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https://www.aimspress.com/journal/era

ERA, 33(11): 7085–7102.
DOI: 10.3934/era.2025313
Received: 21 August 2025
Revised: 12 November 2025
Accepted: 17 November 2025
Published: 21 November 2025

Research article

On multiple-cost optimization and extended controlled vector inequalities

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Abstract: In this study, we established several relations between generalized (weak) vector controlled inequalities of Minty and Stampacchia type and the associated multi-cost models. To this end, we introduced the updated concepts of preconvexity and (strictly) strong convexity for functionals governed by controlled simple integrals and a mean-value-type result. Also, we introduced the corresponding multiobjective extremization models. The theoretical notions and the main results were justified by suitable numerical examples that were non-trivial.

Keywords: multiple-cost optimization; control inequalities; convexity; efficient point; preconvexity

1. Introduction

The most important part of the study on product and process design in the aviation, financial, and gas industries is represented by the vector optimization problem. This field represents a subfield of mathematical optimization in which vector optimization problems are evaluated under certain constraints. For an extensive development in this mathematical field, Hanson [1] conceived the relationship between mathematical and classical programming with variational calculus. In this way, problems involving variational and optimal control have arisen. Important applications associated with variational problems have been well highlighted by Kim [2].

Variational inequalities have been shown to be applicable to many practical models and in other areas including mechanics, physics, control, optimization, transport, etc. Giannessi [3, 4] presented a variational inequality of vector type and formulated numerous relationships between solutions of certain extremization problems and solutions of some given variational inequalities of vector type. This increased interest in vector problems, both theoretic and applied, is due to various mathematical problems that require multiple forms of variational models, such as the Stampacchia or Minty variational inequalities. As far as optimization is concerned, it is effectively used to determine the relationship of variational inequalities of vector type to multiple-objective optimization problems. Oveisiha and Zafarani [5] are some of the authors who took part in the development of this direction. On the same research side, we mention Yu and Yao [6] and Lee [7]. Yang et al. [8] proved that for a vector optimization problem we can identify two different types of solutions, called ideal (or absolute) efficient points and weakly efficient points. They also showed that the Minty variational principle cannot be extended to quasi-convex functions for the vector case. Using the same ideas as their predecessors, Santos et al. [9] started from the case of weakly efficient solutions, under the weak hypothesis of compactness, and demonstrated a result on the existence of weakly efficient solutions associated with the considered nonlinear invex vector optimization problem. For other connected ideas on this topic, we mention Ansari and Lee [10], Al-Homidan and Ansari [11], Arana-Jiménez et al. [12], Miholca [13], and Jayswal et al. [14]. To support the connection between the two classes of solutions, the invexity hypothesis of the involved multitime functionals has been used. Later et al. [15] established several relationships between Stampacchia and Minty generalized variational inequalities and the associated multiple objective optimization problems. For connected ideas and improvements in this research direction, interested readers can study the papers of Cebuc and Treanţă [16], Crespi et al. [17, 18], Ruiz-Garzon et al. [19], Treanţă et al. [20], Yu and Lu [21], Lu et al. [22], and Zhu et al. [23]. Recently, Jurdjevic [24] studied optimal problems in time on Lie groups and provided some applications to quantum control. In addition, Yang et al. [25] investigated an optimal time two-mesh mixed finite element method associated with a nonlinear fractional hyperbolic wave model.

In this paper, having in mind all the above-mentioned developments, we extend vector variational inequalities, in Minty and Stampacchia forms, to the controlled variational inequalities of vector type. More precisely, we add a new variable to these classes of variational problems, namely, the control function. Also, in this new framework, we introduce the notions of preconvexity and (strictly) strong convexity for functionals driven by controlled simple integrals, and a result of mean value type. Next, we introduce the corresponding multiobjective extremization models and establish various connections between the considered classes of variational problems. In addition, the theoretical notions and the main results are justified by suitable numerical examples that are non-trivial.

This study contains four sections. Section 2 includes some preliminary notions, definitions, lemmas, and theorems, which will be used to prove the main results of the current paper. Section 3 establishes numerous relations between Stampacchia and Minty controlled inequalities of vector type and the corresponding multi-cost variational models. In addition, some suitable non-trivial applications are provided in order to support the theoretical notions.

2. Problem formulation

Consider $\mathcal{Y} = [a, b]$ to be a real interval, $\mathbb{R}^n_+$ to be the nonnegative orthant of the Euclidean space $\mathbb{R}^n$ with dimension n, and $\psi : \mathcal{Y} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^r$ to be a continuous differentiable vector-valued function. Let ω and $\dot{\omega}$ denote $\omega(t)$ and $\dot{\omega}(t)$, respectively, with $\omega : \mathcal{Y} \mapsto \mathbb{R}^n$ as a piecewise smooth function having the derivative $\dot{\omega}$. Also, we state θ for $\theta(t)$, with $\theta : \mathcal{Y} \mapsto \mathbb{R}^m$ being a continuous function (piecewise). For a scalar function $q : \mathcal{Y} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, we write the partial derivatives related to ω and $\dot{\omega}$, respectively, as

$$q_\omega = \left(\frac{\partial q}{\partial \omega_1}, \ldots, \frac{\partial q}{\partial \omega_n}\right), \quad q_{\dot\omega} = \left(\frac{\partial q}{\partial \dot\omega_1}, \ldots, \frac{\partial q}{\partial \dot\omega_n}\right),$$

and in a similar way for the partial derivative with respect to θ.

In the following, consider X the family of piecewise differentiable functions $\omega : \mathcal{Y} \mapsto \mathbb{R}^n$ (states) so that $\omega(a) = \upsilon$, $\omega(b) = \chi$, and $\|\omega\| = \|\omega\|_\infty + \|\dot\omega\|_\infty$. Also, consider U the family of piecewise continuous functions $\theta : \mathcal{Y} \mapsto \mathbb{R}^m$ (controls), endowed with $\|\cdot\|_\infty$.

Now, we state the following multiobjective minimization model:

$$(P) \quad \min_{(\omega,\theta)} \int_{\mathcal{Y}} \psi(t,\omega,\theta)dt = \left(\int_{\mathcal{Y}} \psi^1(t,\omega,\theta)dt, \cdots, \int_{\mathcal{Y}} \psi^r(t,\omega,\theta)dt\right),$$

subject to

$$\omega(a) = \upsilon, \quad \omega(b) = \chi,$$

$$\pi(t,\omega,\theta) \leq 0, \quad t \in \mathcal{Y},$$

$$H(t,\omega,\dot\omega,\theta) := \dot\omega - h(t,\omega,\theta) = 0, \quad t \in \mathcal{Y},$$

where $\pi : \mathcal{Y} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ and $h : \mathcal{Y} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are considered to be continuous differentiable functionals. Let $\Omega \subset X \times U$ be the convex set of feasible points of P.

Definition 2.1. *We say $(\vartheta,\mu) \in \Omega$ is an efficient point of P if the inequality*

$$\int_{\mathcal{Y}} \psi^\kappa(t,\omega,\theta)dt \leq \int_{\mathcal{Y}} \psi^\kappa(t,\vartheta,\mu)dt$$

is not satisfied, for all $(\omega,\theta) \in \Omega$, with $<$ for at least one $\kappa \in \mathcal{B}$, with $\mathcal{B} = \{1,\cdots,r\}$.

Definition 2.2. *We say $(\vartheta,\mu) \in \Omega$ is a weak efficient point of P if the inequality*

$$\int_{\mathcal{Y}} \psi^\kappa(t,\omega,\theta)dt < \int_{\mathcal{Y}} \psi^\kappa(t,\vartheta,\mu)dt, \quad \forall\kappa \in \mathcal{B},$$

is not satisfied, for all $(\omega,\theta) \in \Omega$.

Definition 2.3. *The functional of simple integral type*

$$Q : \Omega \to \mathbb{R},\ Q(\omega,\theta) = \int_{\mathcal{Y}} q(t,\omega,\theta)dt$$

is strongly preconvex on Ω if there exists a constant $\lambda > 0$ such that, for all $(\omega,\theta),(\vartheta,\mu) \in \Omega$ and $\rho \in [0,1]$, the following inequality holds:

$$\int_{\mathcal{Y}} q\left(t,\vartheta+\rho(\omega-\vartheta),\mu+\rho(\theta-\mu)\right)dt \leq \rho\int_{\mathcal{Y}} q(t,\omega,\theta)dt + (1-\rho)\int_{\mathcal{Y}} q(t,\vartheta,\mu)dt - \lambda\rho(1-\rho)\|(\omega,\theta)-(\vartheta,\mu)\|^2.$$

Remark 2.1. *We must notice that the norm used in the previous definition for $\|(\omega,\theta)-(\vartheta,\mu)\|$ is the Euclidean-type norm on the space $\mathbb{R}^{n+m}$.*

Definition 2.4. *The functional of simple integral type*

$$Q : \Omega \to \mathbb{R},\ Q(\omega, \theta) = \int_{\mathcal{Y}} q(t, \omega, \theta) dt$$

is strongly convex on Ω *if there exists a constant* $\lambda > 0$ *such that, for all* $(\omega, \theta), (\vartheta, \mu) \in \Omega$*, the following inequality holds:*

$$\int_{\mathcal{Y}} [q_{\omega}(t, \vartheta, \mu)(\omega - \vartheta) + q_{\theta}(t, \vartheta, \mu)(\theta - \mu)] dt + \lambda \|(\omega, \theta) - (\vartheta, \mu)\|^2 \\ \leq \int_{\mathcal{Y}} q(t, \omega, \theta) dt - \int_{\mathcal{Y}} q(t, \vartheta, \mu) dt.$$

We formulate an example of a convex functional that is not a strong convex functional. However, the converse is always true.

Example 2.1. *Define* $q : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*,* $q(t, \omega, \theta) = \omega + \theta$ *and* $\omega(t) = t,\ \theta(t) = c \in \mathbb{R}$*. The simple integral functional* $Q(\omega, \theta) = \int_0^1 q(t, \omega, \theta) dt$ *is convex at* $(\vartheta, \mu) = (0, 0)$ *since, for all* $(\omega, \theta) \in \mathbb{R}^2$*, we have*

$$\int_0^1 q(t, \omega, \theta) dt - \int_0^1 q(t, \vartheta, \mu) dt \geq \int_0^1 \left[q_{\omega}(t, \vartheta, \mu)(\omega - \vartheta) + q_{\theta}(t, \vartheta, \mu)(\theta - \mu) \right] dt.$$

Indeed, we get

$$\begin{aligned} \int_0^1 q(t, \omega, \theta) dt - \int_0^1 q(t, \vartheta, \mu) dt &= \int_0^1 (\omega + \theta) dt - \int_0^1 (\vartheta + \mu) dt \\ &= \frac{t^2}{2}\Big|_0^1 + ct\Big|_0^1 - 0 \\ &= \frac{1}{2} + c, \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \left[q_{\omega}(t, \vartheta, \mu)(\omega - \vartheta) + q_{\theta}(t, \vartheta, \mu)(\theta - \mu) \right] dt &= \int_0^1 \left[(\omega - \vartheta) + (\theta - \mu) \right] dt \\ &= \frac{t^2}{2}\Big|_0^1 + ct\Big|_0^1 - 0 \\ &= \frac{1}{2} + c. \end{aligned}$$

Since we have

$$\int_{\mathcal{Y}} [q_{\omega}(t, \vartheta, \mu)(\omega - \vartheta) + q_{\theta}(t, \vartheta, \mu)(\theta - \mu)] dt + \lambda \|(\omega, \theta) - (\vartheta, \mu)\|^2 \\ - \left[\int_{\mathcal{Y}} q(t, \omega, \theta) dt - \int_{\mathcal{Y}} q(t, \vartheta, \mu) dt \right] \\ = \lambda \|(\omega, \theta) - (\vartheta, \mu)\|^2 \nleq 0,$$

for any $\lambda > 0$*, the simple integral functional* Q *is not strongly convex at* $(\vartheta, \mu) = (0, 0)$*.*

Definition 2.5. *The functional of simple integral type*

$$Q : \Omega \to \mathbb{R}, \; Q(\omega, \theta) = \int_{\mathcal{Y}} q(t, \omega, \theta) dt$$

is strictly strongly convex on Ω *if there exists a constant* $\lambda > 0$ *such that, for all* $(\omega, \theta), (\vartheta, \mu) \in \Omega$ *with* $(\omega, \theta) \neq (\vartheta, \mu)$*, the following inequality holds:*

$$\int_{\mathcal{Y}} [q_{\omega}(t, \vartheta, \mu)(\omega - \vartheta) + q_{\theta}(t, \vartheta, \mu)(\theta - \mu)] dt + \lambda \|(\omega, \theta) - (\vartheta, \mu)\|^2 < \int_{\mathcal{Y}} q(t, \omega, \theta) dt - \int_{\mathcal{Y}} q(t, \vartheta, \mu) dt.$$

Definition 2.6. *Consider* (ω, θ) *and* (ϑ, μ) *two arbitrary points in* Ω*. We say that* $P_{(\omega,\theta),(\vartheta,\mu)}$ *is a closed path joining* (ϑ, μ) *and* (ω, θ)*, if*

$$P_{(\omega,\theta),(\vartheta,\mu)} = \{(z, e) = (\omega, \theta) + \rho((\vartheta, \mu) - (\omega, \theta)) : \rho \in [0, 1]\}.$$

Similalry, $P^0_{(\omega,\theta),(\vartheta,\mu)}$ is named *an open path linking* (ω, θ) *and* (ϑ, μ), if

$$P^0_{(\omega,\theta),(\vartheta,\mu)} = \{(z, e) = (\omega, \theta) + \rho((\vartheta, \mu) - (\omega, \theta)) : \rho \in (0, 1)\}.$$

Theorem 2.1. *Consider the differentiable functional* $Q : \Omega \to \mathbb{R}, Q(\omega, \theta) = \int_{\mathcal{Y}} q(t, \omega, \theta) dt$*, and* $P_{(\omega,\theta),(\vartheta,\mu)}$ *a fixed given closed path included in* Ω*. Then* $\exists \, (x_0, u_0) \in P^0_{(\omega,\theta),(\vartheta,\mu)}$ *so that the following relation holds:*

$$\int_{\mathcal{Y}} q(t, \vartheta, \mu) dt - \int_{\mathcal{Y}} q(t, \omega, \theta) dt = \int_{\mathcal{Y}} [q_{\omega}(t, x_0, u_0)(\vartheta - \omega) + q_{\theta}(t, x_0, u_0)(\mu - \theta)] \, dt.$$

Proof. Let us define $f : [0, 1] \to \mathbb{R}$, with

$$f(\rho) = \int_{\mathcal{Y}} q(t, \omega + \rho(\vartheta - \omega), \theta + \rho(\mu - \theta)) dt - \int_{\mathcal{Y}} q(t, \omega, \theta) - \rho \left[\int_{\mathcal{Y}} q(t, \vartheta, \mu) dt - \int_{\mathcal{Y}} q(t, \omega, \theta) dt \right]. \tag{2.1}$$

By $f(0) = f(1) = 0$, and using the Rolle's theorem, we obtain $\exists \, \epsilon \in (0, 1)$ so that $f'(\epsilon) = 0$. By considering the relation (2.1), we get

$$\begin{aligned} 0 = f'(\epsilon) = &\int_{\mathcal{Y}} q_{\omega}(t, \omega + \epsilon(\vartheta - \omega), \theta + \epsilon(\mu - \theta))(\vartheta - \omega) dt \\ &+ \int_{\mathcal{Y}} q_{\theta}(t, \omega + \epsilon(\vartheta - \omega), \theta + \epsilon(\mu - \theta))(\mu - \theta) dt \\ &- \int_{\mathcal{Y}} q(t, \vartheta, \mu) dt + \int_{\mathcal{Y}} q(t, \omega, \theta) dt, \end{aligned}$$

that is,

$$\int_{\mathcal{Y}} q(t,\vartheta,\mu)dt - \int_{\mathcal{Y}} q(t,\omega,\theta)dt = \int_{\mathcal{Y}} [q_\omega(t,\omega+\epsilon(\vartheta-\omega),\theta+\epsilon(\mu-\theta))(\vartheta-\omega) + q_\theta(t,\omega+\epsilon(\vartheta-\omega),\theta+\epsilon(\mu-\theta))(\mu-\theta)]dt.$$

Taking $(x_0,\theta_0) := (\omega+\epsilon(\vartheta-\omega),\theta+\epsilon(\mu-\theta))$, it follows that

$$\int_{\mathcal{Y}} q(t,\vartheta,\mu)dt - \int_{\mathcal{Y}} q(t,\omega,\theta) = \int_{\mathcal{Y}} \left[q_\omega(t,x_0,u_0)(\vartheta-\omega) + q_\theta(t,x_0,u_0)(\mu-\theta)\right] dt.$$

Lemma 2.1. *Consider the differentiable functional* $Q : \Omega \to \mathbb{R},\ Q(\omega,\theta) = \int_{\mathcal{Y}} q(t,\omega,\theta)dt$. *Then, Q is strongly preconvex on Ω if Q is strongly convex on Ω.*

Proof. The convexity property of Ω implies

$$(x_1,u_1) = (\omega,\theta) + \rho((\vartheta,\mu) - (\omega,\theta)) \in \Omega, \quad \forall(\omega,\theta),\ (\vartheta,\mu) \in \Omega,\ \rho \in [0,1].$$

By the strong convexity assumption of $Q(\omega,\theta) = \int_{\mathcal{Y}} q(t,\omega,\theta)dt$, we obtain $\exists\ \lambda > 0$ so that, for all $(x_1,u_1),\ (\vartheta,\mu) \in \Omega$, we have

$$\int_{\mathcal{Y}} \left[q_\omega(t,x_1,u_1)(\vartheta-x_1) + q_\theta(t,x_1,u_1)(\mu-u_1)\right] dt + \lambda\|(\vartheta,\mu)-(x_1,u_1)\|^2 \leq \int_{\mathcal{Y}} q(t,\vartheta,\mu)dt - \int_{\mathcal{Y}} q(t,x_1,u_1)dt. \tag{2.2}$$

Similarly, we apply the strong convexity of $Q(\omega,\theta) = \int_{\mathcal{Y}} q(t,\omega,\theta)dt$ and we obtain $\exists\ \lambda > 0$ so that, for all $(x_1,u_1),\ (\omega,\theta) \in \Omega$, we have

$$\int_{\mathcal{Y}} \left[q_\omega(t,x_1,u_1)(\omega-x_1) + q_\theta(t,x_1,u_1)(\theta-u_1)\right] dt + \lambda\|(\omega,\theta)-(x_1,u_1)\|^2 \leq \int_{\mathcal{Y}} q(t,\omega,\theta)dt - \int_{\mathcal{Y}} q(t,x_1,u_1)dt. \tag{2.3}$$

We multiply (2.2) and (2.3) by ρ and $1-\rho$, respectively, and, by adding both relations, we get $\exists$ $\lambda > 0$ so that, for all $(\omega,\theta),\ (\vartheta,\mu) \in \Omega$ and $\rho \in [0,1]$, it follows that

$$\int_{\mathcal{Y}} q(t,\omega+\rho(\vartheta-\omega),\theta+\rho(\mu-\theta))dt \leq \rho\int_{\mathcal{Y}} q(t,\vartheta,\mu)dt + (1-\rho)\int_{\mathcal{Y}} q(t,\omega,\theta) -\lambda\rho(1-\rho)\|(\vartheta,\mu)-(\omega,\theta)\|^2.$$

The proof is complete.

Next, considering the research line of [5], we state the generalized vector variational controlled inequalities of Minty and Stampacchia type:

GMVVI$_\xi$: For a given ξ, let us find $(\vartheta,\mu)\in\Omega$ so that $\exists$ no $(\omega,\theta)\in\Omega$, fulfilling

$$\int_{\mathcal{Y}}[\psi^\kappa_\omega(t,\omega,\theta)(\vartheta-\omega)+\psi^\kappa_\theta(t,\omega,\theta)(\mu-\theta)]\,dt+\xi\|(\vartheta,\mu)-(\omega,\theta)\|^2\geq 0,$$

with $>$ for at least one $\kappa\in\mathcal{B}$;

GSVVI$_\xi$: For a given ξ, let us find $(\vartheta,\mu)\in\Omega$ so that $\exists$ no $(\omega,\theta)\in\Omega$, fulfilling

$$\int_{\mathcal{Y}}[\psi^\kappa_\omega(t,\vartheta,\mu)(\omega-\vartheta)+\psi^\kappa_\theta(t,\vartheta,\mu)(\theta-\mu)]\,dt+\xi\|(\omega,\theta)-(\vartheta,\mu)\|^2\leq 0,$$

with $<$ for at least one $\kappa\in\mathcal{B}$;

GWMVVI$_\xi$: For a given ξ, let us find $(\vartheta,\mu)\in\Omega$ so that $\exists$ no $(\omega,\theta)\in\Omega$, fulfilling

$$\int_{\mathcal{Y}}[\psi^\kappa_\omega(t,\omega,\theta)(\vartheta-\omega)+\psi^\kappa_\theta(t,\omega,\theta)(\mu-\theta)]\,dt+\xi\|(\vartheta,\mu)-(\omega,\theta)\|^2>0,\ \forall\kappa\in\mathcal{B};$$

GWSVVI$_\xi$: For a given ξ, let us find $(\vartheta,\mu)\in\Omega$ so that $\exists$ no $(\omega,\theta)\in\Omega$, fulfilling

$$\int_{\mathcal{Y}}[\psi^\kappa_\omega(t,\vartheta,\mu)(\omega-\vartheta)+\psi^\kappa_\theta(t,\vartheta,\mu)(\theta-\mu)]\,dt+\xi\|(\omega,\theta)-(\vartheta,\mu)\|^2<0,\ \forall\kappa\in\mathcal{B}.$$

Particular cases. If $\xi=0$ in **GSVVI**$_\xi$ (**GWSVVI**$_\xi$), we obtain the studies done by Treanţă et al. [26], or Treanţă and Saeed [27]. If we consider $\xi=0$ and remove the control variable, we obtain the study given by Kim [2].

Remark 2.2. *If a point (ϑ,μ) is a solution of **GMVVI**$_\xi$ or its weak formulation, then (ϑ,μ) is also a solution of **GMVVI**$_{\xi'}$ or its weak formulation, for all parameters $\xi'\leq\xi$.*

Next, in order to show that the above-mentioned variational control inequalities are well-defined, let us formulate the next example to illustrate that $\exists$ a solution for **GMVVI**$_\xi$ (for instance).

Example 2.2. *Consider $\psi:[0,1]\times\mathbb{R}^2\to\mathbb{R}^2$, given by*

$$\psi(t,\omega,\theta)=(\psi^1(t,\omega,\theta),\psi^2(t,\omega,\theta))=(-\omega^2\theta^2,-\omega^4u^4)$$

that determines

$$\int_0^1\psi(t,\omega,\theta)dt=\left(\int_0^1\psi^1(t,\omega,\theta)dt,\int_0^1\psi^2(t,\omega,\theta)dt\right),$$

with $\omega:[0,1]\to\mathbb{R},\ \omega(t)=kt,\ k\in\mathbb{R}$ and $\theta:[0,1]\to\mathbb{R},\ \theta(t)=k$.

*For $\xi<0$, we obtain that $(\vartheta,\mu)=(0,0)$ is a solution of **GMVVI**$_\xi$,*

$$\Big(\int_0^1\Big[\psi^1_\omega(t,\omega,\theta)(\vartheta-\omega)+\psi^1_\theta(t,\omega,\theta)(\mu-\theta)\Big]dt+\xi\|(\vartheta,\mu)-(\omega,\theta)\|^2,$$

$$\int_0^1\Big[\psi^2_\omega(t,\omega,\theta)(\vartheta-\omega)+\psi^2_\theta(t,\omega,\theta)(\mu-\theta)\Big]dt+\xi\|(\vartheta,\mu)-(\omega,\theta)\|^2\Big)\ngeq(0,0).$$

The next example illustrates the existence of a solution to **GSVVI**$_\xi$, but it is not a solution for the classes of inequalities presented by Treanţă and Guo in [28].

Example 2.3. *Consider* $\psi : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$*, defined by*

$$\psi(t,\omega,\theta) = (\psi^1(t,\omega,\theta), \psi^2(t,\omega,\theta)) = (-\omega-\theta, -\omega+\theta^2),$$

that determines

$$\int_0^1 \psi(t,\omega,\theta)dt = \left(\int_0^1 \psi^1(t,\omega,\theta)dt, \int_0^1 \psi^2(t,\omega,\theta)dt\right),$$

with $\theta : [0,1]\to\mathbb{R},\ \theta(t)=1$ *and* $\omega : [0,1]\to\mathbb{R},\ \omega(t)=t.$

For $\xi=\frac{3}{2}$*, we obtain that* $(\vartheta,\mu)=(0,0)$ *is a solution of* ***GSVVI***$_\xi$*,*

$$\Big(\int_0^1 \Big[\psi^1_\omega(t,\vartheta,\mu)\,(\omega-\vartheta)+\psi^1_\theta(t,\vartheta,\mu)(\theta-\mu)\Big]dt+\xi\|(\omega,\theta)-(\vartheta,\mu)\|^2,$$

$$\int_0^1 \Big[\psi^2_\omega(t,\vartheta,\mu)\,(\omega-\vartheta)+\psi^2_\theta(t,\vartheta,\mu)(\theta-\mu)\Big]dt+\xi\|(\omega,\theta)-(\vartheta,\mu)\|^2\Big) \nleq (0,0).$$

On the other hand, for $\xi=0$*, the class of inequalities presented by Treanţă and Guo [28] is not solvable at* $(0,0)$.

3. Main results

In this section, we present various connections between the given multiple-cost optimization model and the considered extended controlled vector inequalities.

Theorem 3.1. *For each* $\kappa\in\mathcal{B}$*, let us assume that the functional* $\Psi^\kappa : \Omega\to\mathbb{R},\ \Psi^\kappa(\omega,\theta)=\int_{\mathcal{Y}}\psi^\kappa(t,\omega,\theta)dt$ *is strongly convex on* Ω *with constant* λ_κ*. Then the pair* $(\vartheta,\mu)\in\Omega$ *is an efficient point of* P *if and only if it is a solution of* ***GMVVI***$_\xi$*, where* $\xi=\min\{\lambda_1,...,\lambda_r\}$*.*

Proof. First of all, we suppose that $(\vartheta,\mu)\in\Omega$ is an efficient point of P but not a solution of **GMVVI**$_\xi$. Therefore, for a given ξ, $\exists\ (\omega_\xi,\theta_\xi)\in\Omega$ so that

$$\int_{\mathcal{Y}}\Big[\psi^\kappa_\omega(t,\omega_\xi,\theta_\xi)(\vartheta-\omega_\xi)+\psi^\kappa_\theta(t,\omega_\xi,\theta_\xi)(\mu-\theta_\xi)\Big]dt+\xi\|(\vartheta,\mu)-(\omega_\xi,\theta_\xi)\|^2\geq 0, \tag{3.1}$$

with $>$ for at least one $\kappa\in\mathcal{B}$. Now, for $\Psi^\kappa=\int_{\mathcal{Y}}\psi^\kappa(t,\cdot,\cdot)dt$, we use the strong convexity assumption. Thus, $\exists\ \lambda_\kappa>0$, so that

$$\int_{\mathcal{Y}}[\psi^\kappa_\omega(t,\omega,\theta)(\vartheta-\omega)+\psi^\kappa_\theta(t,\omega,\theta)(\mu-\theta)]\,dt+\lambda_\kappa\|(\vartheta,\mu)-(\omega,\theta)\|^2$$
$$\leq\int_{\mathcal{Y}}\psi^\kappa(t,\vartheta,\mu)dt-\int_{\mathcal{Y}}\psi^\kappa(t,\omega,\theta)dt,\quad \forall(\omega,\theta)\in\Omega,\ \kappa\in\mathcal{B}.$$

For $\xi = \min\{\lambda_1, \lambda_2, ..., \lambda_r\}$, the previous inequality becomes

$$\begin{aligned}\int_{\mathcal{Y}} [\psi^{\kappa}_{\omega}(t,\omega,\theta)(\vartheta-\omega) + \psi^{\kappa}_{\theta}(t,\omega,\theta)(\mu-\theta)]\, dt + \xi\|(\vartheta,\mu)-(\omega,\theta)\|^2 \\ \leq \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt - \int_{\mathcal{Y}} \psi^{\kappa}(t,\omega,\theta)dt, \quad \forall(\omega,\theta)\in\Omega,\ \kappa\in\mathcal{B}.\end{aligned} \tag{3.2}$$

Using the Eqs (3.1) and (3.2) (the point (ω_ξ, θ_ξ) from (3.1) is substituted into inequality (3.2)), we obtain that

$$\int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt - \int_{\mathcal{Y}} \psi^{\kappa}(t,\omega_\xi,\theta_\xi)dt \geq 0,$$

which is fulfilled as $>$, for at least one $\kappa \in \mathcal{B}$. This is a contradiction with the hypothesis that $(\vartheta,\mu)\in\Omega$ is an efficient point of P. In consequence, $(\vartheta,\mu)\in\Omega$ is a solution of $\mathbf{GMVVI}_\xi$.

In reverse, consider $(\vartheta,\mu)\in\Omega$ to be a solution of $\mathbf{GMVVI}_\xi$ with respect to ξ, but not an efficient point of P. Then $\exists\,(\omega,\theta)\in\Omega$ so that

$$\int_{\mathcal{Y}} \psi^{\kappa}(t,\omega,\theta) \leq \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt, \tag{3.3}$$

with $<$ for at least one $\kappa\in\mathcal{B}$.

We set

$$(\omega(\rho),\theta(\rho)) = (\vartheta+\rho(\omega-\vartheta), \mu+\rho(\theta-\mu)), \quad \forall\rho\in[0,1].$$

For $\rho'\in(0,1)$ and using the mean value theorem, $\exists\,\sigma_\kappa\in(0,\rho']$ for $\kappa\in\mathcal{B}$ so that

$$\begin{aligned}\rho' \int_{\mathcal{Y}} [\psi^{\kappa}_{\omega}(t,\vartheta+\sigma_\kappa(\omega-\vartheta),\mu+\sigma_\kappa(\theta-\mu))(\omega-\vartheta) \\ + \psi^{\kappa}_{\theta}(t,\vartheta+\sigma_\kappa(\omega-\vartheta),\mu+\sigma_\kappa(\theta-\mu))(\theta-\mu)]dt \\ = \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta+\rho'(\omega-\vartheta),\mu+\rho'(\theta-\mu))dt - \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt.\end{aligned} \tag{3.4}$$

Using Lemma 2.1, the functional $\Psi = \int_{\mathcal{Y}} \psi(t,\cdot,\cdot)dt$ is also strongly preconvex on Ω with constant λ_κ, so we have

$$\begin{aligned}\int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta+\rho'(\omega-\vartheta),\mu+\rho'(\theta-\mu)dt - \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt \\ \leq \rho'\left[\int_{\mathcal{Y}} \psi^{\kappa}(t,\omega,\theta)dt - \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt\right] - \lambda_\kappa\rho'(1-\rho')\|(\omega,\theta)-(\vartheta,\mu)\|^2.\end{aligned} \tag{3.5}$$

Considering the relations given in (3.3)–(3.5), by direct computation, we obtain

$$\begin{aligned}\int_{\mathcal{Y}} \Big[\psi^{\kappa}_{\omega}(t,\vartheta+\sigma_\kappa(\omega-\vartheta),\mu+\sigma_\kappa(\theta-\mu))(\omega-\vartheta) \\ + \psi^{\kappa}_{\theta}(t,\vartheta+\sigma_\kappa(\omega-\vartheta),\mu+\sigma_\kappa(\theta-\mu))(\theta-\mu)\Big]dt \\ \leq \lambda_\kappa(1-\rho')\|(\omega,\theta)-(\vartheta,\mu)\|^2, \quad \forall\kappa\in\mathcal{B},\end{aligned} \tag{3.6}$$

with < for at least one $\kappa \in \mathcal{B}$. Since $\sigma_\kappa \in (0,1), \forall\kappa \in \mathcal{B}$, we choose $\rho^* \in (0,1)$ so that $\rho^* < \min\{\sigma_\kappa : \forall\kappa \in \mathcal{B}\}$. Now, for any $\kappa \in \mathcal{B}$, it is obvious that

$$\omega(\rho^*) - \omega(\sigma_\kappa) = (\rho^* - \sigma_\kappa)(\omega - \vartheta),\ \theta(\rho^*) - \theta(\sigma_\kappa) = (\rho^* - \sigma_\kappa)(\theta - \mu), \tag{3.7}$$

and

$$\omega(\sigma_\kappa) - \omega(\rho^*) = (\sigma_\kappa - \rho^*)(\omega - \vartheta),\ \theta(\sigma_\kappa) - \theta(\rho^*) = (\sigma_\kappa - \rho^*)(\theta - \mu). \tag{3.8}$$

By considering relations (3.6) and (3.7), it follows that

$$\begin{aligned}\int_{\mathcal{Y}}[\psi^\kappa_\omega(t, \vartheta + \sigma_\kappa(\omega - \vartheta), \mu + \sigma_\kappa(\theta - \mu))(\omega(\rho^*) - \omega(\sigma_\kappa))\\ + \psi^\kappa_\theta(t, \vartheta + \sigma_\kappa(\omega - \vartheta), \mu + \sigma_\kappa(\theta - \mu))(\theta(\rho^*) - \theta(\sigma_\kappa))]dt\\ \geq \lambda_\kappa(1 - \rho')(\sigma_\kappa - \rho^*)\|(\omega, \theta) - (\vartheta, \mu)\|^2, \quad \forall\kappa \in \mathcal{B},\end{aligned}$$

with > for at least one $\kappa \in \mathcal{B}$. The last inequality can be written as

$$\begin{aligned}\int_{\mathcal{Y}}[\psi^\kappa_\omega(t, \omega(\sigma_\kappa), \theta(\sigma_\kappa))(\omega(\rho^*) - \omega(\sigma_\kappa)) + \psi^\kappa_\theta(t, \omega(\sigma_\kappa), \theta(\sigma_\kappa))(\theta(\rho^*) - \theta(\sigma_\kappa))]dt\\ \geq \lambda_\kappa(1 - \rho')(\sigma_\kappa - \rho^*)\|(\omega, \theta) - (\vartheta, \mu)\|^2,\end{aligned} \tag{3.9}$$

with > for at least one $\kappa \in \mathcal{B}$. Further, the strong convexity of $\Psi^\kappa = \int_{\mathcal{Y}} \psi^\kappa(t, \cdot, \cdot)dt$ leads to

$$\begin{aligned}\int_{\mathcal{Y}}[\psi^\kappa_\omega(t, \omega(\sigma_\kappa), \theta(\sigma_\kappa))(\omega(\rho^*) - \omega(\sigma_\kappa)) + \psi^\kappa_\theta(t, \omega(\sigma_\kappa), \theta(\sigma_\kappa))(\theta(\rho^*) - \theta(\sigma_\kappa))]dt\\ + \lambda_\kappa\|(\omega(\rho^*), \theta(\rho^*)) - (\omega(\sigma_\kappa), \theta(\sigma_\kappa))\|^2\\ \leq \int_{\mathcal{Y}} \psi^\kappa(t, \omega(\rho^*), \theta(\rho^*))dt - \int_{\mathcal{Y}} \psi^\kappa(t, \omega(\sigma_\kappa), \theta(\sigma_\kappa))dt.\end{aligned} \tag{3.10}$$

By interchanging $\omega(\rho^*)$ and $\omega(\sigma_\kappa)$ in (3.10), we have

$$\begin{aligned}\int_{\mathcal{Y}}[\psi^\kappa_\omega(t, \omega(\rho^*), \theta(\rho^*))(\omega(\sigma_\kappa) - \omega(\rho^*)) + \psi^\kappa_\theta(t, \omega(\rho^*), \theta(\rho^*))(\theta(\sigma_\kappa) - \theta(\rho^*))]dt\\ + \lambda_\kappa\|(\omega(\sigma_\kappa), \theta(\sigma_\kappa)) - (\omega(\rho^*), \theta(\rho^*))\|^2\\ \leq \int_{\mathcal{Y}} \psi^\kappa(t, \omega(\sigma_\kappa), \theta(\sigma_\kappa))dt - \int_{\mathcal{Y}} \psi^\kappa(t, \omega(\rho^*), \theta(\rho^*))dt.\end{aligned} \tag{3.11}$$

By adding (3.10) and (3.11) and using (3.7) and (3.8), we get

$$\begin{aligned}\int_{\mathcal{Y}}[\psi^\kappa_\omega(t, \omega(\sigma_\kappa), \theta(\sigma_\kappa))(\omega(\rho^*) - \omega(\sigma_\kappa)) + \psi^\kappa_\theta(t, \omega(\sigma_\kappa), \theta(\sigma_\kappa))(\theta(\rho^*) - \theta(\sigma_\kappa))]dt\\ + \int_{\mathcal{Y}}[\psi^\kappa_\omega(t, \omega(\rho^*), \theta(\rho^*))(\omega(\sigma_\kappa) - \omega(\rho^*)) + \psi^\kappa_\theta(t, \omega(\rho^*), \theta(\rho^*))(\theta(\sigma_\kappa) - \theta(\rho^*))]dt\\ \leq -2\lambda_\kappa(\sigma_\kappa - \rho^*)^2\|(\omega, \theta) - (\vartheta, \mu)\|^2.\end{aligned} \tag{3.12}$$

By (3.9) and (3.12) we obtain

$$\int_{\mathcal{Y}} \psi^{\kappa}_{\omega}(t, \omega(\rho^*)\theta(\rho^*))(\omega(\sigma_\kappa) - \omega(\rho^*)) + \psi^{\kappa}_{\theta}(t, \omega(\rho^*), \theta(\rho^*))(\theta(\sigma_\kappa) - \theta(\rho^*))$$
$$\leq -\lambda_\kappa(\sigma_\kappa - \rho^*)[2(\sigma_\kappa - \rho^*) + (1 - \rho')]\|(\omega, \theta) - (\vartheta, \mu)\|^2,$$

with < for at least one $\kappa \in \mathcal{B}$. For the last inequality we can use the relation (3.8) and we have

$$\int_{\mathcal{Y}} [\psi^{\kappa}_{\omega}(t, \omega(\rho^*), \theta(\rho^*))(\vartheta - \omega(\rho^*)) + \psi^{\kappa}_{\theta}(t, \omega(\rho^*), \theta(\rho^*))(\mu - \theta(\rho^*))]dt$$
$$\geq \frac{\lambda_\kappa}{\rho^*}[2(\sigma_\kappa - \rho^*) + (1 - \rho')]\|(\vartheta, \mu) - (\omega(\rho^*), \theta(\rho^*))\|^2.$$

Next, we consider $\gamma_0 = \min\{\lambda_1, ..., \lambda_r\}$ and $\sigma_0 = \min\{\sigma_1, ..., \sigma_r\}$, and obtain

$$\int_{\mathcal{Y}} [\psi^{\kappa}_{\omega}(t, \omega(\rho^*), \theta(\rho^*))(\vartheta - \omega(\rho^*)) + \psi^{\kappa}_{\theta}(t, \omega(\rho^*), \theta(\rho^*))(\mu - \theta(\rho^*))]dt$$
$$+ \xi'\|(\vartheta, \mu) - (\omega(\rho^*), \theta(\rho^*))\|^2 \geq 0,$$

with > for at least one of $\kappa \in \mathcal{B}$ and $\xi' = \frac{-\gamma_0}{\rho^*}[2(\sigma_0 - \rho^*) + (1 - \rho')]$. It is clear that, if $\rho^* \to 0^+$ implies $\xi' \to -\infty$, then for any $\xi' \leq \xi$, we get that (ϑ, μ) is not a solution of $\mathbf{GMVVI}_\xi$ which in turn, by Remark 2.2, contradicts that (ϑ, μ) is a solution of $\mathbf{GMVVI}_\xi$ with contant ξ.

We build the next numerical example to illustrate the result stated in the above-mentioned theorem.

Example 3.1. *Consider* $\psi : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^2$ *given by*

$$\psi(t, \omega, \theta) = (\psi^1(t, \omega, \theta), \psi^2(t, \omega, \theta)) = (\omega^2, 1 + \omega^2),$$

that determines

$$\int_0^1 \psi(t, \omega, \theta)dt = \left(\int_0^1 \psi^1(t, \omega, \theta)dt, \int_0^1 \psi^2(t, \omega, \theta)dt\right),$$

with $\omega, \vartheta : [0, 1] \to \mathbb{R}$, $\omega(t) = c_1 \cdot t$, *and* $\vartheta(t) = c_2 \cdot t$, $\forall c_1, c_2 \in \mathbb{R}$, *and* $\theta, \mu : [0, 1] \to \mathbb{R}$, $\theta(t) = c_1$, $\mu(t) = c_2$. *For* $\lambda_1 = \frac{1}{24}$, *we obtain*

$$\int_0^1 [\psi^1_\omega(t, \vartheta, \mu)(\omega - \vartheta) + \psi^1_\theta(t, \vartheta, \mu)(\theta - \mu)]dt + \lambda_1\|(\omega, \theta) - (\vartheta, \mu)\|^2$$
$$- \int_0^1 \psi^1(t, \omega, \theta)dt + \int_0^1 \psi^1(t, \vartheta, \mu)dt$$
$$= \int_0^1 [2y \cdot (\omega - \vartheta) + 0 \cdot (\theta - \mu)]dt + \lambda_1\|(\omega, \theta) - (\vartheta, \mu)\|^2 - \int_0^1 \omega^2 dt + \int_0^1 \vartheta^2 dt$$
$$= \int_0^1 [2xy - 2y^2]dt + \lambda_1\|((c_1 - c_2)t, c_1 - c_2)\|^2 - \int_0^1 \omega^2 dt + \int_0^1 \vartheta^2 dt$$
$$= \int_0^1 [2xy - 2y^2]dt + 4\lambda_1(c_1 - c_2)^2 - \int_0^1 \omega^2 dt + \int_0^1 \vartheta^2 dt$$

$$= -\frac{(c_1 - c_2)^2}{6} \leq 0.$$

Similarly, for $\lambda_2 = \frac{1}{24}$, *we obtain*

$$\int_0^1 [\psi^2_\omega(t,\vartheta,\mu)(\omega-\vartheta) + \psi^2_\theta(t,\vartheta,\mu)(\theta-\mu)]dt + \lambda_2\|(\omega,\theta)-(\vartheta,\mu)\|^2$$
$$- \int_0^1 \psi^2(t,\omega,\theta)dt + \int_0^1 \psi^2(t,\vartheta,\mu)dt$$
$$= -\frac{(c_1 - c_2)^2}{6} \leq 0,$$

involving the strong convexity with respect to λ_1 *and* λ_2. *Next, we check that* $(\vartheta,\mu) = (0,0)$ *is a solution of* $\boldsymbol{GMVVI}_\xi$ *for* $\xi = \frac{1}{24}$:

$$\Big(\int_0^1 [\psi^1_\omega(t,\omega,\theta)(\vartheta-\omega) + \psi^1_\theta(t,\omega,\theta)(\mu-\theta)]dt + \xi\|(\vartheta,\mu)-(\omega,\theta)\|^2,$$
$$\int_0^1 [\psi^2_\omega(t,\omega,\theta)(\vartheta-\omega) + \psi^2_\theta(t,\omega,\theta)(\mu-\theta)]dt + \xi\|(\vartheta,\mu)-(\omega,\theta)\|^2\Big)$$
$$= \Big(\int_0^1 [2c_1c_2 - 2c_1^2t^2]dt + \xi\|(\vartheta,\mu)-(\omega,\theta)\|^2,$$
$$\int_0^1 [2c_1c_2t^2 - 2c_1^2t^2]dt + \xi\|(\vartheta,\mu)-(\omega,\theta)\|^2\Big) \ngeq (0,0).$$

For $(\vartheta,\mu) = (0,0)$, *we have*

$$\Big(\int_0^1 \psi^1(t,\omega,\theta)dt - \int_0^1 \psi^1(t,\vartheta,\mu)dt, \int_0^1 \psi^2(t,\omega,\theta)dt - \int_0^1 \psi^2(t,\vartheta,\mu)dt\Big) = \Big(\frac{c_1^2}{3}, \frac{c_1^2}{3}\Big) \nleq (0,0),$$

and, in consequence, $(\vartheta,\mu) = (0,0)$ *is an efficient point to P.*

Theorem 3.2. *Assume that* $\Psi^\kappa = \int_{\mathcal{Y}} \psi^\kappa(t,\cdot,\cdot)dt$ *is strongly convex on* Ω, *for every* $\kappa \in \mathcal{B}$, *with constant* λ_κ. *If* $(\vartheta,\mu) \in \Omega$ *is a solution to* $\boldsymbol{GSVVI}_\xi$, $\xi = \min\{\lambda_1, ..., \lambda_r\}$, *and then* $(\vartheta,\mu) \in \Omega$ *is an efficient point in P.*

Proof. Let (ϑ,μ) be a solution for $\mathbf{GSVVI}_\xi$. Thus, for ξ, $(\nexists)$ $(\omega,\theta) \in \Omega$ satisfying

$$\int_{\mathcal{Y}} \psi^\kappa_\omega(t,\vartheta,\mu)(\omega-\vartheta) + \psi^\kappa_\theta(t,\vartheta,\mu)(\theta-\mu)]dt + \xi\|(\omega,\theta)-(\vartheta,\mu)\|^2 \leq 0, \tag{3.13}$$

with $<$ for at least one $\kappa \in \mathcal{B}$. The strong convexity assumption of $\Psi^\kappa = \int_{\mathcal{Y}} \psi^\kappa(t,\cdot,\cdot)dt$ and we get that $\exists$ $\lambda_\kappa > 0$ so that

$$\int_{\mathcal{Y}} [\psi^\kappa_\omega(t,\vartheta,\mu)(\omega-\vartheta) + \psi^\kappa_\theta(t,\vartheta,\mu)(\theta-\mu)]dt + \lambda_\kappa\|(\omega,\theta)-(\vartheta,\mu)\|^2$$

$$\leq \int_{\mathcal{Y}} \psi^{\kappa}(t,\omega,\theta)dt - \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt, \quad \forall(\omega,\theta) \in \Omega,\ \kappa \in \mathcal{B}.$$

In particular, for $\xi = \min\{\lambda_1, ..., \lambda_r\}$, we get

$$\begin{aligned}\int_{\mathcal{Y}} [\psi^{\kappa}_{\omega}(t,\vartheta,\mu)(\omega-\vartheta) + \psi^{\kappa}_{\theta}(t,\vartheta,\mu)(\theta-\mu)]dt + \xi\|(\omega,\theta)-(\vartheta,\mu)\|^2 \\ \leq \int_{\mathcal{Y}} \psi^{\kappa}(t,\omega,\theta)dt - \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt, \quad \forall(\omega,\theta) \in \Omega,\ \kappa \in \mathcal{B}.\end{aligned} \tag{3.14}$$

Using (3.13) and (3.14), we obtain that there is no $(\omega,\theta) \in \Omega$ so that

$$\int_{\mathcal{Y}} \psi^{\kappa}(t,\omega,\theta)dt \leq \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt,$$

with $<$, for at least one $\kappa \in \mathcal{B}$. We have completed the proof.

Theorem 3.3. *Assume that $\Psi^{\kappa} = \int_{\mathcal{Y}} \psi^{\kappa}(t,\cdot,\cdot)dt$ is strongly convex on Ω with λ_{κ}, $\kappa \in \mathcal{B}$. If $(\vartheta,\mu) \in \Omega$ solves **GWSVVI**$_{\xi}$, then it solves **GWMVVI**$_{\xi}$, where $\xi = \min\{\lambda_1, ..., \lambda_r\}$.*

Proof. If (ϑ,μ) is a solution to **GWSVVI**$_{\xi}$, then, for a given ξ, there is no $(\omega,\theta) \in \Omega$ fulfilling

$$\int_{\mathcal{Y}} [\psi^{\kappa}_{\omega}(t,\vartheta,\mu)(\omega-\vartheta) + \psi^{\kappa}_{\theta}(t,\vartheta,\mu)(\theta-\mu)]dt + \xi\|(\omega,\theta)-(\vartheta,\mu)\|^2 < 0, \quad \kappa \in \mathcal{B}. \tag{3.15}$$

For a real $\lambda > 0$, by using the strong convexity hypothesis of Ψ^{κ}, we obtain

$$\int_{\mathcal{Y}} [\psi^{\kappa}_{\omega}(t,\vartheta,\mu)(\omega-\vartheta) + \psi^{\kappa}_{\theta}(t,\vartheta,\mu)(\theta-\mu)]dt + \lambda_{\kappa}\|(\omega,\theta)-(\vartheta,\mu)\|^2$$

$$\leq \int_{\mathcal{Y}} \psi^{\kappa}(t,\omega,\theta)dt - \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt, \quad \forall(\omega,\theta) \in \Omega,\ \kappa \in \mathcal{B}.$$

For $\xi = \min\{\lambda_1, ..., \lambda_r\}$ we get

$$\begin{aligned}\int_{\mathcal{Y}} [\psi^{\kappa}_{\omega}(t,\vartheta,\mu)(\omega-\vartheta) + \psi^{\kappa}_{\theta}(t,\vartheta,\mu)(\theta-\mu)]dt + \xi\|(\omega,\theta)-(\vartheta,\mu)\|^2 \\ \leq \int_{\mathcal{Y}} \psi^{\kappa}(t,\omega,\theta)dt - \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt, \quad \forall(\omega,\theta) \in \Omega,\ \kappa \in \mathcal{B}.\end{aligned} \tag{3.16}$$

In inequality (3.16) we interchange (ω,θ) with (ϑ,μ) and obtain

$$\begin{aligned}\int_{\mathcal{Y}} [\psi^{\kappa}_{\omega}(t,\omega,\theta)(\vartheta-\omega) + \psi^{\kappa}_{\theta}(t,\omega,\theta)(\mu-\theta)]dt + \xi\|(\vartheta,\mu)-(\omega,\theta)\|^2 \\ \leq \int_{\mathcal{Y}} \psi^{\kappa}(t,\vartheta,\mu)dt - \int_{\mathcal{Y}} \psi^{\kappa}(t,\omega,\theta)dt, \quad \forall(\omega,\theta) \in \Omega,\ \kappa \in \mathcal{B}.\end{aligned} \tag{3.17}$$

Using (3.16) and (3.17) we have

$$\begin{aligned}&\int_{\mathcal{Y}} \psi^{\kappa}_{\omega}(t,\omega,\theta)(\vartheta-\omega)+\psi^{\kappa}_{\theta}(t,\omega,\theta)(\mu-\theta)]dt+\xi\|(\vartheta,\mu)-(\omega,\theta)\|^2\\ &\le -\left[\int_{\mathcal{Y}} \psi^{\kappa}_{\omega}(t,\vartheta,\mu)(\omega-\vartheta)+\psi^{\kappa}_{\theta}(t,\vartheta,\mu)(\theta-\mu)]dt\right]-\xi\|(\omega,\theta)-(\vartheta,\mu)\|^2.\end{aligned} \tag{3.18}$$

Now, we combine (3.15) and (3.18) and, for $\xi\in\mathbb{R}$, there is no $(\omega,\theta)\in\Omega$ fulfilling

$$\int_{\mathcal{Y}}[\psi^{\kappa}_{\omega}(t,\omega,\theta)(\vartheta-\omega)+\psi^{\kappa}_{\theta}(t,\omega,\theta)(\mu-\theta)]dt+\xi\|(\vartheta,\mu)-(\omega,\theta)\|^2>0,\quad \kappa\in\mathcal{B}.$$

The proof is complete.

Theorem 3.4. *Assume that, for each $\kappa\in\mathcal{B}$, the integral $\Psi^{\kappa}=\int_{\mathcal{Y}}\psi^{\kappa}(t,\cdot,\cdot)dt$ is strongly convex on Ω with respect to λ_{κ}. If (ϑ,μ) is a solution to* ***GWSVVI**$_{\xi}$*, $\xi=\min\{\lambda_1,...,\lambda_r\}$, and then (ϑ,μ) is a weak efficient point in P.*

Proof. We suppose that (ϑ,μ) is not a weak efficient point in P. Therefore, $\exists\,(\omega,\theta)\in\Omega$ so that

$$\int_{\mathcal{Y}}\psi^{\kappa}(t,\omega,\theta)dt<\int_{\mathcal{Y}}\psi^{\kappa}(t,\vartheta,\mu)dt,\quad \forall\kappa\in\mathcal{B}. \tag{3.19}$$

Since the functional $\Psi^{\kappa}=\int_{\mathcal{Y}}\psi^{\kappa}(t,\cdot,\cdot)dt$ is strongly convex, it follows that $\exists\,\lambda_{\kappa}>0$ so that

$$\begin{aligned}&\int_{\mathcal{Y}}[\psi^{p}_{x}(t,\vartheta,\mu)(\omega-\vartheta)+\psi^{p}_{u}(t,\vartheta,\mu)(\theta-\mu)]dt+\lambda_{\kappa}\|(\omega,\theta)-(\vartheta,\mu)\|^2\\ &\qquad\le\int_{\mathcal{Y}}\psi^{\kappa}(t,\omega,\theta)dt-\int_{\mathcal{Y}}\psi^{\kappa}(t,\vartheta,\mu)dt,\quad\forall(\omega,\theta)\in\Omega,\ \kappa\in\mathcal{B}.\end{aligned}$$

If we consider $\xi=\min\{\lambda_1,...,\lambda_r\}$, we have

$$\begin{aligned}&\int_{\mathcal{Y}}[\psi^{p}_{x}(t,\vartheta,\mu)(\omega-\vartheta)+\psi^{\kappa}_{\theta}(t,\vartheta,\mu)(\theta-\mu)]dt+\xi\|(\omega,\theta)-(\vartheta,\mu)\|^2\\ &\qquad\le\int_{\mathcal{Y}}\psi^{\kappa}(t,\omega,\theta)dt-\int_{\mathcal{Y}}\psi^{\kappa}(t,\vartheta,\mu)dt,\quad\forall(\omega,\theta)\in\Omega,\ \kappa\in\mathcal{B}.\end{aligned} \tag{3.20}$$

Taking into account inequalities (3.19) and (3.20), we obtain that $\exists\,(\omega,\theta)\in\Omega$ so that

$$\int_{\mathcal{Y}}[\psi^{p}_{x}(t,\vartheta,\mu)(\omega-\vartheta)+\psi^{\kappa}_{\theta}(t,\vartheta,\mu)(\theta-\mu)]dt+\xi\|(\omega,\theta)-(\vartheta,\mu)\|^2<0,\quad\kappa\in\mathcal{B},$$

which contradicts that (ϑ,μ) is a solution of **GWSVVI**$_{\xi}$. This completes the proof of the theorem.

Theorem 3.5. *Consider, for each $\kappa\in\mathcal{B}$, the integral $\Psi^{\kappa}=\int_{\mathcal{Y}}\psi^{\kappa}(t,\cdot,\cdot)dt$ is strictly strongly convex on Ω with respect to λ_{κ}. If (ϑ,μ) is a weak efficient point in P, then (ϑ,μ) is a solution to* ***GMVVI**$_{\xi}$*, $\xi=\min\{\lambda_1,...,\lambda_r\}$.*

Proof. By contrast, we suppose that (ϑ,μ) is not a solution of $\mathbf{GMVVI}_\xi$. Therefore, $\exists\ \xi$ and $(\omega_\xi,\theta_\xi)\in\Omega$ so that

$$\int_{\mathcal{Y}}[\psi^\kappa_\omega(t,\omega_\xi,\theta_\xi)(\vartheta-\omega_\xi)+\psi^p_u(t,\omega_\xi,\theta_\xi)(\mu-\theta_\xi)]dt+\xi\|(\vartheta,\mu)-(\omega_\xi,\theta_\xi)\|^2\geq 0, \tag{3.21}$$

with $>$ for at least one $\kappa\in\mathcal{B}$. Since $\Psi^\kappa=\int_{\mathcal{Y}}\psi^\kappa(t,\cdot,\cdot)dt$ is strictly strongly convex, then $\exists\ \lambda_\kappa>0$, so that for all $(\omega,\theta)\in\Omega,\ \kappa\in\mathcal{B},\ (\omega,\theta)\neq(\vartheta,\mu)$, we get

$$\begin{aligned}&\int_{\mathcal{Y}}[\psi^\kappa_\omega(t,\omega,\theta)(\vartheta-\omega)+\psi^\kappa_\theta(t,\omega,\theta)(\mu-\theta)]dt+\lambda_\kappa\|(\vartheta,\mu)-(\omega,\theta)\|^2\\&<\int_{\mathcal{Y}}\psi^\kappa(t,\vartheta,\mu)dt\int_{\mathcal{Y}}\psi^\kappa(t,\omega,\theta)dt.\end{aligned}$$

If we consider $\xi=\min\{\lambda_1,...,\lambda_r\}$, we have

$$\begin{aligned}&\int_{\mathcal{Y}}[\psi^\kappa_\omega(t,\omega,\theta)(\vartheta-\omega)+\psi^\kappa_\theta(t,\omega,\theta)(\mu-\theta)]dt+\xi\|(\vartheta,\mu)-(\omega,\theta)\|^2\\&<\int_{\mathcal{Y}}\psi^\kappa(t,\vartheta,\mu)dt-\int_{\mathcal{Y}}\psi^\kappa(t,\omega,\theta)dt.\end{aligned} \tag{3.22}$$

Taking into account (3.21) and (3.22), we obtain

$$\int_{\mathcal{Y}}\psi^\kappa(t,\vartheta,\mu)dt-\int_{\mathcal{Y}}\psi^\kappa(t,\omega,\theta)dt>0,\quad \kappa\in\mathcal{B},$$

which contradicts that $(\vartheta,\mu)\in\Omega$ is a weak efficient point of P. This completes the proof of the theorem.

Theorem 3.6. *Consider* $\Psi^\kappa=\int_{\mathcal{Y}}\psi^\kappa(t,\cdot,\cdot)dt$ *is a strongly convex functional on* Ω *with respect to* λ_κ*, for each* $\kappa\in\mathcal{B}$*. If* (ϑ,μ) *is a weak efficient point of P, then it is a solution of* $\boldsymbol{GWMVVI}_\xi,\ \xi=\min\{\lambda_1,...,\lambda_r\}$.

Proof. Consider that $(\vartheta,\mu)\in\Omega$ is a weakly efficient point of P but not a solution of $\mathbf{GWMVVI}_\xi$. Therefore, for given ξ and $\kappa\in\mathcal{B}$, $\exists\ (\omega_\xi,\theta_\xi)\in\Omega$ so that

$$\int_{\mathcal{Y}}\left[\psi^\kappa_\omega(t,\omega_\xi,\theta_\xi)(\vartheta-\omega_\xi)+\psi^\kappa_\theta(t,\omega_\xi,\theta_\xi)(\mu-\theta_\xi)\right]dt+\xi\|(\vartheta,\mu)-(\omega_\xi,\theta_\xi)\|^2>0. \tag{3.23}$$

Now, for $\Psi^\kappa=\int_{\mathcal{Y}}\psi^\kappa(t,\cdot,\cdot)dt$, we use the strong convexity assumption. Thus, $\exists\ \lambda_\kappa>0$, so that

$$\begin{aligned}&\int_{\mathcal{Y}}[\psi^\kappa_\omega(t,\omega,\theta)(\vartheta-\omega)+\psi^\kappa_\theta(t,\omega,\theta)(\mu-\theta)]\,dt+\lambda_\kappa\|(\vartheta,\mu)-(\omega,\theta)\|^2\\&\leq\int_{\mathcal{Y}}\psi^\kappa(t,\vartheta,\mu)dt-\int_{\mathcal{Y}}\psi^\kappa(t,\omega,\theta)dt,\quad \forall(\omega,\theta)\in\Omega,\ \kappa\in\mathcal{B}.\end{aligned}$$

For $\xi = \min\{\lambda_1, \lambda_2, ..., \lambda_r\}$, the previous inequality becomes

$$\begin{aligned}&\int_{\mathcal{Y}} [\psi^{\kappa}_{\omega}(t, \omega, \theta)(\vartheta - \omega) + \psi^{\kappa}_{\theta}(t, \omega, \theta)(\mu - \theta)]\, dt + \xi \|(\vartheta, \mu) - (\omega, \theta)\|^2 \\ &\quad \leq \int_{\mathcal{Y}} \psi^{\kappa}(t, \vartheta, \mu) dt - \int_{\mathcal{Y}} \psi^{\kappa}(t, \omega, \theta) dt, \quad \forall (\omega, \theta) \in \Omega,\ \kappa \in \mathcal{B}.\end{aligned} \tag{3.24}$$

Using the Eqs (3.23) and (3.24), we obtain that

$$\int_{\mathcal{Y}} \psi^{\kappa}(t, \vartheta, \mu) dt - \int_{\mathcal{Y}} \psi^{\kappa}(t, \omega_{\xi}, \theta_{\xi}) dt > 0, \quad \kappa \in \mathcal{B}.$$

This is a contradiction with the hypothesis that $(\vartheta, \mu) \in \Omega$ is a weak efficient point of P. In consequence, $(\vartheta, \mu) \in \Omega$ is a solution of **GWMVVI**$_{\xi}$.

4. Conclusions and further research directions

The main results derived in this study have been formulated in 6 principal theorems. The proofs associated with these theoretical outcomes are innovative and non-trivial. Concretely, the novel elements associated with the current paper are:
- introduction of generalized Minty and Stampacchia controlled (weak) vector variational inequalities;
- introduction of the corresponding multiobjective extremization models;
- establishing various connections between the considered classes of variational problems;
- adjusting the concepts of (strictly) strong convexity and preconvexity.

As future research directions, strongly connected with the topic investigated in this paper, we can mention the study of well-posedness of such classes of extremization models, the duality theory, and saddle-point criteria associated with the considered optimization problems. Moreover, we can suggest the case of multiple or curvilinear integrals as functionals for the mentioned studies. In these cases, there are various potential applications of the proposed models in engineering, for instance, due to the physical significance of curvilinear integrals (mechanical work).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest.

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