 - D\sigma(\varepsilon) - Q\sigma(\varepsilon) e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right) Iu - Q\sigma(\varepsilon) e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon)) Iv\\&\quad-\varepsilon^{-1}\left[\left(BC + e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon)) PC\right) u + e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon)) PCv\right].\end{aligned}$$

Thus, the following limits hold:

$$\lim_{\varepsilon\to 0}\left(\varphi(\varepsilon)\left(D + Qe^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon))\right)\right) Iv = \lim_{\varepsilon\to 0}\left(\varepsilon^{-1}e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\sin(\varepsilon\tilde{\tau}\varphi(\varepsilon)) PCv\right) \tag{3.8}$$

and

$$\lim_{\varepsilon\to 0}\left(\varphi(\varepsilon)^2 Iu\right) = \lim_{\varepsilon\to 0}\left(-\varepsilon^{-1}\left(BC + e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos(\varepsilon\tilde{\tau}\varphi(\varepsilon)) PC\right) u\right). \tag{3.9}$$

According to Eq (3.8), we can deduce $((D + Q) I - \tilde{\tau}PC) v = 0$, which shows that

$$r\begin{pmatrix} A \\ (D + Q) I - \tilde{\tau}PC \end{pmatrix} = r((D + Q) I - \tilde{\tau}PC) = r(A) < n.$$

Since $\varphi(\varepsilon) = O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ $(\varepsilon \to 0)$, then let $\varphi(\varepsilon) \sim k\frac{1}{\sqrt{\varepsilon}}$ $(\varepsilon \to 0)$, where k is a positive number. Then, Eq (3.9) can be reduced to

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left(a^2 Iu + \left(BC + e^{-\varepsilon\tilde{\tau}\sigma(\varepsilon)}\cos\left(\varepsilon\tilde{\tau}\varphi(\varepsilon)\right)PC\right)u\right) = 0.$$

According to the above limit, we have $\left(k^2 I + (B+P)C\right)u = 0$. Combined with $Au = 0$, we have the following conclusions:

$$r\begin{pmatrix} k^2 I + (B+P)C \\ A \end{pmatrix} = r\left(k^2 I + (B+P)C\right) = r(A) < n. \tag{3.10}$$

Let $B + P = (b_1, b_2, \cdots, b_n)^{\mathrm{T}}$ and $C = (c_1, c_2, \cdots, c_n)$. Then $(B+P)C$ is an n-order matrix with $((B+P)C)_{ij} = b_i c_j$. Let $\tilde{\lambda}$ be the eigenvalue of the matrix $k^2 I + (B+P)C$. Then,

$$\left|\tilde{\lambda} I - \left(k^2 I + (B+P)C\right)\right| = \left|\left(\tilde{\lambda} - k^2\right)I - (B+P)C\right| = 0.$$

Let $\bar{\lambda} = \tilde{\lambda} - k^2$. Then, $\bar{\lambda}$ is the eigenvalue of matrix $(B+P)C$. Since $r((B+P)C) \le \min\{r(B+P), r(C)\}$, where $B+P$ is a column vector and C is a row vector, we have $r((B+P)C) \le 1$.

Let $\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_n$ be the eigenvalues of matrix $k^2 I + (B+P)C$. Then,

$$\left|k^2 I + (B+P)C\right| = \prod_{i=1}^{n} \lambda_i = 0,$$

indicating that there is at least one eigenvalue $\tilde{\lambda}_k = 0$. Let $u_1, u_2, \cdots, u_n$ be the eigenvalues of a matrix $(B+P)C$. Then, $\tilde{\lambda}_i$ has the following relation with u_i:

$$\tilde{\lambda}_i = a^2 + u_i,\ i = 1, 2, \cdots, n.$$

Thus there is $u_k = -k^2$. Since the rank of the matrix is not less than the number of non-zero eigenvalues (at least 1), we have $r((B+P)C) = 1$, and the matrix $(B+P)C$ has only one non-zero eigenvalue, that is, $u_1 = u_2 = \cdots = u_{k-1} = u_{k+1} = \cdots = u_n = 0$ and $u_k = -k^2$.

According to the relationship between the trace of the matrix and the eigenvalue, we have

$$tr((B+P)C) = \sum_{i=1}^{n} u_i = -k^2,$$
$$tr((B+P)C) = \sum_{i=1}^{n} ((B+P)C)_{ii} = \sum_{i=1}^{n} b_i c_i = \left(B+P, C^{\mathrm{T}}\right),$$

where $(\cdot,\cdot)$ denotes the inner product of n-dimensional column vectors. Finally, we have $\left(B+P, C^{\mathrm{T}}\right) = -k^2 < 0$ and

$$r\left(k^2 I + (B+P)C\right) = n - 1.$$

Sufficiency of the conditions. Let $\beta(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}\sqrt{-C(B+P)}$. According to condition (i), there exists a nonzero vector $x_{n\times 1}$ such that $(-C(B+P)I + (B+P)C)x = 0$ and $Ax = 0$. Let $\alpha(\varepsilon) = \tilde{a}\sqrt{\varepsilon}$, where $\tilde{a}$ is the undetermined coefficient. When $\varepsilon \to 0$, we have

$$-\varepsilon\beta(\varepsilon)\Big(\beta(\varepsilon) + Qe^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\sin(\varepsilon\tilde{\tau}\beta(\varepsilon))Ix \to \Big(BC + e^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\cdot\cos(\varepsilon\tilde{\tau}\beta(\varepsilon)PC)x. \tag{3.11}$$

According to condition (ii), there exists a nonzero vector $y_{n\times 1}$ such that $Ay = 0$ and $((D+Q)I - \tilde{\tau}PC)y = 0$. When $\varepsilon \to 0$, we have

$$-\Big(\alpha(\varepsilon) - D - Qe^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\cos(\varepsilon\tilde{\tau}\beta(\varepsilon))\Big)\varepsilon\beta(\varepsilon)Iy \to e^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\sin(\varepsilon\tilde{\tau}\beta(\varepsilon))PCy. \tag{3.12}$$

From $Ax = 0$ and $Ay = 0$, we obtain that there exists a pair of nonzero real vectors (x, y) such that

$$P(\varepsilon)x_{n\times 1} = Q(\varepsilon)y_{n\times 1}, \tag{3.13}$$

where $P(\varepsilon) = -\beta(\varepsilon)\Big(\beta(\varepsilon) + Qe^{-\tau\alpha(\varepsilon)}\sin(\tau\beta(\varepsilon))\Big)I - \varepsilon^{-1}\Big(BC + e^{-\tau\alpha(\varepsilon)}\cos(\tau\beta(\varepsilon))PC\Big) + \Big(\alpha(\varepsilon) - D_{(x)} - Qe^{-\tau\alpha(\varepsilon)}\cos(\tau\beta(\varepsilon))\Big)\Big(\alpha(\varepsilon)I - \varepsilon^{-1}A\Big)$, and $Q(\varepsilon) = \varepsilon^{-1}e^{-\tau\alpha(\varepsilon)}\sin(\tau\beta(\varepsilon))PC + \Big(\beta(\varepsilon) + Qe^{-\tau\alpha(\varepsilon)}\sin(\tau\beta(\varepsilon))\Big)\Big(\alpha(\varepsilon)I - \varepsilon^{-1}A\Big) + \beta(\varepsilon)\Big(\alpha(\varepsilon) - D - Qe^{-\tau\alpha(\varepsilon)}\cos(\tau\beta(\varepsilon))\Big)I$.

Since $r(A) = n-1$ and there are $Ax_{n\times 1} = 0$ and $Ay_{n\times 1} = 0$, where x and y are nonzero vectors, then there is a nonzero real number h such that $x = hy$. Therefore, we take the undetermined coefficient $\tilde{a} = \frac{\tilde{\tau}Q\sqrt{-C(B+P)}}{h}$, so that when $\varepsilon \to 0$, we have

$$-\beta(\varepsilon)Qe^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\sin(\varepsilon\tilde{\tau}\beta(\varepsilon))Iy + \alpha(\varepsilon)\beta(\varepsilon)Ix \to 0.$$

According to the definition of $\beta(\varepsilon)$, when $\varepsilon \to 0$, we have

$$-\beta(\varepsilon)^2 Iy \to \varepsilon^{-1}\Big(BC + e^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\cos(\varepsilon\tilde{\tau}\beta(\varepsilon))PC\Big)y.$$

Similarly, according to conditions (i) and (ii),

$$\begin{aligned}
&\varepsilon^{-1}(-C(B+P)I + (B+P)C)y \to 0,\\
&\beta(\varepsilon)\Big(-D - Qe^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\cos(\varepsilon\tilde{\tau}\beta(\varepsilon))Ix \to -\varepsilon^{-1}e^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\sin(\varepsilon\tilde{\tau}\beta(\varepsilon))PCx.
\end{aligned}$$

are obtained when $\varepsilon \to 0$, respectively.

In summary, we have obtained

$$P(\varepsilon)y_{n\times 1} = -Q(\varepsilon)x_{n\times 1}, \tag{3.14}$$

where $P(\varepsilon)$ and $Q(\varepsilon)$ are given in Eq (3.13).

According to Eqs (3.13) and (3.14), we have that the real block matrix $\begin{pmatrix} P(\varepsilon) & -Q(\varepsilon) \\ Q(\varepsilon) & P(\varepsilon) \end{pmatrix}$ is singular, and further we have $|P(\varepsilon) + iQ(\varepsilon)| \to 0\,(\varepsilon \to 0)$, that is,

$$\left|\Big(\alpha(\varepsilon) + i\beta(\varepsilon) - D - Qe^{-\tau(\alpha(\varepsilon)+i\beta(\varepsilon))}\Big)\Big((\alpha(\varepsilon) + i\beta(\varepsilon))I - \varepsilon^{-1}A\Big) - \varepsilon^{-1}\Big(BC + e^{-\tau(\alpha(\varepsilon)+i\beta(\varepsilon))}PC\Big)\right| \to 0.$$

Similarly, in the proof method of $\lambda - D - Qe^{-\lambda\tau} \neq 0$ in necessity, we can easily prove

$$\alpha(\varepsilon) + i\beta(\varepsilon) - D - Qe^{-\tau(\alpha(\varepsilon)+i\beta(\varepsilon))} \neq 0\,(\varepsilon \to 0).$$

Therefore, we have

$$\begin{vmatrix} (\alpha(\varepsilon)+i\beta(\varepsilon)) I-\varepsilon^{-1}A & -\varepsilon^{-1}B-\varepsilon^{-1}Pe^{-\tau(\alpha(\varepsilon)+i\beta(\varepsilon))} \\ -C & (\alpha(\varepsilon)+i\beta(\varepsilon))-D-Qe^{-\tau(\alpha(\varepsilon)+i\beta(\varepsilon))} \end{vmatrix} \to 0,\ (\varepsilon\to 0),$$

where $\alpha(\varepsilon)=\frac{\tilde{\tau}Q\sqrt{-C(B+P)}}{h}\sqrt{\varepsilon}$ and $\beta(\varepsilon)=\frac{1}{\sqrt{\varepsilon}}\sqrt{-C(B+P)}$.

Let $\tilde{\lambda}(\varepsilon)=\alpha(\varepsilon)+i\beta(\varepsilon)$. Then, there exists an eigenvalue $\lambda(\varepsilon)$ such that

$$\begin{vmatrix} \lambda(\varepsilon) I-\varepsilon^{-1}A & -\varepsilon^{-1}B-\varepsilon^{-1}Pe^{-\lambda(\varepsilon)\tau} \\ -C & \lambda(\varepsilon)-D-Qe^{-\lambda(\varepsilon)\tau} \end{vmatrix} = 0$$

and $\lambda(\varepsilon)\to\tilde{\lambda}(\varepsilon)\ (\varepsilon\to 0)$. Finally, we prove

$$\lim_{\varepsilon\to 0}\mathrm{Re}(\lambda(\varepsilon))=0 \text{ and } \lim_{\varepsilon\to 0}\mathrm{Im}(\lambda(\varepsilon))=+\infty.$$

We consider the (2, 1)-fast-slow system as follows:

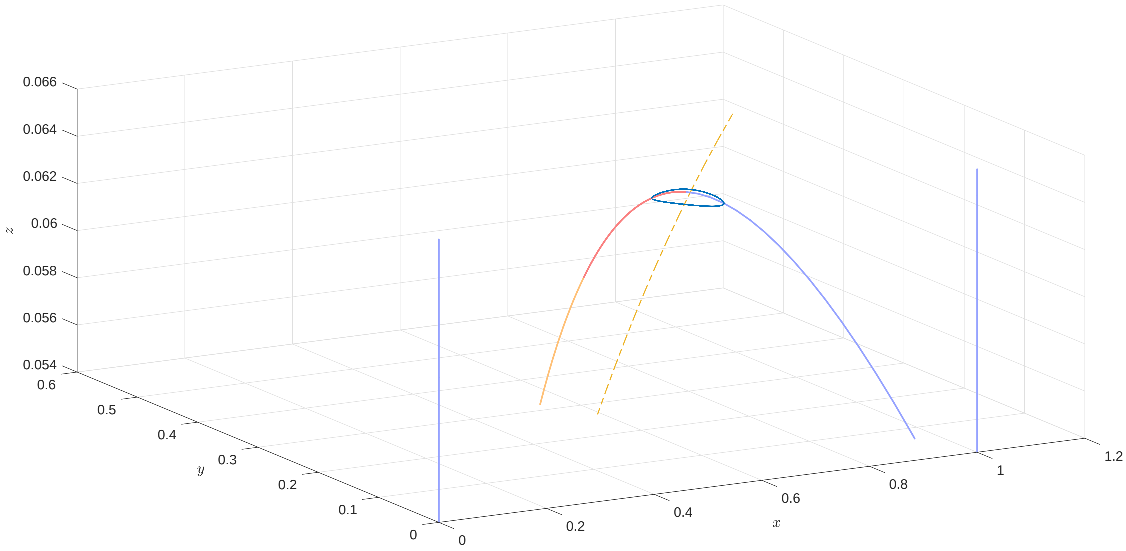
$$\begin{cases} \dfrac{dx}{dt}=x(1-x)(x-A)-\dfrac{xy}{x+\beta_1}, \\ \dfrac{dy}{dt}=\dfrac{b_1xy}{x+\beta_1}-c_1y-\eta xy-\dfrac{yz}{y+\beta_2}, \\ \dfrac{dz}{dt}=\varepsilon\left(\dfrac{yz}{y+\beta_2}-c_2z-\varphi(z)z(t-\tau)\right), \end{cases}$$

where A, η, β_1, β_2, b_1, c_1 and c_2 are the model parameters, ε is the time scale parameter, $\varphi(z)$ is a piecewise smooth function, and τ is the delay parameter.

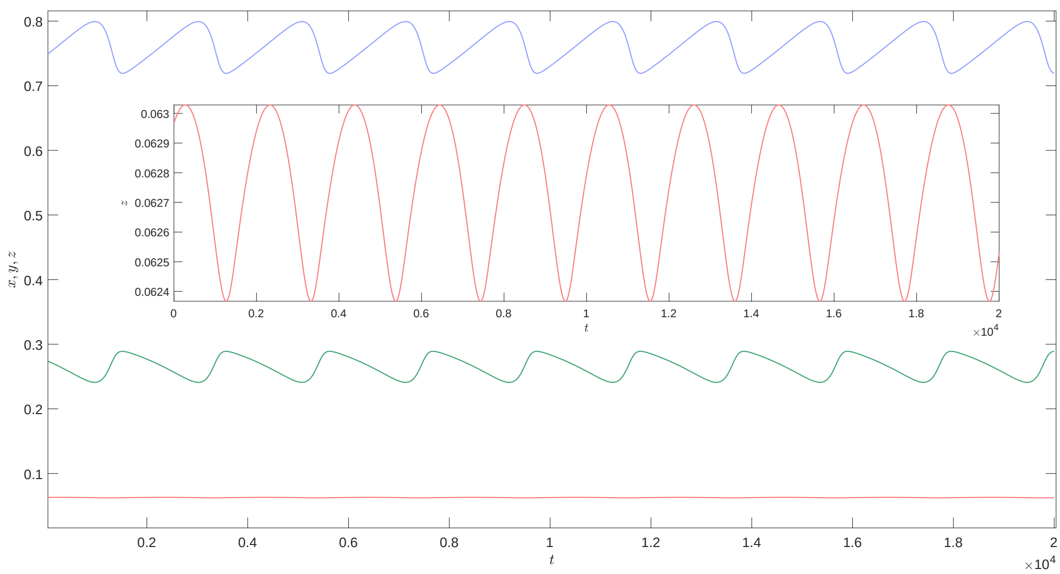
We choose the following parameters:

$$A=-0.125,\ \beta_1=0.5,\ \beta_2=0.62,\ b_1=0.208,\ c_1=0.141,\ \eta=0.169,\ c_2=0.2, \tau=0.12,\ \varepsilon=0.01.$$

These satisfy the conditions of Theorem 3.1, and the system produces a singular Hopf bifurcation near the internal equilibrium as shown in Figure 2.



(a)



(b)

Figure 2. The system generates a singular Hopf bifurcation near the internal equilibrium.

4. Conclusions

The main conclusions of this paper are divided into two parts: In the general nonlinear $(1, 1)$-fast-slow system with delay on slow variables, the necessary and sufficient conditions for the generation of singular pure imaginary eigenvalues are

$$(i)\, f_x = 0;$$
$$(ii)\, \tilde{\tau} g_x f_{y(t-\tilde{\tau})} - g_y - g_{y(t-\tilde{\tau})} = 0;$$
$$(iii)\, g_x \left(f_{y(t-\tilde{\tau})} + f_y \right) < 0,$$

where condition (i) is consistent with the condition of the singular imaginary part eigenvalue in the general nonlinear (1, 1)-fast-slow system, condition (iii) is consistent with the conditional form of the singular imaginary part eigenvalue in the general nonlinear (1, 1)-fast-slow system, and condition (ii) is unique in the delay system. It is worth noting that the codimension of the singular Hopf bifurcation remains 1.

The other part is the general nonlinear $(n, 1)$-fast-slow system ($n > 1$) with delay on the slow variable. The necessary and sufficient conditions for the singular pure imaginary eigenvalue to be generated are

$$(i)\, r\begin{pmatrix} g_x \left(f_{y(t-\tilde{\tau})} + f_y \right) \\ f_x \end{pmatrix} = r\left(g_x \left(f_{y(t-\tilde{\tau})} + f_y \right) \right) = r\left(f_x \right) = n - 1;$$
$$(ii)\, r\begin{pmatrix} f_x \\ \left(g_y + g_{y(t-\tilde{\tau})} \right) g_x I - \tilde{\tau} f_{y(t-\tilde{\tau})} g_x \end{pmatrix} = r\left(\left(g_y + g_{y(t-\tilde{\tau})} \right) g_x I - \tilde{\tau} f_{y(t-\tilde{\tau})} g_x \right) = r\left(f_x \right) = n - 1;$$
$$(iii)\, \left(f_{y(t-\tilde{\tau})} + f_y, g_x^{\mathrm{T}} \right) < 0.$$

Based on the existence of the eigenvalue of the singular pure imaginary part, the singular Hopf bifurcation can be generated by verifying the transversal condition of the Hopf bifurcation. Conversely, the existence and transversality conditions of singular pure imaginary eigenvalues can be obtained by the generation of a singular Hopf bifurcation.

The core of the singular Hopf bifurcation is that when the system parameters cross the critical value, the stability of the equilibrium point suddenly changes, and the singular limit cycle (periodic solution) is generated or disappeared. The transversal condition directly determines the uniqueness of the topological mutation of the phase diagram by the crossing of the canonical eigenvalues. If the transversality condition is not satisfied, the equilibrium point may undergo multiple stability switchings when the eigenvalue slowly slides along the imaginary axis, and multiple limit cycles with different radii appear in the phase diagram, resulting in chaotic topology. In addition, the rank condition guarantees the existence of singular pure imaginary eigenvalues in high-dimensional systems. In future research, we will consider the case of multiple slow variables and discuss the existence of a singular Hopf bifurcation in the (m, n) -fast-slow system and analyze its normal form and its stability.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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Appendix A

For the convergence of Eq (3.11), on the one hand, we have

$$\lim_{\varepsilon\to 0} -\varepsilon\beta(\varepsilon)^2 Ix = \lim_{\varepsilon\to 0} C(B+P) = C(B+P)Ix.$$

On the other hand, we have

$$\lim_{\varepsilon\to 0} -\varepsilon\beta(\varepsilon) Qe^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)} \sin(\varepsilon\tilde{\tau}\beta(\varepsilon)) Ix = -\sqrt{C(B+P)}QIx\lim_{\varepsilon\to 0} \sqrt{\varepsilon}e^{-\tilde{a}\tilde{\tau}\varepsilon^{\frac{2}{3}}} \sin\left(\tilde{\tau}\sqrt{-C(B+P)}\sqrt{\varepsilon}\right).$$

Its convergence is consistent with the convergence of limit $\lim_{\varepsilon\to 0}\sqrt{\varepsilon}e^{-\varepsilon^{\frac{3}{2}}}\sin\left(\sqrt{\varepsilon}\right)$. Therefore, the left side of (3.11) converges with respect to ε, and there is

$$\lim_{\varepsilon\to 0} -\varepsilon\beta(\varepsilon)\left(\beta(\varepsilon) + Qe^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\sin(\varepsilon\tilde{\tau}\beta(\varepsilon))\right)Ix = C(C+P)Ix.$$

The right-hand limit of (3.11) is

$$\lim_{\varepsilon\to 0}\left(BC + e^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\cos(\varepsilon\tilde{\tau}\beta(\varepsilon))PC\right)x = BCx + PCx\lim_{\varepsilon\to 0} e^{-\tilde{a}\tilde{\tau}\varepsilon^{\frac{3}{2}}}\cos\left(\tilde{\tau}\sqrt{-C(B+P)}\sqrt{\varepsilon}\right).$$

Its convergence is consistent with the convergence of limit $\lim_{\varepsilon\to 0} e^{-\varepsilon^{\frac{3}{2}}}\cos\left(\sqrt{\varepsilon}\right)$. Therefore, the right side of (3.11) converges with respect to ε, and there is

$$\lim_{\varepsilon\to 0}\left(BC + e^{-\varepsilon\tilde{\tau}\alpha(\varepsilon)}\cos(\varepsilon\tilde{\tau}\beta(\varepsilon))PC\right)x = (B+P)Cx.$$

From $(-C(B+P)I+(B+P)C)x=0$, it can be obtained that both ends of (3.11) are convergent with respect to ε and the limits are equal.

For the convergence of (3.12), we calculate the limits of the left and right sides are both 0. According to $(-C(B+P)I+(B+P)C)x=0$, the left and right ends of (3.12) are equivalent infinitesimals when $\varepsilon \to 0$.