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Research article

Optimal portfolio of defined contribution pension plan with climate risk under model uncertainty

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Abstract: This manuscript researches the optimal portfolio of the defined contribution (DC) pension plan, comprising a stock index and cash within a dynamic model that incorporates uncertainty regarding climate change. The global temperature is supposed to have an impact on the stock index price, with the probability distribution of the global temperature considered uncertain. To address this uncertainty, we apply Girsanov's theorem for measure transformation. The optimal investment strategy under climate uncertainty and stochastic interest rates is derived in closed form. We further demonstrate that climate uncertainty can result in specific losses in returns. We discover through numerical analysis that the investment in the stock index is especially sensitive to climate uncertainty.

Keywords: optimal portfolio; climate uncertainty; stochastic interest rate; Girsanov's theorem; global temperature

1. Introduction

With the increasing prominence of the aging population, risk management in pension funds has become a growing concern. Among them, the DC pension plan serves as one avenue for pension risk management, wherein participants bear the investment risks associated with uncertainties. The optimization of the portfolio for the DC pension fund has garnered considerable attention in this field. Furthermore, existing literature has demonstrated that climate change can impact the financial market (see e.g., [1,2]). While the portfolio selection study has recently addressed and examined climate risk by Engle et al. [3], its significance for investors in the DC pension plan has not been adequately explored. Moreover, given the substantial uncertainty associated with climate change, the extent to which this uncertainty influences investments remains unclear. Thus, we study the effect of climate uncertainty on the DC pension plan's optimal investment.

The optimal investment strategy problem is the primary topic of the majority of the literature in the field of DC pension plan research. The typical objective is to maximize the expected utility of the terminal wealth, considering various market conditions and constraints. Vigna and Haberman [4] were the first to examine the optimal portfolio issue for the DC pension, deriving the optimal investment strategy in discrete time. Gerrard et al. [5] extended the previous portfolio issue to continuous time. Given that the pension investment entails long-term commitments, and the interest rate undergoes stochastic fluctuations over the investment horizon, investment decisions must account for interest rate risk. The research on the DC pension under stochastic interest rates has been extensively discussed. Boulier et al. [6] studied the DC pension with minimum guarantees, assuming the interest rate follows the Vasicek model. They employed the martingale method to examine how interest rate factors affected the optimum portfolio. By utilizing the martingale approach to further explore the DC pension's optimum asset allocation within a minimum guarantee under an affine rate model in [7]. They obtained explicit solutions for the optimal investment strategy under the power utility function. Guan and Liang [8] studied the optimal management of the DC pension plan in a framework of stochastic interest rate and volatility under the power utility function. In [9], the explicit equilibrium investment strategy was obtained by considering the mean-variance DC pension issue with stochastic interest and volatility. For more research on stochastic interest rates (see e.g., [10–12]).

In light of the substantial uncertainty inherent in climate change (see e.g., [13–15]). Weitzman [16] focused on examining the losses incurred due to temperature changes, analyzing the economic aspects of climate change, and exploring the consequences resulting from variations in temperature. However, to our knowledge, no literature has yet examined the impact of climate risk on the optimal investment of the DC pension. With the dynamic model of the temperature changes, we suppose that the investor is uncertain. The pension fund management is specifically uncertain about the probability distribution of changes in the global temperature and the probability distribution of all other variables impacted by climate change. Indeed, due to market fluctuations, the veracity of the assumed probability measure cannot be precisely determined. Given that the alternative probability measure merely approximates the real measure, namely, fund managers aim to minimize significant deviations between the reference measure and the alternative probability measure.

In this paper, we essentially include a total of three contributions. First, which has not been explored before in the literature, we incorporate climate uncertainty into the portfolio issue in the DC pension. By solving the corresponding HJB equation, we obtain a closed-form solution for the optimal investment strategy. Subsequently, we account for the resulting loss in returns due to climate uncertainty, and the explicit expression for the cost of climate uncertainty is derived. Secondly, we incorporate the stochastic interest rate, establishing the rate model following the Ornstein–Uhlenbeck process. Additionally, we construct a dynamic model for the global temperature, demonstrating the influence of the global temperature variations on the stock index price. Finally, we show that climate uncertainty profoundly affects the DC pension plan investment through numerical examples. In the increasingly volatile financial market, disturbances of relevant internal parameters and the external environment will increase the uncertainty of decision-making models. Therefore, considering issues related to model uncertainty is of great significance in practice. This paper considers the effect of climate uncertainty on investment, which is closely related to the returns of DC pension members and contributes to research on the optimal investment of the DC pension.

The rest of this study is organized as follows. In Section 2, we establish the model for the financial market under climate uncertainty. In Section 3, we obtain the optimal investment strategy and the cost of climate uncertainty. In Section 4 delves into the analysis of the influence of climate uncertainty and other parameters on the optimal portfolio, and conclusions in Section 5.

2. The model

2.1. The financial market

We begin with a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\le t\le T}, P)$, equipped with a filtration $\{\mathscr{F}_t\}_{0\le t\le T}$ denotes the information flow at time t. This probability space defines all stochastic processes that are discussed below. We take into consideration the investor with the capacity to trade the pension account in cash and a stock index. The cash account $B(t)$ is determined by:

$$dB(t) = r(t)B(t)dt, \qquad B(0) = b_0, \tag{2.1}$$

where $r(t)$ is the instantaneous interest rate, which follows the Ornstein–Uhlenbeck model:

$$dr(t) = a(b - r(t))dt - \sigma_r dW_r(t), \qquad r(0) = r_0, \tag{2.2}$$

where a, b, σ_r are positive constants and $W_r(t)$ is a standard Brownian motion on the probability space.

Climate risk has been shown to affect the stock index in [17] due to the prevalence of extreme weather events and the ongoing increase in the global temperature in recent years. The global temperature will be used as a proxy for climate risk in this work. This leads us to believe that there is a higher risk of climate change on occasions with the rising temperature and a lower risk otherwise. The global temperature change is modeled by (cf. [18]):

$$dy(t) = \alpha(\beta - y(t))dt - \sigma_y dW_y(t), \qquad y(0) = y_0, \tag{2.3}$$

where α is the speed of mean reversion, β is the long-run mean level, and σ_y is the volatility of the global temperature. $W_y(t)$ is a standard Brownian motion that is independent of $W_r(t)$.

The dynamic of the stock index price is given by

$$\begin{aligned}\frac{dS(t)}{S(t)} =& (r(t) + \rho y(t))dt + \sigma_1(\lambda_1 dt + dW_r(t)) + \sigma_2(\lambda_2 dt + dW_y(t)) \\ &+ \sigma_3(\lambda_3 dt + dW_s(t)), \qquad S(0) = s_0,\end{aligned} \tag{2.4}$$

where $\rho, \sigma_1, \sigma_2, \sigma_3$ are positive constants. σ_1 and σ_2 are effects of the interest rate volatility and the global temperature volatility on the stock index price, and σ_3 is the stock index volatility, which is influenced by variables other than the interest rate and the global temperature. Consequently, the risky asset volatility is $\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}$. $W_s(t)$ is a standard Brownian motion which is independent of $W_r(t)$ and $W_y(t)$.

The model (2.2)–(2.4) assumes perfect certainty in the probability distribution of the underlying processes concerning the interest rate, the temperature change, and the stock index price. Nevertheless, it is generally acknowledged in the field of climate change studies that there is a great deal of uncertainty about how the global temperature will evolve in the future (see e.g., [13, 19]). When choosing an investment, the investor desires to take into account this uncertainty as well as several other practical models that we will now discuss. Let $\mathbb{P}^w$ denote the alternative measure corresponding to the reference measure $\mathbb{P}$. According to [18], we designate the Radon–Nikodým derivative process as $w(t)$, which is a $\mathbb{R}$-valued stochastic process:

$$\begin{aligned}\xi_t^w =& \mathrm{E}\left[\frac{d\mathbb{P}^w}{d\mathbb{P}}|\mathscr{F}_t\right]\\ =& \exp\Bigg(-\int_0^t \frac{\left(1+m_1^2+m_2^2\right)w^2(u)}{2}du\\ &-\int_0^t w(u)\left(dW_y(u)+m_1 dW_r(u)+m_2 dW_s(u)\right)\Bigg),\end{aligned} \tag{2.5}$$

where m_1, m_2 are parameters. Following Girsanov's theorem, the process:

$$\begin{pmatrix}\widetilde{W}_y(t)\\ \widetilde{W}_r(t)\\ \widetilde{W}_s(t)\end{pmatrix}=\begin{pmatrix}\int_0^t w(u)du+W_y(t)\\ m_1\int_0^t w(u)du+W_r(t)\\ m_2\int_0^t w(u)du+W_s(t)\end{pmatrix} \tag{2.6}$$

is a Brownian motion with respect to the probability measure $\mathbb{P}^w$. Therefore, Eqs (2.2)–(2.4) are rewritten as

$$dr(t)=\left(a\left(b-r(t)\right)+m_1\sigma_r w(t)\right)dt-\sigma_r d\widetilde{W}_r(t), \tag{2.7}$$

$$dy(t)=\left(\alpha\left(\beta-y(t)\right)+\sigma_y w(t)\right)dt-\sigma_y d\widetilde{W}_y(t), \tag{2.8}$$

$$\frac{dS(t)}{S(t)}=\left(r(t)+\rho y(t)+\sigma\lambda-w(t)\sigma m\right)dt+\sigma dW(t), \tag{2.9}$$

where define some vectors $\sigma=(\sigma_1,\sigma_2,\sigma_3)$, $\lambda=(\lambda_1,\lambda_2,\lambda_3)^{\mathrm{T}}$, $m=(m_1,1,m_2)^{\mathrm{T}}$, $dW(t)=(d\widetilde{W}_r(t),d\widetilde{W}_y(t),d\widetilde{W}_s(t))^{\mathrm{T}}$.

2.2. The defined contribution pension process

During the accumulation period $[0,T]$, the pension investor is generally required to make regular premium contributions at a fixed percentage of the salary. Assuming a fixed contribution rate of c, with the salary unit being 1. Let the wealth process at time t be $X(t)$, and the proportion invested in the stock index price is $\pi(t)$. The wealth process $X(t)$ of the pension investor is

$$\begin{aligned}dX(t)=&\left(1-\pi(t)\right)X(t)\frac{dB(t)}{B(t)}+\pi(t)X(t)\frac{dS(t)}{S(t)}+cdt\\ =&\left(r(t)X(t)+c\right)dt+\pi(t)X(t)\left(\rho y(t)+\sigma\lambda-w(t)\sigma m\right)dt\\ &+\pi(t)X(t)\sigma dW(t),\qquad X(0)=x_0.\end{aligned} \tag{2.10}$$

Definition 2.1. A control strategy $\pi(t)$ is said to be an admissible strategy when it fulfills the following constraints:
(i) For $t\in[0,T]$, $\pi(t)$ is $\mathscr{F}_t$-progressively measurable;
(ii) For $t\in[0,T]$, $\mathrm{E}\left(\int_0^T X^2(t)\pi^2(t)\sigma\sigma^{\mathrm{T}}dt\right)<\infty$, where $\sigma\sigma^T<\infty$ and the wealth process $0<X(t)<\infty$;
(iii) The equation (2.10) has a unique solution for any $\pi(t)$.

The set of all admissible strategies is denoted by Π. In this paper, we assume that the pension investor's aim is to maximize the expected utility of the terminal wealth. The fact that most investors are risk averse in real-world investment decisions ensures the concavity of the utility function so that the utility maximizing equilibrium exists. We therefore choose the constant relative risk aversion (CRRA) utility function to describe the pension investor's risk preferences:

$$\Phi^w(t,x,r,y,\pi) = \frac{1}{1-\gamma}\mathrm{E}^{\mathbb{P}^w}_{t,x,r,y}\left[(X(T))^\gamma\right], \tag{2.11}$$

and the value function is

$$V(t,x,r,y) = \sup_{\pi\in\Pi}\inf_{w\in R}\left(\Phi^w(t,x,r,y,\pi) + \mathrm{E}^{\mathbb{P}^w}_{t,x,r,y}\left[\int_t^T \frac{w^2(u)}{2\eta(u)}du\right]\right), \tag{2.12}$$

where $\gamma > 0$ and $\gamma \neq 1$ is the coefficient of risk aversion. As the alternative measure should not deviate too far from the reference measure, a penalty term $\mathrm{E}^{\mathbb{P}^w}_{t,x,r,y}\left[\int_t^T \frac{w^2(u)}{2\eta(u)}du\right]$ is introduced to constrain the difference between the reference measure and the alternative measure. As in [18], for the sake of analysis, we suppose that

$$\eta(t) = \frac{\theta}{(1-\gamma)V(t,x,r,y)}, \tag{2.13}$$

where θ is the ambiguity aversion coefficient. A larger value of θ implies more uncertainty for the pension investor.

3. Optimal investment strategy under uncertainty

3.1. Optimal investment strategy

We utilize the stochastic dynamic programming principle to determine the optimal investment strategy $\pi^*(t)$ that addresses the issue (2.12) mentioned in the preceding part. The HJB equation is

$$\begin{aligned}\sup_{\pi\in\Pi}\inf_{w\in R}\Bigg\{&V_t + a(b-r)V_r + \alpha(\beta-y)V_y + (c+xr)V_x + \frac{1}{2}\sigma_r^2V_{rr} + \frac{1}{2}\sigma_y^2V_{yy}\\ &+ m_1\sigma_r w(t)V_r + \sigma_y w(t)V_y + \pi(t)x(\rho y + \sigma\lambda - w(t)\sigma m)V_x\\ &+ \frac{1}{2}\pi^2(t)x^2\sigma\sigma^{\mathrm{T}}V_{xx} - \pi(t)x\sigma_y\sigma_2V_{xy} - \pi(t)x\sigma_r\sigma_1V_{xr} + \frac{w^2(t)}{2\eta(t)}\Bigg\} = 0.\end{aligned} \tag{3.1}$$

Theorem 3.1. The optimal investment strategy $\pi^*(t)$ and the value function are given by

$$\begin{aligned}\pi^*(t) =&\frac{\rho y + \sigma\lambda}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2} - \frac{\gamma((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r}{(1-\gamma)(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)}i(t)\\ &- \frac{\gamma((1-\gamma)\sigma_2 - \theta\sigma m)\sigma_y}{(1-\gamma)(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)}(j(t) + 2k(t)y),\end{aligned} \tag{3.2}$$

$$V(t,x,r,y) = \frac{x^{1-\gamma}}{1-\gamma}g(t,r,y)^\gamma, \tag{3.3}$$

where the expression of $g(t,r,y)$ is

$$g(t,r,y) = \exp(h(t) + i(t)r + j(t)y + k(t)y^2), \tag{3.4}$$

$$i(t) = \frac{1-\gamma}{a\gamma}\left(1 - e^{-a(T-t)}\right), \tag{3.5}$$

$$j(t) = e^{-\int_t^T \zeta(u)du}\left(\int_t^T \delta(s)e^{\zeta(u)du}ds\right), \tag{3.6}$$

where $\zeta(t)$ and $\delta(t)$ are given by (3.22) and (3.23).

$$k(t) = \begin{cases} \dfrac{l_1 l_2 \exp\left(-\sqrt{\Delta}(T-t)\right) - l_1 l_2}{l_1 \exp\left(-\sqrt{\Delta}(T-t)\right) - l_2}, & \Delta > 0, \\ \dfrac{u_1}{2u_0}\left(\dfrac{1}{1-u_1(T-t)} - 1\right), & \Delta = 0, \\ \dfrac{\sqrt{-\Delta}}{2u_0}\tan\left\{\arctan\left(\dfrac{u_1}{\sqrt{-\Delta}}\right) - \dfrac{\sqrt{-\Delta}}{2}(T-t)\right\} - \dfrac{u_1}{2u_0}, & \Delta < 0, \end{cases} \tag{3.7}$$

$$\begin{aligned} h(t) = &-\int_t^T \left\{\left(\alpha\beta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)\sigma\lambda\sigma_y}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)j(s)\right. \\ &+ \left(ab - \frac{((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma\lambda\sigma_r}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)i(s) \\ &+ \frac{\gamma\sigma_r^2}{2(1-\gamma)}\left(1 - \gamma - m_1^2\theta + \frac{((1-\gamma)\sigma_1 - \theta\sigma m m_1)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)i^2(s) \\ &+ \frac{\gamma\sigma_y^2}{2(1-\gamma)}\left(1 - \gamma - \theta + \frac{((1-\gamma)\sigma_2 - \theta\sigma m)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)j^2(s) + \sigma_y^2 k^2(s) \\ &\left.- \frac{\gamma}{1-\gamma}\left(m_1\sigma_r\sigma_y\theta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)((1-\gamma)\sigma_1 - \theta\sigma m m_1)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)i(s)j(s)\right\}ds \\ &- \frac{1-\gamma}{\gamma}\left(\frac{c}{x} + \frac{(\sigma\lambda)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)(T-t). \end{aligned} \tag{3.8}$$

Proof. We search for a form (3.3) solution to the HJB equation (3.1) in this part. Following the first-order condition for w, we get

$$w^* = \eta\left(\pi(t)x\sigma m V_x - \sigma_y V_y - m_1\sigma_r V_r\right),$$

plugging it into the HJB equation, then we rewrite it as

$$\begin{aligned} \sup_{\pi\in\Pi}&\left\{V_t + a(b-r)V_r + \alpha(\beta-y)V_y + (xr+c)V_x + \pi(t)x(\rho y + \sigma\lambda)V_x + \frac{1}{2}\sigma_r^2 V_{rr}\right. \\ &+ \frac{1}{2}\sigma_y^2 V_{yy} + \frac{1}{2}\pi^2(t)x^2\sigma\sigma^{\mathrm{T}}V_{xx} - \pi(t)x\sigma_r\sigma_1 V_{xr} - \pi(t)x\sigma_y\sigma_2 V_{xy} - \eta m_1\sigma_r\sigma_y V_r V_y \\ &\left.+ \eta\pi(t)x\sigma m(m_1\sigma_r V_r + \sigma_y V_y)V_x - \frac{\eta}{2}(m_1^2\sigma_r^2 V_r^2 + \sigma_y^2 V_y^2) - \frac{\eta}{2}\pi^2(t)x^2(\sigma m)^2 V_x^2\right\} = 0. \end{aligned} \tag{3.9}$$

Differentiating the function (3.3), we obtain:

$$
\begin{aligned}
&V_t = \frac{\gamma}{1-\gamma}x^{1-\gamma}g_t g^{\gamma-1}, \quad V_x = x^{-\gamma}g^{\gamma},\\
&V_y = \frac{\gamma}{1-\gamma}x^{1-\gamma}g_y g^{\gamma-1}, \quad V_r = \frac{\gamma}{1-\gamma}x^{1-\gamma}g_r g^{\gamma-1},\\
&V_{yy} = \frac{\gamma}{1-\gamma}x^{1-\gamma}g_{yy}g^{\gamma-1} - \gamma x^{1-\gamma}g_y^2 g^{\gamma-2},\\
&V_{rr} = \frac{\gamma}{1-\gamma}x^{1-\gamma}g_{rr}g^{\gamma-1} - \gamma x^{1-\gamma}g_r^2 g^{\gamma-2},\\
&V_{xx} = -\gamma x^{-\gamma-1}g^{\gamma}, \quad V_{xy} = \gamma x^{-\gamma}g_y g^{\gamma-1}, \quad V_{xr} = \gamma x^{-\gamma}g_r g^{\gamma-1},
\end{aligned}
$$

putting the above into (3.9) and eliminating the common factor $\frac{\gamma}{1-\gamma}x^{1-\gamma}g^{\gamma-1}$ gives:

$$
\begin{aligned}
&\sup_{\pi\in\Pi}\Bigg\{g_t + a(b-r)g_r + \alpha(\beta-y)g_y + \frac{1-\gamma}{\gamma}\left(r+\frac{c}{x}\right)g + \frac{1}{2}\sigma_r^2 g_{rr} - \frac{1-\gamma}{2}\pi^2(t)\left(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2\right)g\\
&-\frac{1-\gamma}{2}\left(\sigma_r^2\frac{g_r^2}{g} + \sigma_y^2\frac{g_y^2}{g}\right) - \frac{\gamma\theta}{2(1-\gamma)}(m_1\sigma_r g_r + \sigma_y g_y)^2\frac{1}{g} + \frac{1-\gamma}{\gamma}\pi(t)(\rho y + \sigma\lambda)g\\
&+\frac{1}{2}\sigma_y^2 g_{yy} - \pi(t)((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r g_r - \pi(t)((1-\gamma)\sigma_2 - \theta\sigma m)\sigma_y g_y\Bigg\} = 0,
\end{aligned}
\tag{3.10}
$$

according to the first-order maximum principle, we have:

$$
\pi^*(t) = \frac{\rho y + \sigma\lambda}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2} - \frac{\gamma((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r}{(1-\gamma)(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)}\frac{g_r}{g} - \frac{\gamma((1-\gamma)\sigma_2 - \theta\sigma m)\sigma_y}{(1-\gamma)(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)}\frac{g_y}{g}, \tag{3.11}
$$

substituting $\pi^*(t)$ into (3.10), it simplifies to

$$
\begin{aligned}
&g_t + a(b-r)g_r + \alpha(\beta-y)g_y + \frac{1-\gamma}{\gamma}\left(r+\frac{c}{x}\right)g + \frac{1}{2}\sigma_r^2 g_{rr} + \frac{1}{2}\sigma_y^2 g_{yy} - \frac{1-\gamma}{2}\left(\sigma_r^2\frac{g_r^2}{g} + \sigma_y^2\frac{g_y^2}{g}\right)\\
&-\frac{\gamma\theta}{2(1-\gamma)}(m_1\sigma_r g_r + \sigma_y g_y)^2\frac{1}{g} + \frac{(1-\gamma)(\rho y + \sigma\lambda)^2}{2\gamma(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)}g - \frac{(\rho y + \sigma\lambda)((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}g_r\\
&-\frac{(\rho y + \sigma\lambda)((1-\gamma)\sigma_2 - \theta\sigma m)\sigma_y}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}g_y + \frac{\gamma\left[((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r g_r + ((1-\gamma)\sigma_2 - \theta\sigma m)\sigma_y g_y\right]^2}{(1-\gamma)(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)g}\\
&= 0.
\end{aligned}
\tag{3.12}
$$

Let $g(t,r,y) = \exp(h(t) + i(t)r + j(t)y + k(t)y^2)$; differentiating g can have:

$$
\begin{aligned}
&g_t = (h'(t) + i'(t)r + j'(t)y + k'(t)y^2)g,\\
&g_r = i(t)g, \quad g_y = (j(t) + 2k(t)y)g,\\
&g_{rr} = i^2(t)g, \quad g_{yy} = 2k(t)g + (j(t) + 2k(t)y)^2 g,
\end{aligned}
$$

then putting the above derivatives into (3.12), we have:

$$
\begin{aligned}
&r\left(i'(t) - ai(t) + \frac{1-\gamma}{\gamma}\right) + y\left\{j'(t) - \left(\alpha + \frac{\rho((1-\gamma)\sigma_2 - \theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)j(t)\right.\\
&+ 2\left(\alpha\beta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)k(t) + \frac{2\gamma}{1-\gamma}\sigma_y^2\left(1-\gamma-\theta + \frac{((1-\gamma)\sigma_2 - \theta\sigma m)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)j(t)k(t)\\
&- \frac{2\gamma}{1-\gamma}\left(m_1\sigma_r\sigma_y\theta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r\sigma_y}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)i(t)k(t)\\
&\left.- \frac{\rho((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}i(t) + \frac{(1-\gamma)\rho\sigma\lambda}{\gamma(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)}\right\} + y^2\left\{k'(t)\right.\\
&+ \frac{(1-\gamma)\rho^2}{2\gamma(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)} - 2\left(\alpha - \frac{\rho((1-\gamma)\sigma_2 - \theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)k(t)\\
&\left.+ \frac{2\gamma}{1-\gamma}\sigma_y^2\left(1-\gamma-\theta + \frac{((1-\gamma)\sigma_2 - \theta\sigma m)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)k^2(t)\right\} + h'(t) + \sigma_y^2 k^2(t) + \frac{(1-\gamma)c}{\gamma x}\\
&+ \left(ab - \frac{((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r\sigma\lambda}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)i(t) + \left(\alpha\beta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)\sigma_y\sigma\lambda}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)j(t)\\
&- \frac{\gamma}{1-\gamma}\left(m_1\sigma_r\sigma_y\theta - \frac{((1-\gamma)\sigma_1 - \theta\sigma m m_1)((1-\gamma)\sigma_2 - \theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)i(t)j(t)\\
&+ \frac{\gamma}{2(1-\gamma)}\sigma_r^2\left(1-\gamma-m_1^2\theta + \frac{((1-\gamma)\sigma_1 - \theta\sigma m m_1)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)i^2(t)\\
&+ \frac{\gamma}{2(1-\gamma)}\sigma_y^2\left(1-\gamma-\theta + \frac{((1-\gamma)\sigma_2 - \theta\sigma m)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)j^2(t) + \frac{(1-\gamma)(\sigma\lambda)^2}{2\gamma(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)} = 0,
\end{aligned}
$$

removing the dependence on variables r and y yields the following equations:

$$
i'(t) - ai(t) + \frac{1-\gamma}{\gamma} = 0, \quad i(T) = 0, \tag{3.13}
$$

$$
\begin{aligned}
&j'(t) + \frac{(1-\gamma)\rho\sigma\lambda}{\gamma(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)} - \left(\alpha + \frac{\rho((1-\gamma)\sigma_2 - \theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)j(t)\\
&+ 2\left(\alpha\beta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)k(t) - \frac{\rho((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}i(t)\\
&- \frac{2\gamma}{1-\gamma}\left(m_1\sigma_r\sigma_y\theta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r\sigma_y}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)i(t)k(t)\\
&+ \frac{2\gamma}{1-\gamma}\sigma_y^2\left(1-\gamma-\theta + \frac{((1-\gamma)\sigma_2 - \theta\sigma m)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)j(t)k(t) = 0, \quad j(T) = 0,
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
&k'(t) + \frac{(1-\gamma)\rho^2}{2\gamma(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)} - 2\left(\alpha - \frac{\rho((1-\gamma)\sigma_2 - \theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)k(t)\\
&+ \frac{2\gamma}{1-\gamma}\sigma_y^2\left(1-\gamma-\theta + \frac{((1-\gamma)\sigma_2 - \theta\sigma m)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)k^2(t) = 0, \quad k(T) = 0,
\end{aligned} \tag{3.15}
$$

$$\begin{aligned}
&h'(t)+\sigma_y^2k^2(t)+\frac{(1-\gamma)c}{\gamma x}+\frac{(1-\gamma)(\sigma\lambda)^2}{2\gamma(\gamma\sigma\sigma^{\mathrm{T}}+\theta(\sigma m)^2)}\\
&+\Big(ab-\frac{((1-\gamma)\sigma_1-\theta\sigma mm_1)\sigma_r\sigma\lambda}{\gamma\sigma\sigma^{\mathrm{T}}+\theta(\sigma m)^2}\Big)i(t)+\Big(\alpha\beta-\frac{((1-\gamma)\sigma_2-\theta\sigma m)\sigma_y\sigma\lambda}{\gamma\sigma\sigma^{\mathrm{T}}+\theta(\sigma m)^2}\Big)j(t)\\
&-\frac{\gamma}{1-\gamma}\Big(m_1\sigma_r\sigma_y\theta-\frac{((1-\gamma)\sigma_1-\theta\sigma mm_1)((1-\gamma)\sigma_2-\theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}}+\theta(\sigma m)^2}\Big)i(t)j(t)\\
&+\frac{\gamma}{2(1-\gamma)}\sigma_r^2\Big(1-\gamma-m_1^2\theta+\frac{((1-\gamma)\sigma_1-\theta\sigma mm_1)^2}{\gamma\sigma\sigma^{\mathrm{T}}+\theta(\sigma m)^2}\Big)i^2(t)\\
&+\frac{\gamma}{2(1-\gamma)}\sigma_y^2\Big(1-\gamma-\theta+\frac{((1-\gamma)\sigma_2-\theta\sigma m)^2}{\gamma\sigma\sigma^{\mathrm{T}}+\theta(\sigma m)^2}\Big)j^2(t)=0,\quad h(T)=0.
\end{aligned}\tag{3.16}$$

Next, we solve the Eqs (3.13)–(3.15). By a simple calculation, we derive $i(t)$ as

$$i(t)=\frac{1-\gamma}{a\gamma}\left(1-e^{-a(T-t)}\right).\tag{3.17}$$

For convenience of expression, we denote (3.15) as

$$k'(t)=u_0k^2(t)+u_1k(t)+u_2,\quad k(T)=0,\tag{3.18}$$

where $u_0=\frac{2\gamma}{1-\gamma}\sigma_y^2\left(\theta-(1-\gamma)-\frac{((1-\gamma)\sigma_2-\theta\sigma m)^2}{\gamma\sigma\sigma^{\mathrm{T}}+\theta(\sigma m)^2}\right)$, $u_1=2\left(\alpha-\frac{\rho((1-\gamma)\sigma_2-\theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}}+\theta(\sigma m)^2}\right)$, $u_2=-\frac{(1-\gamma)\rho^2}{2\gamma(\gamma\sigma\sigma^{\mathrm{T}}+\theta(\sigma m)^2)}$, $\Delta=u_1^2-4u_0u_2$.

Case I: $\Delta>0$

Equation (3.18) has two different roots: $l_{1,2}=\frac{-u_1\pm\sqrt{\Delta}}{2u_0}$. By integrating from t to T, we obtain the explicit form of $k(t)$ by transforming (3.18) into a solvable form, namely:

$$\begin{aligned}
&\frac{k'(t)}{(k(t)-l_1)(k(t)-l_2)}=u_0,\\
\Rightarrow\ &\frac{k'(t)}{k(t)-l_1}-\frac{k'(t)}{k(t)-l_2}=\sqrt{\Delta},\\
\Rightarrow\ &\int_t^T\frac{k'(s)}{k(s)-l_1}-\int_t^T\frac{k'(s)}{k(s)-l_2}=\sqrt{\Delta}(T-t),\\
\Rightarrow\ &k(t)=\frac{l_1l_2\exp\left(-\sqrt{\Delta}(T-t)\right)-l_1l_2}{l_1\exp\left(-\sqrt{\Delta}(T-t)\right)-l_2}.
\end{aligned}\tag{3.19}$$

Case II: $\Delta=0$

Equation (3.18) has only one root. Similar to the above method, we can get $k(t)$ by

$$\begin{aligned}
&\frac{k'(t)}{\left(k(t)+\frac{u_1}{2u_0}\right)^2}=u_0,\\
\Rightarrow\ &\frac{1}{k(T)+\frac{u_1}{2u_0}}-\frac{1}{k(t)+\frac{u_1}{2u_0}}=u_0(T-t),\\
\Rightarrow\ &k(t)=\frac{u_1}{2u_0}\left(\frac{1}{1-u_1(T-t)}-1\right).
\end{aligned}\tag{3.20}$$

Case III: $\Delta < 0$

The right side of (3.18) is always greater than 0. We solve $k(t)$ by the following method:

$$
\begin{aligned}
&k'(t) = u_0\left(k(t) + \frac{u_1}{2u_0}\right)^2 - \frac{\Delta}{4u_0},\\
\Rightarrow\ & \frac{k'(t)}{\left(k(t) + \frac{u_1}{2u_0}\right)^2 - \frac{\Delta}{4u_0^2}} = u_0,\\
\Rightarrow\ & \frac{k'(t)\sqrt{\frac{4u_0^2}{-\Delta}}}{\left[\sqrt{\frac{4u_0^2}{-\Delta}}k(t) + \sqrt{\frac{4u_0^2}{-\Delta}}\frac{u_1}{2u_0}\right]^2 + 1} = u_0\sqrt{\frac{-\Delta}{4u_0^2}},\\
\Rightarrow\ & \int_t^T \left(\frac{k'(s)\sqrt{\frac{4u_0^2}{-\Delta}}}{\left[\sqrt{\frac{4u_0^2}{-\Delta}}k(s) + \sqrt{\frac{4u_0^2}{-\Delta}}\frac{u_1}{2u_0}\right]^2 + 1}\right) ds = u_0\sqrt{\frac{-\Delta}{4u_0^2}}(T-t),\\
\Rightarrow\ & k(t) = \frac{\sqrt{-\Delta}}{2u_0}\tan\left\{\arctan\left(\frac{u_1}{\sqrt{-\Delta}}\right) - \frac{\sqrt{\Delta}}{2}(T-t)\right\} - \frac{u_1}{2u_0}.
\end{aligned}
\tag{3.21}
$$

Next, solve (3.14); we assume that:

$$
\begin{aligned}
\zeta(t) = &-\left(\alpha + \frac{\rho((1-\gamma)\sigma_2 - \theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)\\
&+ \frac{2\gamma}{1-\gamma}\sigma_y^2\left(1-\gamma-\theta+\frac{((1-\gamma)\sigma_2 - \theta\sigma m)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)k(t),
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
\delta(t) = &-2\left(\alpha\beta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)k(t) + \frac{\rho((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}i(t)\\
&+ \frac{2\gamma}{1-\gamma}\left(m_1\sigma_r\sigma_y\theta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma_r\sigma_y}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2}\right)i(t)k(t)\\
&- \frac{(1-\gamma)\rho\sigma\lambda}{\gamma(\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2)},
\end{aligned}
\tag{3.23}
$$

then (3.14) can be expressed as

$$
j'(t) + \zeta(t)j(t) = \delta(t), \quad j(T) = 0,
\tag{3.24}
$$

by solving this ordinary differential equation, we obtain:

$$
j(t) = e^{-\int_t^T \zeta(u)du}\left(\int_t^T \delta(s)e^{\zeta(u)du}ds\right).
\tag{3.25}
$$

Finally, applying $i(t), j(t), k(t)$ just obtained and integrating over the left and right sides of (3.16) yields:

$$\begin{aligned} h(t) = &- \int_t^T \left\{ \left(\alpha\beta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)\sigma\lambda\sigma_y}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2} \right) j(s) \right. \\ &+ \left(ab - \frac{((1-\gamma)\sigma_1 - \theta\sigma m m_1)\sigma\lambda\sigma_r}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2} \right) i(s) \\ &+ \frac{\gamma\sigma_r^2}{2(1-\gamma)} \left(1 - \gamma - m_1^2\theta + \frac{((1-\gamma)\sigma_1 - \theta\sigma m m_1)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2} \right) i^2(s) \\ &+ \frac{\gamma\sigma_y^2}{2(1-\gamma)} \left(1 - \gamma - \theta + \frac{((1-\gamma)\sigma_2 - \theta\sigma m)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2} \right) j^2(s) + \sigma_y^2 k^2(s) \\ &- \left. \frac{\gamma}{1-\gamma} \left(m_1\sigma_r\sigma_y\theta - \frac{((1-\gamma)\sigma_2 - \theta\sigma m)((1-\gamma)\sigma_1 - \theta\sigma m m_1)}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2} \right) i(s) j(s) \right\} ds \\ &- \frac{1-\gamma}{\gamma} \left(\frac{c}{x} + \frac{(\sigma\lambda)^2}{\gamma\sigma\sigma^{\mathrm{T}} + \theta(\sigma m)^2} \right) (T - t). \end{aligned} \tag{3.26}$$

□

3.2. *The cost of climate uncertainty*

In this subsection, we discuss the process used for evaluating the influence of climate uncertainty on investment return. We achieve this by making a comparison of the maximum expected utility the investor would receive with and without climate change uncertainty. Define μ as the cost of climate uncertainty, representing the proportion of their maximized terminal expected wealth that the pension investor in an ambiguity-free environment is willing to forgo in exchange for eliminating the climate uncertainty they face. Supposing a degree $\tilde{\theta}$ for the pension investor uncertain about the climate risk, then we have:

$$V^{\theta=0}(t, x(1-\mu), r, y) = V^{\theta=\tilde{\theta}}(t, x, r, y), \tag{3.27}$$

where $0 \le \mu \le 1$.

Theorem 3.2. The cost related to uncertainty in climate is presented in the following:

$$\begin{aligned} \mu = &1 - \exp\left(\frac{\gamma}{1-\gamma} \Big[h^{\theta=\tilde{\theta}}(t) - h^{\theta=0}(t) \right. \\ &\left. + \left(j^{\theta=\tilde{\theta}}(t) - j^{\theta=0}(t) \right) y + \left(k^{\theta=\tilde{\theta}}(t) - k^{\theta=0}(t) \right) y^2 \Big] \right). \end{aligned} \tag{3.28}$$

Proof.

$$\begin{aligned} V^{\theta=0}(t, x(1-\mu), r, y) = &\frac{(x(1-\mu))^{1-\gamma}}{1-\gamma} g^{\theta=0}(t, r, y)^\gamma \\ = &\frac{x^{1-\gamma}}{1-\gamma} g^{\theta=\tilde{\theta}}(t, r, y)^\gamma \\ = &V^{\theta=\tilde{\theta}}(t, x, r, y), \end{aligned}$$

then we derive:

$$(1-\mu)^{1-\gamma} = \frac{g^{\theta=\tilde{\theta}}(t, r, y)^\gamma}{g^{\theta=0}(t, r, y)^\gamma},$$

from (3.4) yields:

$$\mu =1 - \left(\frac{g^{\theta=\tilde{\theta}}(t,r,y)^{\gamma}}{g^{\theta=0}(t,r,y)^{\gamma}}\right)^{\frac{\gamma}{1-\gamma}}$$
$$=1 - \exp\left(\frac{\gamma}{1-\gamma}\left[h^{\theta=\tilde{\theta}}(t) - h^{\theta=0} + \left(i^{\theta=\tilde{\theta}(t)} - i^{\theta=0}(t)\right) r\right.\right.$$
$$\left.\left. + \left(j^{\theta=\tilde{\theta}}(t) - j^{\theta=0}(t)\right) y + \left(k^{\theta=\tilde{\theta}}(t) - k^{\theta=0}(t)\right) y^2\right]\right),$$

since $i(t)$ is not related to θ, it follows that:

$$\mu =1 - \exp\left(\frac{\gamma}{1-\gamma}\left[h^{\theta=\tilde{\theta}}(t) - h^{\theta=0}(t)\right.\right.$$
$$\left.\left. + \left(j^{\theta=\tilde{\theta}}(t) - j^{\theta=0}(t)\right)y + \left(k^{\theta=\tilde{\theta}}(t) - k^{\theta=0}(t)\right)y^2\right]\right).$$

□

Remark 3.3. In this paper, we assume that the pension investor is ambiguity averse; therefore, $V^{\theta=0} \geq V^{\theta=\tilde{\theta}}$, and $g \geq 0$, then we have $0 \leq \mu \leq 1$, which indicates the climate uncertainty is perceived as a source of wealth loss and the cost required by the pension investor to eliminate this uncertainty. Furthermore, if $\mu < 0$, this represents an uncertainty premium. This situation occurs in specific circumstances where climate uncertainty is regarded as beneficial. For instance, when ambiguity is associated with upside risks, the ambiguity-averse pension investor may demand additional wealth to give up the certainty of the ambiguity-free world. In models with ambiguity preference, the pension investor may gain utility from an uncertain environment and thus demand a premium in exchange for certainty.

4. Numerical analysis

In this section, we show some numerical cases to demonstrate the theoretical result of the previous section. Unless specifically stated, the assumed values of the model parameters are as follows: $a = 0.2$, $b = 0.02$, $\alpha = 0.1$, $\beta = 0.01$, $\rho = 0.015$, $\gamma = 0.5$, $\theta = 1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.4$, $\lambda_1 = 0.15$, $\lambda_2 = 0.2$, $\lambda_3 = 0.2$, $\sigma_r = 0.02$, $\sigma_y = 0.2$, $m_1 = 0.2$, $m_2 = 0.3$, $y_0 = 0.2$, $t = 0$, $T = 40$.

Figure 1 shows the effect of θ on $\pi^*(t)$. It is evident that the stock index investment is significantly impacted by climate uncertainty. When θ increases, the demand for the stock decreases by about 25%. Changes in the stock index as a result of heightened climate uncertainty play an essential part in shaping investor perceptions and investment choices. From the perspective of market risk, climate uncertainty, such as extreme weather events like droughts and floods, may significantly impact the daily operations of relevant listed companies. This can result in a decline in the stock prices of these listed companies, leading to a reduction in investment returns for stockholders and an increase in market risk.

Figure 2 suggests the impact of volatility in the global temperature on the investment strategy. We can empirically observe a negative correlation between the increasing volatility of the global temperature and the proportion of investments in the stock index. Increased global temperature volatility may lead to heightened risks associated with climate-related events, such as extreme weather, affecting various industries. Therefore, adapting pension fund investment approaches to the changing volatility of the global temperature is crucial for managing risks and identifying sustainable long-term investment opportunities.

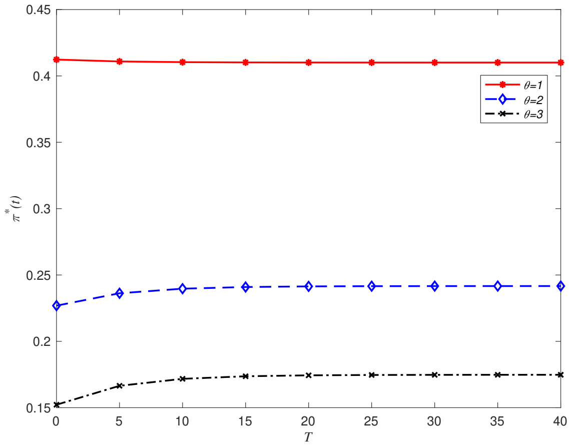

Figure 1. Effect of θ on $\pi^*(t)$.

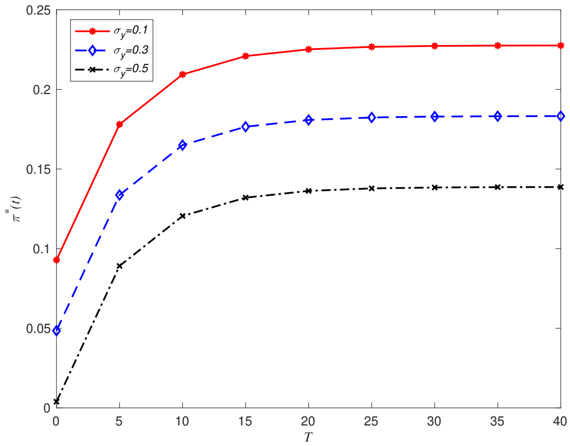

Figure 2. Effect of σ_y on $\pi^*(t)$.

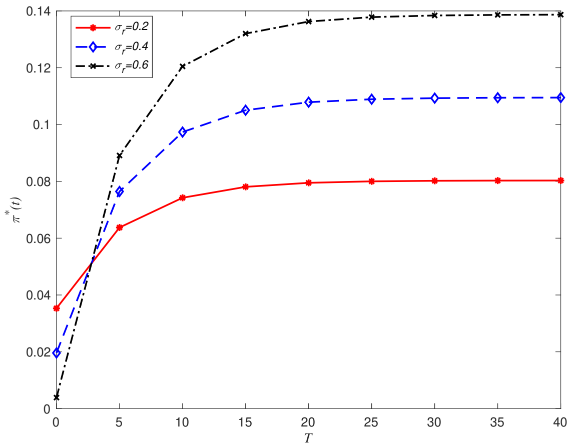

Figure 3. Effect of σ_r on $\pi^*(t)$.

Furthermore, this paper incorporates stochastic volatility into the analysis. As depicted in Figure 3, it is evident that the optimal portfolio is impacted by interest rate volatility. In the initial stages of the investment, the proportion allocated to the stock index tends to decrease with an increase in volatility. However, in the mid-to-late stages, the allocation to the stock index tends to rise with escalating volatility. Moreover, as retirement approaches, the investment tends to stabilize.

The impact of the risk aversion coefficient γ on $\pi^*(t)$ is seen in Figure 4. The graph illustrates that in the initial five years, as the risk aversion coefficient increases, the pension fund managers cut back on the investment in the stock index. However, over time, as assets in the pension accounts accumulate, the pension fund manager increases the investment in the stock index to ensure the quality of life in retirement. Similarly, the investment tends to stabilize as retirement approaches.

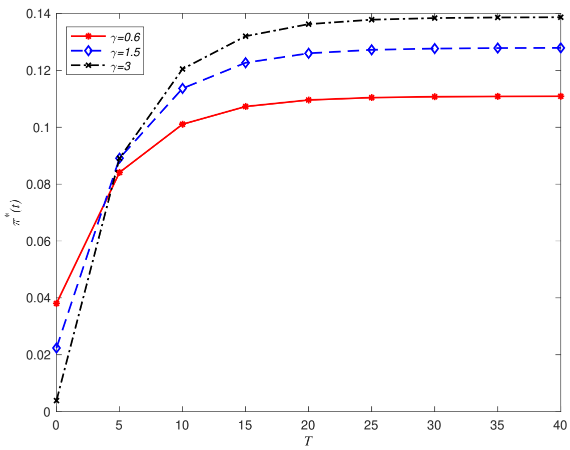

Figure 4. Effect of γ on $\pi^*(t)$.

From Figure 5, it is evident that the optimal investment strategy $\pi^*(t)$ is inversely proportional to ρ. As ρ increases, indicating a greater influence of the global temperature on the stock market, the pension fund manager would consequently reduce the stock investment to mitigate risks and safeguard the post-retirement quality of life for the participant.

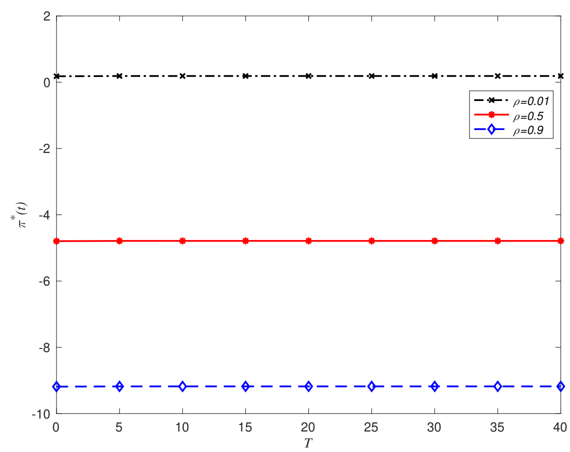

Figure 5. Effect of ρ on $\pi^*(t)$.

Figure 6 reflects the relationship between changes in the investment time T and the investment strategy $\pi^*(t)$. Over time, pension fund managers will progressively decrease their investments. Particularly, beyond the 20th year, investments tend to plateau, ensuring the post-retirement quality of life for employees.

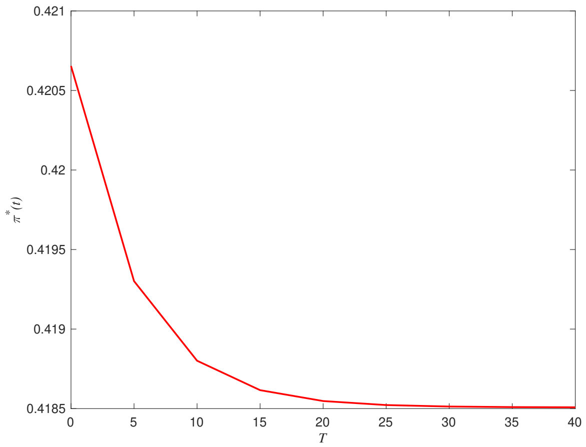

Figure 6. Effect of T on $\pi^*(t)$.

Figure 7 indicates the impact of parameters associated with interest rate on investment. It is observed that the stock investment remains unaffected by b. However, in a broader context, $\pi^*(t)$ decreases with an increase in a. This phenomenon can be attributed to the heightened prudence of the pension fund manager in response to a faster mean reversion speed. The anticipation of the interest rate reverting to the mean more rapidly leads to a reduction in stock investment, reflecting a cautious approach among the pension fund managers.

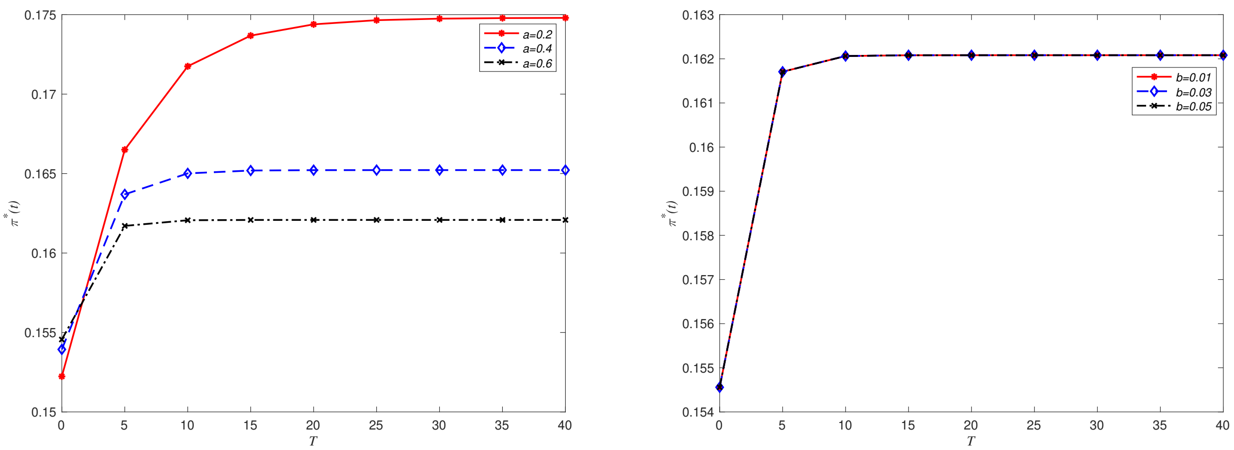

Figure 7. Effect of a and b on $\pi^*(t)$.

Figure 8 illustrates the effect of parameters related to the global temperature on the investment. Specifically, $\pi^*(t)$ decreases with a reduction in α. A slower mean reversion speed implies prolonged uncertainty for climate-related industries, which may adversely affect the stock performance of these sectors. Consequently, the pension fund manager is inclined to decrease investment in the stock. On the other hand, an increase in β corresponds to a concurrent reduction in the stock investment. This is because the long-term mean level of the global temperature may influence investor sentiment and confidence. If the investor harbors concerns regarding the impact of climate change, it may prompt risk-averse behavior, leading to a diminished allocation of investment in the stock.

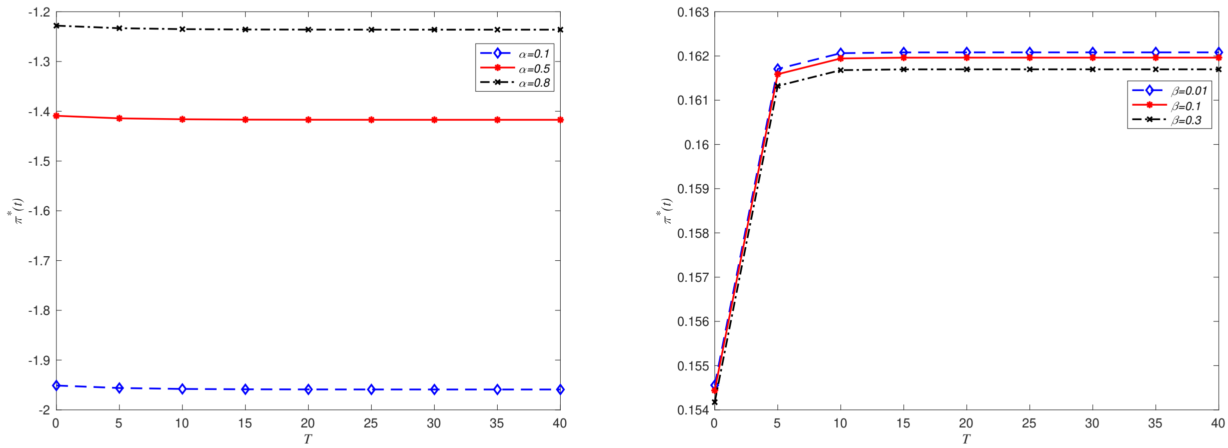

Figure 8. Effect of α and β on $\pi^*(t)$.

5. Conclusions

In this study, we examine the optimal portfolio selection problem for the DC pension plan explicitly considering climate uncertainty. We obtain the closed-form solution for the optimum stock index-cash portfolio strategy under the CRRA utility function, assuming the interest rate exhibits volatility following an Ornstein–Uhlenbeck model. Through numerical analysis, we find that both climate uncertainty and stochastic interest rates significantly impact the optimal portfolio. The research we conducted highlights the substantial influence of climate uncertainty on the decision to invest. Additionally, we emphasize that neglecting climate uncertainty in the investment decision can result in a significant loss of returns, offering valuable information for investors.

In increasingly turbulent financial markets, perturbations from both internal parameters and external environmental factors can exacerbate the uncertainty associated with decision models. Our research underscores the importance of accounting for climate uncertainty when making the portfolio decision for the DC pension fund, as it significantly affects the long-term sustainability and returns of the investment.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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