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Research article

The global existence and blow-up of the solutions for a fractional Kirchhoff hyperbolic equations with viscoelastic term and logarithmic term

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Abstract: This article investigates a type of hyperbolic equation of the fractional Kirchhoff with viscoelastic and logarithmic nonlinear terms subject to homogeneous Dirichlet-boundary:

$$\begin{cases} u_{tt} + M([u]_s^2)(-\Delta)^s u - \int_0^t g(t-\tau)(-\Delta)^s u(\tau)d\tau + u_t = |u|^{h-2}u \ln|u|, & \text{in } \Omega \times (0,\infty), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & \text{in } \Omega, \\ u(\cdot,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

where $[u]_s$ is the Gagliardo semi-norm of u, $(-\Delta)^s$ is the fractional Laplacian with $s \in (0,1)$, $2 < 2\gamma < h < 2_s^*$, u_0 and u_1 are the initial functions, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary. First, the global existence of solutions is established by combining the Galerkin method with the potential well theory. Subsequently, the finite-time blow-up of solutions is derived via the concavity method and a series of peculiar inequalities.

Keywords: hyperbolic; fractional Laplacian; Kirchhoff equations; viscoelastic; logarithmic nonlinear term; global existence; blow-up

1. Introduction

Over a long period of time, research on Kirchhoff's problem has caught a lot of attention. For a start, the Kirchhoff problem dated back to 1883, when Kirchhoff raised a mathematical model to describe a class of inelastic string vibration in [1]:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u(x)}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0.$$

This model accounts for the length changes of strings due to transverse vibrations. Owing to the research of Lions [2], this type of equation has attracted increased attention. In order to solve the existence of external force terms f in the following higher-dimensional problems:

$$\frac{\partial^2 u}{\partial t^2} - \left(a + b\int_\Omega |\nabla u(x)|^2 dx\right)\Delta u = f(x,u),$$

here defines Δ as a Euclidean Laplace operator, they proposed a functional analysis framework. In [3], Ono investigated the global behavior of solutions to hyperbolic nonlinear integro-differential equations. Since then, research on Kirchhoff-type equations has continued to expand into various contexts within the theory of partial differential equations [4–7].

Additionally, it is worth noting that a growing number of researchers have begun to focus on the role of the fractional order. By introducing the concept of s-harmonic extension, Caffarelli and Silvestre [8] defined the fractional Laplace operator so that many important results in classical ellipse problems can be extended and generalized under the background of the fractional Laplace operator. Everyone can take note of [9–12] and the literature it cites.

The fractional Laplacian extends the classical Laplacian to arbitrary real orders, enabling a more accurate representation of nonlocal phenomena and long-range interactions in physical systems. It has become a fundamental tool in modeling anomalous diffusion, financial processes, image analysis, and nonlocal continuum mechanics, due to its ability to capture spatial heterogeneity and memory effects. This operator greatly expands the theoretical framework of partial differential equations and facilitates the integration of local and nonlocal dynamics. Research on the fractional Laplacian therefore holds significant mathematical value while also supporting more realistic approaches to complex scientific and engineering challenges.

In this paper, we are interested in the fractional hyperbolic equation involving the Kirchhoff term, the viscoelastic term, and the logarithmic term:

$$\begin{cases} u_{tt} + M([u]_s^2)(-\Delta)^s u - \int_0^t g(t-\tau)(-\Delta)^s u(\tau)d\tau + u_t = f(u), & \text{in } \Omega\times(0,T),\\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & \text{in } \Omega,\\ u(\cdot,t) = 0, & \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{1.1}$$

where $f(u) = |u|^{h-2}u\ln|u|$. For $s\in(0,1)$ and $\forall\varphi\in C_0^\infty(\mathbb{R}^N)$, give insight into the traits of the fractional Laplacian $(-\Delta)^s$

$$(-\Delta)^s\varphi(x) = 2\lim_{\theta_1\to 0^+}\int_{\mathbb{R}^N\setminus B_{\theta_1(x)}} (\varphi(x)-\varphi(y))K(x-y)dy$$

and it goes up to a normalized constant, $B_{\theta_1(x)}$ here means that the sphere in $\mathbb{R}^N$ where radius $\theta_1 > 0$. $K : \mathbb{R}^{\mathrm{N}}\setminus\{0\}\to(0,\infty)$ is a measurable function with the following property:

$$\begin{cases} mK\in L^1(\mathbb{R}^{\mathrm{N}}), & \text{where } m(x) = \min\{|x|^2,1\};\\ \text{there exists } k_0>0 \text{ such that } K(x)\ge k_0|x|^{-(N+2s)}, & \text{for any } x\in\mathbb{R}^{\mathrm{N}}\setminus\{0\};\\ K(x) = K(-x), & \text{for any } x\in\mathbb{R}^{\mathrm{N}}\setminus\{0\}. \end{cases} \tag{1.2}$$

We now explicitly define $M(t)$ as a function $M : \mathbb{R}_0^+\to\mathbb{R}^+$ that is continuous and satisfies the following assumption:

($\mathscr{M}$) *For* $\forall\mu > 0$, it is freely available that $\varsigma\mathcal{B}(\mu)\ge M(\mu)\mu$ when there is a $\varsigma\ge 1$, where $\mathcal{B}(\mu) = \int_0^\mu M(s)ds$.

A typical example for M is given by $M(t) = a + bt^{\gamma-1}$, where $\gamma > 1, a, b \geq 0$, and $a + b > 0$. In this work, we specifically consider the function of the form $M(t) = 1 + t^{\gamma-1}$, where the parameters satisfy $2 < 2\gamma < h < 2_s^* = \frac{2N}{N-2S}$.

At a time that started not long ago, in the face of certain physical phenomena, it is more convenient to describe them in logarithmic terms. Especially, Shao [13] considered a special class of equations with Dirichlet boundary. The situation is that the logarithmic source fractional hyperbolic equations consisted of $q = 2$ and $f(u) = |u|^{q-2}u \log |u|$:

$$u_{tt} + M([u]_s^2)\mathcal{L}_K u = |u|^{q-2}u \log |u|.$$

For both subcritical and critical initial energy, the author obtained the existence of global weak solutions and proved the finite-time blow-up of solutions. In [14], Xu solved the second-order fractional Kirchhoff problem featuring a logarithmic term:

$$u_{tt} + [u]_s^{2\gamma-2}(-\Delta)^s u + (-\Delta)^s u_t = |u|^{b-2}u \ln |u|.$$

By employing the Galerkin method combined with the potential well theory, Xu established the global existence of solutions in the subcritical energy regime. Furthermore, using the concavity method and a series of delicate inequalities, she proved both the asymptotic behavior and finite-time blow-up of weak solutions. Overall, the logarithmic fractional Sobolev space and associated inequality techniques serve as essential tools for handling the logarithmic nonlinearity. Moreover, the introduction of the Galerkin method and the potential well theory provides an effective framework for studying the existence and dynamics of solutions across different energy levels.

In [15], Kim and Han dealt with a second-order nonlinear hyperbolic equation with a viscoelastic term, and blasting for a limited time of the weak solutions with $p > 2$ is obtained:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = u|u|^{p-2}.$$

In [16], the authors discussed a memory equation featuring a strongly damped term:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \Delta u_t = |u|^{p-2}u,$$

and made a bid of blasting for a limited time by accounting for appropriate assumptions about g. In [17], Xiang and Hu solved the subsequent fractional viscoelastic hyperbolic equation featuring a Kirchhoff term, incorporating strongly damped terms and nonlinear terms with variable coefficients:

$$u_{tt} + M([u]_{\alpha,2}^2)(-\Delta)^\alpha u - \int_0^t g(t-\tau)(-\Delta)^\alpha u(\tau)d\tau + (-\Delta)^s u_t = \lambda|u|^{q-2}u,$$

where $0 < s \leq \alpha < 1, 1 < q < \infty, \lambda > 0$. According to the different values of q and λ, local solutions as well as global solutions can be attained, respectively. More details about the blow-up of solutions for fractional wave equations with viscoelastic terms can be found in [18–20]. References [21–23] provided a better understanding of fractional-order memory equations with damped terms.

Inspired by the previous content, we focus our attention on the problem (1.1), which involves the fractional Kirchhoff hyperbolic equations with viscoelastic and logarithmic terms. In order to overcome the challenges posed by the logarithmic term, we introduce correlation functions and sets of potential wells to control the logarithmic terms and establish Lemma 3.1 to tackle them. The subsequent content consists of five parts: Section 2, we give the definition of fractional Sobolev spaces and make assumptions about g. Section 3, we introduce the potential wells along with essential Lemmas. Section 4, we construct an approximate solution by utilizing the Galerkin method and examine the global existence of the solutions to (1.1). Section 5, we investigate the blow-up of the solutions in finite time. Section 6, we summarize the main results of this article.

2. Preliminaries

This subsection recalls the specific notation and properties of fractional Sobolev spaces [24, 25]. For $\forall r \geq 1$, there is a general Lebesgue space $L^r(\Omega)$, whose norm is defined as

$$\|u\|_r = \left(\int_\Omega |u|^r dx\right)^{\frac{1}{r}}.$$

In the case of $r = 2.$, we have $(u, v) = \int_\Omega uvdx$.

Let $s \in (0, 1)$ run through the paper as well as $2s < N$. Simultaneously, put sights on the sense of characterization of fractional critical exponent $2_s^* = \frac{2N}{N-2s}$.

Furthermore, denote $Q = \mathbb{R}^N \backslash \mathscr{D}$, $\mathscr{D} = (\mathbb{R}^N \backslash \Omega) \times (\mathbb{R}^N \backslash \Omega) \subset \mathbb{R}^{2N}$. Write X as a linear space, and it consists of Lebesgue measurable function u: $\mathbb{R}^n \to \mathbb{R}$, ensuring that $\forall u \in L^2(\Omega)$ holds for the restriction to Ω in X and

$$\int_Q |u(x) - u(y)|^2 K(x - y) dxdy < \infty.$$

The norm that space X possesses is

$$\|u\|_X = \left(\|u\|_{L^2(\Omega)}^2 + \int_Q |u(x) - u(y)|^2 K(x - y) dxdy\right)^{\frac{1}{2}}.$$

And next introduce a closed linear subspace of X

$$X_0 = \left\{u \in X | u(x) = 0 \;\; a.e. \;\text{ in } \mathbb{R}^N \backslash \Omega\right\},$$

and the norm

$$\|u\|_{X_0} = \left(\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy\right)^{\frac{1}{2}}.$$

We recall that the fractional Sobolev space $X^{s,2}(\Omega)$ is

$$X^{s,2}(\Omega) = \left\{u \in L^2(\Omega) : \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \in L^2(\Omega \times \Omega)\right\},$$

endowed with the Hilbertian norm

$$\|u\|_{X^{s,2}(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \iint_{\Omega\times\Omega} |u(x) - u(y)|^2 K(x - y) dxdy\right)^{\frac{1}{2}}.$$

The following properties of X and X_0 , established in [10], underpin our subsequent analysis. In the remainder of this work, we adopt the kernel introduced in [10], namely $K(x-y) = |x-y|^{-(N+2s)}$, and refer the reader to that source for foundational details. In addition, there exist constants $C' > 0$ such that for all $u \in X_0$, $\|u\|_{L^r(\Omega)} \leq C'\|u\|_{X_0}$, where $r \in [1, 2_s^*]$. Moreover, in [26], the embedding $X_0 \hookrightarrow L^r(\Omega)$ is compact for all $r \in [1, 2_s^*)$, but only continuous for the critical exponent $r = 2_s^*$.

Moreover, it is necessary to clarify the usual assumptions about g:

(A_1) *$g(t) : [0, +\infty) \to [0, +\infty)$ is a nonincreasing C^1 function and for $\forall t > 0$ satisfies*

$$k(t) = 1 - \int_0^t g(\tau)d\tau \geq 1 - \int_0^{+\infty} g(\tau)d\tau = k > 0.$$

(A_2) *There exists $\omega > 0$ such that $g'(t) \leq -\omega g(t)$ for all $t > 0$.*

Lemma 2.1. *[27] Let $K : \mathbb{R}^{\mathbb{N}}\backslash\{0\} \to (0, \infty)$ satisfy assumption (1.2). Then*
(1) *there exists $\alpha = \alpha(N, \upsilon, s)$, where $\upsilon \in [1, 2_s^*]$, such that, for all $v \in X_0$*

$$\|v\|^2_{L^{\upsilon}(\Omega)} \leq \alpha \iint_{\Omega\times\Omega} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dxdy \leq \frac{\alpha}{\beta} \int_Q |v(x) - v(y)|^2 K(x-y)dxdy.$$

(2) *there exists $\widetilde{\alpha} = \widetilde{\alpha}(N, s, \beta, \Omega) > 0$ such that, for arbitrary $v \in X_0$,*

$$\int_Q |v(x) - v(y)|^2 K(x-y)dxdy \leq \|v\|_X^2 \leq \widetilde{\alpha} \int_Q |v(x) - v(y)|^2 K(x-y)dxdy.$$

(3) *there exists $v \in L^{\upsilon}(\mathbb{R}^{\mathbb{N}})$ such that up to a subsequence, where $\{v_j\} \in X_0$ is a bounded sequence and $\forall \upsilon \in [1, 2_s^*)$,*

$$v_j \to v \quad \text{strongly in } L^{\upsilon}(\Omega) \text{ as } j \to \infty.$$

3. The potential well

The purpose of this subsection is to establish the notation and lemmas for the next two sections. Primarily, defines

$$\begin{aligned} E(t) =& \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2\gamma}\|u\|_{X_0}^{2\gamma} + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|u\|_{X_0}^2 \\ &+ \frac{1}{2}(g \circ u)(t) - \frac{1}{h}\int_\Omega |u|^h \ln|u|dx + \frac{1}{h^2}\|u\|_h^h, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \mathscr{J}(u) =& \frac{1}{2\gamma}\|u\|_{X_0}^{2\gamma} + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|u\|_{X_0}^2 \\ &+ \frac{1}{2}(g \circ u)(t) - \frac{1}{h}\int_\Omega |u|^h \ln|u|dx + \frac{1}{h^2}\|u\|_h^h, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \mathscr{L}(u) =& \|u\|_{X_0}^{2\gamma} + \left(1 - \int_0^t g(\tau)d\tau\right)\|u\|_{X_0}^2 + (g \circ u)(t) \\ &- \int_\Omega |u|^h \ln|u|dx, \end{aligned} \tag{3.3}$$

for $u \in X_0$, where

$$(g \circ u)(t) = \int_0^t g(t-\tau)\|u(\tau) - u(t)\|_{X_o}^2 d\tau.$$

From (3.2) and (3.3), with the help of computing

$$\begin{aligned}\mathcal{J}(u) =& \frac{1}{h}\mathcal{L}(u) + \frac{h-2\gamma}{2h\gamma}\|u\|_{X_0}^{2\gamma} + \frac{h-2}{2h}(1 - \int_0^t g(\tau)d\tau)\|u\|_{X_0}^2 \\ &+ \frac{h-2}{2h}(g \circ u)(t) + \frac{1}{h^2}\|u\|_h^h.\end{aligned} \tag{3.4}$$

Further, we denote the Nehari manifold $\mathcal{N}$ and the potential well depth d

$$\mathcal{N} = \{u \in X_0 | \mathcal{L}(u) = 0, \|u\|_{X_0} \neq 0\}, \tag{3.5}$$

$$d = \inf_{u \in \mathcal{N}} \mathcal{J}(u). \tag{3.6}$$

The stable set of the potential well

$$W = \left\{u \in X_0 \middle| \mathcal{L}(u) > 0, \mathcal{J}(u) < d\right\} \cup \{0\}, \tag{3.7}$$

and the out set of the potential well

$$V = \left\{u \in X_0 \middle| \mathcal{L}(u) < 0, \mathcal{J}(u) < d\right\}. \tag{3.8}$$

Definition 3.1. *Function $u = u(t) \in L^\infty(0, \infty; X_0)$ is referred to as a weak solution to problem* (1.1)*, in the event of $u_t \in L^\infty(0, \infty; L^2(\Omega))$ and for $\forall \varphi \in X_0$ satisfies*

$$\begin{aligned}&(u_t(\cdot, t), \varphi) + \int_0^t M([u(\cdot, \tau)]_s^2)(u(\cdot, \tau), \varphi)_{X_0} d\tau - \int_0^t \int_0^m g(m-\tau)(u(\cdot, \tau), \varphi)_{X_0} d\tau dm \\ &+ (u(\cdot, t), \varphi) = (u_1, \varphi) + (u_0, \varphi) + \int_0^t (|u(\cdot, \tau)|^{h-2} u(\cdot, \tau) \ln |u(\cdot, \tau)|, \varphi) d\tau,\end{aligned} \tag{3.9}$$

where

$$(u(\cdot, t), \varphi)_{X_0} = \iint_Q [u(x,t) - u(y,t)][\varphi(x) - \varphi(y)] K(x-y) dx dy.$$

Lemma 3.1. *Let $p > 0$ and $\forall a \in [1, +\infty)$, we have*

$$\ln a \leq \frac{1}{ep} a^p.$$

Proof. To begin with, set a function $\mathcal{G}(a)$ and make $\mathcal{G}(a) = \ln a - \frac{1}{ep} a^p$ for $\forall a \in [1, +\infty)$. Obviously, the first derivative of $\mathcal{G}(a)$ with respect to a has the absolute maximum of $\mathcal{G}(a)$ when $a^* = e^{\frac{1}{p}}$; thus, $\mathcal{G}(a) \leq \mathcal{G}(a^*) = 0$ for $\forall a \in [1, +\infty)$. To sum up, the conclusion is proved. □

Lemma 3.2. *For* $\forall u \in \mathcal{N}$, $p > 0$ *and* $2 < 2\gamma < h < h + p < 2_s^*$, *we gain*

$$d = \frac{h-2}{2h} k^{\frac{h+p}{h+p-2}} \left(\frac{ep}{\mathscr{E}_*^{h+p}} \right)^{\frac{2}{h+p-2}}, \tag{3.10}$$

where $\mathscr{E}_*$ *represents the best embedding constant for integrating* X_0 *into* $L^{h+p}(\Omega)$, *i.e.,*

$$\mathscr{E}_* = \sup_{u \in X_0 \setminus \{0\}} \frac{\|u\|_{h+p}}{\|u\|_{X_0}}.$$

Proof. Picking up $u \in \mathcal{N}$ and accounting for assumption (A_1), we get

$$k\|u\|_{X_0}^2 \leq \left(1 - \int_0^t g(\tau)d\tau\right)\|u\|_{X_0}^2 \leq \int_\Omega |u|^h \ln|u| dx,$$

then, by Lemma 2.1 (1) and Lemma 3.1

$$\int_\Omega |u|^h \ln|u| dx \leq \left(\frac{1}{ep}\right)\|u\|_{h+p}^{h+p} \leq \left(\frac{\mathscr{E}_*^{h+p}}{ep}\right)\|u\|_{X_0}^{h+p-2}\|u\|_{X_0}^2.$$

Obviously, it is easy to get

$$\|u\|_{X_0} \geq \left(\frac{kep}{\mathscr{E}_*^{h+p}}\right)^{\frac{1}{h+p-2}}. \tag{3.11}$$

By means of (3.4) and (3.11), we make a conclusion that

$$\begin{aligned}
\mathscr{J}(u) =& \frac{1}{h}\mathscr{L}(u) + \frac{h-2\gamma}{2h\gamma}\|u\|_{X_0}^{2\gamma} + \frac{h-2}{2h}(1 - \int_0^t g(\tau)d\tau)\|u\|_{X_0}^2 \\
&+ \frac{h-2}{2h}(g \circ u)(t) + \frac{1}{h^2}\|u\|_h^h \\
\geq& \frac{h-2}{2h} k \left(\frac{kep}{\mathscr{E}_*^{h+p}}\right)^{\frac{2}{h+p-2}} \\
=& \frac{h-2}{2h} k^{\frac{h+p}{h+p-2}} \left(\frac{ep}{\mathscr{E}_*^{h+p}}\right)^{\frac{2}{h+p-2}}.
\end{aligned}$$

Thus, we can clearly see that (3.10) is valid. □

Overall, Sections 4 and 5 contain the proofs of our main results.

4. Global existence of solutions

Theorem 4.1. *Making assumptions* $E(0) < d$, $\mathscr{L}(u_0) > 0$, *or* $\|u_0\|_{X_0} = 0$, *for* $u_0 \in X_0, u_1 \in L^2(\Omega)$, *the problem* (1.1) *admits a global solution u, with* $u \in L^\infty(0, \infty; X_0), u_t \in L^\infty(0, \infty; L^2(\Omega))$, *and* $u(\cdot, t) \in W$ *for all* $t \in (0, \infty)$ *under hypothesis* (A_1) *and* (A_2).

Proof. To start with, the characteristic function that defines the fractional Laplace operator is $\{\varrho_j\} \subset C_0^\infty$. In addition, $\{\varrho_j\}$ is an orthogonal basis for X_0 and $L^2(\Omega)$. Afterwards, an approximate solution is constructed:

$$u_n(x,t) = \sum_{j=1}^{n} \xi_{jn}(t)\varrho_j(x), \quad n = 1, 2, \cdots, \tag{4.1}$$

meet with

$$\begin{aligned}(u_{ntt}(\cdot,t),\varrho_j) + M([u_n(\cdot,t)]_s^2)(u_n(\cdot,t),\varrho_j)_{X_0} - \int_0^t g(t-\tau)(u_n(\cdot,\tau),\varrho_j)_{X_0}d\tau \\ + (u_{nt}(\cdot,t),\varrho_j) = (f(u_n(\cdot,t)),\varrho_j), \quad j = 1, 2, \cdots, n,\end{aligned} \tag{4.2}$$

where $f(u_n(\cdot,t)) = |u_n(\cdot,t)|^{h-2}u_n(\cdot,t)\ln|u_n(\cdot,t)|$.

$$u_n(\cdot,0) = \sum_{j=1}^{n} \xi_{jn}(0)\varrho_j \to u_0(x) \text{ in } X_0, \text{ as } n \to \infty, \tag{4.3}$$

$$u_{nt}(\cdot,0) = \sum_{j=1}^{n} \xi'_{jn}(0)\varrho_j \to u_1(x) \text{ in } L^2(\Omega), \text{ as } n \to \infty. \tag{4.4}$$

Next, both sides of (4.2) are multiplied by $\xi'_{jn}(t)$ and summed over j, giving

$$\begin{aligned}&(u_{ntt}(\cdot,t),u_{nt}(\cdot,t)) + M([u_n(\cdot,t)]_s^2)(u_n(\cdot,t),u_{nt}(\cdot,t))_{X_0} \\ &- \int_0^t g(t-\tau)(u_n(\cdot,\tau),u_{nt}(\cdot,t))_{X_0}d\tau + (u_{nt}(\cdot,t),u_{nt}(\cdot,t)) \\ =&(f(u_n(\cdot,t)),u_{nt}(\cdot,t)).\end{aligned} \tag{4.5}$$

Take notice of which

$$\begin{aligned}&\int_0^t g(t-\tau)(u_n(\cdot,\tau),u_{nt}(\cdot,t))_{X_0}d\tau \\ =&\int_0^t g(t-\tau)(u_n(\cdot,\tau)-u_n(\cdot,t),u_{nt}(\cdot,t))_{X_0}d\tau + \int_0^t g(t-\tau)(u_n(\cdot,t),u_{nt}(\cdot,t))_{X_0}d\tau \\ =&-\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\|u_n(\cdot,\tau)-u_n(\cdot,t))\|_{X_0}^2 d\tau + \frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\|u_n(\cdot,t)\|_{X_0}^2 d\tau \\ =&-\frac{1}{2}\frac{d}{dt}\left((g\circ u_n)(\cdot,t) - \int_0^t g(\tau)d\tau\|u_n(\cdot,t)\|_{X_0}^2\right) + \frac{1}{2}(g'\circ u_n)(\cdot,t) - \frac{1}{2}g(t)\|u_n(\cdot,t)\|_{X_0}^2.\end{aligned}$$

Putting the formula into (4.5) and integrating it with respect to t, we obtain

$$\begin{aligned}&\int_0^t \frac{d}{dt}\left(\frac{1}{2}\|u_{nt}(\cdot,\tau)\|_2^2 + \frac{1}{2\gamma}\|u_n(\cdot,\tau)\|_{X_0}^{2\gamma} + \frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|u_n(\cdot,\tau)\|_{X_0}^2\right. \\ &\quad\left.+\frac{1}{2}(g\circ u_n)(\cdot,\tau)\right)d\tau + \int_0^t \left(\|u_{nt}(\cdot,\tau)\|_2^2 - \frac{1}{2}(g'\circ u_n)(\cdot,\tau) + \frac{1}{2}g(\tau)\|u_n(\cdot,\tau)\|_{X_0}^2\right)d\tau \\ &=\int_0^t \frac{d}{dt}\left(\frac{1}{h}\int_\Omega |u_n(\cdot,\tau)|^h \ln|u_n(\cdot,\tau)|dx - \frac{1}{h^2}\|u_n(\cdot,\tau)\|_h^h\right)d\tau\end{aligned}$$

.

i.e.,

$$\begin{aligned}
&\frac{1}{2}\|u_{nt}(\cdot,t)\|_2^2+\mathscr{J}(u_n(\cdot,t))\\
&\quad+\int_0^t\left(\|u_{nt}(\cdot,\tau)\|_2^2-\frac{1}{2}(g'\circ u_n)(\cdot,\tau)+\frac{1}{2}g(\tau)\|u_n(\cdot,\tau)\|_{X_0}^2\right)d\tau \qquad (4.6)\\
&=E_n(0),
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{J}(u_n(\cdot,t))=&\frac{1}{2\gamma}\|u_n(\cdot,t)\|_{X_0}^{2\gamma}+\frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|u_n(\cdot,t)\|_{X_0}^2\\
&+\frac{1}{2}(g\circ u_n)(\cdot,t)-\frac{1}{h}\int_\Omega |u_n(\cdot,t)|^h\ln|u_n(\cdot,t)|dx+\frac{1}{h^2}\|u_n(\cdot,t)\|_h^h.
\end{aligned} \qquad (4.7)$$

Based on (4.3) and (4.4), we extrapolate that $E_n(0)<d$. Taking a big enough n, there are

$$\begin{aligned}
&\frac{1}{2}\|u_{nt}(\cdot,t)\|_2^2+\mathscr{J}(u_n(\cdot,t))\\
&\quad+\int_0^t\left(\|u_{nt}(\cdot,\tau)\|_2^2-\frac{1}{2}(g'\circ u_n)(\cdot,\tau)+\frac{1}{2}g(\tau)\|u_n(\cdot,\tau)\|_{X_0}^2\right)d\tau \qquad (4.8)\\
&<d.
\end{aligned}$$

By $u_0\in X_0$, (3.1), and (3.2), we get

$$\frac{1}{2}\|u_{nt}(\cdot,0)\|_2^2+\mathscr{J}(u_n(\cdot,0))=E_n(0)<d.$$

On the assumption that $\mathscr{L}(u_0)>0$ or $\|u_0\|_{X_0}=0$, with big enough n, we get $u_n(\cdot,0)\in W$. In the next place, we need to prove that $u_n(\cdot,t)\in W$. Postulate now that $u_n(\cdot,t)\notin W$ and there is a smallest time t_1 such that $u_n(\cdot,t_1)\notin W$. Due to the continuity of $u_n(\cdot,t)$, then, either $\mathscr{J}(u(\cdot,t_1))=d$ or $\mathscr{L}(u(\cdot,t_1))=0$. For one thing, (4.8) contradicts the fact that $\mathscr{J}(u(\cdot,t_1))=d$. For another, the identity $\mathscr{L}(u(\cdot,t_1))=0$, together with (3.5) and (3.6), we get

$$\mathscr{J}(u(t_1))\geq d. \qquad (4.9)$$

Apparently, (4.8) and (4.9) are contradictory. To sum up, we make a conclusion about $u_n(\cdot,t)\in W$.

With the help of (3.4) and (4.8), there are

$$\begin{aligned}
&\frac{1}{2}\|u_{nt}(\cdot,t)\|_2^2+\frac{h-2}{2h}(1-\int_0^t g(\tau)d\tau\|u_n(\cdot,t)\|_{X_0}^2\\
&\quad+\frac{h-2\gamma}{2h\gamma}\|u_n(\cdot,t)\|_{X_0}^{2\gamma}+\frac{h-2}{2h}(g\circ u_n)(\cdot,t)+\frac{1}{h^2}\|u_n(\cdot,t)\|_h^h\\
&\quad+\int_0^t\left(\|u_{nt}(\cdot,\tau)\|_2^2-\frac{1}{2}(g'\circ u_n)(\cdot,\tau)+\frac{1}{2}g(\tau)\|u_n(\cdot,\tau)\|_{X_0}^2\right)d\tau \qquad (4.10)\\
&<d,
\end{aligned}$$

for all t. Thus, we find

$$\|u_{nt}(\cdot,t)\|_2^2<2d, \qquad (4.11)$$

$$\|u_n(\cdot,t)\|_{X_0}^{2\gamma} < \frac{2h\gamma d}{h-2\gamma}, \tag{4.12}$$

$$\|u_n(\cdot,t)\|_{X_0}^{2} < \frac{2hd}{(h-2)k}, \tag{4.13}$$

$$\|u_n(\cdot,t)\|_{h}^{h} < h^2 d. \tag{4.14}$$

Use clear and concise calculations,

$$\begin{aligned}\int_\Omega \left| |u_n(\cdot,t)|^{h-2}u_n(\cdot,t)\ln|u_n(\cdot,t)|\right|^{\frac{h}{h-1}} dx = & \int_{\Omega_1} \left| |u_n(\cdot,t)|^{h-2}u_n(\cdot,t)\ln|u_n(\cdot,t)|\right|^{\frac{h}{h-1}} dx \\ & + \int_{\Omega_2} \left| |u_n(\cdot,t)|^{h-2}u_n(\cdot,t)\ln|u_n(\cdot,t)|\right|^{\frac{h}{h-1}} dx,\end{aligned}$$

where

$$\Omega_1 = \{x\in\Omega \,|\, |u(\cdot,t)|\le 1\}, \Omega_2 = \{x\in\Omega \,|\, |u(\cdot,t)| > 1\}.$$

Since

$$\inf_{a\in(0,1)} a^{h-1}\ln a = a^{h-1}\ln a\Big|_{a=e^{-\frac{1}{h-1}}} = -\frac{1}{(h-1)e},$$

we speculate

$$\int_{\Omega_1} \left| |u_n(\cdot,t)|^{h-2}u_n(\cdot,t)\ln|u_n(\cdot,t)|\right|^{\frac{h}{h-1}} dx \le \left(\frac{1}{(h-1)e}\right)^{\frac{h}{h-1}} |\Omega_1| := D_0, \quad \forall t\in[0,\infty).$$

Putting $a = \frac{(2_s^*-h)(h-1)}{h}$ out of Lemma 3.1, in the next step, draw support from Lemma 2.1 (1) and (4.13), we have

$$\begin{aligned}\int_{\Omega_2} \left| |u_n(\cdot,t)|^{h-2}u_n(\cdot,t)\ln|u_n(\cdot,t)|\right|^{\frac{h}{h-1}} dx &\le C\int_{\Omega_2} |u_n(\cdot,t)|^{2_s^*} dx \\ &\le C\|u_n(\cdot,t)\|_{L^{2_s^*}(\Omega)}^{2_s^*} \\ &\le CC_1\|u_n(\cdot,t)\|_{X_0}^{2_s^*} \\ &\le CC_1\left(\frac{2hd}{(h-2)k}\right)^{\frac{2_s^*}{2}},\end{aligned}$$

where $C_1 = \frac{\alpha}{\beta}$ in Lemma 2.1 (1). Hence, synthesizing the previous proof process, it can be inferred that

$$\int_{\Omega} \left| |u_n(\cdot,t)|^{h-2}u_n(\cdot,t)\ln|u_n(\cdot,t)|\right|^{\frac{h}{h-1}} dx \le D_0 + CC_1\left(\frac{2hd}{(h-2)k}\right)^{\frac{2_s^*}{2}} := D_1. \tag{4.15}$$

The above estimates signify the following:

$$\{u_n\} \text{ is bounded in } L^\infty(0,\infty;X_0),$$

$$\{u_{nt}\} \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)),$$

$$\{f(u_n)\} \text{ is bounded in } L^\infty(0,\infty;L^{\frac{h}{h-1}}(\Omega)).$$

Consequently, there are u, $\varkappa$ and a subsequence of $\{u_n\}_{n=1}^\infty$, still referred to as $\{u_n\}_{n=1}^\infty$, such that $n\to\infty$, the latter result holds:

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,\infty;X_0), \tag{4.16}$$

$$u_{nt} \overset{*}{\rightharpoonup} u_t \text{ in } L^\infty(0,\infty;L^2(\Omega)), \tag{4.17}$$

$$\{f(u_n)\} \overset{*}{\rightharpoonup} \varkappa \text{ in } L^\infty(0,\infty;L^{\frac{h}{h-1}}(\Omega)), \tag{4.18}$$

taking advantage of Lemma 2.1 (3), there are

$$u_n \to u \text{ in } L^2(0,\infty;L^h(\Omega)).$$

Integrating (4.2) with respect to t yields

$$\begin{aligned}
&(u_{nt}(\cdot,t),\varrho_j)+\int_0^t M([u_n(\cdot,\tau)]_s^2)(u_n(\cdot,\tau),\varrho_j)_{X_0}d\tau\\
&-\int_0^t\int_0^m g(m-\tau)(u_n(\cdot,\tau),\varrho_j)_{X_0}d\tau dm+(u_n(\cdot,t),\varrho_j)\\
=&(u_{nt}(\cdot,0),\varrho_j)+(u_n(\cdot,0),\varrho_j)+\int_0^t(f(u_n(\cdot,\tau)),\varrho_j)d\tau.
\end{aligned}$$

As $n\to\infty$, we get

$$\begin{aligned}
&(u_t(\cdot,t),\varrho_j)+\int_0^t M([u(\cdot,\tau)]_s^2)(u(\cdot,\tau),\varrho_j)_{X_0}d\tau\\
&-\int_0^t\int_0^m g(m-\tau)(u(\cdot,\tau),\varrho_j)_{X_0}d\tau dm+(u(\cdot,t),\varrho_j)\\
=&(u_1,\varrho_j)+(u_0,\varrho_j)+\int_0^t(\varkappa,\varrho_j)d\tau.
\end{aligned}$$

Therefore, since C_0^∞ is dense in X_0, as is shown [25], and we know that $L^2(\Omega)$ has an orthonormal basis $\{\varrho_j\}\subset C_0^\infty$. Then, for all $v\in X_0$, there are

$$\begin{aligned}
&(u_t(\cdot,t),v)+\int_0^t M([u(\cdot,\tau)]_s^2)(u(\cdot,\tau),v)_{X_0}d\tau\\
&-\int_0^t\int_0^m g(m-\tau)(u(\cdot,\tau),v)_{X_0}d\tau dm+(u(\cdot,t),v)\\
=&(u_1,v)+(u_0,v)+\int_0^t(\varkappa,v)d\tau.
\end{aligned}$$

Using the method in [28], we arrive at the conclusion that $\varkappa=|u|^{h-2}u\ln|u|$. By virtue of (4.3)-(4.4), $u(\cdot,0)=u_0$ in X_0 is obtained, and $u_t(\cdot,0)=u_1$ in $L^2(\Omega)$ as well. Besides, setting $v(x)=\varphi(\cdot,t)$, fixing t here, as well as integrating with respect to t for $\forall\varphi\in L^1(0,\infty;X_0)$. By reason of the foregoing, there exists a consequence that $u(\cdot,t)$ serves as a global solution to question (1.1).

At last, due to $u_n\in W$ for any n and for any $t\in(0,\infty)$, it is clearly see that $u(\cdot,t)\in W$ for any $t\in(0,\infty)$. □

5. Blow-up of the solutions

Lemma 5.1. *[29] Postulate that the nonnegativity of function $\Lambda(t) \in C^2[0,T)$ for any $T > 0$ satisfies*

$$\Lambda''(t)\Lambda(t) - (1+\varpi)(\Lambda'(t))^2 \geq 0,$$

where $\varpi > 0$ is a constant. In the event of $\Lambda(0) > 0$ and $\Lambda'(0) > 0$, we have

$$T \leq \frac{\Lambda(0)}{\xi\Lambda'(0)} < +\infty$$

as well as $\Lambda(t) \to +\infty$ as $t \to T$.

Lemma 5.2. *Assume that the problem* (1.1) *has a weak solution u; then E(t) is not only monotonically decreasing but also satisfies the inequality*

$$E(t) + \int_0^t \|u_t(\tau)\|_2^2 d\tau \leq E(0), \quad t \geq 0. \tag{5.1}$$

Proof. The first equation for problem (1.1) is to multiply both sides by u_t and integrate over Ω; because of assumption (A_1), we can deduce that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|u_t(\cdot,t)\|_2^2 + \frac{1}{2\gamma}\|u(\cdot,t)\|_{X_0}^{2\gamma} + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|u(\cdot,t)\|_{X_0}^2\right.\\
&\quad\left. + \frac{1}{h^2}\|u(\cdot,t)\|_h^h + \frac{1}{2}(g\circ u)(\cdot,t) - \frac{1}{h}\int_\Omega |u(\cdot,t)|^h \ln|u(\cdot,t)|dx\right) + \|u_t(\cdot,t)\|_2^2\\
&= \frac{1}{2}(g'\circ u)(\cdot,t) - \frac{1}{2}g(t)\|u(\cdot,t)\|_{X_0}^2\\
&\leq 0.
\end{aligned}$$

The above equation is integral over $[0,t]$ with respect to τ; we attain

$$\begin{aligned}
&\int_0^t \frac{d}{dt}\left(\frac{1}{2}\|u_t(\cdot,\tau)\|_2^2 + \frac{1}{2\gamma}\|u(\cdot,\tau)\|_{X_0}^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|u(\cdot,\tau)\|_{X_0}^2\right.\\
&\quad\left. + \frac{1}{h^2}\|u(\cdot,\tau)\|_h^h + \frac{1}{2}(g\circ u)(\cdot,\tau) - \frac{1}{h}\int_\Omega |u(\cdot,\tau)|^h \ln|u(\cdot,\tau)|dx\right)d\tau + \int_0^t \|u_t(\cdot,\tau)\|_2^2 d\tau\\
&= \frac{1}{2}\int_0^t (g'\circ u)(\cdot,\tau)d\tau - \frac{1}{2}\int_0^t g(\tau)\|u(\cdot,\tau)\|_{X_0}^2 d\tau.
\end{aligned}$$

that is,

$$E(t) + \int_0^t \left(\|u_t(\cdot,\tau)\|_2^2 - \frac{1}{2}(g'\circ u)(\cdot,\tau) + \frac{1}{2}g(\tau)\|u(\cdot,\tau)\|_{X_0}^2\right)d\tau = E(0).$$

Finally, using assumption (A_1), we get (5.1). To sum up, the conclusion is proved. □

Theorem 5.1. *Suppose that the relaxation function g satisfies (A_1) and the Kirchhoff function M satisfies $(\mathscr{M})$. Assume that $u_0 \in X_0$, $u_1 \in L^2(\Omega)$, $h > 2\varsigma$ and the following inequality holds:*

$$\iota := \int_0^\infty g(\tau)d\tau < \frac{h(h-2\varsigma)}{h(h-2)+1}. \tag{5.2}$$

The case of $E(t)$ of problem (1.1) *still fulfills*

$$E(0) < \frac{c_m}{2h}\left(2\int_\Omega u_0u_1dx + \|u_0\|_2^2\right), \tag{5.3}$$

then the solutions of problem (1.1) *blow-up in finite time, where*

$$c_m = \max_{\eta_1\in(0,1)} c(\eta_1),$$

$$c(\eta_1) = \min\left\{\sqrt{\frac{(h+2)\vartheta\eta_1}{E_2^2}}, \frac{\vartheta(1-\eta_1)}{E_2^2}\right\},$$

and

$$\vartheta = (h-2\varsigma) + (2-h-\frac{1}{h})\iota.$$

Proof. Let $u(\cdot,t)$ represent a global solution to problem (1.1). As a matter of convenience, let us replace $u(\cdot,t)$ with $u(t)$. Define

$$\mathscr{A}(t) = 2\mathscr{H}_1 + \varpi_1 - \rho E(0), \tag{5.4}$$

it is momentous to point out

$$\mathscr{H}_1 = \int_\Omega u(t)u_t(t)dx, \mathscr{H}_2 = \int_0^t (u(\tau),u_t(\tau))_2 d\tau,$$
$$\varpi_1 = \|u(t)\|_2^2, \varpi_2 = \|u_t(t)\|_2^2.$$

We'll give the form of the positive number ρ later. Further, we obtain

$$\begin{aligned}\frac{d}{dt}\mathscr{A}(t) =& 2\varpi_2 + 2\int_\Omega u(t)u_{tt}(t)dx + 2\mathscr{H}_1\\ =& 2\varpi_2 - 2M([u]_s^2)\|u(t)\|_{X_0}^2 + 2\int_\Omega |u|^h \ln|u|dx\\ &+ 2\int_0^t g(t-\tau)(u(\tau),u(t))_{X_0}d\tau.\end{aligned} \tag{5.5}$$

$$\begin{aligned}&\int_0^t g(t-\tau)(u(\tau),u(t))_{X_0}d\tau\\ =& \int_0^t g(t-\tau)(u(\tau)-u(t),u(t))_{X_0}d\tau + \int_0^t g(t-\tau)(u(t),u(t))_{X_0}d\tau\\ =& \int_0^t g(t-\tau)(u(\tau)-u(t),u(t))_{X_0}d\tau + \int_0^t g(\tau)d\tau\|u(t)\|_{X_0}^2.\end{aligned} \tag{5.6}$$

Inserting (5.6) into (5.5) and using assumption ($\mathscr{M}$), we get

$$\begin{aligned}\frac{d}{dt}\mathscr{A}(t) =&2\varpi_2 - 2M([u]_s^2)\|u(t)\|_{X_0}^2 + 2\int_\Omega |u|^h \ln|u| dx\\ &+ 2\int_0^t g(t-\tau)(u(\tau)-u(t), u(t))_{X_0} d\tau + 2\int_0^t g(\tau)d\tau\|u(t)\|_{X_0}^2\\ \geq&2\varpi_2 - 2\varsigma\mathscr{B}([u]_s^2) + 2\int_\Omega |u|^h \ln|u| dx\\ &+ 2\int_0^t g(t-\tau)(u(\tau)-u(t), u(t))_{X_0} d\tau + 2\int_0^t g(\tau)d\tau\|u(t)\|_{X_0}^2.\end{aligned}$$

By the Young's inequality, it leads to

$$\begin{aligned}&\int_0^t g(t-\tau)(u(\tau)-u(t), u(t))_{X_0} d\tau\\ &\geq -\int_0^t g(t-\tau)\|u(\tau)-u(t)\|_{X_0}\|u(t)\|_{X_0} d\tau\\ &\geq -\frac{h}{2}\int_0^t g(t-\tau)\|u(\tau)-u(t)\|_{X_0}^2 d\tau - \frac{1}{2h}\int_0^t g(\tau)d\tau\|u(t)\|_{X_0}^2.\end{aligned} \tag{5.7}$$

Taking it one step further and making the most of (5.7), we deduce

$$\begin{aligned}\frac{d}{dt}\mathscr{A}(t) \geq&2\varpi_2 - 2\varsigma\mathscr{B}([u]_s^2) + 2\int_\Omega |u|^h \ln|u| dx + 2\int_0^t g(\tau)d\tau\|u(t)\|_{X_0}^2\\ &- h\int_0^t g(t-\tau)\|u(\tau)-u(t)\|_{X_0}^2 d\tau - \frac{1}{h}\int_0^t g(\tau)d\tau\|u(t)\|_{X_0}^2\\ =&(h+2)\varpi_2 - 2\varsigma\mathscr{B}([u]_s^2) + \frac{h}{\gamma}\|u(t)\|_{X_0}^{2\gamma} + h\|u(t)\|_{X_0}^2 + \frac{2}{h}\|u(t)\|_h^h\\ &+ \left(2 - h - \frac{1}{h}\right)\int_0^t g(\tau)d\tau\|u(t)\|_{X_0}^2 - 2hE(t)\\ \geq&(h+2)\varpi_2 + \left(\frac{h-2\varsigma}{\gamma}\right)\|u(t)\|_{X_0}^{2\gamma} + (h-2\varsigma)\|u(t)\|_{X_0}^2 + \frac{2}{h}\|u(t)\|_h^h\\ &+ \left(2 - h - \frac{1}{h}\right)\iota\|u(t)\|_{X_0}^2 - 2hE(t).\end{aligned} \tag{5.8}$$

By Lemma 5.2, we gain

$$\begin{aligned}\frac{d}{dt}\mathscr{A}(t) \geq&(h+2)\varpi_2 + \left(\frac{h-2\varsigma}{\gamma}\right)\|u(t)\|_{X_0}^{2\gamma} + (h-2\varsigma)\|u(t)\|_{X_0}^2 + \frac{2}{h}\|u(t)\|_h^h\\ &+ \left(2 - h - \frac{1}{h}\right)\iota\|u(t)\|_{X_0}^2 - 2hE(0) + 2h\int_0^t \|u(\tau)\|_2^2 d\tau\\ \geq&(h+2)\varpi_2 + \left[(h-2\varsigma) + \left(2 - h - \frac{1}{h}\right)\iota\right]\|u(t)\|_{X_0}^2 - 2hE(0)\\ &+ 2h\int_0^t \|u_t(\tau)\|_2^2 d\tau.\end{aligned} \tag{5.9}$$

By assumption, we know that

$$(h(h-2)+1)\,\iota < h(h-2\varsigma), h > 2\varsigma,$$

as a consequence

$$\vartheta = (h-2\varsigma) + \left(2-h-\frac{1}{h}\right)\iota > 0.$$

Therefore, using Sobolev embedding theorem, we have

$$\begin{aligned}\frac{d}{dt}\mathscr{A}(t) \geq& (h+2)\varpi_2 + \vartheta\|u(t)\|_{X_0}^2 - 2hE(0)\\ =& (h+2)\varpi_2 + \vartheta\eta_1\|u(t)\|_{X_0}^2 + \vartheta(1-\eta_1)\|u(t)\|_{X_0}^2 - 2hE(0)\\ \geq& (h+2)\varpi_2 + \frac{\vartheta\eta_1}{E_2^2}\varpi_1 + \frac{\vartheta(1-\eta_1)}{E_2^2}\varpi_1 - 2hE(0).\end{aligned}$$

It is significant to note that constant $\eta_1 \in [0,1]$ will be defined soon after.

Using Cauchy's inequality, we gain

$$(h+2)\varpi_2 + \frac{\vartheta\eta_1}{E_2^2}\varpi_1 \geq 2\sqrt{\frac{(h+2)\vartheta\eta_1}{E_2^2}}(\varpi_2\varpi_1)^{\frac{1}{2}} \geq 2\sqrt{\frac{(h+2)\vartheta\eta_1}{E_2^2}}|\mathscr{H}_1|.$$

It can be said that

$$\begin{aligned}\frac{d}{dt}\mathscr{A}(t) \geq& 2\sqrt{\frac{(h+2)\vartheta\eta_1}{E_2^2}}|\mathscr{H}_1| + \frac{\vartheta(1-\eta_1)}{E_2^2}\varpi_1 - 2hE(0)\\ \geq& c(\eta_1)\left(2|\mathscr{H}_1| + \varpi_1 - \frac{2h}{c(\eta_1)}E(0)\right)\\ \geq& c(\eta_1)\left(2\mathscr{H}_1 + \varpi_1 - \frac{2h}{c(\eta_1)}E(0)\right).\end{aligned}$$

where $c(\eta_1) = \min\left\{\sqrt{\frac{(h+2)\vartheta\eta_1}{E_2^2}}, \frac{\vartheta(1-\eta_1)}{E_2^2}\right\} = \min\left\{\mathscr{P}(\eta_1), \mathscr{Q}(\eta_1)\right\}.$

For $0 \leq \eta_1 \leq 1$, with h, ϑ, and E_2 being positive constants, it is straightforward to show that the function $\mathscr{P}(\eta_1) = \sqrt{\frac{(h+2)\vartheta\eta_1}{E_2^2}}$ is strictly increasing. This follows from the fact that the expression inside the square root $\frac{(h+2)\vartheta}{E_2^2}\eta_1$ is a positive constant multiple of η_1, and the square root function itself is monotonically increasing. Therefore, $\mathscr{P}(\eta_1)$ grows as η_1 increases. Simultaneously, we get $Q(\eta_1) = \frac{\vartheta(1-\eta_1)}{E_2^2}$ is strictly decreasing. Since η_1 appears with a negative coefficient in the numerator, $Q(\eta_1)$ is clearly strictly decreasing as η_1 increases. Thus, get the maximum value of $c(\eta)$ when $\eta = \eta_1$. Shortly afterward, let

$$\rho = \frac{2h}{c_m}.$$

In virtue of (5.3), there are

$$\mathscr{A}(0) > 0.$$

Besides, we have

$$\frac{d}{dt}\mathscr{A}(t) \geq c_m \left(2\mathscr{H}_1 + \varpi_1 - \rho E(0)\right) = c_m \mathscr{A}(t).$$

It follows that

$$\mathscr{A}(t) \geq \mathscr{A}(0)e^{c_m t}, t \geq 0.$$

In conclusion,

$$\mathscr{A}(t) \to +\infty, \text{ as } n \to \infty. \tag{5.10}$$

Set

$$\mathscr{B}(t) = \varpi_1 + \int_0^t \|u(\tau)\|_2^2 d\tau + (T - t)\|u_0\|_2^2, \quad \forall t \in [0, T].$$

where $T > 0$ is large enough. We attain

$$\begin{aligned}\mathscr{B}'(t) &= 2\mathscr{H}_1 + \varpi_1 - \|u_0\|_2^2 \\ &= 2\mathscr{H}_1 + 2\mathscr{H}_2.\end{aligned}$$

Obviously,

$$\mathscr{B}''(t) = \frac{d}{dt}\mathscr{A}(t).$$

Making use of (5.5) and (5.9), we extrapolate

$$\begin{aligned}\mathscr{B}''(t) =& \frac{d}{dt}\left(2\mathscr{H}_1 + 2\mathscr{H}_2\right) \\ =& 2\varpi_2 - 2M([u]_s^2)\|u(t)\|_{X_0}^2 + 2\int_\Omega |u|^h \ln|u| dx \\ &+ 2\int_0^t g(t-\tau)(u(\tau), u(t))_{X_0} d\tau \\ \geq& (h+2)\varpi_2 + \vartheta\|u(t)\|_{X_0}^2 - 2hE(0) + 2h\int_0^t \|u_t(\tau)\|_2^2 d\tau.\end{aligned}$$

By the Cauchy inequality, the definition of $\mathscr{H}(t)$, and the embedding X_0 into $L^2(\Omega)$, we infer

$$\mathscr{A}(t) \leq \varpi_2 + 2E_2^2\|u(t)\|_{X_0}^2.$$

Using (5.10), it is easy to get

$$\varpi_2 + 2E_2^2\|u(t)\|_{X_0}^2 \to +\infty$$

as $t \to +\infty$.

Then, we have

$$(h+2)\varpi_2 + \vartheta\|u(t)\|_{X_0}^2 - 2qE(0) \to +\infty.$$

There is $0 < \kappa < 1, \epsilon > 0, T_B > 0$ and when $t > T_B$, we make conclusions about

$$\kappa(h+2) > 4 + \epsilon, \tag{5.11}$$

and

$$(h+2)\varpi_2 + \vartheta\|u(t)\|_{X_0}^2 - 2hE(0) > \kappa((h+2)\varpi_2 + \vartheta\|u(t)\|_{X_0}^2), \tag{5.12}$$

According to (5.11) and (5.12), we obtain

$$\begin{aligned} \mathscr{B}''(t) &> \kappa((h+2)\varpi_2 + \vartheta\|u(t)\|_{X_0}^2) + 2h\int_0^t \|u_t(\tau)\|_2^2 d\tau \\ &> (4+\epsilon)\varpi_2 + 2h\int_0^t \|u_t(\tau)\|_2^2 d\tau \\ &> (4+\epsilon)\left(\varpi_2 + \int_0^t \|u_t(\tau)\|_2^2 d\tau\right), \end{aligned} \tag{5.13}$$

for $t > T_B$.

Set $\theta = 4+\epsilon$. Then,

$$\begin{aligned} \mathscr{B}''(t)\mathscr{B}(t) - \frac{\theta}{4}\mathscr{B}'(t)^2 > &\left(\theta\varpi_2 + \theta\int_0^t \|u_t(\tau)\|_2^2 d\tau\right)\left(\varpi_1 + \int_0^t \|u(\tau)\|_2^2 d\tau + (T-t)\|u_0\|_2^2\right) \\ &- \frac{\theta}{4}(2\mathscr{H}_1 + 2\mathscr{H}_2)^2 . \end{aligned}$$

Using the Hölder inequality, we get the following inequalities:

$$\mathscr{H}_1^2 \le \varpi_1\varpi_2, \tag{5.14}$$

$$\mathscr{H}_2^2 \le \int_0^t \|u(\tau)\|_2^2 d\tau \int_0^t \|u_t(\tau)\|_2^2 d\tau, \tag{5.15}$$

$$\begin{aligned} 2\mathscr{H}_1\mathscr{H}_2 &\le 2\left(\int_\Omega u^2(t)dx\right)^{\frac{1}{2}}\left(\int_\Omega u_t^2(t)dx\right)^{\frac{1}{2}}\left(\int_0^t \|u(\tau)\|_2^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_t(\tau)\|_2^2 d\tau\right)^{\frac{1}{2}} \\ &\le \varpi_1\int_0^t \|u_t(\tau)\|_2^2 d\tau + \varpi_2\int_0^t \|u(\tau)\|_2^2 d\tau. \end{aligned} \tag{5.16}$$

Gathering (5.14)–(5.16), we obtain

$$\mathscr{B}''(t)\mathscr{B}(t) - \frac{\theta}{4}\mathscr{B}'(t)^2 > 0, \quad \text{for } t > T_B.$$

Postulate $\Lambda(\sigma) = c(\sigma + T_B), \sigma = t - T_B$; we check and testify that

$$\Lambda''(\sigma)\Lambda(\sigma) - \frac{\theta}{4}\Lambda'(\sigma)^2 > 0, \quad \text{for } \sigma > 0.$$

To sum up, combine the very best of $\theta > 4$ and Lemma 5.1, the conclusion holds. □

6. Conclusions

This work investigates the well-posedness of fractional viscoelastic Kirchhoff equations featuring both damping and nonlinear logarithmic source terms. First, fractional Sobolev spaces are introduced, along with the potential well theory and several essential lemmas. On this basis, the global existence of solutions is established using the Galerkin method. Furthermore, the finite-time blow-up of solutions is proved via the concavity method.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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