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Theory article

Existence of positive solution for critical Schrödinger-Poisson system on the first Heisenberg group

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Abstract: In this article, we consider a class of critical Schrödinger-Poisson type systems on the first Heisenberg group. We demonstrate that the system admits a positive ground state solution by utilizing the concentration compactness principle and critical point theory. The system has double critical nonlinearity on the Heisenberg group, which is the novelty and peculiarity of this paper. To some extent, we extend some previous results.

Keywords: Heisenberg group; Schrödinger-Poisson system; concentration-compactness principle; variational methods

1. Introduction and main results

In the present paper, we are interested in the following critical Schrödinger-Poisson type system on the first Heisenberg group:

$$\begin{cases} -\Delta_H u - \phi|u|u = \gamma|u|^{q-2}u + |u|^2u & \text{in } \Omega, \\ -\Delta_H \phi = |u|^3 & \text{in } \Omega, \\ \phi = u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_H is denoted as the Kohn-Laplacian on the first Heisenberg group $\mathbb{H}^1$, $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $q \in (1,2)$, and γ is a positive real parameter. The index $Q^* := 2Q/(Q-2) = 4$ is the Sobolev critical exponent on the first Heisenberg group, and $Q = 4$ is regarded as the homogeneous dimension of $\mathbb{H}^1$.

Our study of Schrödinger-Poisson systems (1.1) is based on theoretical and practical needs. On the one hand, Schrödinger-Poisson systems were first studied by [1]. For further information on the physical background of Schrödinger-Poisson systems, the reader is referred to [2, 3]. In contrast, the

Heisenberg group has close connections with multiple fields in mathematics and physics, including complex variables, quantum theory, theta function theory, signal theory, and Cauchy-Leray geometry (see [4,5]). In addition, systems with additional nonlinear terms are discussed in [6], and issues related to dimensionality reduction or constrained boundaries are addressed in [7]. Based on the rich physical background of such systems, many scholars have focused on this topic. In [8], An and Liu considered the following system on the Heisenberg group:

$$\begin{cases} -\Delta_H u + \lambda\phi u = \mu|u|^{q-2}u + |u|^2 u & \text{in } \Omega, \\ -\Delta_H \phi = u^2 & \text{in } \Omega, \\ \phi = u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

By employing the variational method in conjunction with Green's representation formula, it has been established that systems (1.2) admit at least two positive solutions. Bai et al. [9] investigated critical Schrödinger-Poisson type systems involving the p-Laplacian operator, utilizing the Krasnoselskii genus theory as a theoretical framework. For the non-local term satisfying critical growth, there are some results on this topic. On the Euclidean framework, He and Wang [10] were interested in the fractional Schrödinger-Poisson systems; using variational methods and Brouwer degree theory, some existence results of positive bounded solutions are obtained when $\lambda > 0$ is small enough. For more related results, see [11–14]. When shifting the focus to the Heisenberg group, it is apparent that research on this topic remains limited. Guo and Shi [15] studied the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta_H u - \phi|u|u = \mu|u|^{q-2}u & \text{in } \Omega, \\ -\Delta_H \phi = |u|^3 & \text{in } \Omega, \\ \phi = u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $1 < q < 2$ and $Q^* := 2Q/(Q-2) = 4$ is the Sobolev critical exponent on the first Heisenberg group, they obtained similar results as An and Liu [8].

Building on the aforementioned references, the primary objective of this study is to investigate the existence of positive ground state solutions for system (1.1). It is worth emphasizing that the choice of the cubic source term is not arbitrary; rather, it is dictated by the geometric structure of the Heisenberg group. Consequently, system (1.1) exhibits a double-critical nature that is both intrinsic and natural, necessitating the development of new estimates and analytical techniques. Below, we state the main result of this paper.

Theorem 1.1. *Let $q \in (1,2)$ be satisfied. Then, there exists a positive constant Λ_0 such that $\gamma \in (0, \Lambda_0)$, system* (1.1) *has a positive ground state solution.*

Remark 1.1. *Compared with the previous results, there are some difficulties when we consider the results of system* (1.1)*:*

(a) Different from the systems in [10, 15], system (1.1) *simultaneously has a critical non-local term and critical nonlinearity. Therefore, we need to make more accurate estimations in this paper. In addition, the loss of compactness shall occur; we will prove it by means of the principle of concentration and compactness on the Heisenberg group to overcome this difficulty.*

(b) *Although certain properties of the operator Δ_H resemble those of the classical Laplacian Δ, these similarities may be misleading; see, e.g., [16]. A notable example is the critical exponent on the Heisenberg group $\mathbb{H}^1$: it is $Q^* = 4$, whereas on $\mathbb{R}^3$, it is $2^* = 6$. This discrepancy has posed significant obstacles in establishing the existence of solutions to system* (1.1).

(c) *Compared with the result of Liang et al. [17], they showed some existence results of normalized solutions to a class of the critical Schrödinger-Poisson system, that is, λ is a Lagrange multiplier. However, λ is a fixed frequency in this paper. This fixed frequency setting alters the variational structure and requires distinct techniques for establishing the existence of ground states. Besides, in [18], they consider systems* (1.1) *in the case of perturbations that are superlinear, i.e., $q \in (2,4)$, which is essentially different from what we consider. In our sublinear case $q \in (1,2)$, the energy functional exhibits a more complex behavior near zero, and we must develop a new threshold condition (as in Lemma 3.5) to restore compactness. Consequently, the variational framework and the analysis of Palais-Smale sequences in our work are distinct from those in [17, 18].*

This article is well organized: Section 2 presents the foundational concepts pertaining to the Heisenberg group, establishing the necessary theoretical foundation. Section 3 introduces a series of crucial technical lemmas and provides a detailed proof of Theorem 1.1.

2. Variational setting and preliminaries

This section provides an overview of the fundamental properties of the first Heisenberg group and introduces the workspace $S_0^1(\Omega)$. To better understand this space, we refer to [16, 19–21]. Let $\mathbb{H}^1 = (\mathbb{R}^3, \circ)$ denote the first Heisenberg group. If $\xi = (x, y, t) \in \mathbb{H}^1$, the group law is defined by

$$\tau_\xi(\xi') = \xi \circ \xi' = (x + x', y + y', t + t' + 2(x'y - y'x)).$$

The inverse is given by $\xi^{-1} = -\xi$, and therefore, $(\xi)^{-1} \circ \xi^{-1} = (\xi \circ \xi')^{-1}$. For all $s > 0$, the natural group of dilations on $\mathbb{H}^1$ is given by

$$\delta_s(\xi) = (sx, sy, s^2t).$$

Moreover, it satisfies the property, $\delta_s(\xi_0 \circ \xi) = \delta_s(\xi_0) \circ \delta_s(\xi)$. An anisotropic dilation on the Heisenberg group yields the Korányi norm, which is defined as follows:

$$|\xi|_H = \left[(x^2 + y^2)^2 + t^2\right]^{\frac{1}{4}},$$

for any $\xi \in \mathbb{H}^1$. The Kohn-Laplacian is a highly degenerate elliptic operator on $\mathbb{H}^1$ given by

$$\Delta_H u = \mathrm{div}_H(\nabla_H u)$$

and the horizontal gradient

$$\nabla_H u = (X, Y),$$

where X and Y are a basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^1$

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}.$$

The Heisenberg ball at ξ_0 with radius r is the set

$$B_H(\xi_0, r) = \left\{\xi \in \mathbb{H}^1 : d_H(\xi_0, \xi) < r\right\}.$$

The natural volume in $\mathbb{H}^1$ is the Haar measure, which coincides with the Lebesgue measure L^3 in $\mathbb{R}^3$ (see [22]); hence $|B_H(\xi_0, r)| = \alpha_Q r^Q$, where $\alpha_Q = |B_H(0, 1)|$. Consequently, considering the measurable function f: $[a, b] \to \mathbb{R}$, we obtain

$$\int_{B_H(0,b)\setminus B_H(0,a)} f(d_0(\xi))\, d\xi = Q|B_H(0, 1)| \int_a^b f(r) r^{Q-1}\, dr,$$

for any $0 \le a < b$. Form [22], the Folland–Stein space $S^1_0(\Omega)$ is a Hilbert space endowed with the norm

$$\|u\|^2_{S^1_0(\Omega)} = \int_\Omega |\nabla_H u|^2\, d\xi.$$

In our study, we adopt the notation $\|u\| = \|u\|_{S^1_0(\Omega)}$. To keep things concise, let $|\cdot|_p$ represent the standard L^p norm, which is essentially,

$$|u|^p_p = \int_\Omega |u|^p\, d\xi.$$

Moreover, we represent the closed ball of radius ρ centered at zero by B_ρ, and its relative boundary by S_ρ,

$$B_\rho = \{u \in S^1_0(\Omega) : \|u\| \le \rho\}, \quad S_\rho = \{u \in S^1_0(\Omega) : \|u\| = \rho\}.$$

Furthermore, the embedding $S^1_0(\Omega) \hookrightarrow L^p(\Omega)$ is only continuous if $p = 4$, yet it is compact when $1 \le p < 4$ (see [23]). Specifically, Jerison and Lee [24] proved that the best Sobolev constant S, defined as

$$S = \inf_{u \in S^1_0(\mathbb{H}^1)\setminus\{0\}} \frac{\int_{\mathbb{H}^1} |\nabla_H u|^2\, d\xi}{\left(\int_{\mathbb{H}^1} |u|^{Q^*}\, d\xi\right)^{\frac{2}{Q^*}}}$$

is attained by the function C^∞

$$\mathcal{U}(x, y, t) = \frac{c_0}{\sqrt{(1 + x^2 + y^2)^2 + t^2}},$$

where $c_0 > 0$ is an appropriate constant. Specifically, for function $\mathcal{U}$ satisfied

$$\int_{\mathbb{H}^1} |\nabla_H \mathcal{U}|^2\, d\xi = \int_{\mathbb{H}^1} |\mathcal{U}|^4\, d\xi = S^2,$$

as the following Eq (2.1) has a positive solution.

$$-\Delta_H u = u^3, \quad u \in S^1_0(\Omega) \tag{2.1}$$

3. Proof of Theorem 1.1

Let's make a few clarifications at the beginning of this section. We assert that for $(u, \phi) \in S_0^1(\Omega) \times S_0^1(\Omega)$, the system (1.1) has a solution , meaning that for any $v, w \in S_0^1(\Omega)$,

$$\int_\Omega \nabla_H u \nabla_H v \, d\xi - \int_\Omega \phi |u| u v \, d\xi - \gamma \int_\Omega |u|^{q-2} u v \, d\xi - \int_\Omega |u|^2 u v \, d\xi = 0$$

and

$$\int_\Omega \nabla_H \phi \nabla_H w \, d\xi - \int_\Omega u w^3 \, d\xi = 0.$$

We say that (u, ϕ) is a positive solution of system (1.1) if both u and ϕ are positive. To utilize critical point theory, it's essential to define the functional $J(u, \phi) : S_0^1(\Omega) \times S_0^1(\Omega) \to \mathbb{R}$,

$$J(u, \phi) = \frac{1}{2} \int_\Omega |\nabla_H u|^2 \, d\xi + \frac{1}{6} \int_\Omega |\nabla_H \phi|^2 \, d\xi - \frac{1}{3} \int_\Omega \phi |u|^3 \, d\xi - \frac{\gamma}{q} \int_\Omega |u|^q \, d\xi - \frac{1}{4} \int_\Omega |u|^4 \, d\xi$$

for all $(u, \phi) \in S_0^1(\Omega) \times S_0^1(\Omega)$. Using the standard proof as [25], the functional J is C^1 on $S_0^1(\Omega) \times S_0^1(\Omega)$ and its critical points are equivalent to the solutions of system (1.1). Specifically, for any $(v, w) \in S_0^1(\Omega) \times S_0^1(\Omega)$, the partial derivatives of J at (u, ϕ) are denoted by $J'_u(u, \phi)$, $J'_\phi(u, \phi)$,

$$J'_u(u, \phi)[v] = \int_\Omega \nabla_H u \nabla_H v \, d\xi - \int_\Omega \phi |u| u v \, d\xi - \gamma \int_\Omega |u|^{q-2} u v \, d\xi - \int_\Omega |u|^2 u v \, d\xi$$

and

$$J'_\phi(u, \phi)[w] = \frac{1}{3} \int_\Omega \nabla_H \phi \nabla_H w \, d\xi - \frac{1}{3} \int_\Omega w |u|^3 \, d\xi.$$

With the help of the Sobolev embedding theorem and standard computational arguments, we have that J is of class C^1 on $S_0^1(\Omega) \times S_0^1(\Omega)$, and

$$J'_u(u, \phi) = J'_\phi(u, \phi) = 0,$$

if and only if u satisfy system (1.1) and $\phi = \phi_u$.

Before proving Theorem 1.1, we will present several lemmas (see [8, 15]).

Lemma 3.1. *If $u \in S_0^1(\Omega)$. Then there exists a unique nonnegative function $\phi_u \in S_0^1(\Omega)$ such that*

$$\begin{cases} -\Delta_H \phi = |u|^3 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\phi_u > 0$ if $u \neq 0$ and:

(a) For every $s > 0$, then $\phi_{su} = s^3 \phi_u$ and

$$\int_\Omega \phi_u |u|^3 d\xi = \int_\Omega |\nabla_H \phi_u|^2 d\xi \leq S^{-1} |u|_4^6. \tag{3.1}$$

(b) If $u_n \rightharpoonup u$ in $S_0^1(\Omega)$. We have $\phi_{u_n} \to \phi_u$ in $S_0^1(\Omega)$ and

$$\int_\Omega \phi_{u_n} |u_n|^3 v d\xi \to \int_\Omega \phi_u |u|^3 v d\xi \quad \text{for every } v \in S_0^1(\Omega). \tag{3.2}$$

(c) If $\{u_n\} \subset S_0^1(\Omega)$ *and* $u \in S_0^1(\Omega)$ *satisfies* $u_n \rightharpoonup u$ *in* $S_0^1(\Omega)$*, then, up to subsequences,* $\phi_{u_n} \to \phi_u$ *in* $S_0^1(\Omega)$ *and strongly in* $L^p(\Omega)$ *for all* $1 \le p < 4$*. Moreover,*

$$\int_\Omega \phi_{u_n}|u_n|^3 d\xi - \int_\Omega \phi_{u_n-u}|u_n-u|^3 d\xi = \int_\Omega \phi_u |u|^3 d\xi + o_n(1). \tag{3.3}$$

Lemma 3.2. *Let* $\Psi(u) = \phi_u$ *and for any* $u \in S_0^1(\Omega)$*, where* ϕ_u *is as in Lemma 3.1. Let*

$$\Gamma = \{(u,\phi) \in S_0^1(\Omega) \times S_0^1(\Omega) : J'_\phi(u,\phi) = 0\}.$$

Then Ψ *is* C^1 *and* Γ *is the graph of* Ψ*.*

For any $u \in S_0^1(\Omega)$, let $I_\gamma(u) = J(u,\phi_u)$. The functional I_γ is given by

$$I_\gamma(u) := \frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{1}{6}\int_\Omega \phi_u|u|^3 d\xi - \frac{\gamma}{q}\int_\Omega |u|^q d\xi - \frac{1}{4}\int_\Omega |u|^4 d\xi.$$

Drawing upon Lemma 3.1, it is evident that both the non-local term and the critical nonlinearity within I_γ are four homogeneous functionals.

Lemma 3.3. *Assume that* $(u,\phi) \in S_0^1(\Omega) \times S_0^1(\Omega)$*. Then* (u,ϕ) *is a critical point of* J *if and only if* u *is a critical point of* I_γ *and* $\phi = \Psi(u)$*, where* Ψ *is as in Lemma 3.2.*

From the preceding discussion, it can be concluded that the critical point u of I_γ with $\phi = \Psi(u)$, constitutes a solution (u,ϕ_u) of system (1.1) that satisfies

$$I'_\gamma(u)[v] = \int_\Omega \nabla_H u \nabla_H v d\xi - \int_\Omega \phi_u |u| u v d\xi - \gamma \int_\Omega |u|^{q-2} u v d\xi - \int_\Omega |u|^2 u v d\xi.$$

We now proceed to demonstrate the existence of such critical points for I_γ by employing techniques from critical point theory and a variety of analytical methods. More precisely, (u,ϕ_u) represents a ground state solution of system (1.1), which is characterized by having the lowest energy level among all solutions of the system (1.1). This solution satisfies the conditions $I'_\gamma(u) = 0$ and

$$I_\gamma(u) = \inf\{I_\gamma(v) : v \in S_0^1(\Omega) \setminus \{0\},\ I'_\gamma(v) = 0\}.$$

Lemma 3.4. *Assume that* $1 < q < 2$*. Let* $\{u_n\}_n \subset S_0^1(\Omega)$ *be a Palais-Smale sequence of* I_γ*, that is,*

$$I_\gamma(u_n) \to c \quad and \quad I'_\gamma(u_n) \to 0 \quad in \quad (S_0^1(\Omega))'$$

as $n \to \infty$*, where* $(S_0^1(\Omega))'$ *is the dual of* $S_0^1(\Omega)$*. And*

$$c < \left(\frac{q-2}{2q}\right)\left(\frac{(\sqrt{S^{-4} + 4S_{HG}^{-3}} - S^{-2})S_{HG}^3}{2}\right), \tag{3.4}$$

where

$$S_{HG} = \inf_{S_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_H u|^2 d\xi}{\int_\Omega\int_\Omega \frac{|u(\eta)|^3|u(\xi)|^3}{|\eta^{-1}\xi|^2} d\eta d\xi},$$

then there exists a subsequence of $\{u_n\}_n$ *strongly convergent in* $S_0^1(\Omega)$*.*

Proof. Assume that $\{u_n\}_n \subset S_0^1(\Omega)$ be a PS_c sequence for I_γ, with c satisfying (3.4) such that

$$\begin{aligned}1+c+o(1)\|u_n\| &= I_\gamma(u_n)-\frac{1}{4}I'_\gamma(u_n)[u_n]\\ &= \frac{1}{4}\|u_n\|^2-\gamma\left(\frac{1}{q}-\frac{1}{4}\right)\int_\Omega |u_n|^q d\xi+\frac{1}{12}\int_\Omega \phi_{u_n}|u_n|^3 d\xi\\ &\geq \frac{1}{4}\|u_n\|^2-\gamma\left(\frac{1}{q}-\frac{1}{4}\right)S^{-\frac{q}{2}}|\Omega|^{\frac{4-q}{4}}\|u_n\|^q+\frac{1}{12}S^{-1}\|u_n\|^6. \end{aligned} \tag{3.5}$$

Inequality (3.5) shows that $\{u_n\}_n$ is bounded in $S_0^1(\Omega)$ since $1 < q < 2$.

Now we recall the concentration compactness principle [26, Theorem 3.1] in the setting of the Heisenberg group

$$\nu=\left(\int_\Omega \frac{|u(\eta)|^3}{|\eta^{-1}\xi|^2}d\eta\right)|u(\xi)|^3 d\xi+\sum_{j\in J}\nu_j\delta_{z_j}, \qquad \sum_{j\in J}\nu_j^{\frac{1}{3}}<\infty,$$

$$\omega \geq |\nabla_H u|^2 d\xi+\sum_{j\in J}\omega_j\delta_{z_j}, \quad \zeta=|u|^4 d\xi+\sum_{j\in J}\zeta_j\delta_{z_j} \tag{3.6}$$

and

$$S_{HG}\nu_j^{\frac{1}{3}} \leq \omega_j, \qquad \nu_j^{\frac{2}{3}} \leq S^{-\frac{2}{3}}\zeta_j, \quad S\zeta_j^{\frac{1}{2}} \leq \omega_j, \tag{3.7}$$

where J is an at most countable index set, $z_j \in \Omega$, and δ_{ξ_j} is the Dirac mass at ξ_j.

Subsequently, we shall demonstrate that $J = \emptyset$.

Assume, for the sake of contradiction, take $j \in J$ and $J \neq \emptyset$. Fix a cut-off function $\varphi \in C_c^\infty(\mathbb{H}^n)$ such that $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, and $\operatorname{supp}\varphi = B_1$. Let $\varepsilon > 0$ and put $\varphi_\varepsilon(\xi) = \varphi(\delta_{1/\varepsilon}(\xi))$, $\xi \in \mathbb{H}^n$. Then, $\{u_n\varphi_\varepsilon\}_n$ is bounded in $S_0^1(\Omega)$. Obviously, $\langle I'_\gamma(u_n), u_n\varphi_\epsilon\rangle \to 0$ as $n \to \infty$, then we obtain that

$$\begin{aligned}&\int_\Omega |\nabla_H u_n|^2\psi_\varepsilon d\xi+\int_\Omega \nabla_H u_n\cdot\nabla_H\psi_\varepsilon u_n d\xi-\int_\Omega \phi_{u_n}|u_n|^4\psi_\varepsilon(\xi)d\xi\\ &\qquad=\int_\Omega u_n^4\psi_\varepsilon(\xi)d\xi+\gamma\int_\Omega u_n^q\psi_\varepsilon(\xi)d\xi+o(1),\end{aligned}$$

where $o_n(1) \to 0$ as $n \to \infty$.

$$\begin{aligned}\left|\int_\Omega u_n^{q-1}\psi_\varepsilon(\xi)u_n\, d\xi\right| &\leq |B_\varepsilon(\xi_j)|^{\frac{4-q}{4}}\left(\int_{B_\varepsilon(\xi_j)}|u_n|^4\, d\xi\right)^{\frac{q}{4}}\\ &\leq \alpha_Q\varepsilon^{4-q}S^{-\frac{q}{2}}\|u_n\|^q.\end{aligned} \tag{3.8}$$

Hence, combing with (3.5), we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega u_n^q\psi_\varepsilon(\xi)d\xi=0. \tag{3.9}$$

Moreover, by the Hölder inequality, we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_\Omega u_n\nabla_H u_n\cdot\nabla_H\psi_\varepsilon\,d\xi\right|\le\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{B_\varepsilon(\xi_j)}|\nabla_H u_n|^2\,d\xi\right)^{\frac{1}{2}}\times\left(\int_{B_\varepsilon(\xi_j)}|u_n|^2|\nabla_H\psi_\varepsilon|^2\,d\xi\right)^{\frac{1}{2}}$$
$$\le C\lim_{\varepsilon\to 0}\left(\int_{B_H(\xi_j)}|u|^4\,d\xi\right)^{\frac{1}{4}}\times\left(\int_{B_\varepsilon(\xi_j)}|\nabla_H\psi_\varepsilon|^4\,d\xi\right)^{\frac{1}{4}}$$
$$=0. \tag{3.10}$$

By (3.6), we get

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega\phi_{u_n}u_n^4\psi_\varepsilon(\xi)d\xi=\lim_{\varepsilon\to 0}\left(\nu_j+\int_{B_\varepsilon(\xi_j)}\phi_{u_n}|u_n|^4\,d\xi\right)=\nu_j, \tag{3.11}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega|\nabla_H u_n|^2\psi_\varepsilon\,d\xi\ge\lim_{\varepsilon\to 0}\left(\omega_j+\int_{B_\varepsilon(\xi_j)}|\nabla_H u|^2\psi_\varepsilon\,d\xi\right)=\omega_j \tag{3.12}$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega\psi_\varepsilon(\xi)u_n^4\,d\xi=\lim_{\varepsilon\to 0}\left(\zeta_j+\int_{B_\varepsilon(\xi_j)}|u|^4\,d\xi\right)=\zeta_j. \tag{3.13}$$

Hence from (3.7) to (3.13), as $\varepsilon\to 0^+$ and $n\to\infty$, we get that

$$\omega_j\le\zeta_j+\nu_j.$$

Furthermore, it follows from (3.7) that

$$\omega_j\ge\frac{(\sqrt{S^{-4}+4S_{HG}^{-3}}-S^{-2})S_{HG}^3}{2}.$$

According to $\mathcal{I}(u_n)\to c$ and $\mathcal{I}'(u_n)\to 0$ as $n\to\infty$, we get

$$c=\lim_{n\to\infty}\mathcal{I}(u_n)=\lim_{n\to\infty}\left(\mathcal{I}(u_n)-\frac{1}{q}\langle\mathcal{I}'(u_n),u_n\rangle\right)$$
$$\ge\lim_{n\to\infty}\left\{(\frac{1}{2}-\frac{1}{q})\|u_n\|^2+(\frac{1}{q}-\frac{1}{6})\int_\Omega\phi_{u_n}|u_n|^3\,d\xi+\left(\frac{1}{q}-\frac{1}{4}\right)\int_\Omega|u_n|^4\,d\xi\right\}$$
$$\ge\lim_{n\to\infty}\left\{(\frac{1}{2}-\frac{1}{q})\|u_n\|^2\right\}$$
$$\ge\frac{q-2}{2q}\omega_j\ge\left(\frac{q-2}{2q}\right)\left(\frac{(\sqrt{S^{-4}+4S_{HG}^{-3}}-S^{-2})S_{HG}^3}{2}\right).$$

Thanks to (3.4), this is impossible. Therefore, we obtain that $J=\emptyset$. Using this and (3.6), one has

$$\int_\Omega u_n^4\,d\xi\to\int_\Omega u^4\,d\xi\quad\textit{and}\quad\int_\Omega\phi u_n^4\,d\xi\to\int_\Omega\phi u^4\,d\xi\quad\text{as } n\to\infty. \tag{3.14}$$

By (3.6) and (3.14), we obtain that

$$\int_\Omega \nabla_H u_n \nabla_H \varphi\, d\xi - \int_\Omega \phi_{u_n}|u_n|^2 u_n \varphi\, d\xi - \int_\Omega u_n^3 \varphi\, d\xi - \gamma \int_\Omega u_n^{q-1}\varphi\, d\xi = o(1). \tag{3.15}$$

Inserting $\varphi = u$ in (3.15), we get

$$\|u\|^2 - \int_\Omega \phi_u |u|^4\, d\xi - \int_\Omega |u|^4\, d\xi - \gamma \int_\Omega |u|^q\, d\xi = 0. \tag{3.16}$$

Using (3.6), (3.14), and Lemma 3.1, it follows that

$$\lim_{n\to\infty}\left(\|u_n\|^2 - \int_\Omega \phi_u |u|^4\, d\xi - \int_\Omega |u|^4\, d\xi - \gamma \int_\Omega |u|^q\, d\xi\right) = 0. \tag{3.17}$$

And so, by combining (3.16) and (3.17)

$$\lim_{n\to\infty} \|u_n\| = \|u\|.$$

Consequently, we obtain $u_n \to u$ in $S_0^1(\Omega)$. As a result, we finish the Lemma 3.4 proof.

Lemma 3.5. *Let* $1 < q < 2$*, there exist* $\Lambda_0, \rho_0 > 0$ *such that for* $\gamma \in (0, \Lambda_0)$*. Then,* $\inf_{u\in B_{\rho_0}} I_\gamma(u) < 0$ *and*

$$I_\gamma(u) > \frac{1}{2} g(\rho_0)\rho_0^q > 0, \quad \forall u \in S_{\rho_0},$$

where g is a given function.

Proof. Useing the Hölder inequality and (3.1), we get

$$\begin{aligned} I_\gamma(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{6}\int_\Omega \phi_u |u|^3 d\xi - \frac{\gamma}{q}\int_\Omega |u|^q d\xi - \frac{1}{4}\int_\Omega |u|^4 d\xi \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\gamma}{q}\int_\Omega |u|^q d\xi - \frac{1}{6}S^{-1}\|u\|^6 - \frac{1}{4}\int_\Omega |u|^4 d\xi \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\gamma}{q}\int_\Omega |u|^q d\xi - \frac{1}{6}S^{-1}\|u\|^6 - \frac{1}{4}S^{-2}\|u\|^4 \\ &\geq \|u\|^q \left\{\frac{1}{2}\|u\|^{2-q} - \frac{1}{4}S^{-2}\|u\|^{4-q} - \frac{1}{6}S^{-1}\|u\|^{6-q} - \frac{\gamma}{q}S^{-\frac{q}{2}}|\Omega|^{\frac{4-q}{4}}\right\}. \end{aligned} \tag{3.18}$$

Let $a_1 = \frac{1}{4}S^{-2}, a_2 = -\frac{1}{6}S^{-1}$, then from (3.18), we get

$$I_\gamma(u) \geq \|u\|^q \left\{\frac{1}{2}\|u\|^{2-q} - a_1\|u\|^{4-q} + a_2\|u\|^{6-q} - \frac{\gamma}{q}S^{-\frac{q}{2}}|\Omega|^{\frac{4-q}{4}}\right\}. \tag{3.19}$$

We define

$$\mathcal{G}(t) := \frac{1}{2}t^{2-q} - a_1 t^{4-q} + a_2 t^{6-q},$$

then ρ_0 is the maximum value point of $\mathcal{G}(t)$; it follows,

$$\mathcal{G}(\rho_0) = \max_{t>0} g(t) > 0.$$

Hence, if $\Lambda_0 = \frac{1}{2}qS^{-\frac{q}{2}}|\Omega|^{\frac{q-4}{4}}\mathcal{G}(\rho_0)$, then from (3.19) we have, for all $\gamma \in (0, \Lambda_0)$,

$$I_\gamma(u) \geq \frac{\mathcal{G}(\rho_0)}{2}\rho_0^q > 0, \quad \forall u \in S_{\rho_0}.$$

In contrast, for any $u \in S_0^1(\Omega)\backslash\{0\}$, we conclude

$$\lim_{s\to 0^+} \frac{I_\gamma(su)}{s^q} = -\frac{\gamma}{q}\int_\Omega |u|^q d\xi < 0$$

indicates that the existence of $u \in B_{\rho_0}$ such that$I_\gamma(u) < 0$. we have that

$$\inf_{u\in B_\rho} I_\lambda(u) < 0.$$

This completes the proof.

Theorem 3.1. *Assume that* $0 < \lambda < \Lambda_0$, *for* $q \in (1, 2)$. *Then system* (1.1) *has a positive solution* $u \in S_0^1(\Omega)$ *enjoying* $I_\gamma(u) < 0$.

Proof. Assume that

$$d = \inf_{u\in B_{\rho_0}} I_\gamma(u) < 0 < \inf_{u\in S_{\rho_0}} I_\gamma(u). \tag{3.20}$$

By applying Ekeland's variational principle (see [27]), there exists a nonnegative minimizing sequence $\{u_n\} \subset B_{\rho_0}$ such that

$$I_\gamma(u_n) \leq \inf_{u\in B_{\rho_0}} I_\gamma(u) + \frac{1}{n}, \quad I_\gamma(v) \geq I_\gamma(u_n) - \frac{1}{n}\|v - u_n\|, \quad \forall v \in B_{\rho_0}.$$

Notice that $I_\gamma(|u|) = I_\gamma(u)$. Combining with (3.20), we deduce that $I'_\gamma(u_n) \to 0$ and $I_\gamma(u_n) \to d$. Due to the boundedness and non-negativity of $\{u_n\}$, there exist $u \in B_{\rho_0}$ and $u \geq 0$ such that $u_n \rightharpoonup u$ in $S_0^1(\Omega)$ as $n \to \infty$. Furthermore, by Lemma 3.4, we deduce that $u_n \to u$ in $S_0^1(\Omega)$ and

$$d = \lim_{n\to\infty} I_\gamma(u_n) = I_\gamma(u) < 0.$$

This implies that $u \geq 0$ and $u \neq 0$. Additionally, (u, ϕ_u) is a solution of problem (1.1). Consequently, it follows from the first equation of (1.1) that

$$-\Delta_H u + \kappa^+\phi_u|u|u = \kappa^-\phi_u u + \gamma u^{q-1} + |u|^4 > 0,$$

where $\kappa^\pm = \max\{\pm\kappa, 0\}$. It is also known that $\phi_u(\xi) > 0$ for all $\xi \in \Omega$. Corresponding to the maximum principle (see [28, 29]), it follows that $u > 0$ in Ω. The proof is complete.

We now proceed to prove Theorem 1.1, which asserts that system (1.1) admits a positive ground state solution. Let

$$\psi = \inf_{u\in\mathcal{N}} I_\gamma(u), \tag{3.21}$$

where $\mathcal{N} = \{u \in S_0^1(\Omega) \setminus \{0\} : \langle I'_\gamma(u), u\rangle = 0\}$.

Proof of Theorem 1.1. Taking into account that, if $u \in \mathcal{N}$, we have $I_\gamma(|u|) = I_\gamma(u)$, a nonnegative minimizing sequence $\{u_n\} \subset \mathcal{N}$ is considered such that

$$I_\gamma(u_n) \to \psi, \quad \text{as } n \to \infty. \tag{3.22}$$

Clearly, by combining $I_\gamma(u_n) < 0$ with Lemma 3.4, it follows that $\{u_n\}$ is bounded in $S_0^1(\Omega)$ and $\psi < 0$. Subsequently, there exists a sequence $\{u_n\}$ in $S_0^1(\Omega)$ such that $u_n \rightharpoonup u_1$ in $S_0^1(\Omega)$; we now claim that $u_1 \neq 0$. Arguing by contradiction, suppose $u_1 \equiv 0$, then $\lim_{n\to\infty} \|u_n\|^2 = 0$. Furthermore, $\lim_{n\to\infty} I_\gamma(u_n) = 0$, which contradicts (3.22). By Lemma 3.4, it follows that $u_n \to u_1$ in $S_0^1(\Omega)$, and u_1 is a positive solution of system (1.1) satisfying $I_\gamma(u_1) \geq \psi$. Furthermore, we show that $I_\gamma(u_1) \leq \psi$.

In addition, by Fatou's lemma, it holds

$$\begin{aligned}
\psi &= \lim_{n\to\infty} \left(I_\gamma(u_n) - \frac{1}{4}\langle I'_\gamma(u_n), u_n\rangle \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{4}\|u_n\|^2 - \frac{\gamma(4-q)}{4q} \int_\Omega |u_n|^q \, d\xi + \frac{1}{12} \int_\Omega \phi_{u_n} |u_n|^3 \, d\xi \right) \\
&\geq \liminf_{n\to\infty} \frac{1}{4}\|u_n\|^2 - \frac{\gamma(4-q)}{4q} \int_\Omega |u_n|^q \, d\xi + \frac{1}{12} \int_\Omega \phi_{u_n} |u_n|^3 \, d\xi \\
&= \frac{1}{4}\|u_1\|^2 - \frac{\gamma(4-q)}{4q} \int_\Omega |u_1|^q \, d\xi + \frac{1}{12} \int_\Omega \phi_{u_1} |u_1|^3 \, d\xi \\
&= I_\gamma(u_1) - \frac{1}{4}\langle I'_\mu(u_1), u_1\rangle \\
&= I_\gamma(u_1).
\end{aligned}$$

This implies that $I_\gamma(u_1) \leq \psi$ and hence $I_\gamma(u_1) = \psi$. It is evident from the above that u_1 is a positive ground state solution of system (1.1).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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