Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(11): 6971–6997.
DOI: 10.3934/era.2025307
Received: 22 September 2025
Revised: 06 November 2025
Accepted: 13 November 2025
Published: 18 November 2025

Research article

Optimal error estimates and superconvergence analysis of an ultra-weak discontinuous Galerkin method for nonlinear second-order initial-value problems for ODEs

Mahboub Baccouch*

Department of Mathematical and Statistical Sciences, University of Nebraska at Omaha, Omaha, NE 68182, USA

* **Correspondence:** Email: mbaccouch@unomaha.edu.

Abstract: The primary focus of this study was to analyze the convergence and superconvergence properties of an ultra-weak discontinuous Galerkin (UWDG) method for nonlinear second-order initial-value problems (IVPs) for ordinary differential equations (ODEs) of the form $u'' + (g(x,u))' = f(x,u)$, $x \in [a,b]$, subject to $u(a) = \alpha$ and $u'(a) = \beta$. By carefully choosing suitable numerical fluxes and employing a special projection, we established optimal error estimates in the L^2-norm. The order of convergence was proved to be $p + 1$, when utilizing piecewise polynomials of degree at most p. We further proved that the UWDG solution was superconvergent of order $p + 2$ for $p \geq 2$ toward a special projection of the exact solution. Additionally, we proved that the p-degree UWDG solution and its derivative were $O(h^{2p})$ superconvergent at the end of each step. Our proofs were valid for arbitrary uniform or non-uniform partitions of the domain using piecewise polynomials with degree $p \geq 2$. Finally, several numerical examples were provided to validate all theoretical results. It is worth noting that the proposed UWDG method offers a significant advantage for second-order differential equations, as it can be applied directly without introducing auxiliary variables or reformulating the equation as a first-order system. This advantage reduces memory and computational costs.

Keywords: ultra-weak discontinuous Galerkin method; nonlinear initial-value problems; optimal error estimate; superconvergence error analysis

1. Introduction

This paper introduces and analyzes an ultra-weak discontinuous Galerkin (UWDG) method for the following nonlinear second-order initial-value problem (IVP):

$$u'' + (g(x,u))' = f(x,u), \quad x \in \Omega = [a,b], \quad u(a) = \alpha, \quad u'(a) = \beta, \tag{1.1}$$

where $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ are smooth functions with respect to their arguments. Details regarding the specific assumptions necessary for our error analysis will be presented in Section 4. It is worth noting that for an ODE of the form $u'' + P(x, u)u' = Q(x, u)$, we can always transform it into the equivalent form given in Eq (1.1), where

$$g(x, u) = \int^{u} P(x, u)du, \quad f(x, u) = Q(x, u) + \frac{\partial g(x, u)}{\partial x}.$$

Nonlinear IVPs of the form (1.1) are used to model a diverse range of physical problems. Given the prevalence of nonlinear IVPs in physical modeling, numerical schemes are often necessary to approximate their solutions. Since the vast majority of numerical methods are designed for first-order systems, to numerically solve the second-order IVP (1.1), one must reduce it into an equivalent first-order system. Basic numerical methods such as the Euler and Runge-Kutta schemes are commonly used to approximate solutions to first-order systems. However, some standard explicit Runge-Kutta methods may face limitations, particularly for stiff differential equations, although there exist high-order implicit Runge-Kutta schemes, such as Gauss-Radau methods, that are well-suited for stiff problems. The finite element method (FEM) is another important numerical technique for solving differential equations that arise in the study of various physical phenomena. It is particularly useful in handling complex boundaries and interfaces, and is therefore widely employed in the simulation of different models. However, despite its widespread usage, the FEM has limitations when the solution exhibits non-smooth behavior, such as high gradients or singularities.

Among the many numerical methods proposed for IVPs, the discontinuous Galerkin (DG) finite element method is an important class. The DG method is a numerical method for solving first- and higher-order ordinary and partial differential equations. It is a finite element method that discretizes the solution space into smaller, non-overlapping elements, and allows for discontinuities across the interfaces between the elements. The method uses piecewise polynomials to approximate the solution within each element, and a numerical flux to enforce the continuity of the solution across the interfaces. This allows for greater flexibility in modeling complex geometries and can handle solutions with discontinuities or shocks. The DG method has been successfully applied to a wide range of problems in physics, engineering, and other fields. Reed and Hill proposed the DG method in [1] to solve hyperbolic conservation laws that involve only first-order spatial derivatives. Since then, several DG methods have been developed and applied to solve a wide range of problems including second-order elliptic, parabolic, and hyperbolic problems. Consult [2–6] for a more complete and current survey of the DG method and its applications.

Various DG schemes have been used to discretize the second-order IVPs for ODEs, such as the classical DG [7], the local DG (LDG) method [8–10], the direct DG (DDG) method [11], and the ultra-weak DG (UWDG) method, which can be traced back to [12]. However, this paper focuses on the UWDG method. These UWDG methods rely on repeatedly applying integration by parts to shift all spatial derivatives from the solution to the test function in the weak formulations. Unlike the LDG method, the UWDG method does not require introducing auxiliary variables or rewriting the original equation into a larger system. In [13], the authors developed several UWDG methods to solve various equations, including the third-order generalized KdV equation, the second-order convection-diffusion equation, the fourth-order biharmonic equation, and some fifth-order equations. They used the UWDG method in space and then applied the total variation diminishing (TVD) high-order Runge-Kutta method to discretize the resulting systems of ODEs in time. For each case, they proved the stability of the semi-discrete schemes by carefully choosing the interface numerical fluxes. Although their error estimates are suboptimal, their numerical examples show that the scheme attains the optimal $(p + 1)$-th order of accuracy.

In 2007, Adjerid and Temimi [14] proposed a UWDG finite element method for solving linear IVPs for ODEs. They conducted a local error analysis to demonstrate that the UWDG solution attains an optimal convergence rate in the L^2-norm of order $p + 1$, when piecewise polynomials of degree at most p are employed. They also showed that the p-degree UWDG solution of m-th-order linear ODEs and its first $m - 1$ derivatives are superconvergent with order $2p + 2 - m$ at the end of each step. Furthermore, they established that the p-degree discontinuous solution is superconvergent with order $p+2$ at the roots of the $(p+1-m)$-degree Jacobi polynomial on each step. Using the superconvergence results, they constructed asymptotically exact *a posteriori* error estimates. We would like to emphasize that the results in [14] are based on a local error analysis. Here, for the first time, we prove the global superconvergence behavior of the UWDG solution for more general IVPs. This work continues the research and investigates the global superconvergence of the UWDG scheme for more general problems of the form (1.1).

In 2016, Baccouch and Temimi [15] analyzed the convergence and superconvergence properties of a UWDG method for a linear boundary-value problem. They proved that the UWDG solution and its derivative achieve optimal convergence rates of $O(h^{p+1})$ and $O(h^p)$, respectively, in the L^2-norm for $p \geq 1$. They also demonstrated that the p-degree UWDG solution and its derivative are $O(h^{2p})$ superconvergent at the downwind and upwind points, respectively. Since then several UWDG and ultra-weak LDG (UWLDG) methods were proposed and studied in [16–19] for ODEs, [20, 21] for the Schrödinger equation, [22–25] for convection-diffusion and biharmonic equations, [26–28] for third-order generalized KdV, and [29–33] for other higher-order equations.

In this study, we present a superconvergent UWDG method for the model problem (1.1). We prove that the piecewise p-degree UWDG finite element solution converges to the true solution in the L^2-norm with an order of $O(h^{p+1})$. Furthermore, we prove that the UWDG solution is superconvergent with an order of $p + 2$ for $p \geq 2$ toward a special projection of the exact solution. We also prove that the UWDG solution and its first derivative are $O(h^{2p})$ superconvergent at the end of each step. Our proofs are valid for arbitrary uniform or non-uniform partitions of the one-dimensional domain using P^p polynomials with $p \geq 2$. We provide several numerical experiments to verify the sharpness of our theoretical findings. We would like to mention that the proposed UWDG scheme has several advantages over the standard DG method. The main advantage is that the proposed UWDG method can be applied without introducing an auxiliary variable or rewriting the original equation into a larger system, which reduces memory and computational costs. Another advantage is that the current scheme achieves superconvergence results that can be used to construct *a posteriori* error estimates by solving a local problem on each element. This will be discussed in a separate paper. Although our error analysis is presented for the second-order equation, it can be easily extended to higher-order equations.

The paper is organized as follows. Section 2 presents the formulation of the UWDG method for the second-order IVP (1.1). In Section 3, we introduce the notation and define the projection operators that will be used throughout the paper. The main body of the paper consists of Sections 4 and 5, where convergence and superconvergence results are separately proved with a careful choice of numerical fluxes. Section 6 presents numerical examples that demonstrate the accuracy and capability of the proposed scheme. Finally, Section 7 concludes the paper with some remarks on future work.

2. The UWDG scheme

To define the UWDG method, the interval $[a,b]$ is divided into N subintervals $I_i = [x_{i-1}, x_i]$, where $a = x_0 < x_1 < \cdots < x_N = b$. The length of I_i and the length of the largest interval are, respectively, denoted by $h_i = x_i - x_{i-1}$ and $h = \max_{i=1,2,\ldots,N} h_i$. It is assumed that the partition of the domain is regular, meaning that the ratio between the largest and smallest interval lengths remains bounded under refinement.

Multiplying the ODE in (1.1) by a test function v, integrating over I_i, and using integration by parts twice gives the UWDG weak formulation:

$$\int_{I_i} (-u\, v'' + v' g(x,u) + v f(x,u))\, dx - g(x_i, u(x_i))v(x_i) + g(x_{i-1}, u(x_{i-1}))v(x_{i-1})$$
$$- u'(x_i)v(x_i) + u'(x_{i-1})v(x_{i-1}) + u(x_i)v'(x_i) - u(x_{i-1})v'(x_{i-1}) = 0. \tag{2.1}$$

To construct the UWDG scheme, the discontinuous finite element approximation space V_h^p is introduced as follows:

$$V_h^p = \{v : v|_{I_i} \in \mathbb{P}^p(I_i),\ i = 1, 2, \ldots, N\},$$

where $\mathbb{P}^p(I_i)$ denotes the space of polynomials of degree at most p in the interval I_i.

The values of v at $x = x_i$ from the right cell I_{i+1} and from the left cell I_i are denoted by $v(x_i^+)$ and $v(x_i^-)$, respectively, i.e.,

$$v(x_i^\pm) = \lim_{s\to 0^\pm} v(x_i + s), \quad i = 0, 1, \ldots, N.$$

Using the weak formulation (2.1), we define the UWDG scheme as follows: Find $u_h \in V_h^p$ such that for all test functions $v \in V_h^p$, we have

$$\int_{I_i} (-u_h v'' + v' g(x,u_h) + v f(x,u_h))\, dx - g(x_i, \widehat{u_h}(x_i))v(x_i^-) + g(x_{i-1}, \widehat{u_h}(x_{i-1}))v(x_{i-1}^+)$$
$$- \widehat{u_h'}(x_i)v(x_i^-) + \widehat{u_h'}(x_{i-1})v(x_{i-1}^+) + \widehat{u_h}(x_i)v'(x_i^-) - \widehat{u_h}(x_{i-1})v'(x_{i-1}^+) = 0, \tag{2.2a}$$

for all $i = 1, 2, \ldots, N$. The terms $\widehat{u_h}$ and $\widehat{u_h'}$ are called the numerical fluxes, which are approximations to the exact solutions at the nodes. These numerical fluxes need to be designed according to specific guiding principles to maintain the stability and accuracy of the numerical solution. In order to fully specify the UWDG scheme, it is necessary to select the numerical fluxes $\widehat{u_h}$ and $\widehat{u_h'}$. It should be noted that the accuracy of the UWDG method heavily relies on the choice of these fluxes. To define these fluxes, we introduce some notations. The mean value $\{\!\{\cdot\}\!\}$ and the jump $[\![\cdot]\!]$ of a function $v \in V_h^p$ at a specific point x_i are defined as follows:

$$\{\!\{v\}\!\}(x_i) = \frac{1}{2}(v(x_i^+) + v(x_i^-)), \quad [\![v]\!](x_i) = v(x_i^+) - v(x_i^-).$$

We are now ready to define the numerical fluxes. In this paper, we take the following numerical fluxes that are dependent on the boundary conditions:

$$\widehat{u_h}(x_0) = \alpha, \quad \widehat{u_h'}(x_0) = \beta, \quad \widehat{u_h}(x_i) = u_h(x_i^-), \quad \widehat{u_h'}(x_i) = u_h'(x_i^-), \tag{2.2b}$$

for $i = 1, 2, \ldots, N$. Once the numerical fluxes are defined, the discrete problem (2.2) can be formulated as an algebraic system of nonlinear equations for the unknown coefficients in the UWDG solution u_h. This resulting system can be solved numerically using one of the standard methods for solving nonlinear systems of equations.

Remark 2.1. *Here, we use upwind fluxes, since for IVPs for ODEs information propagates from left to right, making the upwind choice natural and consistent. Other numerical fluxes can also be used by incorporating the jump and average of the solution at element interfaces; however, a new choice of numerical flux generally leads to a corresponding suitable projection to eliminate the boundary terms.*

3. Notation and projections

In this section, we shall introduce several notations and define projection operators that will be utilized in the paper.

3.1. Norms

In this paper, we adopt the standard notations and conventions for the Sobolev spaces and their norms. The L^2-inner product over the interval I_i is denoted by $(u, v)_{I_i} = \int_{I_i} u(x)v(x)\,dx$, and the L^2-norm of a function $u(x)$ on I_i is given by $\|u\|_{0,I_i} = (u, u)_{I_i}^{1/2}$. The k-th derivative of u with respect to x is denoted by $D^k u = \frac{d^k u}{dx^k} = \partial_x^k u$. For $s = 1, 2, \ldots$, we use $H^s(I_i) = \left\{u \in L^2(I_i) : D^k u \in L^2(I_i),\ \forall\, k \leq s\right\}$ to represent the standard Sobolev space on I_i. The $H^s(I_i)$-norm for a real-valued function $u \in H^s(I_i)$ is denoted by $\|u\|_{s,I_i} = \left(\sum_{k=0}^{s} \left\|D^k u\right\|_{0,I_i}^2\right)^{1/2}$ for $s \geq 0$. Additionally, we define the L^2-norm of u on the entire domain Ω as $\|u\| = \left(\sum_{i=1}^{N} \|u\|_{0,I_i}^2\right)^{1/2}$, and the H^s-norm of u on the whole domain Ω is denoted by $\|u\|_s = \left(\sum_{i=1}^{N} \|u\|_{s,I_i}^2\right)^{1/2}$ for $s \geq 1$.

3.2. Inverse properties

Here, we recall some classical inverse properties of the finite element space V_h^p. We summarize them in the following lemma.

Lemma 3.1. *If $v \in V_h^p$ then there exists a positive constant C independent of h such that*

$$\|v'\| \leq Ch^{-1}\|v\|, \quad \|v\|_\infty \leq Ch^{-1/2}\|v\|, \quad \sum_{i=1}^{N}\left(v^2(x_{i-1}^+) + v^2(x_i^-)\right) \leq Ch^{-1}\|v\|^2. \tag{3.1}$$

Throughout the paper, we utilize the symbol C (with or without subscripts or superscripts) to represent a general constant that remains independent of h. Nonetheless, it may assume different values in distinct locations. All constants may rely on the exact solution u of the model problem (1.1).

3.3. Projections

For $p \geq 1$ and $i = 0, 1, \ldots, N$, we define a special projection operator $P_h^- : H^1(I_i) \to \mathbb{P}^p(I_i)$ by the following $p + 1$ condition [31]:

$$\int_{I_i} (u - P_h^- u)v\,dx = 0,\ \forall\, v \in \mathbb{P}^{p-2}(I_i), \quad (u - P_h^- u)(x_i^-) = 0, \quad (u - P_h^- u)'(x_i^-) = 0. \tag{3.2}$$

Similarly, we define the special projection P_h^+ into V_h^p using the following conditions:

$$\int_{I_i}(u - P_h^+ u)v\,dx = 0,\ \forall\ v \in \mathbb{P}^{p-2}(I_i), \quad (u - P_h^+ u)(x_{i-1}^+) = 0, \quad (u - P_h^+ u)'(x_{i-1}^+) = 0. \tag{3.3}$$

It is easy to verify that the projections $P_h^{\pm} u$ satisfy the following approximation properties [31]:

$$\left\|u - P_h^{\pm} u\right\| + h\left\|(u - P_h^{\pm} u)'\right\| + h^{1/2}\left\|u - P_h^{\pm} u\right\|_{\Gamma_h} \leq C h^{p+1}\|u\|_{p+1}, \tag{3.4}$$

where Γ_h represents the collection of boundary points of all elements I_i, and C is a positive constant that depends on p, but not on the step size h. Here, the norm $\|\cdot\|_{\Gamma_h}$ is defined by $\|v\|_{\Gamma_h} = \left(\sum_{i=0}^{N} v^2(x_i)\right)^{1/2}$.

4. Convergence error analysis

In this section, we present the error estimates of our UWDG scheme for the model problem (1.1). Let e_u be the error between the exact solution and the UWDG solution defined by $e_u = u - u_h$. As the general treatment of the finite element method, we split the actual error into two terms as

$$e_u = \xi_u + \eta_u, \tag{4.1}$$

where the projection error is defined by

$$\eta_u = u - P_h^- u,$$

and $\xi_u \in V_h^p$ is the error between the UWDG solution and the projection of the exact solution i.e.,

$$\xi_u = P_h^- u - u_h.$$

Let $D = \{(x,u) |\ x \in \overline{\Omega} = [a,b],\ u \in \mathbb{R}\} \subset \overline{\Omega} \times \mathbb{R}$. In our error analysis, we always assume that the functions $f(x,u) : D \to \mathbb{R}$ and $g(x,u) : D \to \mathbb{R}$ are sufficiently differentiable functions with respect to its arguments. To be more precise, we impose the following conditions on $f(x,u)$ and $g(x,u)$: $f(x,u),\ g(x,u) \in C_b^1(D)$, that is, both f and g belong to $C^1(D)$ and the functions themselves together with their first-order partial derivatives are bounded on D. In particular, we assume

$$\left|\frac{\partial f(x,u)}{\partial u}\right| \leq L, \quad \left|\frac{\partial g(x,u)}{\partial u}\right| \leq M, \quad \forall\ (x,u) \in D. \tag{4.2}$$

Using the mean-value theorem, it follows that f satisfies the following uniform Lipschitz condition on D in the variable u with uniform Lipschitz constant L:

$$|f(x,u) - f(x,v)| \leq L\,|u - v|, \quad \text{for all } (x,u),\ (x,v) \in D = [a,b] \times \mathbb{R}. \tag{4.3}$$

Using Taylor's theorem, we write

$$g(x,u) - g(x,u_h) = G\, e_u, \quad G = G(x) = \int_0^1 \frac{\partial g\,(x, u(x) - t e_u(x))}{\partial u}\,dt, \tag{4.4a}$$

$$g(x_i, u(x_i)) - g(x_i, u_h(x_i^-)) = G(x_i^-)\, e_u(x_i^-), \quad G(x_i^-) = \int_0^1 \frac{\partial g\left(x_i, u(x_i) - te_u(x_i^-)\right)}{\partial u}\, dt, \tag{4.4b}$$

$$f(x,u) - f(x,u_h) = R\, e_u, \quad R = R(x) = \int_0^1 \frac{\partial f\,(x, u(x) - te_u(x))}{\partial u}\, dt. \tag{4.4c}$$

In our error analysis, we need a regularity estimate which will be used to estimate the errors $\|\xi_u\|$. We present it in the following lemma.

Lemma 4.1. *Suppose that $V \in H^2(\Omega)$ and $\phi \in L^2(\Omega)$ satisfy the following linear terminal initial-value problem:*

$$-V'' + G(x)\, V' + R(x)\, V = \phi(x), \quad x \in \Omega, \quad V(b) = V'(b) = 0. \tag{4.5}$$

Then the following regularity estimate holds:

$$\|V\|_2 \le C\, \|\phi\|\,, \tag{4.6}$$

where C is a positive constant independent of ϕ.

Proof. The IVP (4.5) can be converted into the linear system of equations

$$\mathbf{v}' = A(x)\mathbf{v} + \mathbf{c}(x), \quad x \in \Omega \quad \text{subject to} \quad \mathbf{v}(b) = \mathbf{0}, \tag{4.7}$$

where

$$\mathbf{v} = \begin{bmatrix} V \\ V' \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ R & G \end{bmatrix}, \; \mathbf{c} = \begin{bmatrix} 0 \\ -\phi \end{bmatrix}.$$

The solution to (4.7) can be expressed in terms of its fundamental matrix Φ as [34]

$$\mathbf{v}(x) = \Phi(x) \int_b^x \Phi^{-1}(t)\mathbf{c}(t)\, dt, \tag{4.8}$$

where $\Phi(x)$ is the 2×2 fundamental matrix whose columns consist of the entries of the solution vectors of the system $\mathbf{v}' = A(x)\mathbf{v}$. It satisfies $\Phi'(x) = A(x)\Phi(x)$.

Under the assumptions on f and g, the entries of the 2×2 matrix $A(x)$ are bounded on $[a, b]$. Using (4.8), we can deduce that there exists a constant C such that (see [34, Lemma 4.2])

$$\|\mathbf{v}\|_1 \le C\, \|\mathbf{c}\| = C\, \|\phi\|\,.$$

Since $\|\mathbf{v}\|_1^2 = \|\mathbf{v}\|^2 + \|\mathbf{v}'\|^2 = \|V\|^2 + 2\, \|V'\|^2 + \|V''\|^2$, we have

$$\|V\|^2 \le C_1\, \|\phi\|^2\,, \quad \|V'\|^2 \le C_2\, \|\phi\|^2\,, \quad \|V''\|^2 \le C_3\, \|\phi\|^2\,.$$

Thus, we deduce

$$\|V\|_2 = \left(\|V\|^2 + \|V'\|^2 + \|V''\|^2\right)^{1/2} \le C\, \|\phi\|\,,$$

which completes the proof of the estimate (4.6).

Now, we are ready to prove our *a priori* error estimate for e_u in the L^2-norm.

Theorem 4.1. *Suppose that $u \in H^{p+1}(\Omega)$ solves (1.1). Let $p \geq 2$ and u_h be the UWDG solution defined in (2.2a). If the functions f and g satisfy assumptions (4.2) and (4.3), then, for sufficiently small h, there exists a positive constant C independent of h such that*

$$\|e_u\| \leq Ch^{p+1}. \tag{4.9}$$

Proof. First, we derive some error equations for the UWDG method. Subtracting (2.2a) from (2.1) with $v \in V_h^p$, we get the error equation on each element I_i: $\forall\, v \in V_h^p$ and $i = 1, 2, \ldots, N$,

$$\begin{aligned}
&-\int_{I_i} e_u v'' dx + \int_{I_i} v'(g(x,u) - g(x,u_h))\, dx + \int_{I_i} v(f(x,u) - f(x,u_h))\, dx \\
&- (g(x_i, u(x_i)) - g(x_i, u_h(x_i^-)))v(x_i^-) + (g(x_{i-1}, u(x_{i-1})) - g(x_{i-1}, u_h(x_{i-1}^-)))v(x_{i-1}^+) \\
&- e_u'(x_i^-)v(x_i^-) + e_u'(x_{i-1}^-)v(x_{i-1}^+) + e_u(x_i^-)v'(x_i^-) - e_u(x_{i-1}^-)v'(x_{i-1}^+) = 0.
\end{aligned} \tag{4.10}$$

Applying (4.2), we immediately deduce that

$$|R(x)| \leq L, \quad |G(x)| \leq M, \quad \forall\, x \in \Omega. \tag{4.11}$$

Combining (4.4) and (4.10), we obtain: $\forall\, v \in V_h^p$,

$$\begin{aligned}
&\int_{I_i} e_u(-v'' + G\, v' + R\, v)dx - G(x_i^-)e_u(x_i^-)v(x_i^-) + G(x_{i-1}^-)e_u(x_{i-1}^-)v(x_{i-1}^+) \\
&- e_u'(x_i^-)v(x_i^-) + e_u'(x_{i-1}^-)v(x_{i-1}^+) + e_u(x_i^-)v'(x_i^-) - e_u(x_{i-1}^-)v'(x_{i-1}^+) = 0.
\end{aligned} \tag{4.12}$$

Define the following bilinear form:

$$\begin{aligned}
A_i(e_u, V) = &\int_{I_i} e_u(-V'' + G\, V' + R\, V)dx - G(x_i^-)e_u(x_i^-)V(x_i^-) + G(x_{i-1}^-)e_u(x_{i-1}^-)V(x_{i-1}^+) \\
&- e_u'(x_i^-)V(x_i^-) + e_u'(x_{i-1}^-)V(x_{i-1}^+) + e_u(x_i^-)V'(x_i^-) - e_u(x_{i-1}^-)V'(x_{i-1}^+).
\end{aligned} \tag{4.13}$$

Then (4.12) can be written compactly as

$$A_i(e_u, v) = 0, \quad \forall\, v \in V_h^p. \tag{4.14}$$

On the other hand, integrating by parts twice, we get

$$\begin{aligned}
A_i(e_u, V) = &\int_{I_i} (-e_u'' - (Ge_u)' + R\, e_u)Vdx - [\![Ge_u]\!](x_{i-1})V(x_{i-1}^+) - [\![e_u']\!](x_{i-1})V(x_{i-1}^+) \\
&+ [\![e_u]\!](x_{i-1})V'(x_{i-1}^+).
\end{aligned} \tag{4.15}$$

Adding and subtracting $P_h^+ V$ to V and using (4.14), we obtain

$$A_i(e_u, V) = A_i(e_u, V - P_h^+ V) + A_i(e_u, P_h^+ V) = A_i(e_u, V - P_h^+ V). \tag{4.16}$$

Combining (4.16) and (4.15), and using the property of the projection P_h^+, we obtain

$$
\begin{aligned}
A_i(e_u, V) = &\int_{I_i} (-e_u'' - (Ge_u)' + R\, e_u)(V - P_h^+ V)dx - [\![Ge_u]\!](x_{i-1})(V - P_h^+ V)(x_{i-1}^+) \\
&- [\![e_u']\!](x_{i-1})(V - P_h^+ V)(x_{i-1}^+) + [\![e_u]\!](x_{i-1})(V - P_h^+ V)'(x_{i-1}^+) \\
= &\int_{I_i} (-e_u'' - (Ge_u)' + R\, e_u)(V - P_h^+ V)dx. \qquad (4.17)
\end{aligned}
$$

Substituting $e_u = \xi_u + \eta_u$ into (4.17) and using the property of the projection P_h^+, we get

$$
A_i(e_u, V) = \int_{I_i} (-\eta_u'' - (G\eta_u)' - (G\xi_u)' + R\, e_u)(V - P_h^+ V)dx. \qquad (4.18)
$$

Integrating by parts and using the fact that $\eta_u(x_i^-) = \eta_u'(x_i^-) = (V - P_h^+ V)(x_{i-1}^+) = 0$, we arrive at

$$
A_i(e_u, V) = \int_{I_i} (\eta_u' + Ge_u)(V - P_h^+ V)'\, dx + \int_{I_i} R\, e_u(V - P_h^+ V)\, dx - G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-). \qquad (4.19)
$$

Equating (4.13) and (4.19), we obtain

$$
\begin{aligned}
&\int_{I_i} e_u(-V'' + G\, V' + R\, V)dx - G(x_i^-)e_u(x_i^-)V(x_i^-) + G(x_{i-1}^-)e_u(x_{i-1}^-)V(x_{i-1}^+) \\
&- e_u'(x_i^-)V(x_i^-) + e_u'(x_{i-1}^-)V(x_{i-1}^+) + e_u(x_i^-)V'(x_i^-) - e_u(x_{i-1}^-)V'(x_{i-1}^+) - e_u(x_{i-1}^-)V'(x_{i-1}^+) \\
&= \int_{I_i} (\eta_u' + Ge_u)(V - P_h^+ V)'\, dx + \int_{I_i} R\, e_u(V - P_h^+ V)\, dx - G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-). \qquad (4.20)
\end{aligned}
$$

Summing over all elements yields

$$
\begin{aligned}
&\int_{\Omega} e_u(-V'' + G\, V' + R\, V)dx - G(x_N^-)e_u(x_N^-)V(x_N^-) + G(x_0^-)e_u(x_0^-)V(x_0^+) \\
&+ \sum_{i=1}^{N-1} G(x_i^-)e_u(x_i^-)[\![V]\!](x_i) - e_u'(x_N^-)V(x_N^-) + e_u'(x_0^-)V(x_0^+) - e_u(x_0^-)V'(x_0^+) + e_u(x_N^-)V'(x_N^-) \\
&+ \sum_{i=1}^{N-1} (e_u'(x_i^-)[\![V]\!](x_i) - e_u(x_i^-)[\![V']\!](x_i)) = \int_{\Omega} (\eta_u' + Ge_u)(V - P_h^+ V)'\, dx \\
&+ \int_{\Omega} R\, e_u(V - P_h^+ V)\, dx - \sum_{i=0}^{N} G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-).
\end{aligned}
$$

Since $e_u(x_0^-) = e_u'(x_0^-) = 0$, we get

$$
\begin{aligned}
&\int_{\Omega} e_u(-V'' + G\, V' + R\, V)dx - G(x_N^-)e_u(x_N^-)V(x_N^-) + \sum_{i=1}^{N-1} G(x_i^-)e_u(x_i^-)[\![V]\!](x_i) \\
&- e_u'(x_N^-)V(x_N^-) + e_u(x_N^-)V'(x_N^-) + \sum_{i=1}^{N-1} (e_u'(x_i^-)[\![V]\!](x_i) - e_u(x_i^-)[\![V']\!](x_i)) \\
&= \int_{\Omega} (\eta_u' + Ge_u)(V - P_h^+ V)'\, dx + \int_{\Omega} R\, e_u(V - P_h^+ V)\, dx - \sum_{i=1}^{N} G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-). \qquad (4.21)
\end{aligned}
$$

We will be using a duality argument to prove the estimate (4.9). Suppose that V satisfies the auxiliary problem (4.5) with $\phi = e_u$ and then the error equation (4.21) reduces to

$$\|e_u\|^2 = \int_\Omega (\eta_u' + Ge_u)(V - P_h^+ V)'\,dx + \int_\Omega R\, e_u(V - P_h^+ V)\,dx - \sum_{i=1}^{N} G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-), \quad (4.22)$$

since $V(x_N^-) = V'(x_N^-) = [\![V]\!](x_i) = [\![V']\!](x_i) = 0$.
Next, we write (4.22) as

$$\|e_u\|^2 = T_1 + T_2 + T_3 + T_4, \quad (4.23a)$$

where

$$T_1 = \int_\Omega \eta_u'(V - P_h^+ V)'\,dx, \quad (4.23b)$$

$$T_2 = \int_\Omega Ge_u(V - P_h^+ V)'\,dx, \quad (4.23c)$$

$$T_3 = \int_\Omega R\, e_u(V - P_h^+ V)\,dx, \quad (4.23d)$$

$$T_4 = -\sum_{i=1}^{N} G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-). \quad (4.23e)$$

We will estimate T_i, $i = 1, 2, 3, 4$, one by one.
Estimate of T_1: Using the Cauchy-Schwarz inequality, the interpolation error estimate (3.4), and the regularity estimate (4.6), we obtain

$$T_1 \le \left\|\eta_u'\right\| \left\|(V - P_h^+ V)'\right\| \le (C_2 h^p)(C_3 h \|V\|_2) \le C_1 h^{p+1} \|e_u\|. \quad (4.24)$$

Estimate of T_2: Using (4.11), the Cauchy-Schwarz inequality, the interpolation error estimate (3.4), and the regularity estimate (4.6), we obtain

$$T_2 \le M \|e_u\| \left\|(V - P_h^+ V)'\right\| \le M \|e_u\| (C_1 h \|V\|_2) \le MC_1 h \|e_u\| (C_3 \|e_u\|) \le C_2 h \|e_u\|^2. \quad (4.25)$$

Estimate of T_3: Using (4.11), the Cauchy-Schwarz inequality, the interpolation error estimate (3.4), and the regularity estimate (4.6), we obtain

$$T_3 \le L \|e_u\| \left\|V - P_h^+ V\right\| \le L \|e_u\| (C_1 h^2 \|V\|_2) \le LC_1 h^2 \|e_u\| (C_2 \|e_u\|) \le C_3 h^2 \|e_u\|^2. \quad (4.26)$$

Estimate of T_4: Using (4.11), the Cauchy-Schwarz inequality, the inverse inequality, the interpolation error estimate (3.4), the relation $\xi_u = e_u - \eta_u$, the triangle inequality, and the regularity estimate (4.6), we obtain

$$\begin{aligned} T_4 &\le M \sum_{i=1}^{N} \left|\xi_u(x_i^-)\right| \left|(V - P_h^+ V)(x_i^-)\right| \le M \left(\sum_{i=1}^{N} \left|\xi_u(x_i^-)\right|^2\right)^{1/2} \left(\sum_{i=1}^{N} \left|(V - P_h^+ V)(x_i^-)\right|^2\right)^{1/2} \\ &\le M \left(C_1 h^{-1/2} \|\xi_u\|\right)\left(C_2 h^{3/2} \|V\|_2\right) \\ &\le MC_1C_2 h (\|e_u\| + \|\eta_u\|)(C_3 \|e_u\|) \\ &\le MC_1C_2C_3 h \left(\|e_u\| + C_5 h^{p+1}\right) \|e_u\| \\ &\le C_4 h \|e_u\|^2 + C_4 h^{p+2} \|e_u\|. \end{aligned} \quad (4.27)$$

Combining (4.23) and the estimates (4.24)–(4.27), we get

$$\begin{aligned}\|e_u\|^2 &\le C_1h^{p+1}\|e_u\| + C_2h\|e_u\|^2 + C_3h^2\|e_u\|^2 + C_4h\|e_u\|^2 + C_4h^{p+2}\|e_u\| \\ &\le Ch^{p+1}\|e_u\| + Ch\|e_u\|^2 .\end{aligned}$$

Assuming $\|e_u\| \neq 0$ and dividing both sides by $\|e_u\|$, we obtain

$$\|e_u\| \le Ch^{p+1} + Ch\|e_u\| .$$

Rearranging terms gives $(1 - Ch)\|e_u\| \le Ch^{p+1}$. To ensure that the coefficient $(1 - Ch)$ is positive, we require $h < \frac{1}{C}$. For a concrete threshold, we can take $0 < h \le \frac{1}{2C}$, so that $1 - Ch \ge \frac{1}{2}$. Dividing both sides by $(1 - Ch)$ gives

$$\|e_u\| \le \frac{C}{1 - Ch}h^{p+1} \le 2Ch^{p+1}. \tag{4.28}$$

Absorbing the factor 2 into the constant C yields the final estimate:

$$\|e_u\| \le Ch^{p+1}, \quad 0 < h \le \frac{1}{2C},$$

which completes the proof of (4.9).

In the following corollary, we establish an optimal estimate for e_u in the infinity norm.

Corollary 4.1. *Assume that the conditions of Theorem 4.1 are satisfied. Then, for sufficiently small h, there exists a positive constant C independent of h such that:*

$$\|e_u\|_\infty \le Ch^p. \tag{4.29}$$

Proof. The estimate (4.29) follows from the inverse property, the triangle inequality, the interpolation property (3.4), and the estimate (4.9)

$$\begin{aligned}\|e_u\|_\infty = \|\xi_u + \eta_u\|_\infty \le \|\xi_u\|_\infty + \|\eta_u\|_\infty &\le C_1h^{-1}\|\xi_u\| + C_2h^p \\ &\le C_1h^{-1}\|e_u - \eta_u\| + C_2h^p \\ &\le C_1h^{-1}(\|e_u\| + \|\eta_u\|) + C_2h^p \\ &\le C_1h^{-1}(C_3h^{p+1} + C_4h^{p+1}) + C_2h^p \\ &\le Ch^p,\end{aligned}$$

which completes the proof of the theorem.

5. Superconvergence error analysis

In this section, we will investigate the superconvergence properties of the UWDG solution. We will study the superconvergence between the special projection of the exact solution $P_h^- u$ and the UWDG solution u_h, the superconvergence of the cell averages, and the function and derivative values at the node x_i of each subinterval I_i.

5.1. Superconvergence toward the projection $P_h^- u$

Here, we will prove that the UWDG solution u_h is superconvergent toward the projection of the exact solution $P_h^- u$ in the L^2-norm. The order of superconvergence is $p+2$ for $p \geq 2$, i.e.,

$$\|\xi_u\| = O(h^{p+2}), \quad p \geq 2.$$

Theorem 5.1. *Suppose that the assumptions of Theorem 4.1 are satisfied. In addition, we assume that the functions $g_1(x) = \frac{\partial f(x,u(x))}{\partial u}$ and $g_2(x) = \frac{\partial g(x,u(x))}{\partial u}$ for $x \in [a,b]$ are sufficiently smooth functions. To be more precise, we assume $\left|g_s^{(k)}(x)\right| \leq C$ for $k = 1,2$ and $s = 1,2$. Then there exists a positive constant C independent of h such that*

$$\|\xi_u\| \leq Ch^{p+2}, \quad p \geq 2. \tag{5.1}$$

Proof. We begin with the error equation (4.12). Applying (4.1) and the properties of the projection P_h^-, we obtain

$$\int_{I_i} \xi_u(-v'' + G\,v' + R\,v)dx + \int_{I_i} \eta_u(G\,v' + R\,v)dx - \left(G(x_i^-)\xi_u(x_i^-) + \xi_u'(x_i^-)\right) v(x_i^-)$$
$$+ \left(G(x_{i-1}^-)\xi_u(x_{i-1}^-) + \xi_u'(x_{i-1}^-)\right) v(x_{i-1}^+) + \xi_u(x_i^-)v'(x_i^-) - \xi_u(x_{i-1}^-)v'(x_{i-1}^+) = 0. \tag{5.2}$$

We define the bilinear form $B_i(u,V)$ as

$$B_i(u,V) =$$
$$\int_{I_i} \xi_u(-V' + G\,V' + R\,V)dx + \int_{I_i} \eta_u(G\,V' + R\,V)dx - \left(G(x_i^-)\xi_u(x_i^-) + \xi_u'(x_i^-)\right) V(x_i^-)$$
$$+ \left(G(x_{i-1}^-)\xi_u(x_{i-1}^-) + \xi_u'(x_{i-1}^-)\right) V(x_{i-1}^+) + \xi_u(x_i^-)V'(x_i^-) - \xi_u(x_{i-1}^-)V'(x_{i-1}^+). \tag{5.3}$$

We remark that (5.2) can be written as

$$B_i(u,v) = 0, \quad \forall\, v \in V_h^p. \tag{5.4}$$

Using integration by parts twice and applying (4.1), we get

$$B_i(u,V) = \int_{I_i} \left(-\xi_u'' - (G\xi_u)' + R\,e_u\right)\, Vdx + \int_{I_i} G\,\eta_u V'dx$$
$$- [\![G\xi_u]\!](x_{i-1})V(x_{i-1}^+) - [\![\xi_u']\!](x_{i-1})V(x_{i-1}^+) + [\![\xi_u]\!](x_{i-1})V'(x_{i-1}^+). \tag{5.5}$$

Adding and subtracting $P_h^+ V$ to V and using (5.4), we get

$$B_i(u,V) = B_i(u, V - P_h^+ V) + B_i(u, P_h^+ V) = B_i(u, V - P_h^+ V). \tag{5.6}$$

Combining (5.6) and (5.5), and using the property of the projection P_h^+, yields

$$B_i(u,V) = \int_{I_i} \left(-\xi_u'' - (G\xi_u)' + R\,e_u\right)(V - P_h^+ V)dx + \int_{I_i} G\,\eta_u(V - P_h^+ V)'dx$$
$$- [\![G\xi_u]\!](x_{i-1})(V - P_h^+ V)(x_{i-1}^+) - [\![\xi_u']\!](x_{i-1})(V - P_h^+ V)(x_{i-1}^+)$$
$$+ [\![\xi_u]\!](x_{i-1})(V - P_h^+ V)'(x_{i-1}^+)$$
$$= \int_{I_i} \left(-(G\xi_u)' + R\,e_u\right)(V - P_h^+ V)dx + \int_{I_i} G\,\eta_u(V - P_h^+ V)'dx.$$

Integrating by parts and using the fact that $\eta_u(x_i^-) = \eta_u'(x_i^-) = (V - P_h^+ V)(x_{i-1}^+) = 0$, we arrive at

$$B_i(u, V) = \int_{I_i} G\, e_u (V - P_h^+ V)' dx + \int_{I_i} R\, e_u (V - P_h^+ V) dx - G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-). \tag{5.7}$$

Equating (5.3) and (5.7), we obtain

$$\begin{aligned}
&\int_{I_i} \xi_u(-V' + G\, V' + R\, V)dx + \int_{I_i} \eta_u(G\, V' + R\, V)dx - \left(G(x_i^-)\xi_u(x_i^-) + \xi_u'(x_i^-)\right) V(x_i^-)\\
&+ \left(G(x_{i-1}^-)\xi_u(x_{i-1}^-) + \xi_u'(x_{i-1}^-)\right) V(x_{i-1}^+) + \xi_u(x_i^-)V'(x_i^-) - \xi_u(x_{i-1}^-)V'(x_{i-1}^+)\\
&= \int_{I_i} G\, e_u (V - P_h^+ V)' dx + \int_{I_i} R\, e_u (V - P_h^+ V) dx - G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-).
\end{aligned} \tag{5.8}$$

Summing over all elements yields

$$\begin{aligned}
&\int_{\Omega} \xi_u(-V'' + G\, V' + R\, V)dx + \int_{\Omega} \eta_u(G\, V' + R\, V)dx + \sum_{i=1}^{N-1} G(x_i^-)\xi_u(x_i^-)[\![V]\!](x_i)\\
&+ \sum_{i=1}^{N-1} \xi_u'(x_i^-)[\![V]\!](x_i) - \sum_{i=1}^{N-1} \xi_u(x_i^-)[\![V']\!](x_i) - \left(G(x_N^-)\xi_u(x_N^-) + \xi_u'(x_N^-)\right) V(x_N^-)\\
&+ \left(G(x_0^-)\xi_u(x_0^-) + \xi_u'(x_0^-)\right) V(x_0^+) + \xi_u(x_N^-)V'(x_N^-) - \xi_u(x_0^-)V'(x_0^+)\\
&= \int_{\Omega} G\, e_u (V - P_h^+ V)' dx + \int_{\Omega} R\, e_u (V - P_h^+ V) dx - \sum_{i=0}^{N} G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-).
\end{aligned}$$

Since $\xi_u(x_0^-) = \xi_u'(x_0^-) = 0$, we deduce

$$\begin{aligned}
&\int_{\Omega} \xi_u(-V'' + G\, V' + R\, V)dx + \int_{\Omega} \eta_u(G\, V' + R\, V)dx + \sum_{i=1}^{N-1} G(x_i^-)\xi_u(x_i^-)[\![V]\!](x_i)\\
&+ \sum_{i=1}^{N-1} \xi_u'(x_i^-)[\![V]\!](x_i) - \sum_{i=1}^{N-1} \xi_u(x_i^-)[\![V']\!](x_i) - \left(G(x_N^-)\xi_u(x_N^-) + \xi_u'(x_N^-)\right) V(x_N^-)\\
&+ \xi_u(x_N^-)V'(x_N^-) = \int_{\Omega} G\, e_u (V - P_h^+ V)' dx + \int_{\Omega} R\, e_u (V - P_h^+ V) dx\\
&- \sum_{i=1}^{N} G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-).
\end{aligned} \tag{5.9}$$

Suppose that V satisfies the auxiliary problem (4.5) with $\phi = \xi_u$ and then the error equation (5.9) reduces to

$$\begin{aligned}
\|\xi_u\|^2 &= \int_{\Omega} G\, e_u (V - P_h^+ V)' dx + \int_{\Omega} R\, e_u (V - P_h^+ V) dx\\
&- \int_{\Omega} \eta_u(G\, V' + R\, V)dx - \sum_{i=1}^{N} G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-),
\end{aligned} \tag{5.10}$$

since $V(x_N^-) = V(x_N^-)' = [\![V]\!](x_{i-1}) = [\![V']\!](x_{i-1}) = 0$.

Thus, we have

$$\|\xi_u\|^2 = S_1 + S_2 + S_3 + S_4, \tag{5.11}$$

where

$$\begin{aligned} S_1 &= \int_\Omega G\, e_u (V - P_h^+ V)' dx, \\ S_2 &= \int_\Omega R\, e_u (V - P_h^+ V) dx, \\ S_3 &= -\int_\Omega \eta_u (G\, V' + R\, V) dx, \\ S_4 &= -\sum_{i=1}^{N} G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-). \end{aligned}$$

Next, we will estimate $S_k,\ k = 1, 2, 3, 4$, one by one.

Estimate of S_1: Using (4.11), the Cauchy-Schwarz inequality, the estimate (3.4), the estimate (4.9), and the regularity estimate (4.6), we obtain

$$\begin{aligned} S_1 &\le M \|e_u\| \left\|(V - P_h^+ V)'\right\| \\ &\le M(C_2 h^{p+1})(C_3 h \|V\|_2) \\ &\le C_1 h^{p+2} \|\xi_u\|. \end{aligned} \tag{5.12}$$

Estimate of S_2: Using (4.11), the Cauchy-Schwarz inequality, the estimate (4.9), and the regularity estimate (4.6), we obtain

$$\begin{aligned} S_2 &\le L \|e_u\| \left\|V - P_h^+ V\right\| \\ &\le L(C_1 h^{p+1}) C_3 h^2 \|V\|_2 \\ &\le C_2 h^{p+3} \|\xi_u\|. \end{aligned} \tag{5.13}$$

Estimate of S_3: For $p \ge 2$, let $\hat{P}_h$ be the standard L^2-projection onto $\mathbb{P}^{p-2}(\Omega)$, i.e., for $u \in L^2(\Omega)$, the projection $\hat{P}_h u$ is the unique polynomial in $\mathbb{P}^{p-2}(\Omega)$ satisfying

$$\int_{I_i} (\hat{P}_h u - u) v\, dx = 0,\ \forall\ v \in \mathbb{P}^{p-2}(I_i).$$

The following well-known projection result holds: For any $V \in H^{p-1}(\Omega)$, there exists a constant C independent of h such that

$$\left\|V - \hat{P}_h V\right\| \le C h^{p-1} \|V\|_{p-1}. \tag{5.14}$$

Adding and subtracting $\hat{P}_h(G\, V' + R\, V)$ to $G\, V' + R\, V$ and using the fact that η_u is orthogonal to $\hat{P}_h(G\, V' + R\, V)$ (this is due to the property of the projection $\hat{P}_h^-$), we obtain

$$\begin{aligned} S_3 &= -\sum_{i=1}^{N} \int_{I_i} \eta_u (G\, V' + R\, V - \hat{P}_h(G\, V' + R\, V))\, dx - \sum_{i=1}^{N} \int_{I_i} \eta_u \hat{P}_h (G\, V' + R\, V)\, dx \\ &= -\sum_{i=1}^{N} \int_{I_i} \eta_u (G\, V' + R\, V - \hat{P}_h(G\, V' + R\, V))\, dx. \end{aligned}$$

Applying the Cauchy-Schwarz inequality, the estimate (5.14), the standard interpolation error estimates (5.14) and (3.4), (4.11), and the estimate (4.6), we obtain for $p \geq 2$

$$\begin{aligned} S_3 \leq \|\eta_u\| \left\|G\,V' + R\,V - \hat{P}_h(G\,V' + R\,V)\right\| &\leq (C_0 h^{p+1})(C_1 h \,\|G\,V' + R\,V\|_1) \\ &\leq C_2 h^{p+2} \|V\|_2 \\ &\leq C_3 h^{p+2} \|\xi_u\|. \end{aligned} \tag{5.15}$$

Estimate of S_4: Using (4.11), the Cauchy-Schwarz inequality, the inverse inequality, the interpolation error estimate (3.4), and the regularity estimate (4.6), we obtain

$$\begin{aligned} S_4 \leq M \sum_{i=1}^{N} \left|\xi_u(x_i^-)\right| \left|(V - P_h^+ V)(x_i^-)\right| &\leq M \left(\sum_{i=1}^{N} \left|\xi_u(x_i^-)\right|^2 \right)^{1/2} \left(\sum_{i=1}^{N} \left|(V - P_h^+ V)(x_i^-)\right|^2 \right)^{1/2} \\ &\leq M \left(C_1 h^{-1/2} \|\xi_u\|\right) \left(C_2 h^{3/2} \|V\|_2\right) \\ &\leq M C_1 C_2 h \|\xi_u\| \,(C_3 \|\xi_u\|) \\ &\leq C_4 h \|\xi_u\|^2. \end{aligned} \tag{5.16}$$

Combining (5.11) and the estimates (5.12), (5.13), (5.15), and (5.16), we get

$$\|\xi_u\|^2 \leq C_1 h^{p+2} \|\xi_u\| + C_2 h^{p+3} \|\xi_u\| + C_3 h^{p+2} \|\xi_u\| + C_4 h \|\xi_u\|^2 \leq C h^{p+2} \|\xi_u\| + C h \|\xi_u\|^2.$$

Dividing by $\|\xi_u\|$ gives

$$\|\xi_u\| \leq C h^{p+2} + C h \|\xi_u\|.$$

Thus, for small h (the choice $h < \frac{1}{C}$ is enough), we get

$$\|\xi_u\| \leq C h^{p+2},$$

which completes the proof of (5.1).

5.2. Superconvergence at the nodes

In this subsection, we prove that the UWDG solution and its derivative are $O(h^{2p})$ superconvergent at the downwind points. In addition, we prove a $2p$-th-order superconvergence rate for the cell averages.

Theorem 5.2. *Assuming that the assumptions of Theorem 5.1 are satisfied, we further assume that $g_1(x) = \frac{\partial f(x,u(x))}{\partial u}$ and $g_2(x) = \frac{\partial g(x,u(x))}{\partial u}$ are sufficiently smooth functions. Specifically, we assume that $g_s(x) \in C^p(\Omega)$ for $s = 1, 2$. Under these conditions, there exists a positive constant C such that*

$$\left|e_u(x_k^-)\right| \leq C h^{2p}, \quad k = 0, 1, \ldots, N. \tag{5.17}$$

$$\left|e_u'(x_k^-)\right| \leq C h^{2p}, \quad k = 0, 1, \ldots, N. \tag{5.18}$$

$$\frac{1}{N+1} \left(\sum_{k=0}^{N} \left|e_u(x_k^-)\right|^2 \right)^{1/2} \leq C h^{2p}. \tag{5.19}$$

$$\frac{1}{N+1} \left(\sum_{k=0}^{N} \left|e_u'(x_k^-)\right|^2 \right)^{1/2} \leq C h^{2p}. \tag{5.20}$$

Proof. We begin by considering the error equation (4.20). By summing (4.20) over the elements I_i for $i = 1, 2, \ldots, k$, where $k = 1, 2, \ldots, N$, we obtain

$$\int_a^{x_k} e_u(-V'' + G\, V' + R\, V)dx + \sum_{i=1}^{k-1} \left[(e'_u(x_i^-) + G(x_i^-)e_u(x_i^-)) [\![V]\!](x_i) - e_u(x_i^-)[\![V']\!](x_i) \right]$$
$$- (G(x_k^-)e_u(x_k^-) + e'_u(x_k^-))\, V(x_k^-) + (G(x_0^-)e_u(x_0^-) + e'_u(x_0^-))\, V(x_0^+) - e_u(x_0^-)V'(x_0^+) + e_u(x_k^-)V'(x_k^-)$$
$$= \int_a^{x_k} (\eta'_u + G\, e_u)(V - P_h^+ V)' dx + \int_a^{x_k} R\, e_u(V - P_h^+ V)dx - \sum_{i=1}^{k} G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-).$$

Using $e_u(x_0^-) = e'_u(x_0^-) = 0$, we obtain

$$\int_a^{x_k} e_u(-V'' + G\, V' + R\, V)dx + \sum_{i=1}^{k-1} \left[(e'_u(x_i^-) + G(x_i^-)e_u(x_i^-)) [\![V]\!](x_i) - e_u(x_i^-)[\![V']\!](x_i) \right]$$
$$- (G(x_k^-)e_u(x_k^-) + e'_u(x_k^-))\, V(x_k^-) + e_u(x_k^-)V'(x_k^-)$$
$$= \int_a^{x_k} (\eta'_u + G\, e_u)(V - P_h^+ V)' dx + \int_a^{x_k} R\, e_u(V - P_h^+ V)dx - \sum_{i=1}^{k} G(x_i^-)\xi_u(x_i^-)(V - P_h^+ V)(x_i^-). \quad (5.21)$$

Let W be the solution of the following auxiliary problem:

$$-W'' + G\, W' + R\, W = 0, \quad x \in \Omega_k = [a, x_k] \quad \text{subject to} \quad W(x_k) = 0, \quad W'(x_k) = 1. \quad (5.22)$$

The assumptions $g_s(x) \in C^p(\Omega),\ s = 1, 2$, imply the existence of a constant C such that

$$\|W\|_{p+1,\Omega_k} \le C. \quad (5.23)$$

Choosing $V = W$ in (5.21), we get

$$e_u(x_k^-) = \int_a^{x_k} (\eta'_u + G\, e_u)(W - P_h^+ W)' dx + \int_a^{x_k} R\, e_u(W - P_h^+ W)dx$$
$$- \sum_{i=1}^{k} G(x_i^-)\xi_u(x_i^-)(W - P_h^+ W)(x_i^-), \quad (5.24)$$

since $-W'' + G\, W' + R\, W = [\![W]\!](x_i) = [\![W']\!](x_i) = W(x_k^-) = 0$ and $W'(x_k^-) = 1$.
Applying (4.11), the Cauchy-Schwarz inequality, and the inverse inequality, we obtain

$$\left|e_u(x_k^-)\right| \le \left(\left\|\eta'_u\right\|_{0,\Omega_k} + M\, \|\eta_u\|_{0,\Omega_k} \right) \left\|(W - P_h^+ W)'\right\|_{0,\Omega_k} + L\, \|e_u\|_{0,\Omega_k} \left\|W - P_h^+ W\right\|_{0,\Omega_k}$$
$$+ M \left(\sum_{i=1}^{k} \left|\xi_u(x_i^-)\right|^2 \right)^{1/2} \left(\sum_{i=1}^{k} \left|(W - P_h^+ W)(x_i^-)\right|^2 \right)^{1/2}$$
$$\le \left(\left\|\eta'_u\right\| + M\, \|\eta_u\| \right) \left\|(W - P_h^+ W)'\right\|_{0,\Omega_k} + L\, \|e_u\| \left\|W - P_h^+ W\right\|_{0,\Omega_k}$$
$$+ M \left(C_1 h^{-1/2}\, \|\xi_u\| \right) \left(\sum_{i=1}^{k} \left|(W - P_h^+ W)(x_i^-)\right|^2 \right)^{1/2}.$$

Applying the estimates (4.9), (3.4), (5.1), and (5.23), we get

$$\begin{aligned}\left|e_u(x_k^-)\right| &\le \left(C_0 h^p + MC_2 h^{p+1}\right)(C_3 h^p \|W\|_{p+1,\Omega_k}) + LC_4 h^{p+1}(C_5 h^{p+1} \|W\|_{p+1,\Omega_k}) \\ &\quad + MC_1 h^{-1/2}\left(C_6 h^{p+2}\right)(C_7 h^{p+1/2} \|W\|_{p+1,\Omega_k}) \\ &\le C_8 h^{2p} + C_9 h^{2p+1} + C_{10} h^{2p+2} \\ &\le Ch^{2p},\end{aligned}$$

for all $k = 1, \ldots, N$, which completes the proof of (5.17) since $e_u(x_0^-) = 0$.
By using the auxiliary problem

$$-W'' + G\,W' + R\,W = 0, \quad x \in \Omega_k = [a, x_k] \quad \text{subject to} \quad W(x_k) = 1, \quad W'(x_k) = 0, \tag{5.25}$$

and following the same steps employed in the proof of equation (5.17), we obtain the estimate (5.18). Finally, the estimates (5.19) and (5.20) can be deduced immediately from equations (5.17) and (5.18). For the sake of brevity, we omit the details.

Remark 5.1. *The main analytical challenges of this work stem from establishing optimal and superconvergent error estimates for the proposed UWDG method without introducing auxiliary variables or reformulating the model equation into a first-order system. The unified weak formulation involves coupled interface jump terms, which require delicate estimates to control their contribution to the global error. Moreover, achieving the* $\mathcal{O}(h^{p+1})$ *convergence in the* L^2*-norm and proving* $\mathcal{O}(h^{p+2})$ *superconvergence results for general polynomial degrees* $p \ge 2$ *on arbitrary nonuniform partitions demand careful projection analysis and the use of trace inequalities. These difficulties distinguish our analysis from standard DG approaches and contribute to the theoretical depth of the present work.*

6. Numerical examples

In this section, we present several numerical examples to validate our theoretical results. To solve the system of nonlinear algebraic equations arising from the UWDG scheme (2.2), we employ the Newton-Raphson method. We set the stopping criterion for Newton's iteration to 10^{-15}. For all the examples, we calculate the L^2 errors $\|e_u\|$ and $\|\xi_u\|$, as well as the following errors:

$$\left\|e_u^{(k)}\right\|_\infty^* = \max_{i=1,2,\ldots,N} \left|e_u^{(k)}(x_i^-)\right|, \quad \left\|\bar{e}_u^{(k)}\right\|^* = \frac{1}{N}\left(\sum_{i=1}^{N} \left|e_u^{(k)}(x_i^-)\right|^2\right)^{1/2}, \quad k = 0, 1.$$

In all numerical experiments, the numerical order of convergence is determined using the formula: $-\frac{\ln\left(\left\|e_u^{N_1}\right\|/\left\|e_u^{N_2}\right\|\right)}{\ln(N_1/N_2)}$, where $e_u^{N_1}$ and $e_u^{N_2}$ represent the errors obtained with N_1 and N_2 elements, respectively. For better visualization, we present the L^2 errors using a logarithmic scale. For each degree p, we perform a least-squares fitting of the data sets with a linear polynomial function. Subsequently, we calculate the slope of the fitting line.

Example 6.1. In this example, we consider the forced *van der Pol* equation:

$$u'' + (u^2 - 1)u' + u = k(x), \quad x \in [0, 1], \quad u(0) = \alpha, \quad u'(0) = \beta. \tag{6.1}$$

This equation arises in the modeling of an electrical circuit with a triode whose resistance changes with the current. It also arises in certain chemical reactions and wind-induced motions of structures. We note that (6.1) can be written as (1.1), where

$$g(x, u) = \frac{u^3}{3} - u, \quad f(x, u) = k(x) - u.$$

We select $k(x)$ and the initial conditions such that $u(x) = e^x$, i.e.,

$$k(x) = e^x + e^{3x}, \quad \alpha = \beta = 1.$$

We solve problem (6.1) using the UWDG scheme presented in Section 2. We employ a uniform partition with $N = 3, 4, \ldots, 8$ elements. The UWDG solution $u_h \in V_h^p$ is computed using finite element spaces V_h^p with $p = 2, 3, 4, 5$. The left figure of Figure 1 shows the L^2-norm of the errors and the corresponding orders of accuracy. We observe that the proposed UWDG scheme achieves optimal $(p+1)$-th-order accuracy for this problem. In the right figure of Figure 1, we present the errors $\|\xi_u\|$ and their convergence orders. We observe that $\|\xi_u\| = O(h^{p+2})$ for $p = 2, 3, 4, 5$. Figures 2 and 3 show the errors $\|e_u\|_\infty^*$, $\left\|e_u'\right\|_\infty^*$, $\|\bar{e}_u\|^*$, and $\left\|\bar{e}_u'\right\|^*$. We observe that $\|e_u\|_\infty^* = O(h^{2p})$, $\left\|e_u'\right\|_\infty^* = O(h^{2p})$, $\|\bar{e}_u\|^* = O(h^{2p})$, and $\left\|\bar{e}_u'\right\|^* = O(h^{2p})$. These results indicate optimal convergence and superconvergence rates. Overall, this numerical example demonstrates the sharpness of the convergence and superconvergence results.

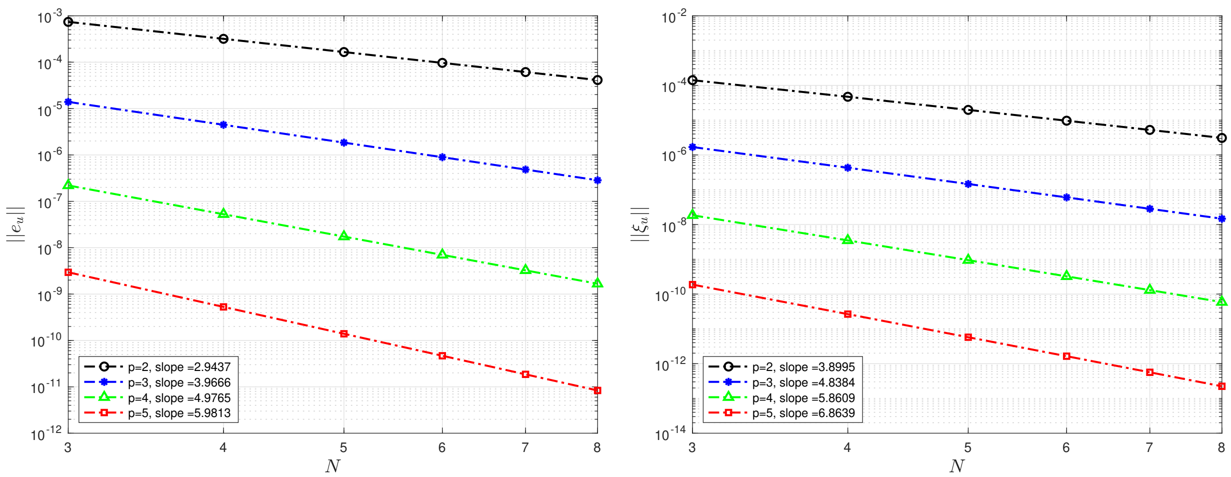

Figure 1. Convergence rates for $\|e_u\|$ (left) and $\|\xi_u\|$ (right) for the IVP (6.1) on uniform partitions having $N = 3, 4, \ldots, 8$ elements using $p = 2$–5.

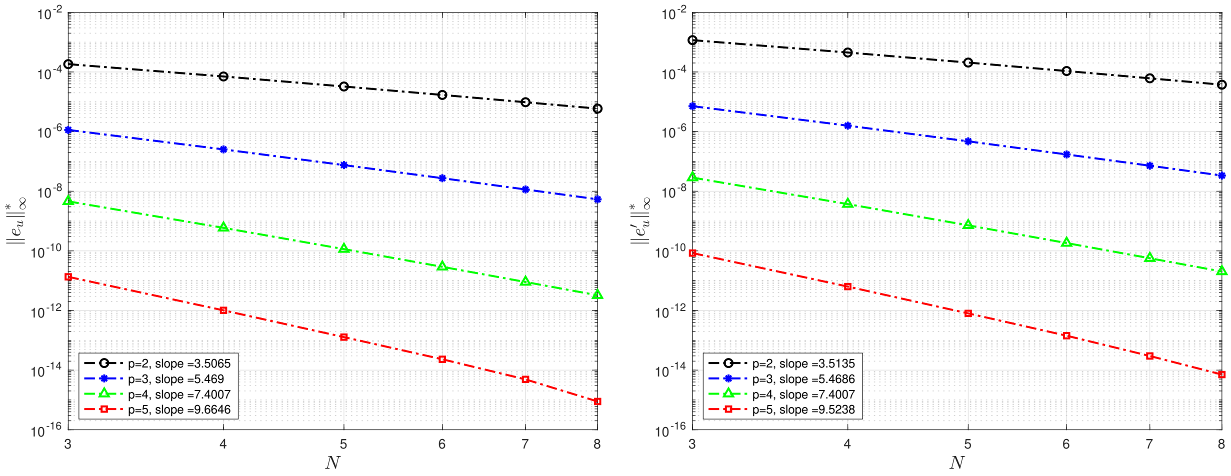

Figure 2. Convergence rates for $\|e_u\|_\infty^*$ (left) and $\left\|e_u'\right\|_\infty^*$ (right) for the IVP (6.1) on uniform partitions having $N = 3, 4, \ldots, 8$ elements using $p = 2$–5.

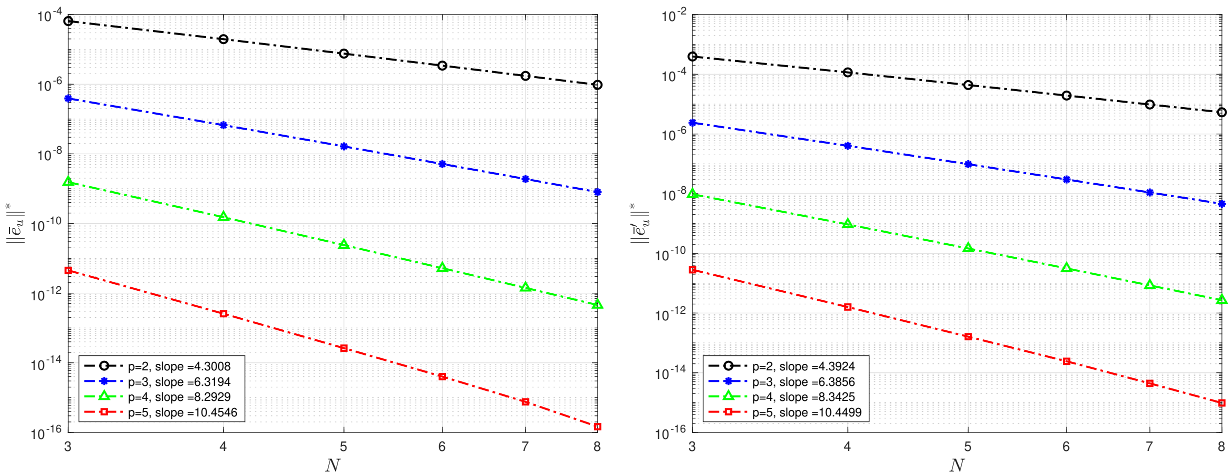

Figure 3. Convergence rates for $\|\bar{e}_u\|^*$ (left) and $\left\|\bar{e}_u'\right\|_\infty^*$ (right) for the IVP (6.1) on uniform partitions having $N = 3, 4, \ldots, 8$ elements using $p = 2$–5.

Example 6.2. In this example, we consider the pendulum problem

$$mu'' + cu' + k\sin(u) = r(x), \quad x \in [0, 2\pi], \quad u(0) = \alpha, \quad u'(0) = \beta. \tag{6.2}$$

This equation describes the damped oscillations of a rigid pendulum that rotates on a pivot subject to a uniform gravitational force in the vertical direction. The unknown function $u(x)$ measures the angle of the pendulum from the vertical. The constant $m > 0$ is the mass of the pendulum bob, $c > 0$

is the coefficient of friction, assumed here to be strictly positive, and $k > 0$ represents the gravitational force. We note that (6.2) can be written as (1.1), where

$$g(x, u) = cu, \quad f(x, u) = r(x) - k\sin(u).$$

We choose $m = c = k = 1$ and we select $r(x)$ and the initial conditions such that $u(x) = \sin(x)$, i.e.,

$$r(x) = -\sin(x) + \cos(x) + \sin(\sin(x)), \quad \alpha = 0, \quad \beta = 1.$$

We solve problem (6.2) using the UWDG scheme presented in Section 2. We employ a uniform partition with $N = 6, 8, \dots, 20$ elements. The UWDG solution $u_h \in V_h^p$ is computed using finite element spaces V_h^p with $p = 2, 3, 4, 5$. The left figure of Figure 4 shows the L^2-norm of the errors and their corresponding orders of accuracy. We observe that the proposed UWDG scheme achieves optimal $(p+1)$-th-order accuracy for this problem. In the right figure of Figure 4, we present the errors $\|\xi_u\|$ and their convergence orders. We observe that $\|\xi_u\| = O(h^{p+2})$. The computed errors are consistent with the theoretical estimates. Figure 5 displays the errors $\|\bar{e}_u\|^*$ and $\left\|\bar{e}_u'\right\|^*$. We observe that $\|\bar{e}_u\|^* = O(h^{2p})$ and $\left\|\bar{e}_u'\right\|^* = O(h^{2p})$. These results are consistent with the theoretical estimates.

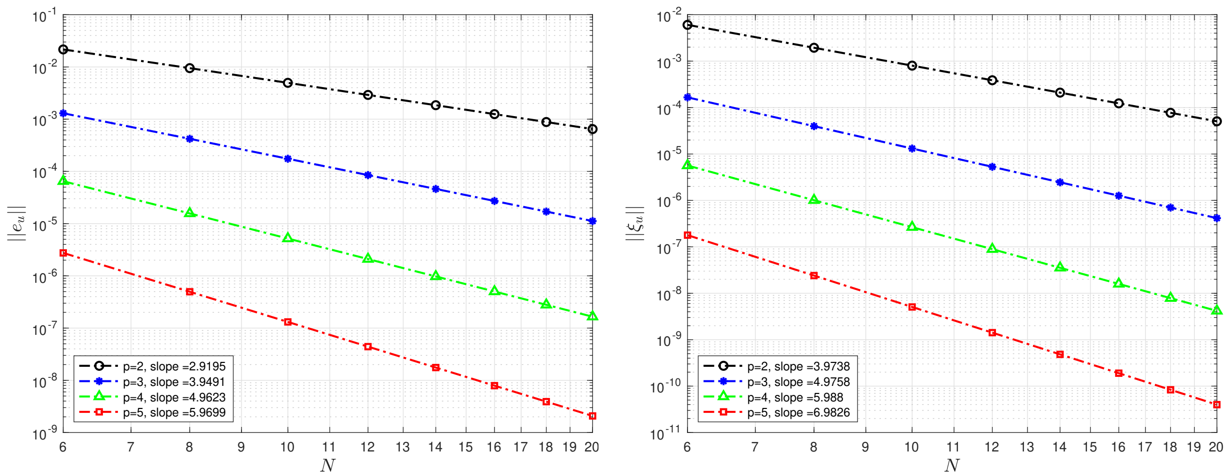

Figure 4. Convergence rates for $\|e_u\|$ (left) and $\|\xi_u\|$ (right) for the IVP (6.2) on uniform partitions having $N = 6, 8, \dots, 20$ elements using $p = 2$–5.

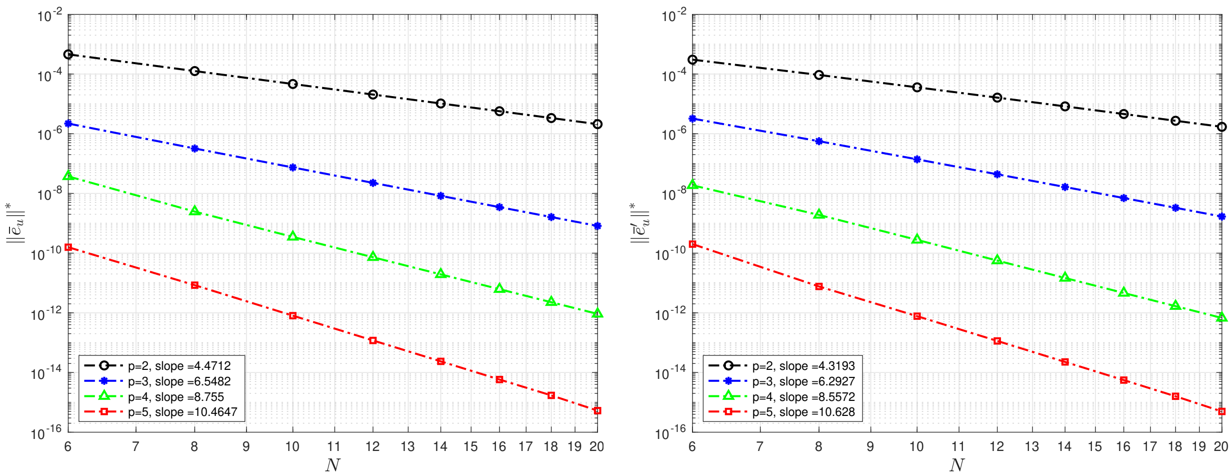

Figure 5. Convergence rates for $||\bar{e}_u||^*$ (left) and $\left\|\bar{e}'_u\right\|_\infty^*$ (right) for the IVP (6.2) on uniform partitions having $N = 6, 8, \ldots, 20$ elements using $p = 2$–5.

Example 6.3. In this example, we consider the following problem:

$$u'' + \frac{1}{8}uu' = \frac{x}{4}u, \quad x \in [1, 3], \quad u(1) = 17, \quad u'(1) = -14. \tag{6.3}$$

The exact solution is given by $u(x) = x^2 + \frac{16}{x}$. We note that (6.3) can be written as (1.1), where

$$g(x, u) = \frac{1}{16}u^2, \quad f(x, u) = \frac{x}{4}u.$$

We utilize the UWDG scheme presented in Section 2 to solve (6.3) on a uniform partition with $N = 4, 5, \ldots, 12$ elements. The UWDG solution $u_h \in V_h^p$ is computed for $p = 2, 3, 4, 5$. The L^2-norm of the errors and their corresponding orders of accuracy are depicted in the left figure of Figure 6. It is observed that the convergence orders match the optimal predictions derived from (4.9). In the right figure of Figure 6, the errors $||\xi_u||$ and their convergence orders are presented. Consistent with the conclusions of Theorem 5.1, it is observed that $||\xi_u|| = O(h^{p+2})$ for $p = 2-5$. These findings align with the theoretical predictions. The errors $||\bar{e}_u||^*$ and $\left\|\bar{e}'_u\right\|^*$ are presented in Figure 7. It can be observed that $||\bar{e}_u||^* = O(h^{2p})$ and $\left\|\bar{e}'_u\right\|^* = O(h^{2p})$. These results are consistent with the theoretical findings. In summary, the numerical results validate the optimal convergence and superconvergence rates predicted by the theoretical analysis.

Remark 6.1. Numerous numerical schemes have been developed in the literature for solving higher-order IVPs. Typically, such problems can be reformulated as systems of first-order ODEs, for which a variety of numerical methods have been proposed. Among these, Taylor and Runge-Kutta methods are the most prominent. Taylor methods are derived by truncating the Taylor series expansions; however, their practical implementation is often hindered by the need to evaluate multiple higher-order derivatives. In contrast, Runge-Kutta methods avoid the explicit computation of derivatives and are therefore among the most widely used schemes due to their simplicity, relatively high accuracy, and broad applicability. Nevertheless, high-order Runge-Kutta methods are known to suffer from stringent stability-imposed time-step restrictions when applied to stiff problems. In this study, we proposed a UWDG method for solving problem (1.1), which offers several notable advantages over classical numerical approaches. In particular, the method exhibits strong superconvergence properties that can be exploited for a posteriori error estimation, supports natural h-adaptivity, and can be systematically extended to PDEs.

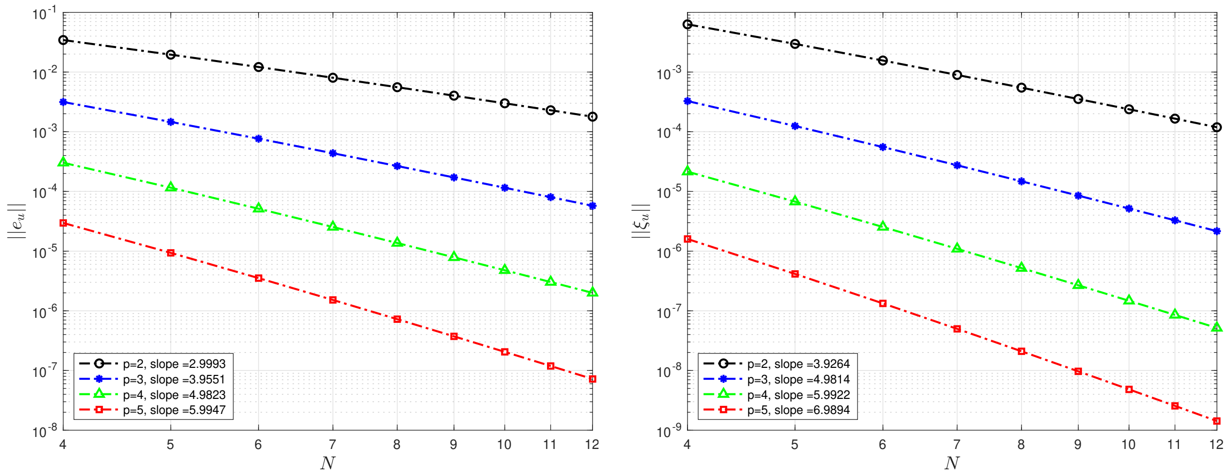

Figure 6. Convergence rates for $\|e_u\|$ (left) and $\|\xi_u\|$ (right) for the IVP (6.3) on uniform partitions having $N = 4, 5, \ldots, 12$ elements using $p = 2$–5.

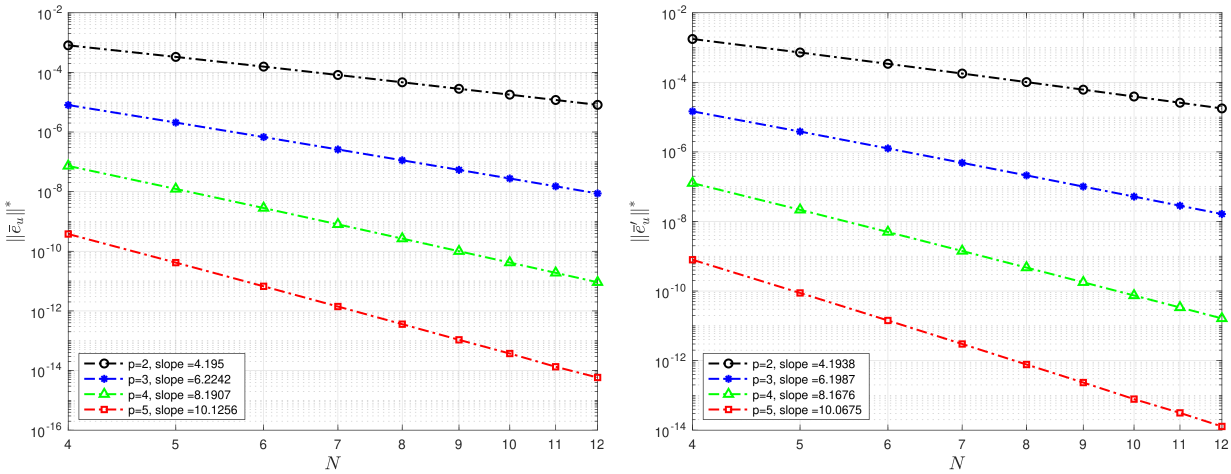

Figure 7. Convergence rates for $\|\bar{e}_u\|^*$ (left) and $\left\|\bar{e}'_u\right\|^*_\infty$ (right) for the IVP (6.3) on uniform partitions having $N = 4, 5, \ldots, 12$ elements using $p = 2$–5.

Remark 6.2. The proposed UWDG scheme offers several advantages over the standard methods. The main advantages of the UWDG method are as follows:

a) One key advantage of our method over the classical DG method [35] is the elimination of the auxiliary variable, resulting in reduced memory and computational costs. In the classical DG method, to solve our second-order IVP, it is necessary to transform the second-order ODE into a first-order system of ODEs before applying the DG method. As a result, the classical DG method requires twice the number of degrees of freedom compared to our proposed scheme. By avoiding the introduction of the auxiliary variable and the subsequent conversion to a first-order system, we effectively reduce memory and computational costs by a factor of two.
b) In comparison to other DG methods, our approach ensures optimal convergence and superconvergence without the need for internal penalty terms. Specifically, our method offers a distinct advantage over the direct DG (DDG) method proposed in [11] as it achieves optimal convergence and superconvergence without the requirement of internal penalty terms.
c) Another advantage of the proposed UWDG scheme over the classical finite element method is its ability to achieve superconvergence toward the projection $P_h^- u$. This supercloseness property is highly significant in constructing *a posteriori* error estimates by solving local problems on individual elements. The details and implications of this aspect will be explored in a separate paper.
d) While our error analysis is focused on second-order IVPs, it can be easily extended to higher-order IVPs and boundary value problems (BVPs). For instance, the error analysis presented for second-order IVPs can be extended to higher-order IVPs by rewriting the higher-order problem as a system of second-order equations and applying the same UWDG framework. The main ideas and estimates remain analogous, with appropriate modifications to account for the increased system size.
e) Extension of our error analysis to higher-order BVPs is possible, but it involves the introduction of new projections and modifications to the numerical fluxes. Our research aims to ultimately extend the method to solve two-dimensional (2-D) and three-dimensional (3-D) elliptic, parabolic, and hyperbolic problems. These extensions will be addressed in a series of papers, where we will explore the convergence, superconvergence, and *a posteriori* error estimation of each scheme.

These advantages make the UWDG scheme a powerful and versatile approach for numerical simulations of partial differential equations.

7. Concluding remarks

In this paper, we investigated the convergence and superconvergence properties of an ultra-weak discontinuous Galerkin (UWDG) method applied to nonlinear second-order initial-value problems (IVPs) for ordinary differential equations (ODEs) of the form $u'' + (g(x, u))' = f(x, u)$. We established optimal L^2 error estimates for the UWDG solution when suitable numerical fluxes are employed. Specifically, we proved that the UWDG solution u_h achieves $(p + 1)$-th-order convergence in the L^2-norm when employing piecewise polynomials of degree up to $p \geq 2$. Additionally, we established a supercloseness result in the L^2-norm toward a special projection of the exact solution. More precisely, we proved that the UWDG solution exhibits superconvergence of order $p + 2$ toward the special projection $P_h^- u$ of the exact solution. Furthermore, we demonstrated that the p-degree

UWDG solution and its derivative are superconvergent with an error of $O(h^{2p})$ at the end of each step. Numerical experiments confirmed the sharpness of the theoretical results, and our proofs hold for general partitions and piecewise polynomials of degree $p \geq 2$. While our error analysis focused on second-order IVPs, it can be readily extended to higher-order IVPs and boundary-value problems (BVPs). We are currently investigating the superconvergence properties and the asymptotic exactness of a posteriori error estimators for UWDG methods applied to two-dimensional elliptic, parabolic, and hyperbolic problems on both rectangular and triangular meshes. Future work will focus on extending this a posteriori error analysis to three-dimensional problems defined on tetrahedral meshes.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Mahboub Baccouch is the Guest Editor of special issue "Numerical methods for partial differential equations: recent developments, analysis, and applications" for the Electronic Research Archive and was not involved in the editorial review or the decision to publish this article. The author declares that there are no competing interests.

References

1. W. H. Reed, T. R. Hill, *Triangular Mesh Methods for the Neutron Transport Equation*, Los Alamos Scientific Laboratory, Los Alamos, 1973.
2. B. Cockburn, G. E. Karniadakis, C. W. Shu, *Discontinuous Galerkin Methods–Theory, Computation and Applications*, Springer, Berlin, 2000.
3. D. Di Pietro, A. Ern, *Mathematical Aspects of Discontinuous Galerkin Methods*, Springer Berlin Heidelberg, 2011.
4. X. Feng, O. Karakashian, Y. Xing, *Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations*, Springer, 2014. https://doi.org/10.1007/978-3-319-01818-8
5. J. Hesthaven, T. Warburton, *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*, Texts in Applied Mathematics, Springer New York, 2008. https://doi.org/10.1007/978-0-387-72067-8
6. B. Rivière, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*, SIAM, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2008. https://doi.org/10.1137/1.9780898717440
7. M. Baccouch, A *posteriori* error estimates and adaptivity for the discontinuous Galerkin solutions of nonlinear second-order initial-value problems, *Appl. Numer. Math.*, **121** (2017), 18–37. https://doi.org/10.1016/j.apnum.2017.06.001

8. M. Baccouch, A superconvergent local discontinuous Galerkin method for nonlinear two-point boundary-value problems, *Numer. Algorithms*, **79** (2018), 697–718. https://doi.org/10.1007/s11075-017-0456-0

9. M. Baccouch, Analysis of optimal superconvergence of a local discontinuous Galerkin method for nonlinear second-order two-point boundary-value problems, *Appl. Numer. Math.*, **145** (2019), 361–383. https://doi.org/10.1016/j.apnum.2019.05.003

10. M. Baccouch, An adaptive local discontinuous Galerkin method for nonlinear two-point boundary-value problems, *Numer. Algorithms*, **84** (2020), 1121–1153. https://doi.org/10.1007/s11075-019-00794-8

11. H. Liu, J. Yan, The direct discontinuous Galerkin (DDG) methods for diffusion problems, *SIAM J. Numer. Anal.*, **47** (2009), 675–698. https://doi.org/10.1137/080720255

12. O. Cessenat, B. Despres, Application of an ultra-weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz problem, *SIAM J. Numer. Anal.*, **35** (1998), 255–299. https://doi.org/10.1137/S0036142995285873

13. Y. Cheng, C. W. Shu, A discontinuous Galerkin finite element method for time dependent partial differential equations with higher order derivatives, *Math. Comput.*, **77** (2008), 699–730. Available from: https://www.jstor.org/stable/40234530.

14. S. Adjerid, H. Temimi, A discontinuous Galerkin method for higher-order ordinary differential equations, *Comput. Methods Appl. Mech. Eng.*, **197** (2007), 202–218. https://doi.org/10.1016/j.cma.2007.07.015

15. M. Baccouch, H. Temimi, Analysis of optimal error estimates and superconvergence of the discontinuous Galerkin method for convection-diffusion problems in one space dimension, *Int. J. Numer. Anal. Model.*, **13** (2016), 403–434.

16. M. Baccouch, Superconvergence of an ultra-weak discontinuous Galerkin method for nonlinear second-order initial-value problems, *Int. J. Comput. Methods*, **20** (2023), 2250042. https://doi.org/10.1142/S0219876222500426

17. M. Baccouch, A superconvergent ultra-weak discontinuous Galerkin method for nonlinear second-order two-point boundary-value problems, *J. Appl. Math. Comput.*, **69** (2023), 1507–1539. https://doi.org/10.1007/s12190-022-01803-1

18. M. Baccouch, A superconvergent ultra-weak local discontinuous Galerkin method for nonlinear fourth-order boundary-value problems, *Numer. Algorithms*, **92** (2023), 1983–2023. https://doi.org/10.1007/s11075-022-01374-z

19. H. Bi, C. Hu, F. Zhao, F. Fu, Optimal error estimates of the ultra weak local discontinuous Galerkin method for nonlinear sixth-order boundary value problems, *Math. Comput. Simul.*, **239** (2026), 96–114. https://doi.org/10.1016/j.matcom.2025.04.040

20. A. Chen, Y. Cheng, Y. Liu, M. Zhang, Superconvergence of ultra-weak discontinuous Galerkin methods for the linear Schrödinger equation in one dimension, *J. Sci. Comput.*, **82** (2020), 22. https://doi.org/10.1007/s10915-020-01124-0

21. A. Chen, F. Li, Y. Cheng, An ultra-weak discontinuous Galerkin method for Schrödinger equation in one dimension, *J. Sci. Comput.*, **78** (2019), 772–815. https://doi.org/10.1007/s10915-018-0789-4

22. X. Chen, Y. Chen, The ultra-weak discontinuous Galerkin method for time-fractional Burgers equation, *J. Anal.*, **33** (2025), 795–817. https://doi.org/10.1007/s41478-024-00862-w

23. Y. Chen, Y. Xing, Optimal error estimates of ultra-weak discontinuous Galerkin methods with generalized numerical fluxes for multi-dimensional convection-diffusion and biharmonic equations, *Math. Comput.*, **93** (2024), 2135–2183. https://doi.org/10.1090/mcom/3927

24. Y. Liao, L. B. Liu, X. Luo, G. Long, A crank–nicolson ultra-weak discontinuous Galerkin method for solving a unsteady singularly perturbed problem with a shift in space, *Appl. Math. Lett.*, **172** (2026), 109736. https://doi.org/10.1016/j.aml.2025.109736

25. H. Wang, A. Xu, Q. Tao, Analysis of the implicit-explicit ultra-weak discontinuous Galerkin method for convection-diffusion problems, *J. Comput. Math.*, **42** (2024), 1–23. https://doi.org/10.4208/jcm.2202-m2021-0290

26. G. Fu, C. W. Shu, An energy-conserving ultra-weak discontinuous Galerkin method for the generalized Korteweg–de Vries equation, *J. Comput. Appl. Math.*, **349** (2019), 41–51. https://doi.org/10.1016/j.cam.2018.09.021

27. J. Huang, Y. Liu, Y. Liu, Z. Tao, Y. Cheng, A class of adaptive multiresolution ultra-weak discontinuous Galerkin methods for some nonlinear dispersive wave equations, *SIAM J. Sci. Comput.*, **44** (2022), A745–A769. https://doi.org/10.1137/21M1411391

28. Y. Li, C. W. Shu, S. Tang, An ultra-weak discontinuous Galerkin method with implicit–explicit time-marching for generalized stochastic KdV equations, *J. Sci. Comput.*, **82** (2020), 61. https://doi.org/10.1007/s10915-020-01162-8

29. Y. Liu, Q. Tao, C. Shu, Analysis of optimal superconvergence of an ultraweak-local discontinuous Galerkin method for a time dependent fourth-order equation, *ESAIM. Math. Model. Numer. Anal.*, **54** (2020), 1797–1820. https://doi.org/10.1051/m2an/2020023

30. Q. Tao, W. Cao, Z. Zhang, Superconvergence analysis of the ultra-weak local discontinuous Galerkin method for one dimensional linear fifth order equations, *J. Sci. Comput.*, **88** (2021), 63. https://doi.org/10.1007/s10915-021-01579-9

31. Q. Tao, Y. Xu, C. W. Shu, An ultraweak-local discontinuous Galerkin method for PDEs with high order spatial derivatives, *Math. Comput.*, **89** (2020), 2753–2783. https://doi.org/10.1090/mcom/3562

32. Z. Xue, F. Yan, Y. Xia, Exactly divergence-free ultra-weak discontinuous Galerkin method for Brinkman–Forchheimer equations, *J. Comput. Appl. Math.*, **477** (2026), 117124. https://doi.org/10.1016/j.cam.2025.117124

33. L. Zhou, W. Chen, R. Guo, Stability and error estimates of ultra-weak local discontinuous Galerkin method with spectral deferred correction time-marching for PDEs with high order spatial derivatives, *J. Comput. Math.*, (2024), 1–21. https://doi.org/10.4208/jcm.2410-m2024-0092

34. M. Delfour, W. Hager, F. Trochu, Discontinuous Galerkin methods for ordinary differential equations, *Math. Comput.*, **36** (1981), 455–473. https://doi.org/10.1090/S0025-5718-1981-0606506-0

35. M. Baccouch, Superconvergence of the discontinuous Galerkin method for nonlinear second-order initial-value problems for ordinary differential equations, *Appl. Numer. Math.*, **115** (2017), 160–179. https://doi.org/10.1016/j.apnum.2017.01.007

AIMS Press