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Research article

Well-posedness for a 2D/3D fluid-shell interaction model

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Abstract: The 2D/3D interaction systems consisting of the incompressible Navier-Stokes equations in a bounded domain and the classical full von Kármán shallow shell equation were investigated in this paper, where the fluid and the shell were coupled via both the transverse and longitudinal displacements. The global existence of weak solutions in 2D/3D was proved by using a special choice of basis functions for the related elliptic equations, which allows one to tackle the delicate estimates for the interaction between the fluid and the shell. Moreover, the uniqueness of the weak solution in 2D was also established via a localization technique.

Keywords: fluid-structure interaction model; von Kármán equation; well-posedness

1. Introduction

In fluid-structure interaction (FSI) problems consisting of the Navier-Stokes equations and hyperbolic systems (such as wave, elastic, viscoelastic, or beam equations), significant developments have been made. Coutand and Shkoller [1–3] proposed the Lagrangian coordinate transformation method, which converts the moving boundary problem into a more tractable mathematical form via coordinate transformation, thereby achieving optimal hyperbolic regularity and establishing the existence and uniqueness of weak solutions. In [4, 5], Chueshov systematically investigated two typical linear fluid-structure interaction models, and his work not only established the global existence of weak solutions for these two classes of models but also employed the quasi-stability method to further prove the existence of their global attractors. Desjardins [6] proved interaction results between a fluid and an elastic structure with finitely many deformation modes, while [7] treated the case of finitely many rigid structures embedded in a fluid. Čanić [8] provided a comprehensive presentation of a benchmark problem in fluid-multilayer-structure interaction and established the corresponding numerical simulation scheme. For more research results, refer to [9–13]. Although there are fruitful results on fluid-structure interface problems, few developments have been achieved for the fluid-shell interaction model, which describes the interactions between an incompressible fluid (governed by 2D/3D Navier-Stokes equations in a container) and the deformations of a thin elastic shell. The shell is located on a part of the fluid boundary and modeled via the full von Kármán shallow shell system, see [14–19]. This model is rather general as it describes both longitudinal and transverse oscillations of the shell. To the best of our knowledge, few studies for fluid-structure interface problems consider the influence of the convection term in the fluid equations and the longitudinal displacements of the thin structure, see [5, 20, 21] and the related references.

Motivated by the recent developments in [22], which considered a fluid flow with a convection term and a thin structure boundary undergoing displacements only in the transverse direction, our main interest in this paper is to study the fluid-shell interaction model described by the Navier-Stokes equations coupled with the shell equation accounting for both transverse and longitudinal displacements, which is formulated in the following two- and three-dimensional modes:

- **3D case**

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a sufficiently smooth boundary as $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, where Γ_0 and Γ_1 are nonempty and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, Γ_0 is rigid, and Γ_1 is flat and flexible.

Assume that a thin elastic plate of thickness h is placed on Γ_1, so that the mid-plane of the plate coincides with $x_3 = 0$ in the equilibrium state. In accordance with the configuration of the plate's motion, we consider both the transverse and longitudinal displacements of the plate surface. To ensure that the container's shape is preserved during deformation, i.e., to allow mild changes in the plate's shape due to deformations and shearing by the fluid, without altering the fluid domain, it is assumed that the plate has a narrow bellow-like rim attached to its edges. This rim is expandable or compressible as required to preserve the configuration [21].

The classical 3D Navier-Stokes equations in Ω for the fluid velocity field u and the pressure p are written as:

$$u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = G_f, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

where $\nu > 0$ is the viscosity and G_f is a body force. The boundary condition on the rigid part of Ω and the data for (1.1) and (1.2) are given by

$$u = 0 \quad \text{on } \Gamma_0$$

and

$$u(0) = u_0 \quad \text{in } \Omega.$$

The full von Kármán model is described by

$$\begin{aligned} \rho h\eta_{tt} - \gamma\Delta\eta_{tt} + \Delta^2\eta = {} & (N_1\eta_x)_x + (N_{12}\eta_x)_y \\ & + (N_2\eta_y)_y + (N_{12}\eta_y)_x - N_1k_1 - N_2k_2 - (T_f)_3 + G_3, \end{aligned} \tag{1.3}$$

$$\begin{aligned} v^1_{tt} - \Delta v^1 - \frac{1+\mu}{1-\mu}\theta_x = {} & \frac{2}{1-\mu}[(k_1\eta)_x + \eta_{xx}\eta_x + \mu(k_2\eta)_x + \mu\eta_{xy}\eta_y] \\ & - \eta_{xy}\eta_y - \eta_x\eta_{yy} - (T_f)_1 + G_1, \end{aligned} \tag{1.4}$$

$$v_{tt}^2 - \Delta v^2 - \frac{1+\mu}{1-\mu}\theta_y = \frac{2}{1-\mu}[(k_2\eta)_y + \eta_{yy}\eta_y + \mu(k_1\eta)_y + \mu\eta_{xy}\eta_x] - \eta_{xy}\eta_x - \eta_y\eta_{xx} - (T_f)_2 + G_2. \tag{1.5}$$

Here, N_1, N_2, N_{12} are the longitudinal stresses and $\varepsilon_1, \varepsilon_2, \varepsilon_{12}$ denote the deformations of the middle surface and read

$$N_1 = Eh(1-\mu^2)^{-1}(\varepsilon_1 + \mu\varepsilon_2), \tag{1.6}$$

$$N_2 = Eh(1-\mu^2)^{-1}(\varepsilon_2 + \mu\varepsilon_1), \tag{1.7}$$

$$N_{12} = \frac{1}{2}Eh(1+\mu)^{-1}\varepsilon_{12}, \tag{1.8}$$

$$\varepsilon_1 = v_x^1 + k_1\eta + \frac{1}{2}(\eta_x)^2, \tag{1.9}$$

$$\varepsilon_2 = v_y^2 + k_2\eta + \frac{1}{2}(\eta_y)^2, \tag{1.10}$$

$$\varepsilon_{12} = v_y^1 + v_x^2 + \eta_x\eta_y, \tag{1.11}$$

$$\theta = v_x^1 + v_y^2. \tag{1.12}$$

The quantities v^1 and v^2 denote longitudinal point displacements on the middle surface of the shell, η is the analogous transverse displacement, E and μ are Young's modulus and Poisson's ratio, respectively, h is the thickness of the shell, γ is a constant proportional to h, k_1 and k_2 are the curvatures of the shell (which is assumed to be sufficiently smooth), $G_1, G_2,$ and G_3 are the given body forces on the shell, and $(T_f)_1, (T_f)_2,$ and $(T_f)_3$ are the corresponding component influence of fluid on the shell. Furthermore the surface force T_f exerted by the fluid on the plate is defined by

$$T_f(u) = \left(\nu(u_{x_3}^1 + u_{x_1}^3) - \frac{1}{2}u^1u^3, \nu(u_{x_3}^2 + u_{x_2}^3) - \frac{1}{2}u^2u^3, 2\nu(u_{x_3}^3) - p - \frac{1}{2}(u^3)^2\right)^T \tag{1.13}$$

which satisfies

$$\int_{\Gamma_1} T_f \cdot \bar{v}\mathrm{d}S = \int_{\Gamma_1}\left(2\nu D(u)\cdot n - pn - \frac{1}{2}(u\cdot n)u\right)\cdot u\mathrm{d}S, \tag{1.14}$$

where $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the strain tensor, and $n = (0,0,1)^T$ denotes the outer unit normal to Γ_1, and $\bar{v}(t,x,y) = (v_t^1(t,x,y), v_t^2(t,x,y), \eta_t(t,x,y))^T$. It is further assumed that the mass density and the linear sizes are measured in units such that

$$\rho h = 1, \qquad 2\rho(1+\mu)E^{-1} = 1, \tag{1.15}$$

where ρ is the mass density of the shell, with these identities already being taken into account in (1.3)–(1.5). Without loss of generality, we assume that $\gamma = 1$.

Clamped boundary conditions are imposed on the shell:

$$v^1|_{\partial\Gamma_1} = v^2|_{\partial\Gamma_1} = \eta|_{\partial\Gamma_1} = \left.\frac{\partial\eta}{\partial n}\right|_{\partial\Gamma_1} = 0 \tag{1.16}$$

and the initial conditions read

$$v^1(0) = v_{01}, \quad v^2(0) = v_{02}, \quad \eta(0) = \eta_0, \tag{1.17}$$

$$v_t^1(0) = v_{11}, \quad v_t^2(0) = v_{12}, \quad \eta_t(0) = \eta_1. \tag{1.18}$$

The following compatibility condition at the interface is based on physical assumptions:

$$u(t,x,y) = (v_t^1(t,x,y), v_t^2(t,x,y), \eta_t(t,x,y))^T, \quad \text{on } \Gamma_1. \tag{1.19}$$

- **2D case**

Let $\Omega \subset \mathbb{R}^2$. The key difference from the 3D case lies in the 1D version of the dynamical von Kármán system, which is formulated as follows:

$$\eta_{tt} + \eta_{xxxx} - \eta_{xxtt} = (f(v,\eta))_x - g(v,\eta) - (T_f)_1 + G_1, \tag{1.20}$$

$$v_{tt} = \frac{2}{1-\mu}(v_x + \frac{1}{2}\eta_x^2 + k_1\eta)_x - (T_f)_2 + G_2, \tag{1.21}$$

where

$$f(v,\eta) = \frac{2}{1-\mu}\eta_x(v_x + \frac{1}{2}\eta_x^2 + k_1\eta) \tag{1.22}$$

and

$$g(v,\eta) = \frac{2k_1}{1-\mu}(v_x + \frac{1}{2}\eta_x^2 + k_1\eta). \tag{1.23}$$

Without loss of generality, the quantities $\eta = \eta(x,t)$ and $v = v(x,t)$ represent, respectively, the longitudinal and transverse displacements of the beam at the point x at time t. Here μ is a constant satisfying $0 < \mu < 1$, and k_1 denotes the curvature of the beam. G_1 and G_2 are given body forces. Additionally, $(T_f)_1$ and $(T_f)_2$ are the corresponding components of the fluid's influence on the beam, and the definition of T_f will be specified precisely in (1.28).

We consider clamped boundary conditions and initial data on $\Gamma_1 = [0, L]$ as follows:

$$v(0,t) = v(L,t) = 0, \tag{1.24}$$

$$\eta(0,t) = \eta(L,t) = \eta_x(0,t) = \eta_x(L,t) = 0, \tag{1.25}$$

$$(\eta(0), \eta_t(0), v(0), v_t(0)) = (\eta_0, \eta_1, v_0, v_1). \tag{1.26}$$

As in (1.14)–(1.19), the following hypotheses are imposed:

$$u(t,x) = (v_t(t,x), \eta_t(t,x))^T, \ \text{ on } \ [0, L], \tag{1.27}$$

$$T_f(u) = \left(\nu(u^1_{x_2} + u^2_{x_1}) - \frac{1}{2}u^1u^2, 2\nu(u^2_{x_2}) - p - \frac{1}{2}(u^2)^2\right)^T, \tag{1.28}$$

$$\int_{\Gamma_1} T_f \cdot \bar{v}\,\mathrm{d}S = \int_{\Gamma_1}\left(2\nu D(u)\cdot n - pn - \frac{1}{2}(u\cdot n)u\right)\cdot u\,\mathrm{d}S, \tag{1.29}$$

where $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the strain tensor, $n = (0,1)^T$ denotes the outer unit normal to $[0, L]$, and $\bar{v}(t,x) = (v_t^1(t,x), \eta_t(t,x))^T$.

Remark 1.1. We consider the particular case of the plate that undergoes only infinitesimal elastic displacements but whose velocity is large enough to make the fluid and structure remain fully coupled, which means that the tiny perturbation of the fluid boundary can be neglected compared to the large scale of the bulk region, see [23].

Remark 1.2. The condition in (1.19), together with the incompressibility of the fluid, implies that

$$\int_{\Gamma_1} \eta_t \mathrm{d}S = 0, \tag{1.30}$$

which states that the global volume of the container is preserved.

Remark 1.3. To simplify the following a priori estimates, (1.4) and (1.5) for the in-plane displacement can be rewritten as follows:

$$v^1_{tt} = (N_1)_x + (N_{12})_y - (T_f)_1 + G_1, \tag{1.31}$$

$$v^2_{tt} = (N_{12})_x + (N_2)_y - (T_f)_2 + G_2. \tag{1.32}$$

Equations (1.3)–(1.5) can also be rewritten in vector form:

$$(1-\gamma\Delta)\eta_{tt} + \Delta^2\eta + \mathrm{tr}\{K\mathscr{N}(v,\eta)\} = \mathrm{div}\{\mathcal{N}(v,\eta)\} - (T_f)_3 + G_3 \tag{1.33}$$

and

$$v_{tt} = \mathrm{div}\{\mathcal{N}(v,\eta)\} + \begin{pmatrix} G_1 - (T_f)_1 \\ G_2 - (T_f)_2 \end{pmatrix}, \tag{1.34}$$

where $K = \mathrm{diag}(k_1, k_2)$, $v = (v^1, v^2)^T$, and

$$\mathcal{N}(v,\eta) = \begin{pmatrix} N_1 & N_{12} \\ N_{12} & N_2 \end{pmatrix} = C[\epsilon(v) + \eta K + f(\nabla\eta)].$$

Denote the linear plane strain tensor as $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u)$. Let $f : \mathbb{R}^2 \to S$ be defined by $f(s) = \frac{1}{2} s \otimes s$, where S denotes the set of all fourth-order symmetric tensors on $\mathbb{R}^2$. Define $C \in \mathcal{L}(S)$ by

$$C[\epsilon] = \frac{2}{(1-\mu)}[\mu(\mathrm{tr}\,\epsilon)I_S + (1-\mu)\epsilon], \quad \forall \epsilon \in S,$$

where I_S is the identity of S.

Our aim is to establish the existence of weak solutions for the problem in both 2D and 3D cases, as well as the uniqueness of weak solutions in the 2D case, which involves the following challenges and key features:

(a) We use $((u \cdot \nabla v), w) - \frac{1}{2}((u \cdot n)v, w)$ to replace the classical trilinear operator $((u \cdot \nabla v), w)$. Actually, the (not necessarily small) fluid term $\frac{1}{2}(u \cdot n)u$ on Γ_1 arises because the interface is stationary. This term is essential for deriving a priori estimates on the energy-level weak solutions.

(b) Due to the elastic shell placed on the boundary, the singular nature of the surface force $(T_f(u))$, the complex nonlinearity of the fluid flow equations (e.g., $(((u \cdot \nabla v), w) - \frac{1}{2}((u \cdot n)v, w)))$, and the supercritical vector nonlinearity arising in the shell component (e.g., $((v^1_y + v^2_x + \eta_x\eta_y)\eta_y)_x)$), the semigroup method cannot be used directly. By choosing special approximations, see [24], to handle the coupled system, the Galerkin technique is employed to prove the well-posedness of Problem (1.1)–(1.14).

(c) If the fluid-structure interaction model involves large deflections of the structure—these deflections that exert significant effects on the fluid (i.e., the structural displacement is sufficiently large)—substantial variations in the fluid domain are induced. Therefore, the assumption of a fixed fluid domain is invalid, and a time-dependent domain is required to describe the coupled fluid-structure interface problem (see, e.g., [12, 13, 24]). Little attention has been paid to Problem (1.1)–(1.14) defined on a bounded domain with a time-dependent boundary.

The paper is organized as follows. The global well-posedness of weak solutions in the 2D and 3D cases are stated in Section 2, while the corresponding proofs are given in Sections 3 and 4, respectively.

2. Main results

2.1. Preliminaries

The space $(C_0^\infty(\Omega))^d (d = 2, 3)$ endowed with the topology of locally uniform convergence is denoted by $\mathcal{D}(\Omega)$, with dual space $\mathcal{D}'(\Omega)$. For simplicity, the functional spaces $(H^s(\Omega))^3$, $(L^2(\Omega))^3$ are denoted by $H^s(\Omega)$, $L^2(\Omega)$, respectively, and $(H^s(\Gamma))^2$, $(L^2(\Gamma))^2$ by $H^s(\Gamma)$, $L^2(\Gamma)$, respectively. The homogeneous spaces with zero average of $L^2(\Omega)$ are defined by $\hat{L}^2(\Omega) = \{u \in L^2(\Omega) : \int_\Omega u \mathrm{d}x = 0\}$ and $\hat{H}^s(\Omega) = H^s(\Omega) \cap \hat{L}^2(\Omega)$ for $s > 0$ and are endowed with the standard H^s-norm.

Let H be the closure of $E_1 := \{u \in \mathcal{D}(\Omega) : \operatorname{div} u = 0, u \cdot n = 0 \text{ on } \Gamma_0\}$ in the $L^2(\Omega)$ topology and V be the closure of $\{u \in \mathcal{D}(\Omega) : \operatorname{div} u = 0, u = 0 \text{ on } \Gamma_0\}$ in the $H^1(\Omega)$ topology. We have the embeddings $V \hookrightarrow H = H^* \hookrightarrow V^*$, where H^* and V^* are the dual spaces of H and V, respectively, the injections being dense and continuous.

The bilinear operator $B(\cdot, \cdot) : V \times V \to V^*$ is defined by

$$(B(u, v), w) = \int_\Omega ((u \cdot \nabla)v \cdot w)\mathrm{d}x - \frac{1}{2}\int_{\Gamma_1} ((u \cdot n)v \cdot w)\mathrm{d}S,$$

and the trilinear operator $b(u, v, w)$ is defined as follows:

$$b(u, v, w) = (B(u, v), w) = \int_\Omega ((u \cdot \nabla)v \cdot w)\mathrm{d}x - \frac{1}{2}\int_{\Gamma_1} ((u \cdot n)v \cdot w)\mathrm{d}S.$$

We have the following properties, using $|u|_s$, $|u|_0$ to denote the H^s, L^2 norms, respectively.

Lemma 2.1. *[25] For every $u \in V$, $v, w \in H^1(\Omega)$ in the d-dimensional space, the trilinear operator $b(\cdot, \cdot, \cdot)$ satisfies the following estimates:*

(1) The term $((u \cdot \nabla)v, w)_\Omega$ can be controlled as

$$|((u \cdot \nabla)v, w)_\Omega| \le C|u|_{s_1}|v|_{s_2+1}|w|_{s_3},$$

where $s_1, s_2, s_3 \ge 0$, $s_1 + s_2 + s_3 \ge \frac{d}{2}$ if $s_i \ne \frac{d}{2}$ for all $i = 1, 2, 3$, or $s_1 + s_2 + s_3 > \frac{d}{2}$ if $s_i = \frac{d}{2}$ for some $i = 1, 2, 3$.

(2) The estimate on $((u \cdot n)v, w)_\Gamma$ can be given by

$$|((u \cdot n)v, w)_\Gamma| \le C|u|_{s_1}|v|_{s_2}|w|_{s_3},$$

where $s_1 \ge \frac{1}{2}$, $s_2, s_3 > \frac{1}{2}$, and $s_1 + s_2 + s_3 \ge \frac{d}{2} + 1$.

(3) The trilinear operator $b(\cdot,\cdot,\cdot)$ satisfies

$$|b(u,v,w)| \le C_1|u|_{1/2}|v|_1|w|_{1/2} + C_2|u|_{1/2}|v|_{3/4}|w|_{3/4} \quad \textit{for } n = 2$$

and

$$|b(u,v,w)| \le C_1|u|_{1/2}|v|_1|w|_1 + C_2|u|_{3/4}|v|_1|w|_{3/4} \quad \textit{for } n = 3.$$

In addition, the antisymmetry property

$$b(u,v,w) = -b(u,w,v) \tag{2.1}$$

still holds owing to the equality $b(u,\phi,\phi) = 0$ holding for arbitrary $\phi \in H^1(\Omega)$.

Remark 2.2. In fact, the trilinear operator $b(u,v,w)$ can be rewritten as

$$b(u,v,w) = \frac{1}{2}\int_\Omega ((u\cdot\nabla v)\cdot w)\mathrm{d}x - \frac{1}{2}\int_\Omega ((u\cdot\nabla w)\cdot v)\mathrm{d}x.$$

Lemma 2.3. *(See [26]) Let Q be a bounded domain in $\mathbb{R}^n_x \times \mathbb{R}_t$, and g and g_μ be measurable functions in $L^q(Q)$ for $1 < q < \infty$. If $\|g_\mu\|_{L^q(Q)} \le C$, $g_\mu \to g$ is almost everywhere in Q, then $g_\mu \rightharpoonup g$ weakly in $L^q(Q)$.*

2.2. Weak formulation

Now, a weak solution of the problem is defined as follows.

Let $\eta_0 \in H^2_0(\Gamma_1), \eta_1 \in \hat{H}^1_0(\Gamma_1), v_0 \in H^1_0(\Gamma_1), v_1 \in L^2(\Gamma_1), u_0 \in L^2(\Omega)$ be such that

- $\operatorname{div} u_0 = 0$ in Ω, $u_0 \cdot n = 0$ on Γ_1,
- $\gamma(u_0) = (v_{11}, v_{12}, \eta_1)^T \cdot n$ on Γ_1.

A pair of function u and $(v^1, v^2, \eta)^T$ is said to be a weak solution to Problem (1.1)–(1.14) on $[0,T]$ if

- $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$,
- $\eta \in W^{1,\infty}(0,T;\hat{H}^1_0(\Gamma_1)) \cap L^\infty(0,T;H^2_0(\Gamma_1))$,
- $v^1, v^2 \in L^\infty(0,T;H^1_0(\Gamma_1))$, $v^1_t, v^2_t \in L^\infty(0,T;L^2(\Gamma_1))$,
- $\gamma(u) = (v^1_t, v^2_t, \eta_t)^T$, for a.e. time t,
- for all $\phi \in L^2(0,T;V), \phi_t \in L^2((0,T)\times\Omega)$, $b_i \in L^2(0,T;H^1_0(\Gamma_1))$, $b^i_t \in L^2((0,T)\times\Gamma_1)$, $d \in L^2(0,T;H^2_0(\Gamma_1))$, $d_t \in L^2(0,T;\hat{H}^1_0(\Gamma_1))$, such that $\phi(t,x,y)|_{\Gamma_1} = (b_1, b_2, d)^T$ and

$$\begin{aligned}
&\int_\Omega u\cdot\phi\mathrm{d}x - \int_0^t\int_\Omega u\cdot\phi_t\mathrm{d}x\mathrm{d}s + \nu\int_0^t\int_\Omega \nabla u : \nabla\phi\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^t\int_\Omega (u\cdot\nabla u)\cdot\phi\mathrm{d}x\mathrm{d}s\\
&-\frac{1}{2}\int_0^t\int_\Omega (u\cdot\nabla\phi)\cdot u\mathrm{d}x\mathrm{d}s + \int_{\Gamma_1}(1-\gamma\Delta)\eta_t d\mathrm{d}S - \int_0^t\int_{\Gamma_1}(1-\gamma\Delta)\eta_t d_t\mathrm{d}S\,\mathrm{d}s\\
&+\int_0^t\int_{\Gamma_1}\Delta\eta\Delta d\mathrm{d}S\,\mathrm{d}s + \int_{\Gamma_1} v_t\, b\mathrm{d}S - \int_0^t\int_{\Gamma_1} v_t\, b_t\mathrm{d}S\,\mathrm{d}s + \int_0^t a(v,b)\mathrm{d}s + \int_0^t P(v,\eta,b,d)\mathrm{d}s\\
=&\int_0^t\int_\Omega G_f\cdot\phi\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\Gamma_1} G\cdot b\mathrm{d}x\mathrm{d}s + \int_\Omega u_0\,\phi(0)\mathrm{d}x + \int_{\Gamma_1}(1-\gamma\Delta)\eta_1\, d(0)\mathrm{d}S + \int_{\Gamma_1} v_1\, b(0)\mathrm{d}S
\end{aligned} \tag{2.2}$$

holds, where $b = (b_1, b_2)^T,\ v_0 = (v_{01}, v_{02})^T,\ v_1 = (v_{11}, v_{12})^T,\ G = (G_1, G_2)^T$, and

$$a(v, b) = \sum_{i=1}^{2} \int_{\Gamma_1} \nabla v^i \cdot \nabla b_i \mathrm{d}S + \frac{1+\mu}{1-\mu} \int_{\Gamma_1} \operatorname{div} v \operatorname{div} b \mathrm{d}S, \tag{2.3}$$

$$\begin{aligned} P(v, \eta, b, d) = &\int_{\Gamma_1} (k_1 N_1 + k_2 N_2) d \mathrm{d}S \\ &+ \int_{\Gamma_1} (N_1 \eta_x + N_{12} \eta_y) d_x \mathrm{d}S + \int_{\Gamma_1} (N_{12} \eta_x + N_2 \eta_y) d_y \mathrm{d}S \\ &+ \frac{1}{1-\mu} \int_{\Gamma_1} \Big(((\eta_x)^2 + \mu (\eta_y)^2) b_{1x} + ((\eta_y)^2 + \mu (\eta_x)^2) b_{2y} \Big) \mathrm{d}S \\ &+ \int_{\Gamma_1} \eta_x \eta_y (b_{1y} + b_{2x}) \mathrm{d}S + \frac{2}{1-\mu} \int_{\Gamma_1} \eta (k_1 b_{1x} + k_2 b_{2y}) \mathrm{d}S \\ = &\int_{\Gamma_1} (k_1 N_1 + k_2 N_2) d \mathrm{d}S + \int_{\Gamma_1} C[f(\nabla \eta)] \nabla b \mathrm{d}S \\ &+ \int_{\Gamma_1} C[\epsilon(v) + \eta K + f(\nabla \eta)] \nabla \eta \nabla d \mathrm{d}S. \end{aligned} \tag{2.4}$$

2.3. Global well-posedness in 2D/3D

Our main results are given in the following theorems.

Theorem 2.4. *In the 3D case, assume $G_f \in L^2((0,T) \times \Omega), G_i \in L^2((0,T) \times \Gamma_1),\ i = 1, 2, 3$. There exist $T^* \in (0, T)$ and a pair of weak solution u and $(v^1, v^2, \eta)^T$ of Problem* (1.1)–(1.14) *on* $[0, T]$ *satisfying*

$$\begin{aligned} &\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u\|_{L^2(0,T;L^2(\Omega))} + \|\eta_t\|_{L^\infty(0,T;\hat{H}_0^1(\Gamma_1))} + \|\eta\|_{L^\infty(0,T;H_0^2(\Gamma_1))} \\ &\qquad + \|v_t\|_{L^\infty(0,T;L^2(\Gamma_1))} + \|\nabla v\|_{L^\infty(0,T;L^2(\Gamma_1))} \le C, \end{aligned} \tag{2.5}$$

where C depends on T, the initial data $u_0, \eta_0, \eta_1, v_0, v_1$, and the given forces $G_f, G_i, i = 1, 2, 3$.

Proof. See Section 3 for details. □

Theorem 2.5. *In the 2D case, under the assumption that $G_f \in L^2((0,T)\times\Omega), G_i \in L^2((0,T)\times[0, L]),\ i = 1, 2$, there exist $T^* \in (0, T)$ and a unique weak solution u, $(v, \eta)^T$ on $[0, T]$, satisfying the corresponding energy estimates for all T.*

Proof. The detailed proof is given in Section 4. □

3. Proof of Theorem 2.4

3.1. A priori estimates

Multiply both sides of Eq (1.1) by u and integrate over Ω and by parts. Then, the combination with (1.2) results in

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |u|^2 \mathrm{d}x + 2 \int_\Omega |D(u)|^2 \mathrm{d}x + \int_\Omega ((u \cdot \nabla) u \cdot u) \mathrm{d}x - \frac{1}{2} \int_{\Gamma_1} ((u \cdot n) u \cdot u) \mathrm{d}S \\ &\qquad + \int_{\Gamma_1} \Big(pn - 2D(u) \cdot n + \frac{1}{2} (u \cdot n) u \Big) \cdot u \mathrm{d}S = \int_\Omega G_f \cdot u \mathrm{d}x. \end{aligned} \tag{3.1}$$

Multiply the shell Eqs (1.3)–(1.5) by η_t, v_t^1, v_t^2, respectively, and integrate over Γ_1. This yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|\eta_t|^2\mathrm{d}S+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|\nabla\eta_t|^2\mathrm{d}S+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|\Delta\eta|^2\mathrm{d}S+\int_{\Gamma_1}N_1\eta_{x_1}(\eta_{x_1})_t\mathrm{d}S\\
&+\int_{\Gamma_1}N_{12}\eta_{x_1}(\eta_{x_2})_t\mathrm{d}S+\int_{\Gamma_1}N_1k_1\eta_t\mathrm{d}S+\int_{\Gamma_1}N_2k_2\eta_t\mathrm{d}S+\int_{\Gamma_1}(T_f)_3\eta_t\mathrm{d}S\\
=&\int_{\Gamma_1}G_3\eta_t\mathrm{d}S,
\end{aligned}\tag{3.2}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|v_t^1|^2\mathrm{d}S+\int_{\Gamma_1}N_1(v_{x_1}^1)_t\mathrm{d}S+\int_{\Gamma_1}N_{12}(v_{x_2}^1)_t\mathrm{d}S+\int_{\Gamma_1}(T_f)_1v_t^1\mathrm{d}S=\int_{\Gamma_1}G_1v_t^1\mathrm{d}S,\tag{3.3}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|v_t^2|^2\mathrm{d}S+\int_{\Gamma_1}N_{12}(v_{x_1}^2)_t\mathrm{d}S+\int_{\Gamma_1}N_2(v_{x_2}^2)_t\mathrm{d}S+\int_{\Gamma_1}(T_f)_2v_t^2\mathrm{d}S=\int_{\Gamma_1}G_2v_t^2\mathrm{d}S,\tag{3.4}$$

by using the equivalent forms of in-plane displacement Eqs (1.31) and (1.32).

From (1.19), (1.14), and (2.1), summing of (3.1)–(3.4) implies the energy equality

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|u|^2\mathrm{d}x+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|\eta_t|^2\mathrm{d}S+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|\nabla\eta_t|^2\mathrm{d}S+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|\Delta\eta|^2\mathrm{d}S\\
&+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|v_t|^2\mathrm{d}S+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Q(v,\eta)+2\nu\int_{\Omega}|D(u)|^2\mathrm{d}x\\
=&\int_{\Omega}G_f\cdot u\mathrm{d}x+\int_{\Gamma_1}G\cdot v_t\mathrm{d}S+\int_{\Gamma_1}G_3\eta_t\mathrm{d}S,
\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}
Q(v,\eta)&=\frac{1}{2(1+\mu)}\int_{\Gamma_1}\Big(N_1^2+N_2^2-2\mu N_1N_2+2(1+\mu)N_{12}^2\Big)\mathrm{d}S\\
&=\frac{2}{1-\mu}\int_{\Gamma_1}\Big(\varepsilon_1^2+\varepsilon_2^2+2\mu\varepsilon_1\varepsilon_2+\frac{1}{2}(1-\mu)\varepsilon_{12}^2\Big)\mathrm{d}S.
\end{aligned}\tag{3.6}$$

Hence, using Cauchy-Schwarz's and Young's inequalities in (3.6) results in

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|u|^2\mathrm{d}x+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|\eta_t|^2\mathrm{d}S+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|\nabla\eta_t|^2\mathrm{d}S+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|\Delta\eta|^2\mathrm{d}S\\
&+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}|v_t|^2\mathrm{d}S+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Q(v,\eta)+2\nu\int_{\Omega}|D(u)|^2\mathrm{d}x\\
\le&\frac{1}{2}\|G_f\|_{L^2(\Omega)}^2+\frac{1}{2}\|G_3\|_{L^2(\Gamma_1)}^2+\frac{1}{2}\|G\|_{L^2(\Gamma_1)}^2+\frac{1}{2}\|u\|_{L^2(\Omega)}^2+\frac{1}{2}\|\eta_t\|_{L^2(\Gamma_1)}^2+\frac{1}{2}\|v_t\|_{L^2(\Gamma_1)}^2,
\end{aligned}\tag{3.7}$$

which implies

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}|u|^2\mathrm{d}x+\frac{1}{2}\int_{\Gamma_1}|\eta_t|^2\mathrm{d}S+\frac{\gamma}{2}\int_{\Gamma_1}|\nabla\eta_t|^2\mathrm{d}S+\frac{1}{2}\int_{\Gamma_1}|\Delta\eta|^2\mathrm{d}S+\frac{1}{2}\int_{\Gamma_1}|v_t|^2\mathrm{d}S\\
&+\frac{1}{2}\int_{\Gamma_1}|\nabla v|^2\mathrm{d}S+2\nu\int_0^t\int_{\Omega}|D(u)|^2\mathrm{d}x\mathrm{d}s\\
\le&\, e^t\Big(\frac{1}{2}\|u_0\|_{L^2(\Omega)}^2+\frac{1}{2}\|\eta_1\|_{L^2(\Gamma_1)}^2+\frac{\gamma}{2}\|\nabla\eta_1\|_{L^2(\Gamma_1)}^2+\frac{1}{2}\|\Delta\eta_0\|_{L^2(\Gamma_1)}^2+\frac{1}{2}\|v_1\|_{L^2(\Gamma_1)}^2+\frac{1}{2}\|\nabla v_0\|_{L^2(\Gamma_1)}^2\Big)\\
&+\frac{1}{2}\int_0^t\exp(t-s)\Big(\|G_f\|_{L^2(\Omega)}^2+\|G_3\|_{L^2(\Gamma_1)}+\|G\|_{L^2(\Gamma_1)}\Big),
\end{aligned}\tag{3.8}$$

by virtue of the Gronwall lemma [27] and an inequality involving $Q(v,\eta)$ given by

$$\|v^1\|^2_{H^1_0(\Gamma_1)} + \|v^2\|^2_{H^1_0(\Gamma_1)} \leq C(Q(v,\eta) + \|\eta\|^2_{L^2(\Gamma_1)} + \|\eta\|^4_{H^{3/2}(\Gamma_1)})$$

for every $v \in H^1_0(\Gamma_1), \eta \in H^2_0(\Gamma_1)$ (see Proposition 3.4 in [5]).

Suppose that $G_f \in L^2(0,T;L^2(\Omega)), G \in L^2(0,T;L^2(\Gamma_1)), G_3 \in L^2(0,T;L^2(\Gamma_1))$ and $\eta_0 \in H^2_0(\Gamma_1), \eta_1 \in \hat{H}^1_0(\Gamma_1), v_0 \in H^1_0(\Gamma_1), v_1 \in L^2(\Gamma_1), u_0 \in H$. Then, this yields

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.9}$$

$$\eta_t \in L^\infty(0,T;\hat{H}^1_0(\Gamma_1)),\ \eta \in L^\infty(0,T;H^2_0(\Gamma_1)), \tag{3.10}$$

$$v \in L^\infty(0,T;H^1_0(\Gamma_1)),\ v_t \in L^\infty(0,T;L^2(\Gamma_1)). \tag{3.11}$$

Remark 3.1. We claim that $u \in L^4(0,T;L^4(\Omega))$ in 2D and $u \in L^4(0,T;L^3(\Omega))$ in the 3D case.

Proof. By Sobolev embedding and interpolation, we get

$$L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \hookrightarrow L^4(0,T;H^{\frac{1}{2}}(\Omega)) \hookrightarrow L^4(0,T;L^{\frac{2d}{d-1}}(\Omega)),\ d = 2,3.$$

□

3.2. Galerkin scheme and compactness argument

Step 1. *The local in time approximated solution*

Let $\{\psi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of the space $\{u \in H^1(\Omega), \operatorname{div} u = 0 \text{ in } \Omega\}$ constructed with the eigenfunctions of the Stokes problem:

$$-\Delta\psi_i + \nabla p_i = \mu_i^*\psi_i \ \text{ in } \Omega, \quad \operatorname{div}\psi_i = 0 \ \text{ in } \ \Omega, \quad \psi_i = 0 \ \text{ on } \partial\Omega,$$

where $0 < \mu_1^* \leq \mu_2^* \leq \cdots$ are the corresponding eigenvalues.

Define $\{\xi_i\}_{i\in\mathbb{N}}$ as the basis in $H^2_0(\Gamma_1)$ which consists of the eigenfunctions for the corresponding eigenvalues $0 < \hat{\iota}_1 \leq \hat{\iota}_2 \leq \cdots$ for the following problem:

$$(\Delta\xi_i, \Delta u)_{\Gamma_1} = \hat{\iota}_i((1-\gamma\Delta)\xi_i, u)_{\Gamma_1}, \qquad \forall u \in H^2_0(\Gamma_1),$$

where $((1-\gamma\Delta)\xi_i, \xi_j)_{\Gamma_1} = \delta_{ij}$.

Let $\{\tau_i\}_{i\in\mathbb{N}}$ be the unit basis in $H^1_0(\Gamma_1) \times H^1_0(\Gamma_1)$ satisfying

$$a(\tau_i, u) = \tilde{\iota}_i(\tau_i, u)_{\Gamma_1}, \quad \forall u \in H^1_0(\Gamma_1) \times H^1_0(\Gamma_1)$$

with eigenvalues $0 < \tilde{\iota}_1 \leq \tilde{\iota}_2 \leq \cdots$.

The functions $\{\hat{\phi}_i\}_{i\in\mathbb{N}}$ satisfy $\operatorname{div}\hat{\phi}_i = 0$, $\hat{\phi}_i|_{\Gamma_1} = (0,0,\xi_i)^T$, and we solve the following Stokes-like problem:

$$\begin{aligned} -\Delta\hat{\phi}_i + \nabla p_i &= 0 \quad \text{in } \Omega, \\ \operatorname{div}\hat{\phi}_i &= 0 \quad \text{in } \Omega, \\ \hat{\phi}_i &= \begin{cases} 0 & \text{on } \Gamma_0, \\ (0,0,\xi_i)^T & \text{on } \Gamma_1. \end{cases} \end{aligned}$$

Similarly, the functions $\{\tilde{\phi}_i\}_{i\in\mathbb{N}}$ solve the following problem:

$$\begin{aligned} -\Delta\tilde{\phi}_i + \nabla p_i &= 0 \quad \text{in } \Omega, \\ \operatorname{div}\tilde{\phi}_i &= 0 \quad \text{in } \Omega, \\ \tilde{\phi}_i &= \begin{cases} 0 & \text{on } \Gamma_0, \\ (\tau_i, 0)^T & \text{on } \Gamma_1. \end{cases} \end{aligned}$$

Define the special basis ϕ_i, π_i as

$$\phi_i = \begin{cases} \tilde{\phi}_i & \text{for } i = 1, \cdots, n, \\ \hat{\phi}_i & \text{for } i = n+1, \cdots, 2n, \end{cases} \qquad \pi_i = \begin{cases} (\tau_i, 0)^T & \text{for } i = 1, \cdots, n, \\ (0, 0, \xi_i)^T & \text{for } i = n+1, \cdots, 2n. \end{cases}$$

Hence, our aim is to find

$$u^{m,n} = \sum_{l=1}^{m} \alpha_l(t)\psi_l + \sum_{k=1}^{2n} \dot{\beta}_k(t)\phi_k, \tag{3.12}$$

$$(v^n, \eta^n) = \sum_{k=1}^{2n} \beta_k(t)\pi_k + (v_0, \eta_0), \tag{3.13}$$

satisfying the approximated problems

$$\begin{aligned} &\int_\Omega u_t^{m,n} \cdot \psi_i \mathrm{d}x + \nu \int_\Omega \nabla u^{m,n} \cdot \nabla\psi_i \mathrm{d}x + \frac{1}{2}\int_\Omega (u^{m,n}\cdot\nabla)u^{m,n}\cdot\psi_i \mathrm{d}x - \frac{1}{2}\int_\Omega (u^{m,n}\cdot\nabla)\psi_i\cdot u^{m,n}\mathrm{d}x \\ = &\int_\Omega G_f \cdot \psi_i \mathrm{d}x, \ \forall 1 \le i \le m \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\int_\Omega u_t^{m,n} \cdot \phi_j \mathrm{d}x + \int_\Omega \nabla u^{m,n} \cdot \nabla\phi_j \mathrm{d}x + \frac{1}{2}\int_\Omega (u^{m,n}\cdot\nabla)u^{m,n}\cdot\phi_j \mathrm{d}x \\ &- \frac{1}{2}\int_\Omega (u^{m,n}\cdot\nabla)\phi_j\cdot u^{m,n}\mathrm{d}x + \int_{\Gamma_1} (1-\gamma\Delta)\eta_{tt}^n \pi_j \mathrm{d}S \\ &+ \int_{\Gamma_1} \Delta\eta^n \Delta\pi_j \mathrm{d}S + \int_{\Gamma_1} v_{tt}^n \pi_j \mathrm{d}S + a(v^n, \pi_j) + P(v^n, \eta^n, \pi_j) \\ = &\int_\Omega G_f \cdot \phi_j \mathrm{d}x + \int_{\Gamma_1} \bar{G}\cdot\pi_j \mathrm{d}S, \ \forall 1 \le j \le 2n, \end{aligned} \tag{3.15}$$

respectively, where $\bar{G} = (G_1, G_2, G_3)^T$.

For our purpose, the approximated problems of (3.14) and (3.15) can be constructed as

$$\sum_{l=1}^{m} \dot{\alpha}_l(t)\int_\Omega \psi_l\psi_i \mathrm{d}x + \sum_{k=1}^{2n}\ddot{\beta}_k(t)\int_\Omega \phi_k\psi_i\mathrm{d}x + \sum_l^m\sum_k^{2n} g_1(\alpha_l(t), \dot{\beta}_k(t)) = \int_\Omega G_f\cdot\psi_i\mathrm{d}x \tag{3.16}$$

and

$$\begin{aligned} &\sum_{l=1}^{m} \dot{\alpha}_l(t)\int_\Omega \psi_l\phi_j \mathrm{d}x + \sum_{k=1}^{2n}\ddot{\beta}_k(t)\int_{\Gamma_1} \phi_k\phi_j\mathrm{d}S + \sum_{k=1}^{2n}\ddot{\beta}_k(t)\delta_{kj} + \sum_l^m\sum_k^{2n} g_2(\alpha_l, \beta_k, \dot{\beta}_k) \\ = &\int_\Omega G_f\cdot\phi_j\mathrm{d}x + \int_{\Gamma_1}\bar{G}\cdot\pi_j\mathrm{d}S \end{aligned} \tag{3.17}$$

with Lipschitz continuous functions g_1 and g_2 and initial data

$$u^{m,n}(0) = u_0^{m,n},\ \beta_i(0) = 0,\ \eta_t^n(0) = \eta_1^n,\ v_t^n(0) = v_1^n,$$

where $u_0^{m,n}$, η_1^n, and v_1^n denote the projections of u_0, η_1, and v_1 onto the finite-dimensional spaces $\mathrm{span}(\psi_i, \phi_j)_{1\le i\le m, 1\le j\le 2n}$ and $\mathrm{span}(\pi_j)_{1\le j\le 2n}$.

For symmetric matrix $M(t)$ with non-negative smooth functions and symmetric positive semi-definite matrix N, denote

$$M(t) + N = \begin{pmatrix} \int_\Omega \psi_l \psi_i \mathrm{d}x & \int_\Omega \phi_k \psi_i \mathrm{d}x \\ \int_\Omega \psi_l \phi_j \mathrm{d}x & \int_\Omega \phi_k \phi_j \mathrm{d}x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

with $1 \le i \le m,\ 1 \le j \le 2n,\ 1 \le k \le 2n,\ 1 \le l \le m.$ Then, the problems (3.16) and (3.17) can be rewritten as the following abstract ordinary differential equation:

$$(M(t) + N)\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \alpha_l(t) \\ \dot\beta_k(t) \end{pmatrix} + g(\alpha_l(t), \beta_k(t), \dot\beta_k(t)) = F(t)$$

for some locally Lipschitz function $g : \mathbb{R}^{m+4n} \mapsto \mathbb{R}^{m+2n}$ and $F \in L^2(0, T; \mathbb{R}^{m+2n})$. By the Cauchy-Peano theorem, the local in time approximated solutions of (3.14) and (3.15) exist on $[0, T_{m,n}]$ for some $T_{m,n} > 0$. Our goal in the sequel is to prove that $T_{m,n}$ can be an arbitrary time in $\mathbb{R}^+$.

Step 2. *Energy estimates and a compact argument*

Multiply (3.14) by α_i and sum over $i = 1 \cdots m$, multiply (3.15) by $\dot\beta_j$ and sum over $j = 1 \cdots 2n$, add resulting equations. Then, this yields

$$\begin{aligned} &\int_\Omega u_t^{m,n} \cdot u^{m,n} \mathrm{d}x + \nu \int_\Omega u^{m,n} \cdot \nabla u^{m,n} \mathrm{d}x + \int_{\Gamma_1} (1 - \gamma\Delta)\eta_{tt}^n \eta_t^n \mathrm{d}S \\ &+ \int_{\Gamma_1} \Delta\eta^n \Delta\eta_t^n \mathrm{d}S + \int_{\Gamma_1} v_{tt}^n\, v_t^n \mathrm{d}S + a(v^n, v_t^n) + P(v^n, \eta^n, v_t^n, \eta_t^n) \\ = &\int_\Omega G_f \cdot u^{m,n} \mathrm{d}x + \int_{\Gamma_1} G_3\, \eta_t^n \mathrm{d}S + \int_{\Gamma_1} G \cdot v_t^n \mathrm{d}S. \end{aligned}$$

From the definition of $a(v, \eta)$ in (2.3) and $P(v, \eta, b, d)$ in (2.4), using the same technique as in Section 3.1 results in

$$\begin{aligned} &\|u^{m,n}\|_{L^\infty(0,T;L^2(\Omega)} + \|\nabla u^{m,n}\|_{L^2(0,T;L^2(\Omega)} + \|\eta_t^n\|_{L^\infty(0,T;L^2(\Gamma_1))} + \|\Delta\eta^n\|_{L^\infty(0,T;L^2(\Gamma_1))} \\ &\qquad + \|v_t^n\|_{L^\infty(0,T;L^2(\Gamma_1))} + \|\nabla v^n\|_{L^\infty(0,T;L^2(\Gamma_1))} \le C, \end{aligned} \tag{3.18}$$

where the constant C depends on the given body forces and initial data but is independent on the index m, n. Thus $T_{m,n} = T$.

Next, the compact argument is given for the limiting procedure in the sequel.

From the Banach-Alaoglu theorem, a standard diagonal argument for choosing the subsequence of $u^{m,n}, v^n, \eta^n$ (relabeled to avoid confusion) is given. Hence, for every time $T > 0$, the convergences

$$u^{m,n} \overset{*}{\rightharpoonup} u \quad \text{weakly-}* \text{ in} \quad L^\infty(0, T; H), \tag{3.19}$$

$$u^{m,n} \rightharpoonup u \quad \text{weakly in} \quad L^2(0, T; V), \tag{3.20}$$

$$\eta^n \overset{*}{\rightharpoonup} \eta \quad \text{weakly-}* \text{ in} \quad L^\infty(0, T; H_0^2(\Gamma_1)), \tag{3.21}$$

$$\eta_t^n \overset{*}{\rightharpoonup} \eta_t \quad \text{weakly-}* \text{ in } \quad L^\infty(0,T;\hat{H}_0^1(\Gamma_1)), \tag{3.22}$$

$$v^n \overset{*}{\rightharpoonup} v \quad \text{weakly-}* \text{ in } \quad L^\infty(0,T;H_0^1(\Gamma_1)), \tag{3.23}$$

$$v_t^n \overset{*}{\rightharpoonup} v_t \quad \text{weakly-}* \text{ in } \quad L^\infty(0,T;L^2(\Gamma_1)) \tag{3.24}$$

hold.

The application of the Aubin-Lions lemma results in

$$v^n \to v \quad \text{strongly in} \quad C(0,T;H_0^{1-\epsilon}(\Gamma_1)), \tag{3.25}$$

$$\eta^n \to \eta \quad \text{strongly in} \quad C(0,T;H_0^{2-\epsilon}(\Gamma_1)) \tag{3.26}$$

for every $\epsilon > 0$.

Choose the test function $\zeta \in V$ with $\zeta|_{\Gamma_1} = 0$. Then, system (2.2) is decoupled and the classical result $u_t \in L^{4/3}(0,T;V^*)$ can be obtained (see [28] for more details).

The combination of the above convergences and the Aubin-Lions lemma implies the following strong convergence:

$$u^{m,n} \to u \quad \text{strongly in} \quad L^2(0,T;H). \tag{3.27}$$

Remark 3.2. Owing to the compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow H^{1-\epsilon}(\Omega)$, using the trace theorem, we see that $u^{m,n}|_{\Gamma_1} \to u|_{\Gamma_1}$ strongly in $L^2(0,T;L^2(\Gamma_1))$.

Step 3. *Passage to the limit to the limit*

Let $\wp$ and ϖ be two smooth functions on $[0,T]$ with $\wp(T)=0, \varpi(T)=0$. Multiply (3.14) and (3.15) by $\wp(t)$ and $\varpi(t)$, respectively, and integrate by parts. Then, this procedure leads to

$$\begin{aligned}
&-\int_0^T\int_\Omega u^{m,n}\cdot\Phi_t^{m,n}\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega\nabla u^{m,n}:\nabla\Phi^{m,n}\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^T\int_\Omega(u^{m,n}\cdot\nabla)u^{m,n}\cdot\Phi^{m,n}\mathrm{d}x\mathrm{d}t\\
&-\frac{1}{2}\int_0^T\int_\Omega(u^{m,n}\cdot\nabla)\Phi^{m,n}\cdot u^{m,n}\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\Gamma_1}(1-\gamma\Delta)\eta_t^n\Psi_t^n\mathrm{d}S\,\mathrm{d}t+\int_0^T\int_{\Gamma_1}\Delta\eta^n\Delta\Psi^n\mathrm{d}S\,\mathrm{d}t\\
&-\int_0^T\int_{\Gamma_1}v_t^n\Psi_t^n\mathrm{d}x\mathrm{d}t+\int_0^T a(v^n,\Psi^n)\mathrm{d}t+\int_0^T P(v^n,\eta^n,\Psi^n)\mathrm{d}t\\
=&\int_0^T\int_\Omega G_f\cdot\Phi^{m,n}\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\Gamma_1}G\Psi^n\mathrm{d}S\,\mathrm{d}t+\int_\Omega u_0^{m,n}\Phi^{m,n}(0)\mathrm{d}x\\
&+\int_{\Gamma_1}(1-\gamma\Delta)\eta_1^n\Psi^n(0)\mathrm{d}S+\int_{\Gamma_1}v_1^n\Psi^n(0)\mathrm{d}S.
\end{aligned}\tag{3.28}$$

Set

$$\Phi^{m,n}=\sum_{l=1}^m\wp(t)\psi_l+\sum_{k=1}^{2n}\varpi(t)\phi_k,\ \Psi^n=\sum_{k=1}^{2n}\varpi(t)\pi_k,$$

$$\Phi^{m,n}(0)=\sum_{l=1}^m\psi_l\wp(0)+\sum_{k=1}^{2n}\phi_k\varpi(0),\ \Psi^n(0)=\sum_{k=1}^{2n}\pi_k\varpi(0).$$

Then, concerning the nonlinear term $b(u^{m,n},u^{m,n},\Psi^{m,n})$, the conditions in (3.20) and (3.27) imply

$$u_i^{m,n}u_j^{m,n}\to u_iu_j \quad \text{in} \quad \mathcal{D}'(\Omega).$$

According to Remark 3.1, $u_i^{m,n}u_j^{m,n}$ are bounded in $L^2(0,T;L^{3/2}(\Omega))$. Thus, Lemma 2.3 implies that

$$u_i^{m,n}u_j^{m,n} \rightharpoonup u_i u_j \quad \text{weakly in} \;\; L^2(0,T;L^{3/2}(\Omega)).$$

Similarly, the convergence of $(u^{m,n}\cdot n)u^{m,n}$ can be deduced from Remark 3.2.

The passage to the limit for $P(v^n,\eta^n,\Psi^n)$ can be done by using the same technique. For simplicity, we only consider the term $\eta_x^n(\eta_y^n)^2$ here, and other terms can be dealt with similarly. In fact, the convergences in (3.21) and (3.26) imply that $\eta_x^n(\eta_y^n)^2$ is bounded in $L^\infty(0,T;L^2(\Gamma_1))$ and

$$\eta_x^n(\eta_y^n)^2 \to \eta_x(\eta_y)^2 \quad \text{in} \;\; \mathcal{D}'(\Omega).$$

Using of Lemma 2.3 leads to

$$\eta_x^n(\eta_y^n)^2 \rightharpoonup \eta_x(\eta_y)^2 \;\; \text{weakly-* in} \;\; L^\infty(0,T;L^2(\Gamma_1)).$$

Passing to the limit as $m,n\to\infty$ for $u^{m,n},\eta^n,v^n$, we conclude that

$$\begin{aligned}
&-\int_0^T\int_\Omega u\cdot\Phi_t^{m,n}\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega\nabla u:\nabla\Phi^{m,n}\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^T\int_\Omega(u\cdot\nabla)u\cdot\Phi^{m,n}\mathrm{d}x\mathrm{d}t\\
&-\frac{1}{2}\int_0^T\int_\Omega(u\cdot\nabla)\Phi^{m,n}\cdot u\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\Gamma_1}(1-\gamma\Delta)\eta_t\Psi_t^n\mathrm{d}S\,\mathrm{d}t+\int_0^T\int_{\Gamma_1}\Delta\eta\Delta\Psi^n\mathrm{d}S\,\mathrm{d}t\\
&-\int_0^T\int_{\Gamma_1}v_t\Psi_t^n\mathrm{d}S\,\mathrm{d}t+\int_0^T a(v,\Psi^n)\mathrm{d}t+\int_0^T P(v,\eta,\Psi^n)\mathrm{d}t\\
=&\int_0^T\int_\Omega G_f\cdot\Phi^{m,n}\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\Gamma_1}G\Psi^n\mathrm{d}S\,\mathrm{d}t+\int_\Omega u_0\Phi^{m,n}(0)\mathrm{d}x\mathrm{d}t\\
&+\int_{\Gamma_1}(1-\gamma\Delta)\eta_1\Psi^n(0)\mathrm{d}S\,\mathrm{d}t+\int_{\Gamma_1}v_1\Psi^n(0)\mathrm{d}S\,\mathrm{d}t
\end{aligned}\tag{3.29}$$

holds for test functions $\Phi^{m,n},\Psi^n$ composed by finite linear combinations of $\psi_l,\phi_k,$ and π_k.

The continuity argument implies that $\Phi^{m,n}\to\phi$ holds true in (3.29), which results in our desired conclusion:

$$\begin{aligned}
&-\int_0^T\int_\Omega u\cdot\phi_t\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega\nabla u:\nabla\phi\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^T\int_\Omega(u\cdot\nabla)u\cdot\phi\mathrm{d}x\mathrm{d}t\\
&-\frac{1}{2}\int_0^T\int_\Omega(u\cdot\nabla)\phi\cdot u\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\Gamma_1}(1-\gamma\Delta)\eta_t\,d_t\mathrm{d}S\,\mathrm{d}t+\int_0^T\int_{\Gamma_1}\Delta\eta\Delta d\mathrm{d}S\,\mathrm{d}t\\
&-\int_0^T\int_{\Gamma_1}v_t\,b_t\mathrm{d}S\,\mathrm{d}t+\int_0^T a(v,b)\mathrm{d}t+\int_0^T P(v,\eta,b,d)\mathrm{d}t\\
=&\int_0^T\int_\Omega G_f\cdot\phi\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\Gamma_1}G\,b\mathrm{d}S\,\mathrm{d}t+\int_\Omega u_0\,\phi(0)\mathrm{d}x\mathrm{d}t\\
&+\int_{\Gamma_1}(1-\gamma\Delta)\eta_1\,d(0)\mathrm{d}S\,\mathrm{d}t+\int_{\Gamma_1}v_1\,b(0)\mathrm{d}S\,\mathrm{d}t,
\end{aligned}\tag{3.30}$$

where ϕ,b,d satisfy the corresponding conditions in (2.2). This finishes the proof.

Remark 3.3. The test function of (2.2) can also be chosen as

$$\phi = \sum_{i=1}^{p} \wp_i^1(t)\psi_i + \sum_{j=1}^{q} \wp_j^2(t)\tilde{\phi}_j + \sum_{j=1}^{q} \wp_j^3(t)\hat{\phi}_j,$$

where $p \le m,\ q \le n$, and $\wp^i$ $(i = 1, 2, 3)$ are scalar absolutely continuous functions on $[0, T]$ with time derivatives in $L^2([0, T])$, which can be used for the construction of approximated solutions $u^{m,n}, v^n, \eta^n$ also. Moreover, the procedure in the above proof still holds true.

4. Proof of Theorem 2.5

The proof of the global existence of weak solutions for the 3D case in Section 3 remains valid for the 2D Navier-Stokes equations coupled with (1.20)–(1.29). For simplicity, we omit the details and only present the proof of the uniqueness of the weak solution in this section.

Let $(\tilde{u}, \tilde{\eta}, \tilde{v})$ and $(\bar{u}, \bar{\eta}, \bar{v})$ be two weak solutions to our problem with the same data, satisfying the following weak forms:

$$\begin{aligned}
&\int_\Omega \tilde{u} \cdot \psi \mathrm{d}x + \int_0^L (1 - \partial_{xx})\tilde{\eta}_t \beta^1 \mathrm{d}x + \int_0^L \tilde{v}_t \beta^2 \mathrm{d}x \\
&= -\nu \int_0^s \int_\Omega \nabla\tilde{u} : \nabla\psi \mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^s \int_\Omega (\tilde{u} \cdot \nabla\tilde{u}) \cdot \psi \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^s \int_\Omega (\tilde{u} \cdot \nabla\psi) \cdot \tilde{u} \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^s \int_0^L \tilde{\eta}_{xx}(\beta^1)_{xx} \mathrm{d}x\mathrm{d}t - \int_0^s P(\tilde{\eta}, \tilde{v}, \beta)\mathrm{d}t - \int_\Omega G_f \cdot \psi \mathrm{d}x - \int_0^L G_1\beta^1 \mathrm{d}x - \int_0^L G_2\beta^2 \mathrm{d}x
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\int_\Omega \bar{u} \cdot \psi \mathrm{d}x + \int_0^L (1 - \partial_{xx})\bar{\eta}_t \beta^1 \mathrm{d}x + \int_0^L \bar{v}_t \beta^2 \mathrm{d}x \\
&= -\nu \int_0^s \int_\Omega \nabla\bar{u} : \nabla\psi \mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^s \int_\Omega (\bar{u} \cdot \nabla\bar{u}) \cdot \psi \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^s \int_\Omega (\bar{u} \cdot \nabla\psi) \cdot \bar{u} \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^s \int_0^L \bar{\eta}_{xx}(\beta^1)_{xx} \mathrm{d}x\mathrm{d}t - \int_0^s P(\bar{\eta}, \bar{v}, \beta)\mathrm{d}t - \int_\Omega G_f \cdot \psi \mathrm{d}x - \int_0^L G_1\beta^1 \mathrm{d}x - \int_0^L G_2\beta^2 \mathrm{d}x
\end{aligned} \tag{4.2}$$

for every $s \in [0, T]$, respectively, where $\psi \in V$, $\beta^1 \in \hat{H}_0^2(0, L)$, $\beta^2 \in H_0^1(0, L)$ satisfy $\psi(t, x)|_{\Gamma_1} = (\beta^1, \beta^2)^T$ and

$$P(\eta, v, \beta) = \frac{2}{1 - \mu}\int_0^L \Big(\eta_x(v_x + \frac{1}{2}\eta_x^2 + k_1\eta)(\beta^1)_x + k_1(v_x + \frac{1}{2}\eta_x^2 + k_1\eta)\beta^1 + (\frac{1}{2}\eta_x^2 + k_1\eta)(\beta^2)_x\Big)\mathrm{d}x.$$

Set $\mathcal{U} = \tilde{u} - \bar{u},\ \mathcal{H} = \tilde{\eta} - \bar{\eta},\ \mathcal{V} = \tilde{v} - \bar{v}$, which satisfy

$$\mathcal{U} \in L^2(0, T; V) \cap L^\infty(0, T; H), \tag{4.3}$$

$$\mathcal{H} \in L^\infty(0, T; H_0^2(0, L)),\ \mathcal{H}_t \in L^\infty(0, T; \hat{H}_0^1(0, L)), \tag{4.4}$$

$$\mathcal{V} \in L^\infty(0, T; H_0^1(0, L)),\ \mathcal{V}_t \in L^\infty(0, T; L^2(0, L)) \tag{4.5}$$

and

$$
\begin{aligned}
&\int_\Omega \mathcal{U}\cdot\psi \mathrm{d}x + \int_0^L (1-\partial_{xx})\mathcal{H}_t\beta^1 \mathrm{d}x + \int_0^L \mathcal{V}_t\beta^2 \mathrm{d}x \\
={}& -\nu\int_0^s\int_\Omega \nabla\mathcal{U}:\nabla\psi \mathrm{d}x\mathrm{d}t - \int_0^s b(\mathcal{U},\tilde{u},\psi)\mathrm{d}t - \int_0^s\int_0^L \mathcal{H}_{xx}(\beta^1)_{xx}\mathrm{d}x\mathrm{d}t \\
&- \int_0^s P(\tilde{\eta},\tilde{v},\beta)\mathrm{d}t + \int_0^s P(\bar{\eta},\bar{v},\beta)\mathrm{d}t.
\end{aligned}
\tag{4.6}
$$

Take $\psi = \int_0^s \mathcal{U}(\sigma)\mathrm{d}\sigma$ with $\psi|_{\Gamma_1} = \beta = (\mathcal{H},\mathcal{V})^T$ for each $s\in[0,T]$ in (4.6) for fixed s and integrate from 0 to t over s. Then, this yields

$$
\begin{aligned}
&\int_\Omega\left|\int_0^t \mathcal{U}(\sigma)\mathrm{d}\sigma\right|^2\mathrm{d}x + \int_0^L(\mathcal{H}^2+(\mathcal{H}_x)^2)\mathrm{d}x + \int_0^L\mathcal{V}^2\mathrm{d}x + \frac{2}{1-\mu}\int_0^s\int_0^L(\mathcal{V}_x)^2\mathrm{d}x \\
&+ 2\nu\int_0^t\int_\Omega\left|\nabla\int_0^s\mathcal{U}(\sigma)\mathrm{d}\sigma\right|^2\mathrm{d}x\mathrm{d}s + 2\int_0^t\int_0^s b\Big(\mathcal{U},\tilde{u},\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big)\mathrm{d}\sigma\mathrm{d}s \\
&+ 2\int_0^t\int_0^s\int_0^L(\mathcal{H}_{xx})^2\mathrm{d}x\mathrm{d}\tau\mathrm{d}s \\
={}& 2\int_0^t\int_0^s(P(\tilde{\eta},\tilde{v},\beta)-P(\bar{\eta},\bar{v},\beta))\mathrm{d}\tau\mathrm{d}s,
\end{aligned}
\tag{4.7}
$$

which implies

$$
\begin{aligned}
&2\int_0^t\Big(P\Big(\tilde{\eta},\tilde{v},\int_0^s\beta(\sigma)\mathrm{d}\sigma\Big) - P\Big(\bar{\eta},\bar{v},\int_0^s\beta(\sigma)\mathrm{d}\sigma\Big)\Big)\mathrm{d}s \\
={}& \frac{4}{1-\mu}\int_0^t\Big\{\int_0^L(k_1)^2\mathcal{H}\int_0^s\mathcal{H}(\sigma)\mathrm{d}\sigma\mathrm{d}x + \frac{1}{2}\int_0^L k_1(\tilde{\eta}_x+\bar{\eta}_x)\mathcal{H}_x\int_0^s\mathcal{H}(\sigma)\mathrm{d}\sigma\mathrm{d}x \\
&+ \int_0^L\bar{v}_x\mathcal{H}_x\int_0^s\mathcal{H}_x(\sigma)\mathrm{d}\sigma\mathrm{d}x + \int_0^L k_1(\tilde{\eta}_x\mathcal{H}+\bar{\eta}_x\mathcal{H}_x)\int_0^s\mathcal{H}_x(\sigma)\mathrm{d}\sigma\mathrm{d}x \\
&+ \frac{1}{2}\int_0^L((\tilde{\eta}_x)^2+\tilde{\eta}_x\bar{\eta}_x+(\bar{\eta}_x)^2)\mathcal{H}_x\int_0^s\mathcal{H}_x(\sigma)\mathrm{d}\sigma\mathrm{d}x \\
&+ \int_0^L\Big(k_1\mathcal{V}_x\int_0^s\mathcal{H}(\sigma)\mathrm{d}\sigma + k_1\mathcal{H}_x\int_0^s\mathcal{V}_x(\sigma)\mathrm{d}\sigma\Big)\mathrm{d}x \\
&+ \int_0^L\tilde{\eta}_x\mathcal{V}_x\int_0^s\mathcal{H}_x(\sigma)\mathrm{d}\sigma\mathrm{d}x + \frac{1}{2}\int_0^L(\tilde{\eta}_x+\bar{\eta}_x)\mathcal{H}_x\int_0^s\mathcal{V}_x(\sigma)\mathrm{d}\sigma\mathrm{d}x\Big\}\mathrm{d}s.
\end{aligned}
\tag{4.8}
$$

By a similar technique and Fubini's theorem, the left-hand side of (4.7) can be expressed as

$$
\mathcal{E}(t) + 2\nu\int_0^t\int_\Omega\left|\nabla\int_0^s\mathcal{U}(\sigma)\mathrm{d}\sigma\right|^2\mathrm{d}x\mathrm{d}s + 2\int_0^t b\Big(\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau,\tilde{u},\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big)\mathrm{d}s,
\tag{4.9}
$$

where

$$
\begin{aligned}
\mathcal{E}(t) ={}& \int_\Omega\left|\int_0^t\mathcal{U}(\sigma)\mathrm{d}\sigma\right|^2\mathrm{d}x + \int_0^L(\mathcal{H}^2+(\mathcal{H}_x)^2)\mathrm{d}x + \int_0^L\Big(\int_0^t H_{xx}(\sigma)\mathrm{d}\sigma\Big)^2\mathrm{d}x \\
&+ \int_0^L\mathcal{V}^2\mathrm{d}x + \frac{2}{1-\mu}\int_0^L\Big(\int_0^t\mathcal{V}_x(\sigma)\mathrm{d}\sigma\Big)^2\mathrm{d}x.
\end{aligned}
\tag{4.10}
$$

Integrating by parts, we can write the sixth and seventh terms in the right-hand side of (4.8) with $\mathcal{V}_x$ as

$$\begin{aligned}&\int_0^t\Big(\int_0^L k_1\mathcal{V}_x\int_0^s\mathcal{H}(\sigma)\mathrm{d}\sigma\mathrm{d}x\Big)\mathrm{d}s\\ =&\int_0^L\Big(\int_0^t k_1\mathcal{V}_x\mathrm{d}\sigma\int_0^t\mathcal{H}(\sigma)\mathrm{d}\sigma\Big)\mathrm{d}x-\int_0^t\Big(\int_0^L\int_0^s k_1\mathcal{V}_x\mathrm{d}\sigma\mathcal{H}\mathrm{d}x\Big)\mathrm{d}s\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}&\int_0^t\Big(\int_0^L\tilde{\eta}_x\mathcal{V}_x\int_0^s\mathcal{H}_x(\sigma)\mathrm{d}\sigma\mathrm{d}x\Big)\mathrm{d}s\\ =&\int_0^L\Big(\int_0^t\tilde{\eta}_x\mathcal{V}_x\mathrm{d}\sigma\int_0^t\mathcal{H}_x(\sigma)\mathrm{d}\sigma\Big)\mathrm{d}x-\int_0^t\Big(\int_0^L\int_0^s\tilde{\eta}_x\mathcal{V}_x\mathrm{d}\sigma\mathcal{H}_x\mathrm{d}x\Big)\mathrm{d}s.\end{aligned}\tag{4.12}$$

Substituting (4.12) and (4.11) into (4.8), and applying Hölder's, Sobolev's, and Young's inequalities, we obtain

$$\int_0^t\Big(P\Big(\tilde{\eta},\tilde{v},\int_0^s\beta(\sigma)\mathrm{d}\sigma\Big)-P\Big(\bar{\eta},\bar{v},\int_0^s\beta(\sigma)\mathrm{d}\sigma\Big)\Big)\mathrm{d}s\le\frac{1}{4}\mathcal{E}(t)+C\int_0^t\mathcal{E}(\tau)\mathrm{d}\tau.\tag{4.13}$$

Using Lemma 2.1, the following estimate:

$$\begin{aligned}&\Big|b\Big(\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau,\tilde{u},\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big)\Big|\\ \le&\, c_1\Big|\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big|_{1/2}^2|\tilde{u}|_1+c_2\Big|\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big|_{1/2}\Big|\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big|_{3/4}|\tilde{u}|_{3/4}\\ \le&\, c_3\Big|\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big|_0\Big|\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big|_1|\tilde{u}|_1+c_4\Big|\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big|_0^{3/4}\Big|\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big|_1^{5/4}|\tilde{u}|_0^{1/4}|\tilde{u}|_1^{3/4}\end{aligned}\tag{4.14}$$

holds.

Integrating (4.14) from 0 to t, by Hölder's and Young's inequalities and considering (4.3)–(4.5), we derive

$$\begin{aligned}&\int_0^t\Big|b\Big(\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau,\tilde{u},\int_0^s\mathcal{U}(\tau)\mathrm{d}\tau\Big)\Big|\mathrm{d}s\\ \le&\, c_5\int_0^t\int_\Omega\Big|\int_0^s\mathcal{U}(\sigma)\mathrm{d}\sigma\Big|^2\mathrm{d}x\mathrm{d}s+\frac{\nu}{8}\int_0^t\int_\Omega\Big|\nabla\int_0^s\mathcal{U}(\sigma)\mathrm{d}\sigma\Big|^2\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{4.15}$$

It is worth pointing out that the above constants $c_i>0,\ i=1,2,3,4,5$, do not depend on t.

The combination of (4.7), (4.13), and (4.15) results in

$$\mathcal{E}(t)\le\mathcal{E}(t)+C_1\int_0^t\int_\Omega\Big|\nabla\int_0^s\mathcal{U}(\sigma)\mathrm{d}\sigma\Big|^2\mathrm{d}x\mathrm{d}s\le C_2\int_0^t\mathcal{E}(\tau)\mathrm{d}\tau,\tag{4.16}$$

which implies that $\mathcal{E}(t)=0$ for all $t\in[0,T]$, owing to the Gronwall lemma. This finishes the proof of uniqueness of the weak solution. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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