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Research article

Semi-classical states for a class of quasilinear Schrödinger equations with a parameter

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Abstract: In this paper, we study the following quasilinear Schrödinger equation with a parameter: $-\varepsilon^2\Delta u+V(x)u-\varepsilon^2\kappa\beta\Delta(|u|^{2\beta})|u|^{2\beta-2}u = |u|^{q-2}u+|u|^{(2\beta)2^*-2}u$ in $\mathbb{R}^N$, where $N \geqslant 3, \beta > \frac{1}{2}, 4\beta < q < (2\beta)2^*$, κ is a constant, and $V : \mathbb{R}^N \to \mathbb{R}$ satisfies the classical global assumption. By using a change of variable, we obtain the existence, multiplicity, and concentration behavior. We will see that the existence of solutions depends heavily on the parameter β.

Keywords: semi-classical states; quasilinear Schrödinger equations; a parameter; critical growth

1. Introduction

In this paper, we are concerned with the existence, multiplicity and concentration behavior for the quasilinear Schrödinger equations as follows:

$$-\varepsilon^2\Delta u + V(x)u - \varepsilon^2\kappa\beta\Delta(|u|^{2\beta})\,|u|^{2\beta-2}\,u = g(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $V(x)$ is a given potential, $\varepsilon > 0$ is a real parameter, κ is a real constant, $g(x,u)$ is a nonlinearity enjoying various hypotheses, $\beta > 0$ is a constant, and $2^* := 2N/(N-2)$ is the critical Sobolev exponent. The solutions associated with problem (1.1) have a close connection with the standing wave solutions of certain quasilinear Schrödinger equations. These equations frequently emerge in diverse physical phenomena, such as the theory of superfluid films within the realm of plasma physics (see [1–3]).

In recent decades, the existence and qualitative properties of solutions for the quasilinear Schrödinger equation (1.1) has been widely investigated, especially for the case $\varepsilon = 1$. In [4], by employing a change of variable, Liu et al. transformed the equation into a semilinear form, and hence the existence of solutions was established through variational methods under various types of potentials $V(x)$. This method has proven to be crucial and has been widely adopted in the study of these types of problems. In [5], Liu et al. have obtained the existence of positive ground states of Eq (1.1) as $g(x,u) = |u|^{p-2}u$. In [6], the existence of standing wave solutions was proved by combining the concentration-compactness principle of Lions with the classical arguments of Brézis and Nirenberg (see [7]). In particular, we recall the concentration behavior of positive ground state solutions, which was first established by Rabinowitz in [8] and Wang [9] when $\kappa = 0$. More precisely, they proved that the positive ground state solutions must concentrate at the global minima of V as $\varepsilon \to 0$. By using the penalization method developed by del Pino and Felmer in [10], Wang and Zou [11] studied the existence and concentration of positive bound states to (1.1) under the assumptions of local potential. For the global potential case, He et al. [12] dealt with the existence, multiplicity and concentration behavior of the ground states to (1.1) when $g(x, u) = u^{22^*-1} + h(u)$. We also refer interested readers to [11, 13–16] and the references therein.

On the other hand, there are some works that focus on the existence of solutions to (1.1) for general β. We also remark that most of them mainly dealt with the case for $\beta > \frac{1}{2}$. In [3], when $g(x, u) = \lambda|u|^{q-2}u$ with $2 < q + 1 < 2^*(2\beta)$, Liu and Wang proved the existence of solution by using the approach of Lagrange multiplier. In [17], by employing the fibering method, Moameni proved (1.1) has at least a standing wave solution when $g(x, u) = \mu g(x)|u|^{q-2}u$. In [18], Li and Zhang proved the existence of a positive spike solution of Eq (1.1) when $g(x, u) = |u|^{q-2}u + |u|^{(2\beta)2^*-2}u$, where $4\beta \leqslant q \leqslant (2\beta)2^*$. In [19], the second author, Tang, and Zhang proved that (1.1) has at least a positive ground state solution when $g(x, u) = |u|^{q-2}u + |u|^{(2\beta)2^*-2}u$. We also cite [20, 21] and the references therein for more details. As far as we know, there is only one work related to (1.1) with $\beta \in (0, \frac{1}{2})$: Li [18] proved the existence of positive solutions for Eq (1.1) when $g(x, u) = |u|^{q-2}u + |u|^{2^*-2}u$ with $2 \leqslant q \leqslant 2^*$.

However, to our knowledge, there are no works which focus on the multiplicity and concentration behavior of (1.1) when $\beta \neq 0, 1$. In [19], the authors proved that there is a sharp gap between the two cases $\frac{1}{2} < \beta < 1$ and $\beta \geqslant 1$. Inspired by the above-mentioned works, especially by [12, 19], we are concerned with the semi-classical solutions for the quasilinear Schrödinger equations with critical growth in this paper:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u - \varepsilon^2\kappa\beta\Delta(|u|^{2\beta})\,|u|^{2\beta-2}\,u = |u|^{q-2}\,u + |u|^{(2\beta)2^*-2}\,u & \text{in } \mathbb{R}^N, \\ u(x) > 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{Q_ε}$$

where $2^* := 2N/(N-2)$, $N \geqslant 3$, $\beta > \frac{1}{2}$, $4\beta < q < (2\beta)2^*$, κ is a positive constant, and $V : \mathbb{R}^N \to \mathbb{R}$ satisfies the global assumption as follows:

$$(V) \quad 0 < V_0 = \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x) = V_\infty < \infty.$$

To state our main result, we introduce several notations. Let

$$M = \{x \in \mathbb{R}^N : V(x) = V_0\}$$

and

$$M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leqslant \delta\} \quad \text{for } \delta > 0.$$

$cat_{M_\delta}(M)$ denotes Ljusternik-Schnirelmann category of the sets M and M_δ. For $\beta \geqslant 1$, denote

$$Q_{N,\beta} = \begin{cases} \frac{4\beta N - 2N + 4}{N-2}, & N < [4\beta + 2], \\ 4\beta, & N \geqslant [4\beta + 2]. \end{cases}$$

For $\frac{1}{2} < \beta < 1$, denote

$$Q_{N,\beta} = \begin{cases} \frac{8\beta}{N-2} + 2, & N \leqslant [4 + \frac{2}{2\beta-1}], \\ 4\beta, & N > [4 + \frac{2}{2\beta-1}], \end{cases}$$

where $\left[x\right] := \sup\{n \in \mathbb{Z} | n \leqslant x\}$. Our main result in this paper is given as follows.

Theorem 1.1. *Suppose that* (V) *and* $Q_{N,\beta} < q < (2\beta)2^*$ *are satisfied. Then, for any given* $\delta > 0$*, there exists* $\varepsilon^* = \varepsilon^*(\delta) > 0$ *such that equation* (Q_ε) *has at least* $cat_{M_\delta}(M)$ *positive solutions for all* $\varepsilon \in (0, \varepsilon^*)$*. Moreover, each solution decays to zero at infinity. If* u_ε *represents one of these positive solutions, and we denote its global maximum point by* η_ε*, then*

$$\lim_{\varepsilon \to 0} V(\eta_\varepsilon) = V_0.$$

Remark 1.2. *Clearly, our result partly generalizes the one in [12]. The hypothesis* (V) *was originally introduced by Rabinowitz in [8], and in this work, we only focus on the case for* $V_\infty < \infty$*. In [12], He et al. dealt with both* $V_\infty < \infty$ *and* $V_\infty = \infty$ *for the case* $\beta = 1$*. However, under the assumption of* $V_\infty = \infty$*, one can not verify the boundness of* (PS) *sequence for modified functional if we choose the workspace as* $H^1(\mathbb{R}^N)$*. On the other hand, the Orlicz space given by [12] is not even a linearly normed space if* $\beta \neq 1$*. Thus, it is an interesting question whether similar results in Theorem 1.1 can be obtained when* $\beta \neq 1$ *and* $V_\infty = \infty$*.*

The organization of this paper is as follows. In Section 2, we prove the existence of a positive ground state for (Q_ε). In Section 3, we focus on the multiplicity and concentration of solutions to (Q_ε), and we give the proof of Theorem 1.1.

Notation. Throughout this paper, $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence, respectively. $\|\cdot\|$ denotes the norm in $H^1\left(\mathbb{R}^N\right)$. $\|\cdot\|_p$ denotes the norm in $L^p\left(\mathbb{R}^N\right)$ for $1 \leqslant p \leqslant \infty$. C_0, C, C_i denote various positive constants whose value may change from line to line but are not essential to the analysis of the proof.

2. Existence of a positive ground state

2.1. Variational framework and Mountain pass geometry

In order to simplify the notation, we denote $\alpha := 2\beta$ and $\kappa\beta = 1$. By making a variable change $\varepsilon z = x$, (Q_ε) is equivalent to (Q_ε^*) as follows:

$$\begin{cases} -\Delta u + V(\varepsilon x)u - \kappa\beta\Delta(|u|^{2\beta})\,|u|^{2\beta-2}\,u = |u|^{q-2}\,u + |u|^{(2\beta)2^*-2}\,u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), u(x) > 0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{Q_ε^*}$$

To find the positive solutions to (Q_ε), it is natural to consider the following functional:

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1 + \alpha|u|^{2(\alpha-1)})|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2 - \frac{1}{q}\int_{\mathbb{R}^N}|u^+|^q - \frac{1}{2^*\alpha}\int_{\mathbb{R}^N}|u^+|^{2^*\alpha},$$

which is not well-defined in the usual Sobolev space $H^1(\mathbb{R}^N)$. To overcome this difficulty, inspired by [13], we make a change of variables $v = g^{-1}(u)$, where g is defined by

$$\begin{cases} g(0) = 0, \\ g'(t) = \left(1 + \alpha|g(t)|^{2(\alpha-1)}\right)^{-\frac{1}{2}}, & \forall t > 0, \\ -g(t) = g(-t), & \forall t < 0. \end{cases} \tag{2.1}$$

After the change of variables, we focus on the functional

$$J_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(\varepsilon x)g^2(v)\right) - \frac{1}{q}\int_{\mathbb{R}^N}|g(v^+)|^q - \frac{1}{2^*\alpha}\int_{\mathbb{R}^N}|g(v^+)|^{2^*\alpha}$$

and the following semilinear problem:

$$-\Delta v + V(\varepsilon x)g(v)g'(v) = |g(v^+)|^{q-1}g'(v) + |g(v^+)|^{2^*\alpha-1}g'(v).$$

Observe that $v(x)$ is a critical point of J_ε if and only if $g(v(\varepsilon x))$ solves equation (Q_ε). Moreover, by Lemma 2.1 below, we have $J_\varepsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

Lemma 2.1. *The function g satisfies the following properties:*
(a) $g \in C^\infty(\mathbb{R}, \mathbb{R})$ *is a uniquely defined function and invertible.*
(b) $|g'(t)| \leqslant 1$ *for all* $t \in \mathbb{R}$.
(c) $|g(t)| \leqslant |t|$ *for all* $t \in \mathbb{R}$.
(d) $\lim_{t\to 0} \frac{g(t)}{t} = 1$.
(e) $\lim_{t\to\infty} \frac{|g(t)|^\alpha}{|t|} = \sqrt{\alpha}$.
(f) $\frac{1}{\alpha}g(t) \leqslant tg'(t) \leqslant g(t)$ *for all* $t \geqslant 0$.
(g) $|g(t)| \leqslant (\alpha)^{\frac{1}{2\alpha}}|t|^{\frac{1}{\alpha}}$ *for all* $t \in \mathbb{R}$.
(h) there exists a positive constant C such that

$$g(t) \geqslant \begin{cases} C|t|, & \text{if } |t| \leqslant 1, \\ C|t|^{\frac{1}{\alpha}}, & \text{if } |t| > 1. \end{cases}$$

(i) $|g(t)|^{\alpha-1}g'(t) \leqslant \frac{1}{\sqrt{\alpha}}$ *for all* $t \in \mathbb{R}$.
(j) the function $g(t)g'(t)t^{-1}$ *is decreasing for* $t > 0$.
(k) the function $g^p(t)g'(t)t^{-1}$ *is increasing for* $p \geqslant 2\alpha - 1$ *and* $t > 0$.
(l) the function $G(t) := \frac{1}{2}g^{\alpha 2^*-1}(t)g'(t)t - \frac{1}{\alpha 2^*}g^{\alpha 2^*}(t)$ *is increasing for* $t > 0$.
(m) the function $\frac{1}{2}g^{q-1}(t)g'(t)t - \frac{1}{q}g^q(t)$ *is increasing for* $t > 0$.

Proof. (a)–(i) can be found in [13, 19]. For (j) and (k), we refer to [11, 22]. Here, we only prove (l), because (m) is similar. By (f) and (i), we get

$$
\begin{aligned}
G'(t) &:= \frac{\alpha 2^*-1}{2} g^{\alpha 2^*-2}(t) g'(t)^2 t + \frac{1}{2} g^{\alpha 2^*-1}(t) g''(t) t + \frac{1}{2} g^{\alpha 2^*-1}(t) g'(t) - g^{\alpha 2^*-1}(t) g'(t) \\
&= \frac{\alpha 2^*-1}{2} g^{\alpha 2^*-2}(t) g'(t)^2 t - \frac{1}{2} g^{\alpha 2^*-1}(t)(\alpha(\alpha-1) g'(t)^4 g(t)^{2\alpha-3}) t - \frac{1}{2} g^{\alpha 2^*-1}(t) g'(t) \\
&= g^{\alpha 2^*-2}(t) g'(t) \left[\frac{\alpha 2^*-1}{2} g'(t) t - \frac{(\alpha(\alpha-1))}{2} g'(t)^3 g^{2(\alpha-1)}(t) t - \frac{1}{2} g(t) \right] \\
&\geqslant g^{\alpha 2^*-2}(t) g'(t) \left[\frac{\alpha 2^*-1}{2} g'(t) t - \frac{\alpha-1}{2} g'(t) t - \frac{1}{2} g(t) \right] \\
&= g^{\alpha 2^*-2}(t) g'(t) \left[\frac{\alpha(2^*-1)}{2} g'(t) t - \frac{1}{2} g(t) \right] \\
&\geqslant g^{\alpha 2^*-2}(t) g'(t) \left[\frac{2^*-1}{2} g(t) - \frac{1}{2} g(t) \right] \\
&= g^{\alpha 2^*-1}(t) g'(t) \left[\frac{2^*-2}{2} \right] \\
&> 0.
\end{aligned}
$$

Lemma 2.2. *Assume* (V) *and* $Q_{N,\beta} < q < (2\beta)2^*$ *hold. Then*
(i) *there exist* $\alpha_0 > 0$ *and* $\rho_0 > 0$ *such that* $J_\varepsilon(v) \geqslant \alpha_0$ *for all* $\|v\| = \alpha_0$.
(ii) *there exists a* $v \in H^1(\mathbb{R}^N)$ *with* $\|v\| > \rho_0$ *such that* $J_\varepsilon(v) < 0$.

Proof. For any $\rho > 0$, by Claim 2 of Lemma 2.2 in [19] and the boundedness of V, there exists $C_\rho > 0$ such that

$$
\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(\varepsilon x) g^2(v) \right) \geqslant C_\rho \|v\|^2, \forall\, v \in \Sigma_\rho, \tag{2.2}
$$

where $\Sigma_\rho = \{v \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} \left(|\nabla v|^2 + g^2(v) \right) \leqslant \rho\}$. We may assume that $0 < \rho_0 < 1$. From Lemma 2.1 and the Sobolev embedding theorem, we get

$$
\begin{aligned}
J_\varepsilon(v) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(\varepsilon x) g^2(v) \right) - \frac{1}{q} \int_{\mathbb{R}^N} |g(v^+)|^q - \frac{1}{2^*\alpha} \int_{\mathbb{R}^N} |g(v^+)|^{2^*\alpha} \\
&\geqslant C_1 \|v\|^2 - \frac{\alpha^{\frac{q}{2\alpha}}}{q} \int_{\mathbb{R}^N} |v|^{\frac{q}{\alpha}} - \frac{\alpha^{\frac{2^*}{2}}}{2^*\alpha} \int_{\mathbb{R}^N} |v|^{2^*} \\
&\geqslant C_1 \|v\|^2 - C_2 \|v\|^{\frac{q}{\alpha}} - C_3 \|v\|^{2^*}.
\end{aligned}
$$

Since $2\alpha < q < 2^*\alpha$, (i) is satisfied if we choose ρ_0 small enough.

Next, we prove (ii). Choose a non-negative function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi \geqslant 0$. By Lemma 2.1-(h), we have

$$
\begin{aligned}
J_\varepsilon(t\varphi) &\leqslant t^2 \left(\frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla \varphi|^2 + \frac{V(\varepsilon x) g^2(t\varphi)}{t^2} \right) - \frac{1}{\alpha 2^*} \int_{\mathbb{R}^N} \frac{g^{\alpha 2^*}(t\varphi)}{t^2} \right) \\
&\leqslant t^2 \left(C_1 \|v\|^2 - C_2 t^{2^*-2} \int_{\{t\varphi(x)>1\}} \varphi^{2^*} \right).
\end{aligned}
$$

Hence, $J_\varepsilon(t\varphi) \to -\infty$ as $t \to \infty$. By choosing a large $t^* > 0$ and taking $v = t^*\varphi$, (ii) holds.

By applying Lemma 2.2 with the mountain pass theorem without the $(C)_c$ condition in [23], there exists a $(PS)_c$ sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ such that $J_\varepsilon(v_n) \to c$ and $J'_\varepsilon(v_n) \to 0$ in $H^1(\mathbb{R}^N)^*$ with the minimax level

$$c_\varepsilon = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J_\varepsilon(\gamma(t)) > 0,$$

where $\Gamma = \{\gamma \in C^1([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, J_\varepsilon(\gamma(1)) < 0\}$. Now, we introduce the Nehari manifold method associated with J_ε defined by

$$\mathcal{N}_\varepsilon = \left\{v \in H^1(\mathbb{R}^N)\backslash\{0\} : \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(\varepsilon x)g(v)g'(v)v\right) - \left|g(v^+)\right|^{q-1} g'(v^+)v^+ - \left|g(v^+)\right|^{\alpha 2^*-1} g'(v^+)v^+ = 0\right\}.$$

Lemma 2.3. *For every $v \in H^1(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t(v) > 0$ such that $t(v)v \in \mathcal{N}_\varepsilon$. Moreover, $J_\varepsilon(t(v)v) = \max_{t\geqslant 0} J_\varepsilon(tv)$.*

Proof. For fixed $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, we define the function $f(t) \triangleq J_\varepsilon(tv)$ for $t \geqslant 0$. We observe that $f'(t) = \langle J'_\varepsilon(tv), v\rangle = 0$ if and only if $tv \in \mathcal{N}_\varepsilon$. Since $f'(t) = 0$ is equivalent to

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v|^2 &= \int_{\mathbb{R}^N}\left(\frac{|g(tv^+)|^{q-1}g'(tv^+)v^+}{t} + \frac{|g(tv^+)|^{\alpha 2^*-1}g'(tv^+)v^+}{t} - \frac{V(\varepsilon x)g(tv)g'(tv)v}{t}\right)\\ &= \int_{\{v(x)>0\}} \frac{|g(tv^+)|^{q-1}g'(tv^+)v^2}{tv} + \int_{\{v(x)>0\}} \frac{|g(tv)|^{\alpha 2^*-1}g'(tv)v^2}{tv}\\ &- \int_{\{v(x)\neq 0\}} \frac{V(\varepsilon x)g(t|v|)g'(t|v|)v^2}{t|v|}.\end{aligned} \tag{2.3}$$

According to Lemma 2.1-(l), the right-hand side of Eq (2.3) is an increasing function with respect to t. On the other hand, it is clear that $f(t) > 0$ for $t > 0$ small and $f(t) < 0$ for $t > 0$ large. Thus, there is a unique $t(v) > 0$ such that $f'(t(v)) = 0$, i.e., $t(v)v \in \mathcal{N}_\varepsilon$. Moreover, $J_\varepsilon(t(v)v) = \max_{t\geqslant 0} J_\varepsilon(tv)$.

Define

$$c^*_\varepsilon = \inf_{v\in\mathcal{N}_\varepsilon} J_\varepsilon(v), \qquad c^{**}_\varepsilon = \inf_{v\in H^1(\mathbb{R}^N)\backslash\{0\}} \max_{t\geqslant 0} J_\varepsilon(tv). \tag{2.4}$$

Lemma 2.4. $c_\varepsilon = c^*_\varepsilon = c^{**}_\varepsilon$.

Proof. From Lemma 2.3, we conclude that $c^*_\varepsilon = c^{**}_\varepsilon$. Observe that for any $v \in H^1(\mathbb{R}^N)\backslash\{0\}$, there exists $t_0 > 0$ such that $J_\varepsilon(t_0 v) < 0$. Define a path $\gamma : [0,1] \to H^1(\mathbb{R}^N)$ by $\gamma(t) = tt_0 v$. It is clear that $\gamma \in \Gamma$, and we conclude that $c_\varepsilon \leqslant c^{**}_\varepsilon = c^*_\varepsilon$. Now, it is sufficient to prove $c_\varepsilon \geqslant c^*_\varepsilon$. Clearly, $J_\varepsilon(v) \geqslant 0$ for $0 \leqslant t \leqslant t(v)$.

Using Lemma 2.1, we have

$$
\begin{aligned}
& f'(t) = \langle J'_\varepsilon(tv), v\rangle \\
& = t\left\{\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N}\left(\frac{V(\varepsilon x)g(tv)g'(tv)v}{t} - \frac{|g(tv^+)|^{q-1}g'(tv^+)v^+}{t} - \frac{|g(tv^+)|^{\alpha 2^*-1}g'(tv^+)v^+}{t}\right)\right\} \\
& = t\left\{\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\{v(x)\neq 0\}} \frac{V(\varepsilon x)g(tv)g'(tv)v^2}{tv} - \int_{\{v(x)>0\}} \frac{|g(tv^+)|^{q-1}g'(tv^+)v^2}{tv}\right. \\
& \left. - \int_{\{v(x)>0\}} \frac{|g(tv^+)|^{\alpha 2^*-1}g'(tv^+)v^2}{tv}\right\} \\
& \geqslant t\left\{\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\{v(x)\neq 0\}} \frac{V(\varepsilon x)g(t\,(v)\,v)g'(t\,(v)\,v)v^2}{t\,(v)\,v}\right. \\
& \left. - \int_{\{v(x)>0\}} \frac{|g(t(v)v)|^{\alpha 2^*-1}g'(t(v)v)v^2}{t(v)v} - \int_{\{v(x)>0\}} \frac{|g(t(v)v)|^{q-1}g'(t(v)v)v^2}{t(v)v}\right\} \\
& = \frac{t}{t\,(v)^2}\langle J'_\varepsilon(t\,(v)\,v, t\,(v)\,v\rangle = 0.
\end{aligned}
$$

Consequently, all paths of Γ cross $\mathcal{N}_\varepsilon$ and $c_\varepsilon \geqslant c^*_\varepsilon$. This finishes the proof.

2.2. The Palais-Smale condition

Lemma 2.5. *Let $\{v_n\}$ be a $(C)_c$ sequence of J_ε. Then, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Consider a $(C)_c$ sequence for J_ε, i.e., $J_\varepsilon(v_n) = c + o_n(1), (1 + \|v_n\|)J'_\varepsilon(v_n) \to 0$ in $H^1(\mathbb{R}^N)^*$. By Lemma 2.1-(f), we have

$$
\begin{aligned}
c + o_n(1) &= J_\varepsilon(v_n) - \frac{\alpha}{q}\langle J'_\varepsilon(v_n), v_n\rangle \\
&\geqslant \left(\frac{1}{2} - \frac{\alpha}{q}\right)\int_{\mathbb{R}^N} |\nabla v_n|^2 + \left(\frac{1}{2} - \frac{1}{q}\right)\int_{\mathbb{R}^N} V(\varepsilon x)g^2(v_n) \\
&+ \left(\frac{1}{q} - \frac{1}{\alpha 2^*}\right)\int_{\mathbb{R}^N} g^{\alpha 2^*}(v_n^+) \\
&\geqslant \frac{q - 2\alpha}{2q}\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(\varepsilon x)g^2(v_n)\right).
\end{aligned}
$$

By (2.2), $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Lemma 2.6. *There exists $r^* > 0$ such that $\|v\| \geqslant r^* > 0$ for all $v \in \mathcal{N}_\varepsilon$.*

Proof. Let $v \in \mathcal{N}_\varepsilon$. Without loss of generality, we may assume that $v \in \Sigma_1$. Using Lemma 2.1-(f) and (2.2), we get

$$\begin{aligned}0= & \int_{\mathbb{R}^N}\left(|\nabla v|^2+V(\varepsilon x)g(v)g'(v)v\right)-\int_{\mathbb{R}^N}|g(v^+)|^{\alpha 2^*-1}g'(v^+)v^+-\int_{\mathbb{R}^N}|g(v^+)|^{q-1}g'(v^+)v^+\\ \geqslant & \frac{1}{\alpha}\int_{\mathbb{R}^N}\left(|\nabla v|^2+V(\varepsilon x)g^2(v)\right)-\int_{\mathbb{R}^N}\left|g(v^+)\right|^{\alpha 2^*}-\int_{\mathbb{R}^N}\left|g(v^+)\right|^{q}\\ \geqslant & C_1\|v\|^2-\frac{\alpha^{\frac{q}{2\alpha}}}{q}\int_{\mathbb{R}^N}|v|^{\frac{q}{\alpha}}-\frac{\alpha^{\frac{2^*}{2}}}{2^*\alpha}\int_{\mathbb{R}^N}|v|^{2^*}\\ = & C_1\|v\|^2-C_2\|v\|^{\frac{q}{\alpha}}-C_3\|v\|^{2^*}.\end{aligned}$$

By $2<\frac{q}{\alpha}<2^*$, we have

$$\|v\|\geqslant r^*>0. \tag{2.5}$$

The proof is completed.

To prove J_ε satisfies the $(C)_c$ conditions, we introduce the following problem:

$$\begin{cases}-\Delta v+\mu g(v)g'(v)=g^{\alpha 2^*-1}(v)g'(v)+g^{q-1}(v)g'(v) & \text{in } \mathbb{R}^N,\\ v\in H^1(\mathbb{R}^N),\quad v(x)>0 & \text{in } \mathbb{R}^N,\end{cases} \tag{Q_μ}$$

where $\mu\in\mathbb{R}^+$. The weak solutions of (Q_μ) correspond to the critical points of the functional

$$\mathcal{E}_\mu(v)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2+\mu g^2(v)\right)-\frac{1}{\alpha 2^*}\int_{\mathbb{R}^N}|g(v^+)|^{\alpha 2^*}-\frac{1}{q}\int_{\mathbb{R}^N}|g(v^+)|^{q}.$$

Let

$$m_\mu=\inf_{v\in\mathcal{N}_\mu}\mathcal{E}_\mu(v),$$

where $\mathcal{N}_\mu$ is the Nehari manifold associated with problem (Q_μ). Clearly, m_μ and $\mathcal{N}_\mu$ possess properties similar to those of c_ε and $\mathcal{N}_\varepsilon$.

Lemma 2.7. *Consider whether one of the following cases occurs:*

(a) $\alpha\geqslant 2$, $\frac{2\alpha N-2N+4}{N-2}<q<2^*\alpha$, *and* $N\leqslant[2\alpha+2]$.

(b) $\alpha\geqslant 2, 2\alpha<q<2^*\alpha$ *and* $N>[2\alpha+2]$.

(c) $1<\alpha<2$, $\frac{4\alpha}{N-2}+2<q<2^*\alpha$,*and* $N\leqslant[4+\frac{2}{\alpha-1}]$.

(d) $1<\alpha<2, 2\alpha<q<2^*\alpha$,*and* $N>[4+\frac{2}{\alpha-1}]$.
Then, for any $\mu>0$, *there exists some* $v\in H^1(\mathbb{R}^N)\backslash\{0\}$ *such that*

$$\max_{t\geqslant 0}\mathcal{E}_\mu(tv)<\frac{1}{\alpha N}S^{\frac{N}{2}}.$$

In particular, $m_\mu<\frac{1}{\alpha N}S^{\frac{N}{2}}$.

Proof. The proof can be found in [19].

Lemma 2.8. *Let* $\{v_n\} \subset H^1(\mathbb{R}^N)$ *be a* $(C)_c$ *sequence for* J_ε *with* $c < \frac{1}{\alpha N} S^{\frac{N}{2}}$ *and* $v_n \rightharpoonup 0$ *in* $H^1(\mathbb{R}^N)$*. Then, one of the following conclusions holds.*
(a) $v_n \to 0$ *in* $H^1(\mathbb{R}^N)$.
(b) There exists $R, \delta > 0$ *and a sequence* $\{y_n\} \subset \mathbb{R}^N$ *such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} v_n^2 \geqslant \delta > 0.$$

Proof. Assume that (b) does not occur. Then, for any $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} v^2 = 0.$$

Note that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By Lemma 1.3 of [24], we have

$$v_n \to 0 \quad \text{in } L^r(\mathbb{R}^N) \quad \text{for } r \in [2, 2^*).$$

Since $2\alpha < q < 2^*\alpha$, by Lemma 2.1-(c), (g) and the interpolation inequality, we get

$$\int_{\mathbb{R}^N} |g(v_n)|^q = o_n(1). \tag{2.6}$$

Since $\left\{\frac{g(v_n)}{g'(v_n)}\right\}$ is bounded in $H^1(\mathbb{R}^N)$, it follows from (2.6) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1+\alpha^2|g(v_n)|^{2(\alpha-1)}}{1+\alpha|g(v_n)|^{2(\alpha-1)}}|\nabla v_n|^2 + \int_{\mathbb{R}^N} V(\varepsilon x)|g(v_n)|^2 \\
&= \int_{\mathbb{R}^N} |g(v_n)|^q + \int_{\mathbb{R}^N} |g(v_n)|^{2^*\alpha} + o_n(1) \\
&= \int_{\mathbb{R}^N} |g(v_n)|^{2^*\alpha} + o_n(1).
\end{aligned}$$

Up to a subsequence, we may assume that there exists $l \geqslant 0$ such that

$$\int_{\mathbb{R}^N} \frac{1+\alpha^2|g(v_n)|^{2(\alpha-1)}}{1+\alpha|g(v_n)|^{2(\alpha-1)}} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(\varepsilon x)|g(v_n)|^2 \to l, \quad \int_{\mathbb{R}^N} |g(v_n)|^{2^*\alpha} \to l.$$

Suppose that $l > 0$. Then,

$$\begin{aligned}
S &\leqslant \frac{\int_{\mathbb{R}^N} |\nabla g^\alpha(v_n^+)|^2}{\left(\int_{\mathbb{R}^N} |g^\alpha(v_n^+)|^{2^*}\right)^{2/2^*}} \\
&\leqslant \frac{\int_{\mathbb{R}^N} \frac{1+\alpha^2|g(v_n)|^{2(\alpha-1)}}{1+\alpha|g(v_n)|^{2(\alpha-1)}} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(\varepsilon x)|g(v_n)|^2}{\left(\int_{\mathbb{R}^N} |g(v_n^+)|^{\alpha 2^*}\right)^{2/2^*}} \\
&\to l^{1-2/2^*}
\end{aligned}$$

as $n \to \infty$. Thus, $l \geqslant S^{\frac{N}{2}}$. From (2.6), we can deduce that

$$\begin{aligned} c &= \lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(\varepsilon x)g^2(v_n)\right) - \frac{1}{2^*\alpha}\int_{\mathbb{R}^N}|g(v_n)|^{2^*\alpha}\right) \\ &\geqslant \frac{1}{2\alpha}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\frac{1+\alpha^2|g(v_n)|^{2(\alpha-1)}}{1+\alpha|g(v_n)|^{2(\alpha-1)}}|\nabla v_n|^2 + \int_{\mathbb{R}^N}V(\varepsilon x)g^2(v_n)\right) \\ &- \frac{1}{2^*\alpha}\lim_{n\to\infty}\int_{\mathbb{R}^N}|g(v_n)|^{2^*\alpha} \\ &= \frac{1}{\alpha N}l \\ &\geqslant \frac{1}{\alpha N}S^{\frac{N}{2}}. \end{aligned}$$

This contradicts with $c < \frac{1}{\alpha N}S^{\frac{N}{2}}$, and hence $l = 0$. By (2.2), $v_n \to 0$ in $H^1(\mathbb{R}^N)$.

Lemma 2.9. *Suppose that $\{v_n\}$ is a $(C)_c$ sequence for the functional J_ε with $c < \frac{1}{\alpha N}S^{\frac{N}{2}}$ and $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. If $\|v_n\| \nrightarrow 0$, then $c \geqslant m_{V_\infty}$.*

Proof. Suppose that $\{t_n\} \subset (0,\infty)$ such that $\{t_nv_n\} \subset \mathcal{N}_{V_\infty}$. First, we assert that $\limsup_{n\to\infty} t_n \leqslant 1$. We argue by contradiction that there exists $\delta > 0$ and a subsequence of $\{t_n\}$ (still denoted by $\{t_n\}$) such that

$$t_n \geqslant 1 + \delta \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 2.5, we can conclude that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Moreover, we may assume $v_n \geqslant 0$ for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned} o_n(1) = \langle J_\varepsilon'(v_n), v_n\rangle = &\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(\varepsilon x)g(v_n)g'(v_n)v_n\right) - \int_{\mathbb{R}^N}g^{\alpha 2^*-1}(v_n)g'(v_n)v_n \\ &- \int_{\mathbb{R}^N}g^{q-1}(v_n)g'(v_n)v_n. \end{aligned} \tag{2.7}$$

In view of $t_nv_n \in \mathcal{N}_{V_\infty}$, we have

$$\begin{aligned} \int_{\mathbb{R}^N}t_n^2|\nabla v_n|^2 + \int_{\mathbb{R}^N}V_\infty g(t_nv_n)g'(t_nv_n)t_nv_n = &\int_{\mathbb{R}^N}g^{\alpha 2^*-1}(t_nv_n)g'(t_nv_n)t_nv_n \\ &+ \int_{\mathbb{R}^N}g^{q-1}(t_nv_n)g'(t_nv_n)t_nv_n. \end{aligned} \tag{2.8}$$

By (2.7) and (2.8), we get

$$\begin{aligned} &\int_{\mathbb{R}^N}\frac{V_\infty g(t_nv_n)g'(t_nv_n)t_nv_n}{t_n^2} - \int_{\mathbb{R}^N}V(\varepsilon x)g(v_n)g'(v_n)v_n \\ &= \int_{\mathbb{R}^N}\left(\frac{|g(t_nv_n)|^{\alpha 2^*-1}g'(t_nv_n)t_nv_n}{t_n^2} - |g(v_n)|^{\alpha 2^*-1}g'(v_n)v_n\right) \\ &+ \int_{\mathbb{R}^N}\left(\frac{|g(t_nv_n)|^{q-1}g'(t_nv_n)t_nv_n}{t_n^2} - |g(v_n)|^{q-1}g'(v_n)v_n\right) + o_n(1), \end{aligned}$$

or

$$\int_{\mathbb{R}^N}[V_\infty - V(\varepsilon x)]g(v_n)g'(v_n)v_n + \int_{\mathbb{R}^N} V_\infty\left[\frac{g(t_nv_n)g'(t_nv_n)}{t_nv_n} - \frac{g(v_n)g'(v_n)}{v_n}\right]v_n^2$$
$$= \int_{\mathbb{R}^N}\left[\frac{|g(t_nv_n)|^{\alpha 2^*-1}g'(t_nv_n)}{t_nv_n} - \frac{|g(v_n)|^{\alpha 2^*-1}g'(v_n)}{v_n}\right]v_n^2$$
$$+ \int_{\mathbb{R}^N}\left[\frac{|g(t_nv_n)|^{q-1}g'(t_nv_n)}{t_nv_n} - \frac{|g(v_n)|^{q-1}g'(v_n)}{v_n}\right]v_n^2 + o_n(1).$$

By (V), for any $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that

$$V(\varepsilon x) \geqslant V_\infty - \zeta \qquad \text{for any } |\varepsilon x| \geqslant R. \tag{2.9}$$

By Lemma 2.1-(j) and $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, we conclude that

$$\int_{\mathbb{R}^N}\left[\frac{|g(t_nv_n)|^{q-1}g'(t_nv_n)}{t_nv_n} - \frac{|g(v_n)|^{q-1}g'(v_n)}{v_n}\right]v_n^2 \leqslant \zeta C + o_n(1). \tag{2.10}$$

If $\|v_n\|_\varepsilon \nrightarrow 0$, by Lemma 2.8, there exist $\{y_n\} \subset \mathbb{R}^N$ and $R^*, \delta > 0$ such that

$$\int_{B_{R^*}(y_n)} v_n^2 \geqslant \delta. \tag{2.11}$$

Define $\widetilde{v_n}(x) = v_n(x + y_n)$. Going to a subsequence if necessary, we may assume

$$\widetilde{v_n} \rightharpoonup \widetilde{v} \text{ in } H^1(\mathbb{R}^N),$$
$$\widetilde{v_n} \rightharpoonup \widetilde{v} \text{ in } L^s_{B_{R^*}(0)}(\mathbb{R}^N) \text{ for } s \in [1, 2^*),$$
$$\widetilde{v_n} \to \widetilde{v} \text{ a.e. } x \in \mathbb{R}^N.$$

Furthermore, by (2.11), there is a subset $\Omega \subset B_{R^*}(0)$ with positive measure such that $\widetilde{v} > 0$ a.e. in Ω. It follows from (2.10) and $t_n \geqslant 1 + \delta$ that

$$\int_{\mathbb{R}^N}\left[\frac{|g(1+\delta)\widetilde{v_n}|^{q-1}g'(1+\delta)\widetilde{v_n}}{(1+\delta)\widetilde{v_n}} - \frac{|g(\widetilde{v_n})|^{q-1}g'(\widetilde{v_n})}{\widetilde{v_n}}\right]\widetilde{v_n}^2 \leqslant \zeta C + o_n(1),$$

for any $\zeta > 0$. Applying Fatou's lemma, we have for any $\zeta > 0$,

$$0 < \int_{\Omega}\left[\frac{|g((1+\delta)\widetilde{v})|^{q-1}g'((1+\delta)\widetilde{v})}{(1+\delta)\widetilde{v}} - \frac{|g(\widetilde{v})|^{q-1}g'(\widetilde{v})}{\widetilde{v}}\right]\widetilde{v}^2 \leqslant \zeta C,$$

which is a contradiction. Next, we analyze the following two cases:
Case 1. $\limsup_{n\to\infty} t_n = 1$. In this case, we can find a subsequence, still denoted by $\{t_n\}$, such that $t_n \to 1$ as $n \to \infty$. Hence,

$$c + o_n(1) = J_\varepsilon(v_n) \geqslant m_{V_\infty} + J_\varepsilon(v_n) - \mathcal{E}_{V_\infty}(t_nv_n). \tag{2.12}$$

Based on condition (V), (2.9), and $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, we get

$$\begin{aligned} J_\varepsilon(v_n) - \mathcal{E}_{V_\infty}(t_nv_n) \geqslant{} & \frac{1-t_n^2}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2 + \frac{V_\infty}{2}\int_{\mathbb{R}^N}\left(g^2(v_n) - g^2(t_nv_n)\right) - \zeta C + o_n(1) \\ & + \frac{1}{q}\int_{\mathbb{R}^N}(|g(t_nv_n)|^q - |g(v_n)|^q) + \frac{1}{\alpha 2^*}\int_{\mathbb{R}^N}\left(|g(t_nv_n)|^{\alpha 2^*} - |g(v_n)|^{\alpha 2^*}\right). \end{aligned} \tag{2.13}$$

Using the mean value theorem, it is easy to see that

$$\int_{\mathbb{R}^N}\left(g^2(v_n)-g^2(t_nv_n)\right)=o_n(1),\quad \int_{\mathbb{R}^N}(|g(t_nv_n)|^q-|g(v_n)|^q)=o_n(1),$$
$$\int_{\mathbb{R}^N}\left(|g(t_nv_n)|^{\alpha 2^*}-|g(v_n)|^{\alpha 2^*}\right)=o_n(1). \tag{2.14}$$

From (2.12)–(2.14), we have

$$c\geqslant m_{V_\infty}-\zeta C+o_n(1).$$

Since ζ is arbitrary, we have $c\geqslant m_{V_\infty}$.

Case 2. $\limsup_{n\to\infty}t_n=t_0<1$. Without loss of generality, we suppose that $t_n<1$ for all $n\in\mathbb{N}$. In view of Lemma 2.1 and (2.9), we have

$$\begin{aligned}
m_{V_\infty}&\leqslant \mathcal{E}_{V_\infty}(t_nv_n)-\frac{1}{2}\langle\mathcal{E}'_{V_\infty}(t_nv_n),t_nv_n\rangle\\
&=\frac{1}{2}\int_{\mathbb{R}^N}V_\infty[g^2(t_nv_n)-g(t_nv_n)g'(t_nv_n)t_nv_n]\\
&\quad+\int_{\mathbb{R}^N}\left[\frac{1}{2}|g(t_nv_n)|^{q-1}g'(t_nv_n)t_nv_n-\frac{1}{q}|g(t_nv_n)|^q\right]\\
&\quad+\int_{\mathbb{R}^N}\left[\frac{1}{2}|g(t_nv_n)|^{\alpha 2^*-1}g'(t_nv_n)t_nv_n-\frac{1}{\alpha 2^*}|g(t_nv_n)|^{\alpha 2^*}\right]\\
&\leqslant\frac{1}{2}\int_{\{|\varepsilon x|>R\}}(V(\varepsilon x)+\zeta)[g^2(v_n)-g(v_n)g'(v_n)v_n]\\
&\quad+\frac{1}{2}\int_{\{|\varepsilon x|\leqslant R\}}V_\infty[g^2(v_n)-g(v_n)g'(v_n)v_n]+\int_{\mathbb{R}^N}\left[\frac{1}{2}|g(t_nv_n)|^{q-1}g'(t_nv_n)t_nv_n-\frac{1}{q}|g(t_nv_n)|^q\right]\\
&\quad+\int_{\mathbb{R}^N}\left[\frac{1}{2}|g(v_n)|^{\alpha 2^*-1}g'(v_n)v_n-\frac{1}{\alpha 2^*}|g(v_n)|^{\alpha 2^*}\right]\\
&\leqslant\frac{1}{2}\int_{\mathbb{R}^N}V(\varepsilon x)[g^2(v_n)-g(v_n)g'(v_n)v_n]+\int_{\mathbb{R}^N}\left[\frac{1}{2}|g(t_nv_n)|^{q-1}g'(t_nv_n)t_nv_n-\frac{1}{q}|g(t_nv_n)|^q\right]\\
&\quad+\int_{\mathbb{R}^N}\left[\frac{1}{2}|g(v_n)|^{\alpha 2^*-1}g'(v_n)v_n-\frac{1}{\alpha 2^*}|g(v_n)|^{\alpha 2^*}\right]+\zeta C+o_n(1)\\
&=J_\varepsilon(v_n)-\frac{1}{2}\langle J'_\varepsilon(v_n),v_n\rangle+\zeta C+o_n(1)\\
&\leqslant c+\zeta C+o_n(1).
\end{aligned}$$

Let $\zeta\to 0$ and $n\to\infty$. We infer that $c\geqslant m_{V_\infty}$.

Proposition 2.10. *Let $Q_{N,\beta}<q<(2\beta)2^*$ hold. Then, J_ε satisfies the $(C)_c$ conditions at any level $c<m_{V_\infty}$.*

Proof. Let $\{v_n\}\subset H^1(\mathbb{R}^N)$ be a $(C)_c$ sequence of J_ε, i.e., $J_\varepsilon(v_n)\to c$ and $J'_\varepsilon(v_n)\to 0$. By Lemma 2.5, we conclude that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, and there exists $v\in H^1(\mathbb{R}^N)$ such that

$$\begin{aligned}
&v_n\rightharpoonup v \text{ in } H^1(\mathbb{R}^N),\\
&v_n\rightharpoonup v \text{ in } L^s_{B_{R^*}(0)}(\mathbb{R}^N) \text{ for } s\in[1,2^*),\\
&v_n\to v \text{ a.e. } x\in\mathbb{R}^N.
\end{aligned}$$

Clearly, $J'_\varepsilon(v) = 0$. Let $w_n = v_n - v$. By Lemma 2.4 in [12], we obtain

$$J_\varepsilon(w_n) = J_\varepsilon(v_n) - J_\varepsilon(v) + o_n(1) = c - J_\varepsilon(v) + o_n(1) := d + o_n(1),$$

and

$$J'_\varepsilon(w_n) = J'_\varepsilon(v_n) - J'_\varepsilon(v) + o_n(1) = o_n(1).$$

By Lemma 2.1, we derive that

$$\begin{aligned}
J_\varepsilon(v) &= J_\varepsilon(v) - \frac{1}{2}\langle J'_\varepsilon(v), v\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)\left[g^2(v) - g(v)g'(v)v\right] + \int_{\mathbb{R}^N} \frac{1}{2}|g(v^+)|^{q-1}g'(v^+)v^+ - \int_{\mathbb{R}^N} \frac{1}{q}|g(v^+)|^q \\
&\quad + \int_{\mathbb{R}^N} \frac{1}{2}|g(v^+)|^{\alpha 2^*-1}g'(v^+)v^+ - \int_{\mathbb{R}^N} \frac{1}{\alpha 2^*}|g(v^+)|^{\alpha 2^*} \\
&\geqslant 0.
\end{aligned}$$

Consequently, $\{w_n\} \subset H^1(\mathbb{R}^N)$ is a $(C)_d$ sequence for J_ε with $d < m_{V_\infty} < \frac{1}{\alpha N}S^{\frac{N}{2}}$. By Lemma 2.9, $w_n \to 0$ in $H^1(\mathbb{R}^N)$.

2.3. The proof of existence of a positive ground state

Theorem 2.11. *Suppose that conditions (V) and $Q_{N,\beta} < q < (2\beta)2^*$ hold. Then, there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, problem (Q^*_ε) has a ground state solution.*

Proof. According to Lemma 2.2, the functional J_ε possesses the mountain pass geometry. Then, by utilizing a version of the mountain pass theorem without (PS) condition (e.g., [23]), there exists $\{v_n\}$ in $H^1(\mathbb{R}^N)$ such that

$$J_\varepsilon(v_n) \to c_\varepsilon \quad \text{and} \quad J'_\varepsilon(v_n) \to 0.$$

Without loss of generality, we assume that $V_0 = V(0) = \inf_{x\in\mathbb{R}^N} V(x)$. Taking $\mu \in \mathbb{R}$ such that $V_0 < \mu < V_\infty$, by the result in [19], one can check that $m_{V_0} < m_\mu < m_{V_\infty}$. Furthermore, there exists a non-negative function $w \in H^1(\mathbb{R}^N)$ with compact support such that $\mathcal{E}_\mu(w) = \max_{t\geqslant 0}\mathcal{E}_\mu(tw)$ and $\mathcal{E}_\mu(w) < m_{V_\infty}$. From condition (V), there exists some $\varepsilon^* > 0$ such that

$$V(\varepsilon x) \leqslant \mu \quad \text{for any } \varepsilon \in (0, \varepsilon^*) \quad \text{and} \quad x \in \operatorname{supp} w.$$

Then,

$$J_\varepsilon(tw) \leqslant \mathcal{E}_\mu(tw) \quad \text{for any } \varepsilon \in (0, \varepsilon^*) \quad \text{and } t \geqslant 0$$

and hence

$$\max_{t\geqslant 0} J_\varepsilon(tw) \leqslant \max_{t\geqslant 0}\mathcal{E}_\mu(tw) = \mathcal{E}_\mu(w) < m_{V_\infty} \quad \text{for any } \varepsilon \in (0, \varepsilon^*).$$

Then, $c < m_{V_\infty}$ for $\varepsilon \in (0, \varepsilon^*)$. By Proposition 2.10, there exists $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $J'_\varepsilon(v) = 0$ and $J_\varepsilon(v) = c_\varepsilon$. Using standard arguments similar to those in [25,26], we conclude that $v \in L^\infty(\mathbb{R}^N)$ and $v \in C^{1,\gamma}_{loc}(\mathbb{R}^N)$ with $0 < \gamma < 1$. By Lemma 2.1, we get

$$\begin{aligned}
\left\|g(v^-)\right\|^2 &\leqslant \alpha\int_{\mathbb{R}^N}\left(\left|\nabla v^-\right|^2 + g(v)g'(v)v^-\right) \\
&\leqslant C\left(\int_{\mathbb{R}^N} |g(v^+)|^{q-1}g'(v^+)v^- + \int_{\mathbb{R}^N} |g(v^+)|^{2^*\alpha-1}g'(v^+)v^-\right) \\
&= 0,
\end{aligned}$$

which implies that $g(v^-) = 0$, and hence $v \geqslant 0$. By applying the maximum principle, $v > 0$.

3. Multiplicity and concentration of solutions to (Q_ε^*)

In this section, we focus on the multiplicity of solutions and analyze the behavior of their maximum points.

3.1. Preliminary results

Lemma 3.1. *The critical points of the functional J_ε on $\mathcal{N}_\varepsilon$ are critical points of J_ε in $H^1(\mathbb{R}^N)$.*

Proof. Assuming $v \in \mathcal{N}_\varepsilon$ is a critical point of J_ε on $\mathcal{N}_\varepsilon$, then $\upsilon \neq 0$. By Theorem 8.5 in [23], there exists $\lambda \in \mathbb{R}$ such that

$$J'_\varepsilon(v) = \lambda E'_\varepsilon(v),$$

where $E_\varepsilon(v) = \langle J'_\varepsilon(v), v\rangle$. Notice that

$$\begin{aligned}
\langle E'_\varepsilon(v), v\rangle &= \int_{\mathbb{R}^N} \Big(2|\nabla v_n|^2 + V(\varepsilon x)|g'(v)|^2 v^2 + V(x)g(v)g''(v)v^2 + V(\varepsilon x)g(v)g'(v)v\Big) \\
&- (q-1)\int_{\mathbb{R}^N} |g(v)|^{q-2}|g'(v)|^2 v_n^2 - \int_{\mathbb{R}^N} |g(v)|^{q-2} g(v_n)g''(v)v^2 \\
&- \int_{\mathbb{R}^N} |g(v)|^{q-2}g(v)g'(v)v - (2^*\alpha - 1)\int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2}|g'(v)|^2 v^2 \\
&- \int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2} g(v)g'(v)v - \int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2}g(v)g''(v)v^2 \\
&= \int_{\mathbb{R}^N} V(\varepsilon x)\Big(|g'(v)|^2 v^2 + g(v)g''(v)v^2 - g(v)g'(v)v\Big) \\
&- (q-1)\int_{\mathbb{R}^N} |g(v)|^{q-2}|g'(v)|^2 v^2 - \int_{\mathbb{R}^N} |g(v)|^{q-2}g(v)g''(v)v^2 \\
&+ \int_{\mathbb{R}^N} |g(v)|^{q-2}g(v)g'(v)v - (2^*\alpha - 1)\int_{\mathbb{R}^N} |g(v)|^{2^*\alpha}|g'(v)|^2 v^2 \\
&+ \int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2}g(v)g'(v)v - \int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2}g(v)g''(v)v^2 \\
&\leqslant \int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2}g(v)g'(v)v - \int_{\mathbb{R}^N} |g(v_n)|^{2^*\alpha-2}g(v)g''(v)v^2 \\
&- (2^*\alpha - 1)\int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2}|g'(v)|^2 v^2 \\
&\leqslant \alpha \int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2}|g'(v)|^2 v^2 + (\alpha - 1)\int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2}|g'(v_n)|^2 v^2 \\
&- (2^*\alpha - 1)\int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2}|g'(v)|^2 v^2 \\
&= -\frac{4\alpha}{N-2}\int_{\mathbb{R}^N} |g(v)|^{2^*\alpha-2}|g'(v)|^2 v^2 \\
&\leqslant -\frac{4}{(N-2)\alpha}\int_{\mathbb{R}^N} |g(v)|^{2^*\alpha}.
\end{aligned}$$

Thus, $\lambda = 0$ and $J'_\varepsilon(v) = 0$.

Take $\delta > 0$ fixed, and let w be a ground state solution of problem (Q_{V_0}). Let η be defined on $[0, \infty)$ such that $\eta(s) = 1$ if $0 \leqslant s \leqslant \delta/2$ and $\eta(s) = 0$ if $s \geqslant \delta$. For any $y \in M$, we define

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|)w\left(\frac{\varepsilon x - y}{\varepsilon}\right).$$

Let $t_\varepsilon > 0$ be such that

$$\max_{t\geqslant 0} J_\varepsilon(t\Psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}).$$

Define $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ by

$$\Phi_\varepsilon(y) = t_\varepsilon \Psi_{\varepsilon,y}.$$

Clearly, the function $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Lemma 3.2.

$$\lim_{\varepsilon\to 0} J_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0} \quad \textit{uniformly in } y \in M.$$

Proof. We argue by contradiction that there exists $\delta_0 > 0$, $\{y_n\} \subset M$, and $\varepsilon_n \to 0$ such that

$$|J_\varepsilon(\Phi_{\varepsilon_n}(y_n)) - m_{V_0}| \geqslant \delta_0. \tag{3.1}$$

Next, we denote t_n, Ψ_n, and Φ_n by t_{ε_n}, Ψ_{ε_n,y_n}, and Φ_{ε_n} for notational simplicity. We claim that $\lim_{n\to\infty} t_n = 1$. From the definition of t_n and Lemma 2.6, we get

$$\begin{aligned} \frac{r^*}{\alpha} &\leqslant \int_{\mathbb{R}^N} \left(|\nabla(t_n\Psi_n)|^2 + V(\varepsilon_n x)g(t_n\Psi_n)g'(t_n\Psi_n)t_n\Psi_n\right) \\ &= \int_{\mathbb{R}^N} |g(t_n\Psi_n)|^{q-1}g'(t_n\Psi_n)t_n\Psi_n + \int_{\mathbb{R}^N} |g(t_n\Psi_n)|^{\alpha 2^*-1}g'(t_n\Psi_n)t_n\Psi_n. \end{aligned} \tag{3.2}$$

It is clear that $t_n \geqslant t_0 > 0$ for some $t_0 > 0$. If $t_n \to \infty$, by Lemma 2.1-(e), (f) and the boundedness of $\{\Psi_n\}$, we have

$$\begin{aligned} C &> \int_{\mathbb{R}^N} \left(|\nabla\Psi_n|^2 + V(\varepsilon_n x)\frac{g(t_n\Psi_n)g'(t_n\Psi_n)t_n\Psi_n}{t_n^2}\right) \\ &= \int_{\mathbb{R}^N} \frac{|g(t_n\Psi_n)|^{q-1}\,g'(t_n\Psi_n)t_n\Psi_n}{t_n^2} + \int_{\mathbb{R}^N} \frac{|g(t_n\Psi_n)|^{\alpha 2^*-1}\,g'(t_n\Psi_n)t_n\Psi_n}{t_n^2} \\ &> \frac{1}{\alpha}\int_{\mathbb{R}^N} \frac{|g(t_n\eta(\varepsilon_n z)w(z))|^{\alpha 2^*}}{t_n^2} \\ &\geqslant \frac{1}{\alpha}\int_{B_{\delta/2\varepsilon_n}(0)} \frac{|g(t_n w)|^{\alpha 2^*}}{t_n^2} \\ &\to \alpha^{\frac{2^*-2}{2}} t_n^{2^*-2}\int_{B_{\delta/2\varepsilon_n}(0)} w^{2^*} \\ &\to \infty, \end{aligned}$$

a contradiction. Therefore, we may assume that $t_n \to T > 0$. Next, we claim that $T = 1$. By applying Lebesgue's theorem, one can check that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla\Psi_n|^2 + V(\varepsilon_n x)g(\Psi_n)g'(\Psi_n)\Psi_n\right) = \int_{\mathbb{R}^N}\left(|\nabla w|^2 + V_0 g(w)g'(w)w\right);$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|g(\Psi_n)|^{\alpha 2^*-1}g'(\Psi_n)\Psi_n = \int_{\mathbb{R}^N}|g(w)|^{\alpha 2^*-1}g'(w)w;$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|g(\Psi_n)|^{q-1}g'(\Psi_n)\Psi_n = \int_{\mathbb{R}^N}|g(w)|^{q-1}g'(w)w.$$

From (3.2), we have

$$\int_{\mathbb{R}^N}\left(|\nabla(Tw)|^2 + V_0 g(Tw)g'(Tw)Tw\right) = \int_{\mathbb{R}^N}|g(Tw)|^{q-1}g'(Tw)Tw + \int_{\mathbb{R}^N}|g(Tw)|^{\alpha 2^*-1}g'(Tw)Tw.$$

Consequently,

$$\begin{aligned}
&-\int_{\mathbb{R}^N}V_0\frac{g(w)g'(w)}{w}w^2 + \int_{\mathbb{R}^N}\frac{|g(w)|^{q-1}g'(w)}{w}w^2 + \int_{\mathbb{R}^N}\frac{|g(w)|^{\alpha 2^*-1}g'(w)}{Tw}w^2\\
&= \int_{\mathbb{R}^N}|\nabla w|^2\\
&= -\int_{\mathbb{R}^N}V_0\frac{g(Tw)g'(Tw)Tw}{T^2} + \int_{\mathbb{R}^N}\frac{|g(Tw)|^{q-1}\,g'(Tw)Tw}{T^2} + \int_{\mathbb{R}^N}\frac{|g(Tw)|^{\alpha 2^*-1}\,g'(Tw)Tw}{T^2}\\
&= -\int_{\mathbb{R}^N}V_0\frac{g(Tw)g'(Tw)}{Tw}w^2 + \int_{\mathbb{R}^N}\frac{|g(Tw)|^{q-1}g'(Tw)}{Tw}w^2 + \int_{\mathbb{R}^N}\frac{|g(Tw)|^{\alpha 2^*-1}g'(Tw)}{Tw}w^2.
\end{aligned}\tag{3.3}$$

From Lemma 2.1, the right-hand side of the last equality is increasing with respect to $T > 0$, and hence $T = 1$. This leads to

$$\begin{aligned}
J_{\varepsilon_n}(\Phi_n(y_n)) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla(t_n\eta(|\varepsilon_n z|)w)|^2 + V(\varepsilon_n z + y_n)g^2(t_n\eta(|\varepsilon_n z|)w)\right)\\
&\quad - \frac{1}{q}\int_{\mathbb{R}^N}|g(t_n\eta(|\varepsilon_n z|)w)|^q - \frac{1}{\alpha 2^*}\int_{\mathbb{R}^N}|g(t_n\eta(|\varepsilon_n z|)w)|^{\alpha 2^*}\\
&\to \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla w|^2 + V_0 g^2(w)\right) - \frac{1}{q}\int_{\mathbb{R}^N}|g(w)|^q - \frac{1}{\alpha 2^*}\int_{\mathbb{R}^N}|g(w)|^{\alpha 2^*}\\
&= m_{V_0},
\end{aligned}$$

which contradicts (3.1).

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Define $\chi : \mathbb{R}^N \to \mathbb{R}^N$ as $\chi(x) = x$ for $|x| \leqslant \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \geqslant \rho$. Finally, we consider the map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(v) = \frac{\int_{\mathbb{R}^N}\chi(\varepsilon x)v^2}{\int_{\mathbb{R}^N}v^2}.\tag{3.4}$$

By [27], we have

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \text{uniformly for } y \in M.$$

Lemma 3.3. *Let $\{v_n\} \subset \mathcal{N}_\mu$ be a sequence satisfying $\mathcal{E}_\mu(v_n) \to m_\mu$. Then, one of (a) and (b) hold:*
(a) $\{v_n\}$ has a strongly convergent subsequence in $H^1(\mathbb{R}^N)$.
(b) there exists $\{\widetilde{y_n}\} \subset \mathbb{R}^N$ such that $\widetilde{v_n}(x) = v_n(x + \widetilde{y_n})$ converges strongly in $H^1(\mathbb{R}^N)$.

Proof. By Ekeland's variational principle (Theorem 8.5 in [23]), we may assume that $\{v_n\}$ is a $(C)_{m_\mu}$ sequence for $\mathcal{E}_\mu$. It is easy to see that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Up to a subsequence, we have

$$\begin{aligned} &v_n \rightharpoonup v \text{ in } H^1(\mathbb{R}^N),\\ &v_n \rightharpoonup v \text{ in } L^p_{loc}(\mathbb{R}^N) \text{ for } p \in [1, 2^*),\\ &v_n \to v \text{ a.e. } x \in \mathbb{R}^N. \end{aligned}$$

By a standard argument, we have $\mathcal{E}'_\mu(v) = 0$. To conclude our proof, we analyze two cases.
Case 1. Consider $v \neq 0$. We assert that

$$\int_{\mathbb{R}^N} |\nabla v|^2 = \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2. \tag{3.5}$$

Otherwise, it follows from Fatou's lemma that

$$\begin{aligned} m_\mu &\leqslant \mathcal{E}_\mu(v) = \mathcal{E}_\mu(v) - \frac{\alpha}{q}\langle \mathcal{E}'_\mu(v), v\rangle\\ &= \left(\frac{1}{2} - \frac{\alpha}{q}\right)\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N}\left(\frac{1}{2}\mu g^2(v) - \frac{\alpha}{q}\mu g(v)g'(v)v\right)\\ &+ \int_{\mathbb{R}^N}\left(\frac{\alpha}{q}|g(v^+)|^{q-1}g'(v^+)v^+ - \frac{1}{q}|g(v^+)|^q\right)\\ &+ \int_{\mathbb{R}^N}\left(\frac{\alpha}{q}|g(v^+)|^{\alpha 2^*-1}g'(v^+)v^+ - \frac{1}{\alpha 2^*}|g(v^+)|^{\alpha 2^*}\right)\\ &< \liminf_{n\to\infty}\left\{\left(\frac{1}{2} - \frac{\alpha}{q}\right)\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N}\left(\frac{1}{2}\mu g^2(v_n) - \frac{\alpha}{q}\mu g(v_n)g'(v_n)v_n\right)\right.\\ &+ \int_{\mathbb{R}^N}\left(\frac{\alpha}{q}g^{q-1}(v_n)g'(v_n)v_n - \frac{1}{q}g^q(v_n)\right)\\ &\left.+ \int_{\mathbb{R}^N}\left(\frac{\alpha}{q}g^{\alpha 2^*-1}(v_n)g'(v_n)v_n - \frac{1}{\alpha 2^*}g^{\alpha 2^*}(v_n)\right)\right\}\\ &= \liminf_{n\to\infty}\left(\mathcal{E}_\mu(v_n) - \frac{\alpha}{q}\langle \mathcal{E}'_\mu(v_n), v_n\rangle\right)\\ &= m_\mu, \end{aligned} \tag{3.6}$$

a contradiction. Hence,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 = \int_{\mathbb{R}^N} |\nabla v|^2. \tag{3.7}$$

From the proof of (3.6), we also have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\left(\frac{1}{2}\mu g^2(v_n) - \frac{\alpha}{q}\mu g(v_n)g'(v_n)v_n\right) = \int_{\mathbb{R}^N}\left(\frac{1}{2}\mu g^2(v) - \frac{\alpha}{q}\mu g(v)g'(v)v\right). \tag{3.8}$$

As a result, there is a function $h \in L^1(\mathbb{R}^N)$ satisfying

$$\left(\frac{q-2\alpha}{2q}\right)\mu g\,(v_n)^2 \leqslant \frac{1}{2}\mu g^2(v_n) - \frac{\alpha}{q}\mu g\,(v_n)\,g'(v_n)v_n \leqslant h \quad \text{a.e. in } \mathbb{R}^N.$$

By Lebesgue's theorem, we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\mu g^2(v_n) = \int_{\mathbb{R}^N}\mu g^2(v).$$

By (3.7) and (3.8), $v_n \to v$ in $H^1(\mathbb{R}^N)$.

Case 2. $v \equiv 0$. We claim that there exist $R, r > 0$ and $\widetilde{y_n} \in \mathbb{R}^N$ such that

$$\liminf_{n\to\infty}\int_{B_R(\widetilde{y_n})} v_n^2 \geqslant r. \tag{3.9}$$

If this claim is not true, up to a subsequence, we obtain

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)} v_n^2 = 0.$$

As the proof of Lemma 2.8, we deduce that $\int_{\mathbb{R}^N}|\nabla v_n|^2+\mu g(v_n)^2 \to 0$ as $n\to\infty$. Then, by (2.2), we have $v_n \to 0$ in $H^1(\mathbb{R}^N)$. Consequently, $\mathcal{E}_\mu(v_n) \to 0$ as $n\to\infty$, which contradicts with $\mathcal{E}_\mu(v_n) \to m_\mu > 0$. Let $w_n(x) = v_n(x+\widetilde{y_n})$. Then, $\{w_n\}$ is also a $(C)_{m_\mu}$ sequence of $\mathcal{E}_\mu$. Moreover, by (3.9), there exists $w \in H^1(\mathbb{R}^N)\setminus\{0\}$ such that $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$. Consequently, the proof is completed by employing similar arguments as for case 1.

Lemma 3.4. *Let $\varepsilon_n \to 0$ and $\{v_n\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(v_n) \to m_{V_0}$. Then, there exists a sequence $\{\widetilde{y_n}\} \subset \mathbb{R}^N$ such that $v_n(x) = v_n(x+\widetilde{y_n})$ has a convergent subsequence in $H^1(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n := \varepsilon_n\widetilde{y_n} \to y \in M$.*

Proof. Let $t_n > 0$ be chosen such that $w_n := t_nv_n \in \mathcal{N}_{V_0}$. Since $v_n \in \mathcal{N}_{\varepsilon_n}$, we can infer that

$$\begin{aligned}
m_{V_0} &\leqslant \mathcal{E}_{V_0}(w_n)\\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla w_n|^2 + V(\varepsilon_n(x))g^2(w_n)\right] - \frac{1}{\alpha 2^*}\int_{\mathbb{R}^N}|g(w_n^+)|^{\alpha 2^*} - \frac{1}{q}\int_{\mathbb{R}^N}|g(w_n^+)|^q\\
&= \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla(t_nv_n)|^2 + V(\varepsilon_n(x+y_n))g^2(t_nv_n)\right] - \frac{1}{\alpha 2^*}\int_{\mathbb{R}^N}|g(t_n{v_n}^+)|^{\alpha 2^*} - \frac{1}{q}\int_{\mathbb{R}^N}|g(t_n{v_n}^+)|^q\\
&= J_{\varepsilon_n}(t_nv_n)\\
&\leqslant J_{\varepsilon_n}(v_n)\\
&= m_{V_0} + o_n(1).
\end{aligned}$$

Therefore, $\lim_{n\to\infty}\mathcal{E}_{V_0}(w_n) = m_{V_0}$. Next, we claim that, up to a subsequence, $t_n \to t^* > 0$. Indeed, by Lemma 2.6, there exists some $\delta > 0$ such that $0 < \delta \leqslant \|v_n\|$. Therefore,

$$0 \leqslant t_n\delta \leqslant \|t_nv_n\| = \|w_n\| \leqslant C\|w_n\| \leqslant C.$$

Thus, we may assume that $t_n \to t^* \geqslant 0$. If $t^* = 0$, then $\mathcal{E}_{V_0}(w_n) \to 0$, which contradicts the fact that $m_{V_0} > 0$. By Lemma 3.3, there exist $\{\widetilde{y_n}\} \subset \mathbb{R}^N$ and $w \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that $w_n(x+\widetilde{y_n}) \to w$ in $H^1(\mathbb{R}^N)$. Since $t_n \to t^* > 0$, $v_n(x+\widetilde{y_n}) \to \frac{1}{t^*}w$ in $H^1(\mathbb{R}^N)$. If $\{y_n\}$ is not bounded, then

$$\begin{aligned}
m_{V_0} &= \mathcal{E}_{V_0}(w)\\
&< \mathcal{E}_{V_\infty}(w)\\
&\leqslant \varliminf_{n\to\infty}\Big(\frac{1}{2}\int_{\mathbb{R}^N}\Big[|\nabla w_n(x+\widetilde{y_n})|^2 + V(\varepsilon_n(x+\widetilde{y_n}))g^2(w_n(x+\widetilde{y_n}))\Big]\\
&\quad - \frac{1}{\alpha 2^*}\int_{\mathbb{R}^N}|g(w_n^+(x+\widetilde{y_n}))|^{\alpha 2^*} - \frac{1}{q}\int_{\mathbb{R}^N}|g(w_n^+(x+\widetilde{y_n}))|^q\Big)\\
&= \varliminf_{n\to\infty}\Big(\frac{1}{2}\int_{\mathbb{R}^N}\Big[|\nabla(t_n v_n(x+\widetilde{y_n}))|^2 + V(\varepsilon_n(x+y_n))g^2(t_n v_n(x+\widetilde{y_n}))\Big]\\
&\quad - \frac{1}{\alpha 2^*}\int_{\mathbb{R}^N}|g(t_n v_n(x+\widetilde{y_n})^+)|^{\alpha 2^*} - \frac{1}{q}\int_{\mathbb{R}^N}|g(t_n v_n(x+\widetilde{y_n})^+)|^q\Big)\\
&= J_{\varepsilon_n}(t_n v_n)\\
&\leqslant J_{\varepsilon_n}(v_n)\\
&= m_{V_0} + o_n(1),
\end{aligned} \tag{3.10}$$

a contradiction. As a result, $\{y_n\}$ is bounded and up to a subsequence, $y_n \to y$ in $\mathbb{R}^N$. If $y \notin M$, then $V(y) > V_0$, and we can obtain a contradiction through a similar argument to (3.10). So, $y \in M$, and the proof is completed.

Let $h(\varepsilon) = |J_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}|$. By Lemma 3.2, $h(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. Let

$$\widetilde{\mathcal{N}_\varepsilon} = \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leqslant m_{V_0} + h(\varepsilon)\}.$$

For a fixed $y \in M$, $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}_\varepsilon}$ and $\widetilde{\mathcal{N}_\varepsilon} \neq \emptyset$ for any $\varepsilon > 0$.

Lemma 3.5. *For any $\delta > 0$, there holds that*

$$\lim_{\varepsilon\to 0}\sup_{v\in\widetilde{\mathcal{N}_\varepsilon}} \operatorname{dist}(\beta_\varepsilon(v), M_\delta) = 0.$$

Proof. Let $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$ and $v_n \in \widetilde{\mathcal{N}_{\varepsilon_n}}$ such that

$$\operatorname{dist}(\beta_{\varepsilon_n}(v_n), M_\delta) = \sup_{v\in\widetilde{\mathcal{N}_{\varepsilon_n}}} \operatorname{dist}(\beta_{\varepsilon_n}(v), M_\delta) + o_n(1).$$

Since $\{v_n\} \subset \widetilde{\mathcal{N}_{\varepsilon_n}} \subset \mathcal{N}_{\varepsilon_n}$,

$$m_{V_0} \leqslant c_{\varepsilon_n} \leqslant J_{\varepsilon_n}(v_n) \leqslant m_{V_0} + h(\varepsilon_n),$$

which implies that $J_{\varepsilon_n}(v_n) \to m_{v_0}$. By applying Proposition 3.4, there exists a sequence $\{\widetilde{y_n}\} \subset \mathbb{R}^N$ such that $v_n(x) = v_n(x+\widetilde{y_n})$ has a convergent subsequence in $H^1(\mathbb{R}^N)$, and $y_n = \varepsilon_n\widetilde{y_n} \subset M_\delta$ for n large enough. Note that

$$\beta_{\varepsilon_n}(v_n) = y_n + \frac{\int_{\mathbb{R}^N}(\chi(\varepsilon_n z + y_n) - y_n)\widetilde{v_n}^2(z)}{\int_{\mathbb{R}^N}\widetilde{v_n}^2(z)}$$

where $\widetilde{v_n}(x) = v_n(x + \widetilde{y_n})$. By the Lebesgue dominated convergence theorem, one can check that $\beta_{\varepsilon_n}(v_n) = y_n + o_n(1)$, and hence

$$\operatorname{dist}(\beta_{c\varepsilon_n}(v_n), M_\delta) \leqslant |\beta_{\varepsilon_n}(v_n) - y_n| = o_n(1).$$

The proof is completed.

3.2. Multiplicity and concentration of solutions to (Q_ε^*)

Theorem 3.6. *Suppose that conditions (V) hold. Then, for any $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that for any $\varepsilon \in (0, \varepsilon(\delta))$, problem (Q_ε^*) has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions. Moreover, if v_ε is one of these positive solutions and $z_\varepsilon \in \mathbb{R}^N$ its global maximum, then*

$$\lim_{\varepsilon\to 0} V(\varepsilon z_\varepsilon) = V_0.$$

Proof. For any $\delta > 0$, by Lemmas 3.2 and 3.5, there exists $\varepsilon(\delta) > 0$ such that for any $\varepsilon \in (0, \varepsilon(\delta))$, the map $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the inclusion map $Id : M \to M_\delta$. By Lemmas 3.2 and 3.3 in [25], we have

$$\operatorname{cat}_{\widetilde{\mathcal{N}_\varepsilon}}(\widetilde{\mathcal{N}_\varepsilon}) \geqslant \operatorname{cat}_{M_\delta}(M).$$

Due to the $(C)_c$ condition is satisfied for J_ε if $c \in (m_{V_0}, m_{V_0} + h(\varepsilon))$. In view of the Ljusternik-Schnirelmann theory in [23], J_ε admits at least $\operatorname{cat}_{M_\delta}(M)$ critical points on $\mathcal{N}_\varepsilon$. By Lemma 3.1, J_ε has at least $\operatorname{cat}_{M_\delta}(M)$ positive critical points in $H^1(\mathbb{R}^N)$.

Let u_{ε_n} be a positive solution of $(Q_{\varepsilon_n}^*)$. Then, $v_n(x) = g^{-1}(u_{\varepsilon_n}(x + \widetilde{y_n}))$ is a solution of the problem

$$\begin{cases} -\Delta v_n + V_n(x) g(v_n) g'(v_n) = g^{q-1}(v_n) g'(v_n) + g^{\alpha 2^*-1}(v_n) g'(v_n) & \text{in } \mathbb{R}^N, \\ v_n \in H^1(\mathbb{R}^N), v_n > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n \widetilde{y_n})$ and $\{\widetilde{y_n}\} \subset \mathbb{R}^N$ is obtain by Lemma 3.4. Passing to a subsequence if necessary, we may assume that $\widetilde{v_n} \to v$ in $H^1(\mathbb{R}^N)$ and $y_n \to y$ in M with $y_n = \varepsilon_n \widetilde{y_n}$. We assert that there exists $\delta > 0$ such that $\|v_n\|_\infty \geqslant \delta$, for all $n \in \mathbb{N}$. We argue by contradiction that $\|v_n\|_\infty \to 0$. It follows the interpolation inequality and Lemma 2.1-(f) that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V_0 v_n^2 \right) &\leqslant \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V_n g(v_n) g'(v_n) v_n \right) \\ &= \int_{\mathbb{R}^N} g^{q-1}(v_n) g'(v_n) v_n + \int_{\mathbb{R}^N} g^{\alpha 2^*-1}(v_n) g'(v_n) v_n \\ &\leqslant \int_{\mathbb{R}^N} g^q(v_n) + \int_{\mathbb{R}^N} g^{\alpha 2^*}(v_n) \\ &\leqslant \frac{V_0}{4} \int_{\mathbb{R}^N} |g(v_n)|^2 + C \int_{\mathbb{R}^N} |g(v_n)|^{\alpha 2^*} \\ &\leqslant \frac{V_0}{4} \int_{\mathbb{R}^N} v_n^2 + C \|v_n\|_{L^\infty}^{2^*-2} \int_{\mathbb{R}^N} v_n^2, \end{aligned}$$

which implies that $v_n \to 0$ in $H^1(\mathbb{R}^N)$, a contradiction. By an iterative technique in Lemma 4.6 of [12], $\|v_n\|_\infty \leqslant C$ for all $n \in \mathbb{N}$ and

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \text{uniformly for all } n.$$

Let p_n and z_{ε_n} be the global maximums of v_n and $u_{\varepsilon n}$, respectively. Then,

$$\varepsilon_n z_{\varepsilon_n} = \varepsilon_n p_n + \varepsilon_n \widetilde{y_n} = \varepsilon_n p_n + y_n.$$

Since $p_n \in B_R(0)$ for some $R > 0$, $\varepsilon_n z_{\varepsilon_n} \to y$, and hence

$$\lim_{n\to\infty} V(\varepsilon_n z_{\varepsilon_n}) = V_0.$$

3.3. Proof of Theorem 1.1

Suppose that u_ε is a positive solution of (Q_ε^*). Then, $w_\varepsilon(x) = u_\varepsilon(\frac{x}{\varepsilon})$ is a positive solution of (Q_ε). Let η_ε and z_ε be the maximum points of w_ε and u_ε, respectively. Hence, $\eta_\varepsilon = \varepsilon z_\varepsilon$, and hence

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = \lim_{n\to\infty} V(\varepsilon_n z_{\varepsilon_n}) = V_0.$$

Consequently, Theorem 1.1 can be derived from Theorems 2.11 and 3.6.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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