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Research article

Well-posedness of a triply coupled system of fractional Langevin equations with closed boundary conditions

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Abstract: In this paper, we investigated a class of nonlinear triply coupled systems of fractional Langevin equations subject to closed boundary conditions. The existence of solutions to the proposed boundary value problem was first established by applying Krasnoselskii's fixed point theorem. Furthermore, the uniqueness of the solution was obtained via the Banach contraction mapping principle. To demonstrate the effectiveness of the theoretical results, illustrative examples are provided.

Keywords: triply coupled system; fractional Langevin equation; closed boundary condition; existence and uniqueness; fixed point theorem

1. Introduction

Fractional calculus constitutes an extension of classical differential and integral calculus, wherein the order of differentiation and integration is generalized from integer values to arbitrary real or even complex numbers. The foundational idea is to relax the constraint of integer-order operators, thereby enabling a broader analytical framework for describing nonlocal and memory-dependent phenomena. Since the 1990s, fractional calculus has evolved from a purely theoretical construct into a powerful tool with widespread applications across scientific disciplines, including physics, control theory, biomedical engineering, finance, and economics, as well as computational engineering and image processing [1–3]. For instance, within the framework of Caputo fractional calculus, Lutz and Burov [4, 5] proposed the following fractional Langevin equation (FLE)

$$\mathfrak{x}''(t) + \zeta\, {}^{C}\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}(t) = \Upsilon(t),$$

where $0 < \alpha < 1$, ${}^{C}\mathfrak{D}^{\alpha}_{0+}$ is the Caputo fractional derivative (CFD) of order α. The Langevin equation plays a central role in understanding and describing the behavior of particles in fluids and other macroscopic phenomena. It bridges macroscopic physics and microscopic statistical mechanics, providing a powerful tool for studying complex systems [6].

In recent years, the solvability of anti-periodic boundary value problems (BVPs) for fractional differential equations (FDEs) has attracted extensive attention from scholars [7–10]. In particular, anti-periodic BVPs driven by the FLEs have been intensively studied [11–21]. For example, Baghani et al. [11] used the Banach contraction mapping principle (BCMP) to prove the existence and uniqueness (E&U) of solutions for the anti-periodic BVP of the following coupled system of FLEs

$$\begin{cases} {}^C\mathfrak{D}^{\eta_1}_{0+}({}^C\mathfrak{D}^{\xi_1}_{0+}+\zeta_1)\mathfrak{z}(t)=\Psi(t,\mathfrak{z}(t),\mathfrak{w}(t)),\ t\in(0,1),\ \xi_1\in(0,1],\ \eta_1\in(1,2],\\ {}^C\mathfrak{D}^{\eta_2}_{0+}({}^C\mathfrak{D}^{\xi_2}_{0+}+\zeta_2)\mathfrak{w}(t)=\Phi(t,\mathfrak{z}(t),\mathfrak{w}(t)),\ t\in(0,1),\ \xi_2\in(0,1],\ \eta_2\in(1,2],\\ \mathfrak{z}(0)+\mathfrak{z}(1)=0,\ \mathcal{D}^{\xi_1}\mathfrak{z}(0)+\mathcal{D}^{\xi_1}\mathfrak{z}(1)=0,\ \mathcal{D}^{2\xi_1}\mathfrak{z}(0)+\mathcal{D}^{2\xi_1}\mathfrak{z}(1)=0,\\ \mathfrak{w}(0)+\mathfrak{w}(1)=0,\ \mathcal{D}^{\xi_2}\mathfrak{w}(0)+\mathcal{D}^{\xi_2}\mathfrak{w}(1)=0,\ \mathcal{D}^{2\xi_2}\mathfrak{w}(0)+\mathcal{D}^{2\xi_2}\mathfrak{w}(1)=0, \end{cases}$$

where ${}^C\mathfrak{D}^{\kappa}_{0+}$ is the CFD of order κ, $\kappa\in\{\eta_1,\xi_1,\eta_2,\xi_2\}$. $\mathcal{D}^{m\xi_i}$ $(m,i=1,2)$ denotes the sequential fractional derivative, $\Psi,\Phi\in C([0,1]\times\mathbb{R}^2,\mathbb{R}),\chi_1,\chi_2\in\mathbb{R}$.

Zhang and Ni [12] employed the Krasnoselskii fixed point theorem (FPT) and the BCMP to investigate the E&U of solutions for the cyclic anti-periodic BVPs of the following tripled system of FLEs

$$\begin{cases} {}^C\mathfrak{D}^{\beta}_{0+}({}^C\mathfrak{D}^{\alpha}_{0+}+\zeta)\mathfrak{x}_i(t)=\mathfrak{f}_i(t,\mathfrak{x}_1(t),\mathfrak{x}_2(t),\mathfrak{x}_3(t)),\ t\in(0,1),\ i=1,2,3,\\ \mathfrak{x}_1(0)+\mathfrak{x}_2(1)=0,\quad {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_1(0)+{}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_2(1)=0,\\ \mathfrak{x}_2(0)+\mathfrak{x}_3(1)=0,\quad {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_2(0)+{}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_3(1)=0,\\ \mathfrak{x}_3(0)+\mathfrak{x}_1(1)=0,\quad {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_3(0)+{}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_1(1)=0, \end{cases}$$

where ${}^C\mathfrak{D}^{\kappa}_{0+}$ denotes the CFD of order $\kappa\in\{\alpha,\beta\}$, $\alpha,\beta\in(0,1)$, $\zeta\in\mathbb{R}^+$, $\mathfrak{f}_i\in C([0,1]\times\mathbb{R}^3,\mathbb{R})$, and $i=1,2,3$.

Alsaedi et al. [22] utilized the Leray-Schauder FPT and the BCMP to examine the E&U of solutions for the following coupled system of FDEs with closed boundary conditions (BCs)

$$\begin{cases} {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}(t)=\mathfrak{f}_1(t,\mathfrak{x}(t),\mathfrak{y}(t)),\ t\in(0,T),\ \alpha\in(1,2),\\ {}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{y}(t)=\mathfrak{f}_2(t,\mathfrak{x}(t),\mathfrak{y}(t)),\ t\in(0,T),\ \beta\in(1,2),\\ \mathfrak{x}(T)=p_1\mathfrak{y}(0)+q_1T\mathfrak{y}'(0),\ T\mathfrak{x}'(T)=\gamma_1\mathfrak{y}(0)+\delta_1T\mathfrak{y}'(0),\\ \mathfrak{y}(T)=p_2\mathfrak{x}(0)+q_2T\mathfrak{x}'(0),\ T\mathfrak{y}'(T)=\gamma_2\mathfrak{x}(0)+\delta_2T\mathfrak{x}'(0), \end{cases}$$

where ${}^C\mathfrak{D}^{\kappa}_{0+}$ denotes the CFD of order $\kappa\in\{\alpha,\beta\}$, $p_1,p_2,q_1,q_2,\gamma_1,\gamma_2,\delta_1,\delta_2\in\mathbb{R}$, $T>0$, $\mathfrak{f}_1,\mathfrak{f}_2\in C([0,T]\times\mathbb{R}^2,\mathbb{R})$.

According to the available literature, there is no research on the existence of solutions for the tripled system of FLEs with closed BCs. This constitutes a new class of BVPs for fractional differential systems. Therefore, inspired by the literature, we qualitatively analyze the nonlinear triply coupled system of FLEs with closed BCs. The specific form is as follows:

$$\begin{cases} {}^C\mathfrak{D}^{\beta}_{0+}({}^C\mathfrak{D}^{\alpha}_{0+}+\zeta)\mathfrak{x}_j(t)=\mathfrak{f}_j(t,\mathfrak{x}_1(t),\mathfrak{x}_2(t),\mathfrak{x}_3(t)),\ t\in(0,1),\ \alpha,\beta\in(0,1),\\ \mathfrak{x}_1(1)=\mu_1\mathfrak{x}_1(0)+\eta_1{}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_1(0),\ {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_1(1)=\gamma_1\mathfrak{x}_1(0)+\delta_1{}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_1(0),\\ \mathfrak{x}_2(1)=\mu_2\mathfrak{x}_2(0)+\eta_2{}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_2(0),\ {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_2(1)=\gamma_2\mathfrak{x}_2(0)+\delta_2{}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_2(0),\\ \mathfrak{x}_3(1)=\mu_3\mathfrak{x}_3(0)+\eta_3{}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_3(0),\ {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_3(1)=\gamma_3\mathfrak{x}_3(0)+\delta_3{}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_3(0), \end{cases} \tag{1.1}$$

where $j = 1,2,3$, ${}^C\mathfrak{D}^{\kappa}_{0+}$ denotes the CFD of order $\kappa \in \{\alpha,\beta\}$, $1 < \alpha + \beta = \iota < 2$, $\zeta \in \mathbb{R}^+$, $\mathfrak{f}_j \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$, $\mu_j, \eta_j, \gamma_j, \delta_j \in \mathbb{R}$, satisfying

$$\begin{aligned}\Delta_j &= (1 - \delta_j - \zeta\eta_j)(1 - \mu_j + \zeta\eta_j)\Gamma(\alpha + 1)\\ &\quad + (\gamma_j + \zeta\mu_j - \zeta\delta_j - \zeta^2\eta_j)(1 - \eta_j\Gamma(\alpha + 1)) \neq 0, \quad j = 1,2,3.\end{aligned}$$

Note that the closed BCs are a class of generalized anti-periodic BCs. If the parameters take special values $\mu_j = \delta_j = -1$, $\eta_j = \gamma_j = 0$, and $j = 1,2,3$, then the closed BCs can reduce to the anti-periodic BCs. Moreover, the closed BCs discussed in this paper involve fractional derivatives that are more general than those in reference [22]. Consequently, the results obtained in this paper extend and enrich research findings on anti-periodic BVPs for FLEs.

The organization of the paper is as follows: In Section 2, we provide a concise overview of basic concepts and lemmas in fractional calculus, as well as Krasnoselskii's FPT and the BCMP, which together constitute the analytical foundation for the proofs of the major results. Section 3 is devoted to establishing sufficient conditions for the E&U of solutions to the nonlinear BVP (1.1) by constructing appropriate operator equations based on the Krasnoselskii's FPT and BCMP. In Section 4, we validate the effectiveness and applicability of our main conclusions by constructing two examples. Finally, in Section 5, we given a concise summary of the principal findings and outline several open problems that merit further analytical investigation.

2. Preliminaries

In this section, we collect the essential definitions and properties of fractional calculus, along with the Krasnoselskii's FPT and the BCMP.

Definition 2.1. [3, 23, 24] The Riemann-Liouville fractional integral of order $\hbar$ ($\hbar > 0$) for a function $\vartheta : [0, +\infty) \to \mathbb{R}$ is given by

$$\mathfrak{I}^{\hbar}_{0+}\vartheta(t) = \frac{1}{\Gamma(\hbar)} \int_0^t (t-s)^{\hbar-1}\vartheta(s)ds, \quad t > 0,$$

provided that the right-hand side is pointwise well-defined on $(0, +\infty)$.

Definition 2.2. [3, 23, 24] For $\vartheta(t) \in AC^{\mathfrak{n}}[0, +\infty)$, the CFD of order $\hbar$ ($\hbar > 0$) is given by

$${}^C\mathfrak{D}^{\hbar}_{0+}\vartheta(t) = \frac{1}{\Gamma(\mathfrak{n}-\hbar)} \int_0^t (t-s)^{\mathfrak{n}-\hbar-1}\vartheta^{(\mathfrak{n})}(s)ds, \quad t > 0,$$

where $\mathfrak{n} = [\hbar] + 1$.

Lemma 2.1. [3, 23, 24] Let $\hbar, \aleph > 0$, and $\vartheta(t) \in C[0,1]$, then

$$(\mathfrak{I}^{\hbar}_{0+}\mathfrak{I}^{\aleph}_{0+}\vartheta)(t) = (\mathfrak{I}^{\hbar+\aleph}_{0+}\vartheta)(t), \quad \mathfrak{I}^{\hbar}_{0+}t^{\aleph-1} = \frac{\Gamma(\aleph)}{\Gamma(\hbar+\aleph)}t^{\hbar+\aleph-1},$$

$${}^C\mathfrak{D}^{\hbar}_{0+}\mathfrak{I}^{\hbar}_{0+}\vartheta(t) = \vartheta(t), \quad {}^C\mathfrak{D}^{\hbar}_{0+}t^{\aleph-1} = \frac{\Gamma(\aleph)}{\Gamma(\aleph-\hbar)}t^{\aleph-\hbar-1}.$$

Lemma 2.2. [3, 23, 24] Let $\hbar > 0$ and $\mathfrak{n} = [\hbar] + 1$. If $\vartheta(t) \in AC^{\mathfrak{n}}[0,1]$, then

$$\mathfrak{I}^{\hbar}_{0+}{}^C\mathfrak{D}^{\hbar}_{0+}\vartheta(t) = \vartheta(t) + \sum_{\mathfrak{i}=0}^{\mathfrak{n}-1}\mathfrak{c}_{\mathfrak{i}}t^{\mathfrak{i}}, \ 0 < t < 1,$$

where $\mathfrak{c}_0, \mathfrak{c}_1, \cdots, \mathfrak{c}_{\mathfrak{n}-1} \in \mathbb{R}$.

Theorem 2.1. (Krasnoselskii's FPT) [23,24] Let Ω be a nonempty, bounded, convex, and closed subset of the Banach space $\mathfrak{X}$. Let $\mathcal{G}$ and $\mathcal{F}$ be two operators, satisfying

(¶) $\mathcal{G}\mathfrak{x} + \mathcal{F}\mathfrak{y} \in \Omega$, $\forall \mathfrak{x}, \mathfrak{y} \in \Omega$;
(¶) $\mathcal{G} : \Omega \to \mathfrak{X}$ is compact and continuous;
(¶) $\mathcal{F} : \Omega \to \mathfrak{X}$ is a contraction mapping.

Then, one can find an element $\mathfrak{z} \in \Omega$, such that $\mathfrak{z} = \mathcal{G}\mathfrak{z} + \mathcal{F}\mathfrak{z}$.

Theorem 2.2. (BCMP) [23, 24] Let Ω is a closed nonempty subset in the Banach space $\mathfrak{X}$, and let $\mathfrak{T} : \Omega \subseteq \mathfrak{X} \to \Omega$ is contractive, i.e., for a fixed $\mu \in [0, 1)$,

$$\|\mathfrak{T}\mathfrak{x} - \mathfrak{T}\mathfrak{y}\|_{\mathfrak{X}} \leq \mu \|\mathfrak{x} - \mathfrak{y}\|_{\mathfrak{X}}, \quad \forall\ \mathfrak{x}, \mathfrak{y} \in \Omega.$$

Then $\mathfrak{T}$ has exactly one fixed point on Ω.

3. Major results

In this section, we prove the existence and E&U of solutions to the BVP (1.1) by means of Theorems 2.1 and 2.2, respectively. To begin, we introduce the Banach space $\mathfrak{X} = C[0, 1]$, endowed with the norm

$$\|\mathfrak{x}\|_\infty = \max_{t\in[0,1]} |\mathfrak{x}(t)|.$$

Additionally, we define the space $\mathcal{X} = \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$, endowed with the norm

$$\|(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)\|_{\mathcal{X}} = \|\mathfrak{x}_1\|_\infty + \|\mathfrak{x}_2\|_\infty + \|\mathfrak{x}_3\|_\infty,\ (\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3) \in \mathcal{X}.$$

Thus, $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach space.

To introduce the fixed point approach, we next present an auxiliary lemma that solves a linear variant associated with problem (1.1), which serves as a foundational step in reformulating the original nonlinear problem into an equivalent fixed point framework.

Lemma 3.1. Let $\mathfrak{h}_j \in C([0, 1], \mathbb{R})$, $j = 1, 2, 3$. Then the system of FLEs

$$^C\mathfrak{D}^{\beta}_{0+}(^C\mathfrak{D}^{\alpha}_{0+} + \zeta)\mathfrak{x}_j(t) = \mathfrak{h}_j(t), \quad t \in (0, 1), \tag{3.1}$$

$1 < \iota < 2$, under the closed BCs

$$\mathfrak{x}_j(1) = \mu_j \mathfrak{x}_j(0) + \eta_j {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_j(0),\ {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_j(1) = \gamma_j \mathfrak{x}_j(0) + \delta_j {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_j(0), \tag{3.2}$$

admits a solution of the following form

$$\begin{aligned}\mathfrak{x}_j(t) = &\frac{1}{\Gamma(\iota)}\int_0^t (t-s)^{\iota-1}\mathfrak{h}_j(s)ds - \frac{Q_{j2}t^{\alpha}}{\Gamma(\iota)}\int_0^1 (1-s)^{\iota-1}\mathfrak{h}_j(s)ds\\ &- \frac{Q_{j4}}{\Gamma(\iota)}\int_0^1 (1-s)^{\iota-1}\mathfrak{h}_j(s)ds - \frac{Q_{j1}t^{\alpha}}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\mathfrak{h}_j(s)ds\\ &+ \frac{Q_{j3}}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\mathfrak{h}_j(s)ds - \frac{\zeta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathfrak{x}_j(s)ds\end{aligned} \tag{3.3}$$

$$+\frac{\zeta Q_{j2}t^{\alpha}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\mathfrak{x}_j(s)ds+\frac{\zeta Q_{j4}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\mathfrak{x}_j(s)ds,$$

where

$$Q_{j1}=\frac{1-\mu_j+\zeta\eta_j}{\Delta_j},\quad Q_{j2}=\frac{\gamma_j+\zeta\mu_j-\zeta\delta_j-\zeta^2\eta_j}{\Delta_j},$$
$$Q_{j3}=\frac{1-\eta_j\Gamma(\alpha+1)}{\Delta_j},\quad Q_{j4}=\frac{(1-\delta_j-\zeta\eta_j)\Gamma(\alpha+1)}{\Delta_j}.$$

Proof. Let us apply the operator $\mathfrak{I}^{\beta}_{0+}$ to Eq (3.1). Then, by Lemma 2.2, we deduce

$$({}^C\mathfrak{D}^{\alpha}_{0+}+\zeta)\mathfrak{x}_j(t)=\mathfrak{I}^{\beta}_{0+}\mathfrak{h}_j(t)+\mathfrak{c}^j_0,\ \mathfrak{c}^j_0\in\mathbb{R},\ j=1,2,3. \tag{3.4}$$

By placing the operator $\mathfrak{I}^{\alpha}_{0+}$ on both sides of Eq (3.4) and using Lemmas 2.1 and 2.2, we obtain

$$\mathfrak{x}_j(t)=\mathfrak{I}^{\iota}_{0+}\mathfrak{h}_j(t)-\zeta\mathfrak{I}^{\alpha}_{0+}\mathfrak{x}_j(t)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\mathfrak{c}^j_0+\mathfrak{c}^j_1, \tag{3.5}$$

$\mathfrak{c}^j_0, \mathfrak{c}^j_1\in\mathbb{R}$, $j=1,2,3$. From Eqs (3.4) and (3.5), it can be derived that

$$\begin{cases}\mathfrak{x}_j(0)=\mathfrak{c}^j_1,\ {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_j(0)=-\zeta\mathfrak{c}^j_1+\mathfrak{c}^j_0,\\ \mathfrak{x}_j(1)=\mathfrak{I}^{\iota}_{0+}\mathfrak{h}_j(t)|_{t=1}-\zeta\mathfrak{I}^{\alpha}_{0+}\mathfrak{x}_j(t)|_{t=1}+\dfrac{\mathfrak{c}^j_0}{\Gamma(\alpha+1)}+\mathfrak{c}^j_1,\\ {}^C\mathfrak{D}^{\alpha}_{0+}\mathfrak{x}_j(1)=\mathfrak{I}^{\beta}_{0+}\mathfrak{h}_j(t)|_{t=1}+(1-\zeta\eta_j)\mathfrak{c}^j_0-(\zeta\mu_j-\zeta^2\eta_j)\mathfrak{c}^j_1.\end{cases} \tag{3.6}$$

Substituting (3.6) into (3.2) yields a system of linear equations in the unknown coefficients $\mathfrak{c}^j_0$ and $\mathfrak{c}^j_1$,

$$\begin{cases}\dfrac{1-\eta_j\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\mathfrak{c}^j_0+(1-\mu_j+\zeta\eta_j)\mathfrak{c}^j_1=\zeta\mathfrak{I}^{\alpha}_{0+}\mathfrak{x}_j(t)|_{t=1}-\mathfrak{I}^{\iota}_{0+}\mathfrak{h}_j(t)|_{t=1},\\ (1-\delta_j-\zeta\eta_j)\mathfrak{c}^j_0-(\gamma_j+\zeta\mu_j-\zeta\delta_j-\zeta^2\eta_j)\mathfrak{c}^j_1=-\mathfrak{I}^{\beta}_{0+}\mathfrak{h}_j(t)|_{t=1}.\end{cases} \tag{3.7}$$

By solving the system of Eq (3.7), we obtain

$$\begin{aligned}\mathfrak{c}^j_0=&\frac{(1-\mu_j+\zeta\eta_j)\Gamma(\alpha+1)}{\Delta_j}(-\mathfrak{I}^{\beta}_{0+}\mathfrak{h}_j(t)|_{t=1})\\ &+\frac{(\gamma_j+\zeta\mu_j-\zeta\delta_j-\zeta^2\eta_j)\Gamma(\alpha+1)}{\Delta_j}(\zeta\mathfrak{I}^{\alpha}_{0+}\mathfrak{x}_j(t)|_{t=1}-\mathfrak{I}^{\iota}_{0+}\mathfrak{h}_j(t)|_{t=1}),\\ \mathfrak{c}^j_1=&\frac{(1-\eta_j\Gamma(\alpha+1))}{\Delta_j}\mathfrak{I}^{\beta}_{0+}\mathfrak{h}_j(t)|_{t=1}\\ &+\frac{(1-\delta_j-\zeta\eta_j)\Gamma(\alpha+1)}{\Delta_j}(\zeta\mathfrak{I}^{\alpha}_{0+}\mathfrak{x}_j(t)|_{t=1}-\mathfrak{I}^{\iota}_{0+}\mathfrak{h}_j(t)|_{t=1}),\quad j=1,2,3.\end{aligned}$$

Substituting $\mathfrak{c}^j_0$ and $\mathfrak{c}^j_1$ into Eq (3.5) immediately yields Eq (3.3). Conversely, for any $\mathfrak{x}_j(t)\in C[0,1]$, $(j=1,2,3)$ satisfying Eq (3.3), and it follows from Lemma 2.1 that $\mathfrak{x}_j(t)$ satisfies Eq (3.1) with BCs (3.2). Therefore, the assertion of the lemma holds.

Based on Lemma 3.1, define the operator $\mathfrak{T} : \mathcal{X} \to \mathcal{X}$ as follows:

$$\mathfrak{T}(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t) := (\mathfrak{T}_1(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t), \mathfrak{T}_2(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t), \mathfrak{T}_3(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t)),$$

where

$$\begin{aligned}
&\mathfrak{T}_j(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t) \\
&= \frac{1}{\Gamma(\iota)} \int_0^t (t-s)^{\iota-1} \mathbf{f}_j(s) ds - \frac{Q_{j2} t^{\alpha}}{\Gamma(\iota)} \int_0^1 (1-s)^{\iota-1} \mathbf{f}_j(s) ds \\
&\quad - \frac{Q_{j4}}{\Gamma(\iota)} \int_0^1 (1-s)^{\iota-1} \mathbf{f}_j(s) ds - \frac{Q_{j1} t^{\alpha}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \mathbf{f}_j(s) ds \\
&\quad + \frac{Q_{j3}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \mathbf{f}_j(s) ds - \frac{\zeta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathfrak{x}_j(s) ds \\
&\quad + \frac{\zeta Q_{j2} t^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \mathfrak{x}_j(s) ds + \frac{\zeta Q_{j4}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \mathfrak{x}_j(s) ds, \ \ j = 1, 2, 3,
\end{aligned}$$

and $\mathbf{f}_j(s)$ is denoted by

$$\mathbf{f}_j(s) = \mathfrak{f}_j(s, \mathfrak{x}_1(s), \mathfrak{x}_2(s), \mathfrak{x}_3(s)), \ \ j = 1, 2, 3.$$

Therefore, $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)$ is a solution to the BVP (1.1) if and only if $\mathfrak{x}$ is a fixed point of the operator $\mathfrak{T}$.

In the following, we establish an existence result for the BVP (1.1) by applying Krasnoselskii's FPT.

Theorem 3.1. Suppose that the conditions below are met:

(C_1) $\mathfrak{f}_j \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$, for $j = 1, 2, 3$.

(C_2) There are functions $\tilde{p}_j, \tilde{q}_j, \tilde{r}_j, \tilde{k}_j \in C([0,1], \mathbb{R}^+)$, for $j = 1, 2, 3$, satisfying

$$|\mathfrak{f}_j(t, \phi, \varphi, \psi)| \le \tilde{k}_j(t) + \tilde{p}_j(t)|\phi(t)| + \tilde{q}_j(t)|\varphi(t)| + \tilde{r}_j(t)|\psi(t)|,$$

$(t, \phi, \varphi, \psi) \in [0,1] \times \mathbb{R}^3$. Then BVP (1.1) admits at least one solution on $[0,1]$, provided that

$$A + B < 1, \tag{3.8}$$

where

$$\begin{aligned}
A &= \sum_{j=1}^{3} \left[\frac{1 + |Q_{j2}| + |Q_{j4}|}{\Gamma(\iota+1)} + \frac{|Q_{j1}| + |Q_{j3}|}{\Gamma(\beta+1)} \right] \ell_j, \\
B &= \frac{\zeta}{\Gamma(\alpha+1)} \sum_{j=1}^{3} (1 + |Q_{j2}| + |Q_{j4}|), \\
\mathfrak{p}_j &= \|\tilde{p}_j\|_\infty, \ \ \mathfrak{q}_j = \|\tilde{q}_j\|_\infty, \ \ \mathfrak{r}_j = \|\tilde{r}_j\|_\infty, \\
\mathfrak{k}_j &= \|\tilde{k}_j\|_\infty, \ \ \ell_j = \mathfrak{p}_j + \mathfrak{q}_j + \mathfrak{r}_j, \ \ j = 1, 2, 3.
\end{aligned}$$

Proof. Let $\varepsilon > 0$ and

$$\varepsilon \ge \frac{1}{1 - (A+B)} \sum_{j=1}^{3} \left[\frac{1 + |Q_{j2}| + |Q_{j4}|}{\Gamma(\iota+1)} + \frac{|Q_{j1}| + |Q_{j3}|}{\Gamma(\beta+1)} \right] \mathfrak{k}_j.$$

Define a bounded closed set

$$\mathcal{B}_\varepsilon = \{\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3) \in \mathcal{X} : \|\mathfrak{x}\|_{\mathcal{X}} \leq \varepsilon\}.$$

Define the operators $\mathcal{F}, \mathcal{G} : \mathcal{B}_\varepsilon \to \mathcal{X}$ on $\mathcal{B}_\varepsilon$, respectively, as follows:

$$(\mathcal{F}\mathfrak{x})(t) = (\mathcal{F}_1(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t), \mathcal{F}_2(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t), \mathcal{F}_3(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t)),$$

$$(\mathcal{G}\mathfrak{x})(t) = (\mathcal{G}_1(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t), \mathcal{G}_2(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t), \mathcal{G}_3(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t)),$$

where

$$\begin{aligned}
\mathcal{F}_j(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t) = & -\frac{\zeta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathfrak{x}_j(s)ds + \frac{\zeta Q_{j2} t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \mathfrak{x}_j(s)ds \\
& + \frac{\zeta Q_{j4}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \mathfrak{x}_j(s)ds, \quad j = 1, 2, 3, \\
\mathcal{G}_j(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)(t) = & \frac{1}{\Gamma(\iota)} \int_0^t (t-s)^{\iota-1} \mathbf{f}_j(s)ds - \frac{Q_{j2} t^\alpha}{\Gamma(\iota)} \int_0^1 (1-s)^{\iota-1} \mathbf{f}_j(s)ds \\
& - \frac{Q_{j4}}{\Gamma(\iota)} \int_0^1 (1-s)^{\iota-1} \mathbf{f}_j(s)ds - \frac{Q_{j1} t^\alpha}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \mathbf{f}_j(s)ds \\
& + \frac{Q_{j3}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \mathbf{f}_j(s)ds, \quad j = 1, 2, 3.
\end{aligned}$$

To prove Theorem 3.1 using Theorem 2.1, we divide the proof into three steps.

Step 1. We prove that for $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3)$, $\mathfrak{y} = (\mathfrak{y}_1, \mathfrak{y}_2, \mathfrak{y}_3) \in \mathcal{B}_\varepsilon$, $\mathcal{G}\mathfrak{x} + \mathcal{F}\mathfrak{y} \in \mathcal{B}_\varepsilon$. Indeed, in view of $\mathfrak{x}, \mathfrak{y} \in \mathcal{B}_\varepsilon$, we have $\|\mathfrak{x}\|_{\mathcal{X}} \leq \varepsilon$, $\|\mathfrak{y}\|_{\mathcal{X}} \leq \varepsilon$. From (C_2), it follows that

$$\begin{aligned}
|\mathbf{f}_j(s)| &= |\mathfrak{f}_j(s, \mathfrak{x}_1(s), \mathfrak{x}_2(s), \mathfrak{x}_3(s))| \\
&\leq \mathfrak{k}_j + \mathfrak{p}_j \|\mathfrak{x}_1\|_\infty + \mathfrak{q}_j \|\mathfrak{x}_2\|_\infty + \mathfrak{r}_j \|\mathfrak{x}_3\|_\infty \leq \mathfrak{k}_j + \ell_j \|\mathfrak{x}\|_{\mathcal{X}}, \quad j = 1, 2, 3,
\end{aligned}$$

then

$$\begin{aligned}
|\mathcal{G}_j \mathfrak{x}| \leq & \frac{1}{\Gamma(\iota)} \int_0^t (t-s)^{\iota-1} |\mathbf{f}_j(s)|ds + \frac{|Q_{j2}| t^\alpha}{\Gamma(\iota)} \int_0^1 (1-s)^{\iota-1} |\mathbf{f}_j(s)|ds \\
& + \frac{|Q_{j4}|}{\Gamma(\iota)} \int_0^1 (1-s)^{\iota-1} |\mathbf{f}_j(s)|ds + \frac{|Q_{j1}| t^\alpha}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |\mathbf{f}_j(s)|ds \\
& + \frac{|Q_{j3}|}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |\mathbf{f}_j(s)|ds \\
\leq & \left[\frac{1 + |Q_{j2}| + |Q_{j4}|}{\Gamma(\iota+1)} + \frac{|Q_{j1}| + |Q_{j3}|}{\Gamma(\beta+1)}\right] (\mathfrak{k}_j + \ell_j \|\mathfrak{x}\|_{\mathcal{X}}) \qquad (3.9) \\
\leq & \left[\frac{1 + |Q_{j2}| + |Q_{j4}|}{\Gamma(\iota+1)} + \frac{|Q_{j1}| + |Q_{j3}|}{\Gamma(\beta+1)}\right] (\mathfrak{k}_j + \ell_j \varepsilon), \quad j = 1, 2, 3.
\end{aligned}$$

Besides, for any $t \in [0, 1]$, we obtain

$$|\mathcal{F}_j \mathfrak{y}| \leq \frac{\zeta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\mathfrak{y}_j(s)|ds + \frac{\zeta |Q_{j2}| t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |\mathfrak{y}_j(s)|ds$$

$$+\frac{\zeta|Q_{j4}|}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|\mathfrak{y}_j(s)|ds \tag{3.10}$$
$$\leq\frac{\zeta}{\Gamma(\alpha+1)}(1+|Q_{j2}|+|Q_{j4}|)\|\mathfrak{y}_j\|_\infty,\quad j=1,2,3.$$

Combining Eqs (3.9) and (3.10), one can obtain

$$|\mathcal{G}_j\mathfrak{x}+\mathcal{F}_j\mathfrak{y}|\leq\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big](\mathfrak{k}_j+\ell_j\varepsilon)$$
$$+\frac{\zeta}{\Gamma(\alpha+1)}(1+|Q_{j2}|+|Q_{j4}|)\|\mathfrak{y}_j\|_\infty,\quad j=1,2,3.$$

This implies that

$$\|\mathcal{G}_j\mathfrak{x}+\mathcal{F}_j\mathfrak{y}\|_\infty\leq\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big](\mathfrak{k}_j+\ell_j\varepsilon)$$
$$+\frac{\zeta}{\Gamma(\alpha+1)}(1+|Q_{j2}|+|Q_{j4}|)\|\mathfrak{y}_j\|_\infty,\quad j=1,2,3.$$

Noting that $\|\mathfrak{y}\|_\mathcal{X}\leq\varepsilon$, we conclude that $\|\mathfrak{y}_j\|_\infty\leq\varepsilon$. Therefore, from (3.8), we can derive that

$$\|\mathcal{G}\mathfrak{x}+\mathcal{F}\mathfrak{y}\|_\mathcal{X}=\|\mathcal{G}_1\mathfrak{x}+\mathcal{F}_1\mathfrak{y}\|_\infty+\|\mathcal{G}_2\mathfrak{x}+\mathcal{F}_2\mathfrak{y}\|_\infty+\|\mathcal{G}_3\mathfrak{x}+\mathcal{F}_3\mathfrak{y}\|_\infty$$
$$\leq\sum_{j=1}^{3}\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big](\mathfrak{k}_j+\ell_j\varepsilon)$$
$$+\frac{\zeta\varepsilon}{\Gamma(\alpha+1)}\sum_{j=1}^{3}(1+|Q_{j2}|+|Q_{j4}|)$$
$$=\sum_{j=1}^{3}\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big]\mathfrak{k}_j+(A+B)\varepsilon\leq\varepsilon,$$

that is, $\mathcal{G}\mathfrak{x}+\mathcal{F}\mathfrak{y}\in\mathcal{B}_\varepsilon$.

Step 2. We show that $\mathcal{F}$ is contractive on $\mathcal{B}_\varepsilon$. For $\mathfrak{x}=(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3)$ and $\mathfrak{y}=(\mathfrak{y}_1,\mathfrak{y}_2,\mathfrak{y}_3)$ in $\mathcal{B}_\varepsilon$, and $t\in[0,1]$, the following estimates hold:

$$|\mathcal{F}_j\mathfrak{x}-\mathcal{F}_j\mathfrak{y}|\leq\frac{\zeta}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|\mathfrak{x}_j(s)-\mathfrak{y}_j(s)|ds$$
$$+\frac{\zeta Q_{j2}t^\alpha}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|\mathfrak{x}_j(s)-\mathfrak{y}_j(s)|ds$$
$$+\frac{\zeta Q_{j4}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|\mathfrak{x}_j(s)-\mathfrak{y}_j(s)|ds$$
$$\leq\frac{\zeta}{\Gamma(\alpha+1)}(1+|Q_{j2}|+|Q_{j4}|)\|\mathfrak{x}_j-\mathfrak{y}_j\|_\infty,\quad j=1,2,3.$$

Thus,

$$\|\mathcal{F}\mathfrak{x}-\mathcal{F}\mathfrak{y}\|_\mathcal{X}\leq\frac{\zeta}{\Gamma(\alpha+1)}\sum_{j=1}^{3}(1+|Q_{j2}|+|Q_{j4}|)\|\mathfrak{x}-\mathfrak{y}\|_\mathcal{X}=B\|\mathfrak{x}-\mathfrak{y}\|_\mathcal{X}.$$

It follows from condition (3.8) that $\mathcal{F}$ is a contraction.
Step 3. We verify that $\mathcal{G}$ is completely continuous on $\mathcal{B}_\varepsilon$. Actually, the continuity of the functions $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$ ensures that $\mathcal{G}$ is continuous on $\mathcal{B}_\varepsilon$. Therefore, it remains to verify the compactness of $\mathcal{G}$ on $\mathcal{B}_\varepsilon$. For $\mathfrak{x}(t) \in \mathcal{B}_\varepsilon$, $t \in [0,1]$, conclusion (i) implies that $\mathcal{G}$ is uniformly bounded on $\mathcal{B}_\varepsilon$. Next, we demonstrate that $\mathcal{G}$ is equi-continuous. Given any $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3) \in \mathcal{B}_\varepsilon$ and $t_1, t_2 \in [0,1]$ with $0 \le t_1 < t_2 \le 1$, we infer that

$$\begin{aligned}
&|\mathcal{G}_j\mathfrak{x}(t_2) - \mathcal{G}_j\mathfrak{x}(t_1)| \\
&\le \frac{1}{\Gamma(\iota)}\Big|\int_0^{t_1} [(t_2 - s)^{\iota-1} - (t_1 - s)^{\iota-1}]\mathbf{f}_j(s)ds \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{\iota-1}\mathbf{f}_j(s)ds\Big| + \frac{|Q_{j2}|}{\Gamma(\iota)}\int_0^1 (1 - s)^{\iota-1}|\mathbf{f}_j(s)|ds(t_2^\alpha - t_1^\alpha) \\
&\quad + \frac{|Q_{j1}|}{\Gamma(\beta)}\int_0^1 (1 - s)^{\beta-1}|\mathbf{f}_j(s)|ds(t_2^\alpha - t_1^\alpha) \\
&\le \frac{1}{\Gamma(\iota + 1)}(\mathfrak{k}_j + \ell_j\|\mathfrak{x}\|_{\mathcal{X}})(t_2^\iota - t_1^\iota) \\
&\quad + \Big[\frac{|Q_{j2}|}{\Gamma(\iota + 1)} + \frac{|Q_{j1}|}{\Gamma(\beta + 1)}\Big](\mathfrak{k}_j + \ell_j\|\mathfrak{x}\|_{\mathcal{X}})(t_2^\alpha - t_1^\alpha), \quad j = 1, 2, 3.
\end{aligned}$$

Since t^ι and t^α exhibit uniform continuity on $[0, 1]$, we proceed

$$|\mathcal{G}_j\mathfrak{x}(t_2) - \mathcal{G}_j\mathfrak{x}(t_1)| \to 0, \text{ as } t_1 \to t_2, \ j = 1, 2, 3.$$

Therefore, $\mathcal{G}$ is equi-continuous on $\mathcal{B}_\varepsilon$. It follows from the Arzelà-Ascoli theorem that $\mathcal{G}$ is compact on $\mathcal{B}_\varepsilon$. Hence, invoking Theorem 2.1, we conclude that the BVP (1.1) admits at least one solution on $[0, 1]$.

Having established the existence of solutions to BVP (1.1), we now turn to the issue of E&U. To this end, we impose additional Lipschitz-type conditions and apply BCMP. The following theorem presents the corresponding existence and uniqueness result.
Theorem 3.2. Suppose that the following assumptions are satisfied:

(C_1) $\mathfrak{f}_j \in C([0, 1] \times \mathbb{R}^3, \mathbb{R})$, for $j = 1, 2, 3$.
(C_2) There are constants $\mathcal{L}_j > 0$ $(j = 1, 2, 3)$, such that for all $\mathfrak{x}_j, \mathfrak{y}_j \in \mathbb{R}$ $(j = 1, 2, 3)$ and $t \in [0, 1]$,

$$|\mathfrak{f}_j(t, \mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3) - \mathfrak{f}_j(t, \mathfrak{y}_1, \mathfrak{y}_2, \mathfrak{y}_3)| \le \mathcal{L}_j(|\mathfrak{x}_1 - \mathfrak{y}_1| + |\mathfrak{x}_2 - \mathfrak{y}_2| + |\mathfrak{x}_3 - \mathfrak{y}_3|), \ j = 1, 2, 3.$$

If the condition

$$\Lambda + B < 1, \tag{3.11}$$

is satisfied, then the BVP (1.1) possesses a unique solution on $[0, 1]$, where

$$\Lambda = \sum_{j=1}^{3}\Big[\frac{1 + |Q_{j2}| + |Q_{j4}|}{\Gamma(\iota + 1)} + \frac{|Q_{j1}| + |Q_{j3}|}{\Gamma(\beta + 1)}\Big]\mathcal{L}_j.$$

Proof. Let $\varrho > 0$ and

$$\varrho \geq \frac{1}{1-(\Lambda+B)}\sum_{j=1}^{3}\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big]\mathfrak{w}_j,$$

where $\mathfrak{w}_j = \max_{t\in[0,1]}|\mathfrak{f}_j(t,0,0,0)|,\ j=1,2,3$. Define the set

$$\mathcal{B}_\varrho = \{(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3)\in\mathcal{X} : \|\mathfrak{x}\|_{\mathcal{X}} \leq \varrho\}.$$

We show that $\mathfrak{T}\mathcal{B}_\varrho \subset \mathcal{B}_\varrho$. For $\mathfrak{x}=(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3)\in\mathcal{B}_\varrho$ and $t\in[0,1]$, by condition (C_3), we obtain

$$\begin{aligned}|\mathfrak{f}_j(t,\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3)| &\leq |\mathfrak{f}_j(t,\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3)-\mathfrak{f}_j(t,0,0,0)|+|\mathfrak{f}_j(t,0,0,0)|\\ &\leq \mathcal{L}_j(\|\mathfrak{x}_1\|_\infty+\|\mathfrak{x}_2\|_\infty+\|\mathfrak{x}_3\|_\infty)+\mathfrak{w}_j\\ &= \mathcal{L}_j\|\mathfrak{x}\|_{\mathcal{X}}+\mathfrak{w}_j \leq \mathcal{L}_j\varrho+\mathfrak{w}_j,\ \ j=1,2,3.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}|\mathfrak{T}_j(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3)(t)| \leq{}& \Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big](\mathcal{L}_j\varrho+\mathfrak{w}_j)\\ &+\zeta\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\alpha+1)}\Big]\|\mathfrak{x}_j\|_\infty, \quad j=1,2,3.\end{aligned}$$

Then from (3.11), we deduce that

$$\begin{aligned}\|\mathfrak{T}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3)(t)\|_{\mathcal{X}} ={}& \sum_{j=1}^{3}\|\mathfrak{T}_j(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3)(t)\|_\infty\\ \leq{}& \sum_{j=1}^{3}\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big](\mathcal{L}_j\varrho+\mathfrak{w}_j)\\ &+\zeta\sum_{j=1}^{3}\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\alpha+1)}\Big]\|\mathfrak{x}_j\|_\infty\\ \leq{}& \sum_{j=1}^{3}\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big](\mathcal{L}_j\varrho+\mathfrak{w}_j)\\ &+\zeta\sum_{j=1}^{3}\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\alpha+1)}\Big]\varrho\\ ={}& (\Lambda+B)\varrho+\sum_{j=1}^{3}\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big]\mathfrak{w}_j \leq \varrho.\end{aligned}$$

Consequently, $\mathfrak{T}\mathcal{B}_\varrho\subset\mathcal{B}_\varrho$. We now show that $\mathfrak{T}$ is contractive on $\mathcal{B}_\varrho$. For $\mathfrak{x}=(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3),\ \mathfrak{y}=(\mathfrak{y}_1,\mathfrak{y}_2,\mathfrak{y}_3)\in\mathcal{B}_\varrho$, and denote

$$\mathbf{f}_{j\mathfrak{x}}(s)=\mathfrak{f}_j(s,\mathfrak{x}_1(s),\mathfrak{x}_2(s),\mathfrak{x}_3(s)),\ \mathbf{f}_{j\mathfrak{y}}(s)=\mathfrak{f}_j(s,\mathfrak{y}_1(s),\mathfrak{y}_2(s),\mathfrak{y}_3(s)),\ j=1,2,3.$$

It follows that

$$\begin{aligned}
&|\mathfrak{T}_j(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3)(t)-\mathfrak{T}_j(\mathfrak{y}_1,\mathfrak{y}_2,\mathfrak{y}_3)(t)|\\
&\le \frac{1}{\Gamma(\iota)}\int_0^t (t-s)^{\iota-1}|\mathbf{f}_{j\mathfrak{x}}(s)-\mathbf{f}_{j\mathfrak{y}}(s)|ds+\frac{|Q_{j2}|t^{\alpha}}{\Gamma(\iota)}\int_0^1 (1-s)^{\iota-1}|\mathbf{f}_{j\mathfrak{x}}(s)-\mathbf{f}_{j\mathfrak{y}}(s)|ds\\
&\quad+\frac{|Q_{j4}|}{\Gamma(\iota)}\int_0^1 (1-s)^{\iota-1}|\mathbf{f}_{j\mathfrak{x}}(s)-\mathbf{f}_{j\mathfrak{y}}(s)|ds+\frac{|Q_{j1}|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}|\mathbf{f}_{j\mathfrak{x}}(s)-\mathbf{f}_{j\mathfrak{y}}(s)|ds\\
&\quad+\frac{|Q_{j3}|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}|\mathbf{f}_{j\mathfrak{x}}(s)-\mathbf{f}_{j\mathfrak{y}}(s)|ds+\frac{\zeta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\mathfrak{x}_j(s)-\mathfrak{y}_j(s)|ds\\
&\quad+\frac{\zeta|Q_{j2}|t^{\alpha}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|\mathfrak{x}_j(s)-\mathfrak{y}_j(s)|ds+\frac{\zeta|Q_{j4}|}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|\mathfrak{x}_j(s)-\mathfrak{y}_j(s)|ds\\
&\le\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big]\mathcal{L}_j\|\mathfrak{x}-\mathfrak{y}\|_{\mathcal{X}}\\
&\quad+\frac{\zeta}{\Gamma(\alpha+1)}(1+|Q_{j2}|+|Q_{j4}|)\|\mathfrak{x}_j-\mathfrak{y}_j\|_\infty,\quad j=1,2,3,
\end{aligned}$$

which yields

$$\begin{aligned}
\|\mathfrak{T}\mathfrak{x}(t)-\mathfrak{T}\mathfrak{y}(t)\|_{\mathcal{X}}&=\sum_{j=1}^{3}\|\mathfrak{T}_j\mathfrak{x}(t)-\mathfrak{T}_j\mathfrak{y}(t)\|_\infty\\
&\le\sum_{j=1}^{3}\Big[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)}+\frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\Big]\mathcal{L}_j\|\mathfrak{x}-\mathfrak{y}\|_{\mathcal{X}}\\
&\quad+\frac{\zeta}{\Gamma(\alpha+1)}\sum_{j=1}^{3}(1+|Q_{j2}|+|Q_{j4}|)\|\mathfrak{x}-\mathfrak{y}\|_{\mathcal{X}}=(\Lambda+B)\|\mathfrak{x}-\mathfrak{y}\|_{\mathcal{X}}.
\end{aligned}$$

From condition (3.11), we know that $\mathfrak{T}$ is a contraction operator. By applying the Banach contraction mapping principle, $\mathfrak{T}$ admits a unique fixed point $\mathfrak{x}\in\mathcal{B}_\varrho$, which implies that the BVP (1.1) has a unique solution.

4. Example

To validate the theoretical results established in this paper, we present the following two concrete examples corresponding to Theorems 3.1 and 3.2, respectively:

Example 4.1. Let $\alpha=\frac{1}{2}$, $\beta=\frac{3}{4}$, $\zeta=\frac{1}{25}$. Consider the following BVP:

$$\begin{cases}
{}^C\mathfrak{D}_{0+}^{3/4}({}^C\mathfrak{D}_{0+}^{1/2}+\frac{1}{25})\mathfrak{x}_j(t)=\mathfrak{f}_j(t,\mathfrak{x}_1(t),\mathfrak{x}_2(t),\mathfrak{x}_3(t)),\ t\in(0,1),\ j=1,2,3,\\
\mathfrak{x}_1(1)=\mathfrak{x}_1(0)-{}^C\mathfrak{D}_{0+}^{1/2}\mathfrak{x}_1(0),\ {}^C\mathfrak{D}_{0+}^{1/2}\mathfrak{x}_1(1)=3\mathfrak{x}_1(0)-3{}^C\mathfrak{D}_{0+}^{1/2}\mathfrak{x}_1(0),\\
\mathfrak{x}_2(1)=2\mathfrak{x}_2(0)-2{}^C\mathfrak{D}_{0+}^{1/2}\mathfrak{x}_2(0),\ {}^C\mathfrak{D}_{0+}^{1/2}\mathfrak{x}_2(1)=2\mathfrak{x}_2(0)-2{}^C\mathfrak{D}_{0+}^{1/2}\mathfrak{x}_2(0),\\
\mathfrak{x}_3(1)=3\mathfrak{x}_3(0)-3{}^C\mathfrak{D}_{0+}^{1/2}\mathfrak{x}_3(0),\ {}^C\mathfrak{D}_{0+}^{1/2}\mathfrak{x}_3(1)=\mathfrak{x}_3(0)-{}^C\mathfrak{D}_{0+}^{1/2}\mathfrak{x}_3(0),
\end{cases}\tag{4.1}$$

where

$$\begin{aligned}
&\mu_1 = 1,\ \mu_2 = 2,\ \mu_3 = 3,\ \eta_1 = -1,\ \eta_2 = -2,\ \eta_3 = -3,\\
&\gamma_1 = 3,\ \gamma_2 = 2,\ \gamma_3 = 1,\ \delta_1 = -3,\ \delta_2 = -2,\ \delta_3 = -1,\\
&\mathfrak{f}_1(t, \mathfrak{x}_1(t), \mathfrak{x}_2(t), \mathfrak{x}_3(t)) = e^t + \frac{\sin \mathfrak{x}_1(t)}{16(20 + t^2)} + \frac{\mathfrak{x}_2(t)}{80(6 + 6e^t)} + \frac{\sin \mathfrak{x}_3(t)}{60(8e^t + 8)},\\
&\mathfrak{f}_2(t, \mathfrak{x}_1(t), \mathfrak{x}_2(t), \mathfrak{x}_3(t)) = \cos t + 1 + \frac{\mathfrak{x}_1(t)}{80\sqrt{t^2 + 16}} + \frac{\mathfrak{x}_2(t)}{60(2 + t)^4} + \frac{\mathfrak{x}_3(t)}{240(e^t + 3)},\\
&\mathfrak{f}_3(t, \mathfrak{x}_1(t), \mathfrak{x}_2(t), \mathfrak{x}_3(t)) = \ln(3 + t) + \frac{\mathfrak{x}_1(t)}{(8\sqrt{5} + t)^2} + \frac{\sin \mathfrak{x}_2(t)}{(10e^t)^3 - 40} + \frac{\mathfrak{x}_3(t)}{160\sqrt{15e^t + 21}}.
\end{aligned}$$

From the explicit expressions of $\mathfrak{f}_j$ for $j = 1, 2, 3$ given above, it is evident that each $\mathfrak{f}_j \in C([0, 1] \times \mathbb{R}^3, \mathbb{R})$, and thus condition (C_1) in Theorem 3.1 is satisfied. For $t \in [0, 1]$, choose

$$\begin{aligned}
&\tilde{k}_1(t) = e^t,\ \tilde{k}_2(t) = \cos t + 1,\ \tilde{k}_3(t) = \ln(3 + t),\\
&\tilde{p}_1(t) = \frac{1}{16(20 + t^2)},\ \tilde{p}_2(t) = \frac{1}{80\sqrt{t^2 + 16}},\ \tilde{p}_3(t) = \frac{1}{(8\sqrt{5} + t)^2},\\
&\tilde{q}_1(t) = \frac{1}{80(6 + 6e^t)},\ \tilde{q}_2(t) = \frac{1}{60(2 + t)^4},\ \tilde{q}_3(t) = \frac{1}{(10e^t)^3 - 40},\\
&\tilde{r}_1(t) = \frac{1}{60(8e^t + 8)},\ \tilde{r}_2(t) = \frac{1}{240(e^t + 3)},\ \tilde{r}_3(t) = \frac{1}{160\sqrt{15e^t + 21}}.
\end{aligned}$$

It is then straightforward to verify that condition (C_2) in Theorem 3.1 is verified. Moreover, we can obtain

$$\begin{aligned}
&\mathfrak{p}_1 = \frac{1}{320},\ \mathfrak{p}_2 = \frac{1}{320},\ \mathfrak{p}_3 = \frac{1}{320},\ \mathfrak{q}_1 = \frac{1}{960},\ \mathfrak{q}_2 = \frac{1}{960},\ \mathfrak{q}_3 = \frac{1}{960},\\
&\mathfrak{r}_1 = \frac{1}{960},\ \mathfrak{r}_2 = \frac{1}{960},\ \mathfrak{r}_3 = \frac{1}{960},\ \ell_j = \mathfrak{p}_j + \mathfrak{q}_j + \mathfrak{r}_j = \frac{1}{192}\ (j = 1, 2, 3).
\end{aligned}$$

By calculation, we obtain

$$\begin{aligned}
\Delta_1 &= (1 - \delta_1 - \zeta\eta_1)(1 - \mu_1 + \zeta\eta_1)\Gamma(\alpha + 1)\\
&\quad + (\gamma_1 + \zeta\mu_1 - \zeta\delta_1 - \zeta^2\eta_1)(1 - \eta_1\Gamma(\alpha + 1)) = 3\Gamma(1.5) + 3.1616 \neq 0,\\
\Delta_2 &= (1 - \delta_2 - \zeta\eta_2)(1 - \mu_2 + \zeta\eta_2)\Gamma(\alpha + 1)\\
&\quad + (\gamma_2 + \zeta\mu_2 - \zeta\delta_2 - \zeta^2\eta_2)(1 - \eta_2\Gamma(\alpha + 1)) = \Gamma(1.5) + 2.1632 \neq 0,\\
\Delta_3 &= (1 - \delta_3 - \zeta\eta_3)(1 - \mu_3 + \zeta\eta_3)\Gamma(\alpha + 1)\\
&\quad + (\gamma_3 + \zeta\mu_3 - \zeta\delta_3 - \zeta^2\eta_3)(1 - \eta_3\Gamma(\alpha + 1)) = -\Gamma(1.5) + 1.1648 \neq 0,\\
Q_{11} &= \frac{1 - \mu_1 + \zeta\eta_1}{\Delta_1} = -\frac{1}{25} \cdot \frac{1}{\Delta_1} \approx -0.00687252,\\
Q_{12} &= \frac{\gamma_1 + \zeta\mu_1 - \zeta\delta_1 - \zeta^2\eta_1}{\Delta_1} = \frac{1976}{625} \cdot \frac{1}{\Delta_1} \approx 0.5432,\\
Q_{13} &= \frac{1 - \eta_1\Gamma(\alpha + 1)}{\Delta_1} = \frac{1 + \Gamma(1.5)}{\Delta_1} \approx 0.324078,
\end{aligned}$$

$$Q_{14} = \frac{(1-\delta_1-\zeta\eta_1)\Gamma(\alpha+1)}{\Delta_1} = \frac{101}{25}\cdot\frac{\Gamma(1.5)}{\Delta_1} \approx 0.615152,$$
$$Q_{21} = \frac{1-\mu_2+\zeta\eta_2}{\Delta_2} = -\frac{27}{25}\cdot\frac{1}{\Delta_2} \approx -0.354165,$$
$$Q_{22} = \frac{\gamma_2+\zeta\mu_2-\zeta\delta_2-\zeta^2\eta_2}{\Delta_2} = \frac{1352}{625}\cdot\frac{1}{\Delta_2} \approx 0.709379,$$
$$Q_{23} = \frac{1-\eta_2\Gamma(\alpha+1)}{\Delta_2} = \frac{1+2\Gamma(1.5)}{\Delta_2} \approx 0.909172,$$
$$Q_{24} = \frac{(1-\delta_2-\zeta\eta_2)\Gamma(\alpha+1)}{\Delta_2} = \frac{77}{25}\cdot\frac{\Gamma(1.5)}{\Delta_2} \approx 0.895112,$$
$$Q_{31} = \frac{1-\mu_3+\zeta\eta_3}{\Delta_3} = -\frac{53}{25}\cdot\frac{1}{\Delta_3} \approx -7.61021,$$
$$Q_{32} = \frac{\gamma_3+\zeta\mu_3-\zeta\delta_3-\zeta^2\eta_3}{\Delta_3} = \frac{728}{625}\cdot\frac{1}{\Delta_3} \approx 4.18131,$$
$$Q_{33} = \frac{1-\eta_3\Gamma(\alpha+1)}{\Delta_3} = \frac{1+3\Gamma(1.5)}{\Delta_3} \approx 13.1336,$$
$$Q_{34} = \frac{(1-\delta_3-\zeta\eta_3)\Gamma(\alpha+1)}{\Delta_3} = \frac{53}{25}\cdot\frac{\Gamma(1.5)}{\Delta_3} \approx 6.74437.$$

Therefore,

$$A = \sum_{j=1}^{3}\left[\frac{1+|Q_{j2}|+|Q_{j4}|}{\Gamma(\iota+1)} + \frac{|Q_{j1}|+|Q_{j3}|}{\Gamma(\beta+1)}\right]\ell_j \approx 0.2225,$$
$$B = \frac{\zeta}{\Gamma(\alpha+1)}\sum_{j=1}^{3}(1+|Q_{j2}|+|Q_{j4}|) \approx 0.7532,$$

and

$$A + B \approx 0.9757 < 1.$$

In view of Theorem 3.1, the BVP (4.1) admits at least one solution.

Example 4.2. Let $\alpha = \frac{1}{4}$, $\beta = \frac{4}{5}$, $\zeta = \frac{1}{40}$. Consider the following BVP:

$$\begin{cases} {}^{C}\mathfrak{D}_{0+}^{4/5}({}^{C}\mathfrak{D}_{0+}^{1/4} + \frac{1}{40})\mathfrak{x}_j(t) = \mathfrak{f}_j(t, \mathfrak{x}_1(t), \mathfrak{x}_2(t), \mathfrak{x}_3(t)),\ t \in (0,1),\ \ j = 1, 2, 3, \\ \mathfrak{x}_1(1) = 3\mathfrak{x}_1(0) - 3{}^{C}\mathfrak{D}_{0+}^{1/5}\mathfrak{x}_1(0),\ {}^{C}\mathfrak{D}_{0+}^{1/4}\mathfrak{x}_1(1) = \mathfrak{x}_1(0) - {}^{C}\mathfrak{D}_{0+}^{1/5}\mathfrak{x}_1(0), \\ \mathfrak{x}_2(1) = 2\mathfrak{x}_2(0) - 2{}^{C}\mathfrak{D}_{0+}^{1/5}\mathfrak{x}_2(0),\ {}^{C}\mathfrak{D}_{0+}^{1/4}\mathfrak{x}_2(1) = 2\mathfrak{x}_2(0) - 2{}^{C}\mathfrak{D}_{0+}^{1/5}\mathfrak{x}_2(0), \\ \mathfrak{x}_3(1) = \mathfrak{x}_3(0) - {}^{C}\mathfrak{D}_{0+}^{1/5}\mathfrak{x}_3(0),\ {}^{C}\mathfrak{D}_{0+}^{1/4}\mathfrak{x}_3(1) = 3\mathfrak{x}_3(0) - 3{}^{C}\mathfrak{D}_{0+}^{1/5}\mathfrak{x}_3(0), \end{cases} \tag{4.2}$$

where

$$\mu_1 = 3,\ \mu_2 = 2,\ \mu_3 = 1,\ \eta_1 = -3,\ \eta_2 = -2,\ \eta_3 = -1,$$
$$\gamma_1 = 1,\ \gamma_2 = 2,\ \gamma_3 = 3,\ \delta_1 = -1,\ \delta_2 = -2,\ \delta_3 = -3,$$
$$\mathfrak{f}_1(t, \mathfrak{x}_1(t), \mathfrak{x}_2(t), \mathfrak{x}_3(t)) = \frac{|\mathfrak{x}_1(t)| + |\mathfrak{x}_2(t)| + |\mathfrak{x}_3(t)|}{20(7+3e^t)(1 + |\mathfrak{x}_1(t)| + |\mathfrak{x}_2(t)| + |\mathfrak{x}_3(t)|)},$$

$$\mathfrak{f}_2(t, \mathfrak{x}_1(t), \mathfrak{x}_2(t), \mathfrak{x}_3(t)) = \frac{1}{50\sqrt{t^2+1}}\left[\frac{|\mathfrak{x}_1(t)|}{2+|\mathfrak{x}_1(t)|} + \frac{|\mathfrak{x}_2(t)|}{1+|\mathfrak{x}_2(t)|} + \frac{2|\mathfrak{x}_3(t)|}{\sqrt{2}+3|\mathfrak{x}_3(t)|}\right],$$
$$\mathfrak{f}_3(t, \mathfrak{x}_1(t), \mathfrak{x}_2(t), \mathfrak{x}_3(t)) = \frac{|\mathfrak{x}_1(t)|}{(5\sqrt{2}+t)^2} + \frac{\cos\mathfrak{x}_2(t)}{5e^t+45} + \frac{|\mathfrak{x}_3(t)|}{25\sqrt{e^t+3}}.$$

The explicit forms of $\mathfrak{f}_j$ for $j = 1, 2, 3$ presented above demonstrate that each function belongs to $C([0, 1] \times \mathbb{R}^3, \mathbb{R})$. Therefore, condition (C_1) is fulfilled. On the other hand, for $t \in [0, 1]$, we choose

$$\mathcal{L}_1 = \frac{1}{200},\ \mathcal{L}_2 = \frac{1}{50},\ \mathcal{L}_3 = \frac{1}{50},$$

from which it is evident that condition (C_3) is satisfied. It follows from straightforward computation that

$$\begin{aligned}
\Delta_1 &= (1-\delta_1-\zeta\eta_1)(1-\mu_1+\zeta\eta_1)\Gamma(\alpha+1) \\
&\quad + (\gamma_1+\zeta\mu_1-\zeta\delta_1-\zeta^2\eta_1)(1-\eta_1\Gamma(\alpha+1)) = -\Gamma(1.25)+1.101875 \neq 0, \\
\Delta_2 &= (1-\delta_2-\zeta\eta_2)(1-\mu_2+\zeta\eta_2)\Gamma(\alpha+1) \\
&\quad + (\gamma_2+\zeta\mu_2-\zeta\delta_2-\zeta^2\eta_2)(1-\eta_2\Gamma(\alpha+1)) = \Gamma(1.25)+2.10125 \neq 0, \\
\Delta_3 &= (1-\delta_3-\zeta\eta_3)(1-\mu_3+\zeta\eta_3)\Gamma(\alpha+1) \\
&\quad + (\gamma_3+\zeta\mu_3-\zeta\delta_3-\zeta^2\eta_3)(1-\eta_3\Gamma(\alpha+1)) = 3\Gamma(1.25)+3.100625 \neq 0, \\
Q_{11} &= \frac{1-\mu_1+\zeta\eta_1}{\Delta_1} = -\frac{83}{40}\cdot\frac{1}{\Delta_1} \approx -10.6153, \\
Q_{12} &= \frac{\gamma_1+\zeta\mu_1-\zeta\delta_1-\zeta^2\eta_1}{\Delta_1} = \frac{1763}{1600}\cdot\frac{1}{\Delta_1} \approx 5.63698, \\
Q_{13} &= \frac{1-\eta_1\Gamma(\alpha+1)}{\Delta_1} = \frac{1+3\Gamma(1.25)}{\Delta_1} \approx 19.0268, \\
Q_{14} &= \frac{(1-\delta_1-\zeta\eta_1)\Gamma(\alpha+1)}{\Delta_1} = \frac{83}{40}\cdot\frac{\Gamma(1.25)}{\Delta_1} \approx 9.62174, \\
Q_{21} &= \frac{1-\mu_2+\zeta\eta_2}{\Delta_2} = \frac{-42}{40}\cdot\frac{1}{\Delta_2} \approx -0.349109, \\
Q_{22} &= \frac{\gamma_2+\zeta\mu_2-\zeta\delta_2-\zeta^2\eta_2}{\Delta_2} = \frac{3362}{1600}\cdot\frac{1}{\Delta_2} \approx 0.698635, \\
Q_{23} &= \frac{1-\eta_2\Gamma(\alpha+1)}{\Delta_2} = \frac{1+2\Gamma(1.25)}{\Delta_2} \approx 0.935216, \\
Q_{24} &= \frac{(1-\delta_2-\zeta\eta_2)\Gamma(\alpha+1)}{\Delta_2} = \frac{122}{40}\cdot\frac{\Gamma(1.25)}{\Delta_2} \approx 0.919165, \\
Q_{31} &= \frac{1-\mu_3+\zeta\eta_3}{\Delta_3} = -\frac{1}{40}\cdot\frac{1}{\Delta_3} \approx -0.004295656, \\
Q_{32} &= \frac{\gamma_3+\zeta\mu_3-\zeta\delta_3-\zeta^2\eta_3}{\Delta_3} = \frac{4961}{1600}\cdot\frac{1}{\Delta_3} \approx 0.5327688, \\
Q_{33} &= \frac{1-\eta_3\Gamma(\alpha+1)}{\Delta_3} = \frac{1+\Gamma(1.25)}{\Delta_3} \approx 0.32757, \\
Q_{34} &= \frac{(1-\delta_3-\zeta\eta_3)\Gamma(\alpha+1)}{\Delta_3} = \frac{161}{40}\cdot\frac{\Gamma(1.5)}{\Delta_3} \approx 0.626869.
\end{aligned}$$

It follows that

$$\Lambda = \sum_{j=1}^{3} \Big[\frac{1 + |Q_{j2}| + |Q_{j4}|}{\Gamma(\iota + 1)} + \frac{|Q_{j1}| + |Q_{j3}|}{\Gamma(\beta + 1)} \Big] \mathcal{L}_j \approx 0.37549,$$

$$B = \frac{\zeta}{\Gamma(\alpha + 1)} \sum_{j=1}^{3} (1 + |Q_{j2}| + |Q_{j4}|) \approx 0.5802,$$

and

$$\Lambda + B \approx 0.95569 < 1.$$

By Theorem 3.2, the BVP (4.2) admits a unique solution.

5. Conclusions

In this study, we analyze the E&U of solutions for a nonlinear triply coupled system of FLEs subject to closed BCs. By employing properties of fractional calculus, the original BVP (1.1) has been equivalently transformed into a fixed point problem of a nonlinear operator equation in Banach space $\mathcal{X}$. Within this framework, the results on the existence and E&U of solutions are respectively established by combining Krasnoselskii's FPT and the BCMP. The present work has enriched theoretical findings on BVPs for tripled system of FDEs. In future investigations, we will focus on: Analyzing the well-posedness and stability of solutions for triply coupled system of FLEs with dual BCs; and exploring E&U criteria for triply coupled system of fractional Hybrid-Sturm-Liouville-Langevin equations with Sturm-Liouville BCs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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