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http://www.aimspress.com/journal/era

ERA, 33(11): 6952–6970.
DOI: 10.3934/era.2025306
Received: 24 June 2025
Revised: 17 October 2025
Accepted: 04 November 2025
Published: 18 November 2025

*Research article*

# The symmetric and asymmetric version of Goursat's Lemma

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**Abstract:** Goursat's lemma gives a good method to describe subgroups of the direct product of two groups $G_1, G_2$, and to determine whether subgroups of $G_1 \times G_2$ are direct products. However, the usual symmetric version of Goursat's lemma is difficult to describe subgroups of a direct product of a finite number of groups. Fortunately, the asymmetric version of Goursat's lemma provide a new method to solve the difficulty. In this paper, we used additional conditions $\pi_i(H) = G_i$, the injectivity $\rho_i$, and $H_{i22} = H_{i12} \cap H_{i21}$ for $1 \leq i \leq 3$ to give some related results about groups (resp. $R$-modules, $R$-algebras (rings as corollary)), and then we gave the answer on whether a $R$-submodule $M$ of $M_1 \times \cdots \times M_n$ has the form $M = \widetilde{N} \times N_n$ and $M = N_1 \times \cdots \times N_n$. Further, we extended similar conclusions to $R$-algebras (rings as corollary).

**Keywords:** Goursat's lemma; $R$-modules; $R$-algebras; direct product

## 1. Introduction

Let $G_1$ and $G_2$ be two groups. Goursat's lemma for groups first appeared in 1889 [1], which describes subgroups of the direct product of two groups $G_1, G_2$, and involves isomorphisms among quotient groups of subgroups of $G_1$ and $G_2$. Tóth [2] obtained a simple representation and the invariant factor decompositions of the subgroups of the group $\mathbb{Z}_n \times \mathbb{Z}_m$ by Goursat's lemma for groups. Petrillo [3] used Goursat's lemma as the main tool to solve the problem of the total number of subgroups of the group $\mathbb{Z}_n \times \mathbb{Z}_m$ for all positive integers $m$ and $n$ and used Goursat's lemma to calculate the number of subgroups of the direct product of two finite cyclic groups.

The subgroup (resp. subring) of the form $H_1 \times H_2$ (resp. $T_1 \times T_2$) is called a subproduct of $G_1 \times G_2$ (resp. $R_1 \times R_2$), where $G_1$ and $G_2$ (resp. $R_1$ and $R_2$) are groups (resp. rings). Anderson and Camillo [4] extended Goursat's lemma from groups to rings; that is, a ring version of Goursat's lemma, giving the subrings and ideals of a direct product of two rings, and they consider a question: Whether every subgroup (resp. subring) of $G_1 \times G_2$ (resp. $R_1 \times R_2$) is a subproduct of $G_1 \times G_2$ (resp. $R_1 \times R_2$)? The answer is clearly not, and the authors [4] provide a counterexample: $\mathbb{Z}_2 \times \mathbb{Z}_2$ with normal subgroup $\{(\bar{0}, \bar{0}), (\bar{1}, \bar{1})\}$. The authors in [4] stated Goursat's lemma for $R$-modules without proof (also see [5, 6]). Meng and Guo [7] provided a proof of Goursat's lemma for $R$-modules, and extended Goursat's lemma to $R$-algebras, which characterizes subalgebras of the direct product of two $R$-algebras. The generalization of Goursat's lemma to the direct products of $n$ groups ($n > 2$) leads to a very complicated situation, and the asymmetric version of Goursat's lemma [8] provides a good method to solve this complicated problem.

If $A$ and $B$ are groups, then the direct product of $A$ and $B$ is a group. Since $A \times B \times C \approx (A \times B) \times C$, we can give the asymmetric version of Goursat's lemma for $A \times B \times C$ by giving the asymmetric version of Goursat's lemma of $A \times B$ first and then give the asymmetric version of Goursat's lemma of $A \times B \times C$ by putting $A \times B$ as a whole. Using the same method, we can give the asymmetric version of Goursat's lemma for $n$ groups. Please see the generalization of subgroups of a direct product of a finite number of groups, which involves homomorphisms between subgroups and quotient groups [8]. Consequently, Mbarga [9] generalized Goursat's lemma to submodules of a direct product of a finite number of $R$-modules, which contains homomorphisms between modules and quotient $R$-modules. In [8, Section 4], the authors stated that the subgroup $H \leqslant G_1 \times \cdots \times G_n$ has the form $H = H_1 \times \cdots \times H_n$ if and only if $\theta_i$ is trivial for $1 \leq i \leq n-1$. The motivation of this article is the question: Whether all $R$-submodules $M$ of a direct product $M_1 \times \cdots \times M_n$ of a finite number of $R$-modules $M_1, \ldots, M_n$ can be written as $\widetilde{N} \times N_n$ and $N_1 \times \cdots \times N_n$, where $\widetilde{N}$ is a $R$-submodule of $M_1 \times \cdots \times M_{n-1}$? The answer is obviously not, but how do we know that the $R$-submodule $M$ of a direct product $M_1 \times \cdots \times M_n$ of a finite number of $R$-moudles $M_1, \ldots, M_n$ can be written as $\widetilde{N} \times N_n$ and $N_1 \times \cdots \times N_n$ is our primary aim. In Theorem 3.9 and Corollary 3.10, we give the answer to this question. We also solve a similar problem for $R$-algebras by the asymmetric version of Goursat's lemma (rings as corollary), which we give in Theorem 3.20 and Corollary 3.21 (Corollaries 3.25 and 3.26 for rings), respectively.

This paper is organized as follows. In Section 2, we introduce the symmetric and asymmetric version of Goursat's lemma for groups, and use additional conditions $\pi_i(H) = G_i$, the injectivity $\rho_i$, and $H_{i22} = H_{i12} \cap H_{i21}$ for $1 \leq i \leq 3$ to give some related results for groups. In Section 3, we use similar additional conditions as in Section 2 to give some related results for $R$-modules and $R$-algebras (rings as corollary). Further, we also discuss whether submodules of $R$-modules and subalgebras of $R$-algebras (rings as corollary) can be expressed as a direct product of a finite number of $R$-submodules and $R$-subalgebras, respectively.

## 2. Goursat's lemma for groups

Anderson and Camillo [4] stated that every subgroup of $G_1 \times G_2$ is of the form $H_1 \times H_2$ (called subproduct) if and only if $g_i \in G_i$, $g_i$ has finite order $o(g_i)$ for $i = 1, 2$ and $\gcd(o(g_1), o(g_2)) = 1$, where $G_1$ and $G_2$ are nontrivial groups. Furthermore, they stated that every subring of $R_1 \times R_2$ with identity $(1, 1)$ is a subproduct of $R_1 \times R_2$ if and only if each $R_i$ has nonzero characteristic $\mathrm{char}R_i$ and $\gcd(\mathrm{char}R_1, \mathrm{char}R_2) = 1$. In this section, we first introduce the symmetric versions of Goursat's lemma for groups, and use additional conditions $\pi_i(H) = G_i$, the injectivity $\rho_i$, and $H_{i22} = H_{i12} \cap H_{i21}$ for $1 \leq i \leq 3$ to obtain some related results, and then we use the asymmetric version of Goursat's lemma of groups to state that the subgroup $H$ of $G_1 \times \cdots \times G_n$ can be expressed as $H = \widetilde{H} \times H_n$ and $H = H_1 \times \cdots \times H_n$, where $\widetilde{H}$ is a subgroup of $G_1 \times \cdots \times G_{n-1}$.

*2.1. Symmetric version of Goursat's lemma for groups*

Let $G_1$ and $G_2$ be groups, and $H$ be a subgroup of $G_1 \times G_2$. The identity element of each group $G_i$, with slight abuse of notation, is written as "$e$". Let

$$\begin{aligned} H_{11} &= \{a \in G_1 | (a, e) \in H\}, \\ H_{12} &= \{a \in G_1 | (a, b) \in H \text{ for some } b \in G_2\}, \\ H_{21} &= \{b \in G_2 | (e, b) \in H\}, \\ H_{22} &= \{b \in G_2 | (a, b) \in H \text{ for some } a \in G_1\}. \end{aligned}$$

**Theorem 2.1** (Symmetric version of Goursat's lemma for 2 groups, [4, Theorem 4]). *Let $G_1, G_2$ be groups.*

1) *Let $H$ be a subgroup of $G_1 \times G_2$, then $H_{i1}, H_{i2}$ are subgroups of $G_i$ with $H_{i1} \lhd H_{i2}$ for $i = 1, 2$, and the map*

$$f_H : H_{12}/H_{11} \to H_{22}/H_{21}, aH_{11} \mapsto bH_{21}$$

*is an isomorphism, where $(a, b) \in H$. Moreover, if $H \lhd G_1 \times G_2$, then $H_{i1}, H_{i2} \lhd G_i$ and $H_{i2}/H_{i1} \subseteq C(G_i/H_{i1})$, the center of $G_i/H_{i1}$.*

2) *Let $H_{i1}, H_{i2}$ be subgroups of $G_i$ with $H_{i1} \lhd H_{i2}$ for $i = 1, 2$ and let $f : H_{12}/H_{11} \to H_{22}/H_{21}$ be an isomorphism. Then*

$$H = \{(a, b) \in H_{12} \times H_{22} | f(aH_{11}) = bH_{21}\}$$

*is a subgroup of $G_1 \times G_2$. Furthermore, suppose $H_{i1}, H_{i2} \lhd G_i$ and $H_{i2}/H_{i1} \subseteq C(G_i/H_{i1})$ for $i = 1, 2$, then $H \lhd G_1 \times G_2$.*

3) *The constructions given in* 1) *and* 2) *are inverses to each other.*

Similarly, Bauer et al. [8] used Goursat quintuples to express Theorem 2.1 in a simpler form, which is given as follows.

**Lemma 2.2** (Symmetric version of Goursat's lemma for 2 groups, [8, Theorem 2.1]). *There is a bijective correspondence between subgroups $H$ of $G_1 \times G_2$ and quintuples $\{H_{11}, H_{12}, H_{21}, H_{22}, \theta\}$, where $H_{i1} \lhd H_{i2} \leqslant G_i$ for $i = 1, 2$, and the map $\theta : H_{12}/H_{11} \to H_{22}/H_{21}$ given by $aH_{11} \mapsto bH_{21}$ for $(a, b) \in H$ is an isomorphism.*

Subsequently, for the case $G_1 \times G_2 \times G_3$, $H \leqslant G_1 \times G_2 \times G_3$, we should consider 12 subgroups of $G_1, G_2, G_3$ as follows:

$$\begin{aligned} H_{111} &= \{a \in G_1 | (a, b, c) \in H \text{ for some } b \in G_2, c \in G_3\}, \\ H_{112} &= \{a \in G_1 | (a, b, e) \in H \text{ for some } b \in G_2\}, \\ H_{121} &= \{a \in G_1 | (a, e, c) \in H \text{ for some } c \in G_3\}, \\ H_{122} &= \{a \in G_1 | (a, e, e) \in H\}; \\ H_{211} &= \{b \in G_2 | (a, b, c) \in H \text{ for some } a \in G_1, c \in G_3\}, \\ H_{212} &= \{b \in G_2 | (a, b, e) \in H \text{ for some } a \in G_1\}, \\ H_{221} &= \{b \in G_2 | (e, b, c) \in H \text{ for some } c \in G_3\}, \\ H_{222} &= \{b \in G_2 | (e, b, e) \in H\}; \end{aligned}$$

$$H_{311} = \{c \in G_3 | (a, b, c) \in H \text{ for some } a \in G_1, b \in G_2\},$$
$$H_{312} = \{c \in G_3 | (a, e, c) \in H \text{ for some } a \in G_1\},$$
$$H_{321} = \{c \in G_3 | (e, b, c) \in H \text{ for some } b \in G_2\},$$
$$H_{322} = \{c \in G_2 | (e, e, c) \in H\}.$$

*Remark* 2.3. Note that if $G_3 = \{e\}$, then $H_{111} = H_{112}, H_{121} = H_{122}$, $H_{211} = H_{212}, H_{221} = H_{222}$, $H_{311} = H_{312}$, and $H_{321} = H_{322}$.

According to Theorem 2.1, we know that Goursat's lemma for $G_1 \times G_2$ has a clean and well-known formulation. However, unlike the case $G_1 \times G_2$, the structure of a direct product $G_1 \times G_2 \times G_3$ is more complex and may not always provide a simple classification. As authors argued in [8, Section 5], a symmetric version of Goursat's lemma for the case $G_1 \times G_2 \times G_3$ is most likely impossible. The reason is that $H_{122} \neq H_{112} \cap H_{121}$, and similar for $H_{222}, H_{322}$.

Let $\pi_i : G_1 \times G_2 \times G_3 \twoheadrightarrow G_i$ be the standard projection onto the $i$th factor and $H \leqslant G_1 \times G_2 \times G_3$ such that $\pi_i(H) = G_i$ for $1 \leq i \leq 3$. This means that for any $a \in G_1$, there exist $b \in G_2, c \in G_3$, such that $(a, b, c) \in H$. Define the homomorphism $\rho_1 : H \to G_2 \times G_3, (g_1, g_2, g_3) \mapsto (g_2, g_3)$. Note that $\ker(\rho_1) = \{(e, e, e)\}$. If $\rho_1 : H \to G_2 \times G_3$ is injective, then $H \cong \rho_1(H)$, and for any $(a, b, c), (a', b, c) \in H$, we have $a = a'$. Similar for the homomorphisms $\rho_2 : H \to G_1 \times G_3$ and $\rho_3 : H \to G_1 \times G_2$. In the following, we use the additional conditions $\pi_i(H) = G_i$, the injectivity $\rho_i$, and $H_{i22} = H_{i12} \cap H_{i21}$ $(1 \leq i \leq 3)$ to obtain some related results.

**Theorem 2.4.** *Let $H \leqslant G_1 \times G_2 \times G_3$ satisfying $\pi_i(H) = G_i$ for $1 \leq i \leq 3$. Then*

$$H = \{(a, b, c) \in G_1 \times G_2 \times G_3 | (aH_{122}, bH_{222}, cH_{322}) \in H/(H_{122} \times H_{222} \times H_{322})\}.$$

*Proof.* Let $\Omega := \{(a, b, c) \in G_1 \times G_2 \times G_3 | (aH_{122}, bH_{222}, cH_{322}) \in H/(H_{122} \times H_{222} \times H_{322})\}$ and $N := H_{122} \times H_{222} \times H_{322}$. Since $\pi_i(H) = G_i$, we have $H_{i22} \lhd G_i$ for $1 \leq i \leq 3$. Also, since $N \subseteq H$, so $N \lhd H$.

For any $(h_1, h_2, h_3) \in H$, we have $(h_1H_{122}, h_2H_{222}, h_3H_{322}) \in H/N$, thus $(h_1, h_2, h_3) \in \Omega$. Conversely, for any $(a, b, c) \in \Omega$, i.e., $(aH_{122}, bH_{222}, cH_{322}) \in H/N$, there exists $(h'_1, h'_2, h'_3) \in H$ such that $(aH_{122}, bH_{222}, cH_{322}) = (h'_1H_{122}, h'_2H_{222}, h'_3H_{322})$. This means that $(a, b, c) = (h'_1n_1, h'_2n_2, h'_3n_3)$ for some $(n_1, n_2, n_3) \in N$. Since $N \lhd H$, we have $(a, b, c) \in H$. Therefore, $H = \Omega$.

**Theorem 2.5.** *Let $H \leqslant G_1 \times G_2 \times G_3$ satisfying $\pi_i(H) = G_i$, and $\rho_i$ is injective for $1 \leq i \leq 3$. Let*

$$f_{12} : H_{112} \to H_{212}, a \mapsto b, (a, b, e) \in H,$$
$$f_{13} : H_{121} \to H_{312}, a \mapsto c, (a, e, c) \in H,$$
$$f_{23} : H_{221} \to H_{321}, b \mapsto c, (e, b, c) \in H,$$

*then*

$$f_{12}|_{H_{112} \cap H_{121}} : H_{112} \cap H_{121} \to H_{212} \cap H_{221},$$
$$f_{23}|_{H_{212} \cap H_{221}} : H_{212} \cap H_{221} \to H_{312} \cap H_{321},$$
$$f_{13}|_{H_{112} \cap H_{121}} : H_{112} \cap H_{121} \to H_{312} \cap H_{321},$$

*are isomorphisms and $f_{13}|_{H_{112} \cap H_{121}} = f_{23}|_{H_{212} \cap H_{221}} \circ f_{12}|_{H_{112} \cap H_{121}}$.*

*Proof.* We show only that $f_{12}|_{H_{112}\cap H_{121}} : H_{112} \cap H_{121} \to H_{212} \cap H_{221}$ is an isomorphism. This is similar for $f_{13}|_{H_{112}\cap H_{121}}, f_{23}|_{H_{212}\cap H_{221}}$. Since $\pi_i(H) = G_i$ and $\rho_i$ is injective, so $f_{12}|_{H_{112}\cap H_{121}}$ is well-defined. In fact, for any $a \in H_{112} \cap H_{121}$, there exist unique $b \in H_{212}, c \in H_{312}$, such that $(a, b, e), (a, e, c) \in H$ following the injectivity $\rho_i$ and then $f_{12}(a) = b, f_{13}(a) = c$. Since $(a, b, e) \cdot (a, e, c)^{-1} = (e, b, c^{-1}) \in H$, we have $b \in H_{221}, c \in H_{321}$. Thus, $b \in H_{212} \cap H_{221}$ and $c \in H_{312} \cap H_{321}$.

Suppose that for any $a_1, a_2 \in H_{112} \cap H_{121}$, $f_{12}(a_1) = b_1, f_{12}(a_2) = b_2$, then $(a_1, b_1, e), (a_2, b_2, e) \in H$. Since $H \leqslant G_1 \times G_2 \times G_3$, we have $(a_1, b_1, e) \cdot (a_2, b_2, e) = (a_1a_2, b_1b_2, e) \in H$. This means that $f_{12}(a_1a_2) = b_1b_2 = f_{12}(a_1) \cdot f_{12}(a_2)$. It follows that $f_{12}|_{H_{112}\cap H_{121}}$ is a homomorphism.

Suppose that $f_{12}(a_1) = f_{12}(a_2) = b$ for any $a_1, a_2 \in H_{112} \cap H_{121}$, then $(a_1, b, e), (a_2, b, e) \in H$. Since $\rho_1$ is injective, we have $a_1 = a_2$. Thus, $f_{12}|_{H_{112}\cap H_{121}}$ is injective. For any $b \in H_{212} \cap H_{221}$, we have $(a, b, e) \in H$ for some $a \in H_{112}$ and $(e, b, c) \in H$ for some $c \in H_{321}$. Thus, $(a, b, e)(e, b, c)^{-1} = (a, e, c^{-1}) \in H$, which means that $a \in H_{121}$, and then $a \in H_{112} \cap H_{121}$ and $f_{12}(a) = b$. It tells that $f_{12}|_{H_{112}\cap H_{121}}$ is an isomorphism.

Suppose that $f_{12}(a) = b$ and $f_{13}(a) = c$ for any $a \in H_{112} \cap H_{121}$, then $(a, b, e), (a, e, c) \in H$ and $(a, b, e)(a, e, c)^{-1} = (e, b, c^{-1}) \in H$. It follows that $f_{23}(b) = c^{-1}$. Additionally, following the definition $f_{23}$, if $f_{23}(b) = c$, then $(e, b, c) \in H$. With the injectivity $\rho_i$, we have $c = c^{-1}$. Thus, $f_{13}(a) = f_{23}(f_{12}(a))$ for any $a \in H_{112} \cap H_{121}$. Therefore, $f_{13}|_{H_{112}\cap H_{121}} = f_{23}|_{H_{212}\cap H_{221}} \circ f_{12}|_{H_{112}\cap H_{121}}$.

**Theorem 2.6.** *Let $H \leqslant G_1 \times G_2 \times G_3$, satisfying $\pi_i(H) = G_i$ and $\rho_i$ is injective for $1 \le i \le 3$. Suppose $H_{i12} \cap H_{i21} = H_{i22}$ for $1 \le i \le 3$, then*

$$f_{12} : H_{111}/H_{121} \to H_{211}/H_{221}, g_1H_{121} \mapsto g_2H_{221}, (g_1, g_2, g_3) \in H \text{ for some } g_3 \in G_3,$$
$$f_{23} : H_{211}/H_{212} \to H_{311}/H_{312}, g_2H_{212} \mapsto g_3H_{312}, (g_1, g_2, g_3) \in H \text{ for some } g_1 \in G_1,$$
$$f_{13} : H_{111}/H_{112} \to H_{311}/H_{321}, g_1H_{112} \mapsto g_3H_{321}, (g_1, g_2, g_3) \in H \text{ for some } g_2 \in G_2,$$

*are isomorphisms and*

$$H = \{(g_1, g_2, g_3) \in H_{111}\times H_{211}\times H_{311} | f_{12}(g_1H_{121}) = g_2H_{221}, f_{23}(g_2H_{212}) = g_3H_{312}, f_{13}(g_1H_{112}) = g_3H_{321}\}.$$

*Proof.* Since $H_{i12} \cap H_{i21} = H_{i22}$ for $1 \le i \le 3$, we have $H_{i21} \lhd H_{i11}$ and $H_{i12} \lhd H_{i11}$ for $1 \le i \le 3$. We prove only that $f_{12}$ is an isomorphism. This is similar for $f_{23}, f_{13}$. Suppose that $g_1H_{121} = g_1'H_{121}$ (i.e., $g_1^{-1}g_1' \in H_{121}$) for $(g_1, g_2, g_3), (g_1', g_2', g_3') \in H$, then there exists $(a, e, c) \in H$, such that $g_1^{-1}g_1' = a$, i.e., $g_1' = g_1a$. Thus,

$$(g_1, g_2, g_3)(a, e, c)(g_1', g_2', g_3')^{-1} = (g_1a(g_1')^{-1}, g_2(g_2')^{-1}, g_3c(g_3')^{-1}) = (e, g_2(g_2')^{-1}, g_3c(g_3')^{-1}) \in H.$$

It follows that $g_2H_{221} = g_2'H_{221}$, and then $f_{12}$ is well-defined. Additionally, for any $a, a' \in H_{111}$ with $(a, b, c), (a', b', c') \in H$, we have

$$f_{12}((aH_{121})(a'H_{121})) = f_{12}(aa'H_{121}) = (bH_{221})(b'H_{221}) = f_{12}(aH_{121}) \cdot f_{12}(a'H_{121}),$$

which means that $f_{12}$ is a homomorphism.

For any $bH_{221} \in H_{211}/H_{221}$, there exists $(a, b, c) \in H$ by the fact $\pi_i(H) = G_i$. Thus, $f_{12}(aH_{121}) = bH_{221}$ and then $f_{12}$ is surjective. Consequently, suppose that $f_{12}(g_1H_{121}) = H_{221}$ for $(g_1, g_2, g_3) \in H$, it suffices to prove $g_1 \in H_{121}$ for the injective $f_{12}$. Since $g_2 \in H_{221}$, we have $(e, g_2, c') \in H$. Thus,

$(g_1, g_2, g_3)(e, g_2, c')^{-1} = (g_1, e, g_3(c')^{-1}) \in H$. This means that $g_1 \in H_{121}$ by the injectivity $\rho_i$ and $H_{i12} \cap H_{i21} = H_{i22}$. Therefore, $f_{12}$ is an isomorphism.

Let

$$\Gamma := \{(g_1, g_2, g_3) \in H_{111} \times H_{211} \times H_{311} | f_{12}(g_1 H_{121}) = g_2 H_{221}, f_{23}(g_2 H_{212}) = g_3 H_{312}, f_{13}(g_1 H_{112}) = g_3 H_{321}\},$$

for any $(g_1, g_2, g_3) \in H$, we have $f_{12}(g_1 H_{121}) = g_2 H_{221}, f_{23}(g_2 H_{212}) = g_3 H_{312}, f_{13}(g_1 H_{112}) = g_3 H_{321}$, that is, $(g_1, g_2, g_3) \in \Gamma$. On the other hand, suppose $(g'_1, g'_2, g'_3) \in \Gamma$, i.e.,

$$f_{12}(g'_1 H_{121}) = g'_2 H_{221}, f_{23}(g'_2 H_{212}) = g'_3 H_{312}, f_{13}(g'_1 H_{112}) = g'_3 H_{321},$$

since $\pi_i(H) = G_i$, there exists $(g'_1, b, c) \in H$ such that $bH_{221} = g'_2 H_{221}, cH_{321} = g'_3 H_{321}$. Thus, by the injectivity of $\rho_i$ and $H_{i12} \cap H_{i21} = H_{i22}$, $b = g'_2$ implies $c = g'_3$. We obtain the assertion.

*Remark* 2.7. Note that the condition $H_{i12} \cap H_{i21} = H_{i22}$ is necessary to guarantee that $f_{12}, f_{13}$, and $f_{23}$ are well-defined. For example, let

$$G_1 = G_2 = G_3 = (\mathbb{Z}_2, +) = \{\bar{0}, \bar{1}\}$$

and $H = \{(\bar{0}, \bar{0}, \bar{0}), (\bar{1}, \bar{1}, \bar{0}), (\bar{1}, \bar{0}, \bar{1}), (\bar{0}, \bar{1}, \bar{1})\}$, which is a subgroup of $G_1 \times G_2 \times G_3$. Then

$$H_{111} = \{\bar{0}, \bar{1}\}, H_{122} = \{\bar{0}\}, H_{211} = \{\bar{0}, \bar{1}\}, H_{222} = \{\bar{0}\}, H_{311} = \{\bar{0}, \bar{1}\}, H_{322} = \{\bar{0}\},$$
$$H_{112} = \{\bar{0}, \bar{1}\}, H_{121} = \{\bar{0}, \bar{1}\}, H_{212} = \{\bar{0}, \bar{1}\}, H_{221} = \{\bar{0}, \bar{1}\}, H_{312} = \{\bar{0}, \bar{1}\}, H_{321} = \{\bar{0}, \bar{1}\}.$$

We have $H_{112} \cap H_{121} \neq H_{122}, H_{212} \cap H_{221} \neq H_{222}$, and $H_{111}/H_{121}, H_{211}/H_{221}$, which are trivial. According to the definition $f_{12}$, we have $f_{12}(aH_{121}) = bH_{221}$ for $(a, b, c) \in H$. When $a = 0$, $(0, 0, 0) \in H$, we have $f_{12}(0 + H_{121}) = 0 + H_{221}$; when $a = 1$, $(1, 1, 0) \in H$, we have $f_{12}(1 + H_{121}) = 1 + H_{221}, (1, 1, c) \in H$. Thus $0 + H_{121} = 1 + H_{121}$, but $0 + H_{221} \neq 1 + H_{221}$ since $1 \in H_{221}$ in this case. This means that $f_{12}$ is not well-defined. Furthermore, the conditions $H_{i12} \cap H_{i21} = H_{i22}$ and $\pi_i(H) = G_i$ for $1 \leq i \leq 3$ can guarantee the exsitence of $f_{12}, f_{13}$, and $f_{23}$.

### 2.2. *Asymmetric version of Goursat's lemma for groups*

Similarly, for the case $G_1 \times \cdots \times G_n$, we should consider $n(n+1)$ subgroups of $G_i$ ($i = 1, 2, \ldots, n$). The computations are huge and it is also most likely impossible to give a symmetric version of Goursat's lemma. Fortunately, Bauer-Sen-Zvengrowskithe [8] provided a new method, named asymmetric version of Goursat's lemma, to make it workable to give the subgroups of $G_1 \times \cdots \times G_n$.

**Lemma 2.8** (Asymmetric version of Goursat's lemma for 2 groups, [8, Theorem 2.3])**.** *There is a bijective correspondence between subgroups $H$ of $G_1 \times G_2$ and quadruples $\{H_{11}, H_{21}, H_{22}, \theta_1\}$, where $H_{11} \leqslant G_1$, $H_{22} \lhd H_{21} \leqslant G_2$, and the map $\theta_1 : H_{11} \to H_{21}/H_{22}$ is a surjective homomorphism.*

To generalize the case $G_1 \times G_2$ to the case $G_1 \times \cdots \times G_n$, we need the following notations.

**Definition 2.9** ( [8, Definition 3.1])**.** Let $H$ be a subgroup of $G_1 \times \cdots \times G_n$, where $G_i$ are groups for $1 \leq i \leq n$. Let $S \subsetneq \{1, 2, \ldots, n\} =: [n]$, and $i \in [n] \backslash S$. Then

$$H(i|S) := \{x_i \in G_i | (x_1, \ldots, x_i, \ldots, x_n) \in H \text{ for some } x_j \in G_j,\ 1 \leq j \leq n,\ j \neq i, \text{ with } x_j = e \text{ if } j \in S\}.$$

*Remark* 2.10. If $S = \{2, 3, \ldots, n-1\}$, then

$$H(1|S) = \{x_1 \in G_1 | (x_1, e, \ldots, e, x_n) \in H \text{ for some } x_n \in G_n\}.$$

If $S = \{2, 3, \ldots, n\}$, then

$$H(1|S) = \{x_1 \in G_1 | (x_1, e, \ldots, e) \in H\}.$$

If $S = \emptyset$, then $H(1|\emptyset) = \{x_1 \in G_1 | (x_1, x_2, \ldots, x_n) \in H \text{ for some } x_k \in G_k, 2 \le k \le n\}$. It is clear that the definition of $H(i|S)$ is the extension of $H_{ij}$. In fact, if $n = 2$, then $H(1|\emptyset) = H_{12}, H(1|\{2\}) = H_{11}, H(2|\emptyset) = H_{22}, H(2|\{1\}) = H_{21}$. If $n = 3$, then

$$\begin{aligned}
&H(1|\emptyset) = H_{111}, H(1|\{2\}) = H_{121}, H(1|\{3\}) = H_{112}, H(1|\{2,3\}) = H_{122},\\
&H(2|\emptyset) = H_{211}, H(2|\{1\}) = H_{221}, H(2|\{3\}) = H_{212}, H(2|\{1,3\}) = H_{222},\\
&H(3|\emptyset) = H_{311}, H(3|\{1\}) = H_{321}, H(3|\{2\}) = H_{312}, H(3|\{1,2\}) = H_{322}.
\end{aligned}$$

Note that it is convenient to use $H(i|S)$ to denote the subgroups of $G_i$ for $1 \le i \le n$. For convenience, we omit the brackets { } to denote "$\{\cdots\}$" by "$\cdots$" (similar as in Section 3), for example, we denote $H(2|\{1,3\})$ by $H(2|1,3)$.

Let $\overline{H}_i := H(i|\emptyset)$ for all $i \in [n]$, i.e.,

$$\overline{H}_i := \{x_i \in G_i | (x_1, \ldots, x_i, \ldots, x_n) \in H \text{ for some } x_j \in G_j,\ 1 \le j \le n,\ j \neq i\}.$$

It is obvious that $H(i|S) \subseteq \overline{H}_i$. In [8], the authors defined $\overline{G}_i = \pi_i(H)$, where $\pi_i : G_1 \times \cdots \times G_n \twoheadrightarrow G_i$ is the standard projection onto the $i$th factor and $H \leqslant G_1 \times \cdots \times G_n$. By the definition of $\overline{H}_i$, we also have $\overline{H}_i = \pi_i(H)$. Thus, $\overline{H}_i = \overline{G}_i$. Let $\prod_i : G_1 \times \cdots \times G_n \twoheadrightarrow G_1 \times \cdots \times G_i$ be the standard projection onto the first $i$ factors for $1 \le i \le n$. Note that $\prod_n = \mathrm{id}_{G_1 \times \cdots \times G_n}$.

**Lemma 2.11** (Asymmetric version of Goursat's lemma for $n$ groups with $n \ge 2$, [8, Theorem 3.2])**.** *There is a bijective correspondence between subgroups $H \leqslant G_1 \times \cdots \times G_n$ and $(3n-2)$-tuples*

$$Q_n(H) := \left\{\overline{H}_1, \overline{H}_2, H(2|1), \theta_1, \overline{H}_3, H(3|1,2), \theta_2, \ldots, \overline{H}_n, H(n|1, \ldots, n-1), \theta_{n-1}\right\},$$

*where $\overline{H}_i \leqslant G_i$, $H(i+1|1, \ldots, i) \lhd \overline{H}_{i+1} \leqslant G_{i+1}$ for $1 \le i \le n-1$, and the map $\theta_i : \Lambda_i \twoheadrightarrow \overline{H}_{i+1}/H(i+1|1, \ldots, i)$ is a surjective homomorphism. Here $\Lambda_i \leqslant G_1 \times \cdots \times G_i$, $1 \le i \le n-1$, is defined recursively by setting $\Lambda_1 := \overline{H}_1$ and*

$$\Lambda_{i+1} := \Gamma_2\left(\left\{\Lambda_i, \overline{H}_{i+1}, H(i+1|1, \ldots, i), \theta_i\right\}\right) \leqslant (G_1 \times \cdots \times G_i) \times G_{i+1},$$

*with*

$$\Gamma_2\left(\left\{\Lambda_i, \overline{H}_{i+1}, H(i+1|1, \ldots, i), \theta_i\right\}\right) := p_i^{-1}(\mathcal{G}_{\theta_i}),$$

*where $p_i : \Lambda_i \times \overline{H}_{i+1} \to \Lambda_i \times \left(\overline{H}_{i+1}/H(i+1|1, \ldots, i)\right)$ is the natural surjection and $\mathcal{G}_{\theta_i} \subseteq \Lambda_i \times (\overline{H}_{i+1}/H(i+1|1, \ldots, i))$ is the graph of $\theta_i$.*

*Remark* 2.12. In fact, we have $\Lambda_1 = \overline{H}_1$, $\theta_1 : \Lambda_1 \twoheadrightarrow \overline{H}_2/H(2|1)$, $p_1 : \Lambda_1 \times \overline{H}_2 \to \Lambda_1 \times \overline{H}_2/H(2|1)$. Suppose that $p_1$ is defined by $p_1(a,b) = (a, bH(2|1))$, then $\mathcal{G}_{\theta_1} = \{(a, \theta_1(a)) | a \in \Lambda_1\} \subseteq \Lambda_1 \times \overline{H}_2/H(2|1)$. Thus, $p_1^{-1}(\mathcal{G}_{\theta_1}) = \{(a,b) \in \Lambda_1 \times \overline{H}_2 | p_1(a,b) \in \mathcal{G}_{\theta_1}\} = \{(a,b) \in \overline{H}_1 \times \overline{H}_2 | bH(2|1) = \theta_1(a)\}$. Since $\Lambda_2 = \Gamma_2\left(\left\{\Lambda_1, \overline{H}_2, H(2|1), \theta_1\right\}\right) = p_1^{-1}(\mathcal{G}_{\theta_1})$, we have $\Lambda_2 = \{(a,b) \in \overline{H}_1 \times \overline{H}_2 | bH(2|1) = \theta_1(a)\}$ and $\Lambda_2 \subseteq \overline{H}_1 \times \overline{H}_2$. Following the proof of Theorem 3.2 in [8], we also have $\Lambda_2 = \prod_2(H)$. Subsequently, we have $\Lambda_{n-1} = \prod_{n-1}(H) \subseteq \overline{H}_1 \times \cdots \times \overline{H}_{n-1}$.

In [8, Section 4], the authors stated that the subgroup $H \leqslant G_1 \times \cdots \times G_n$ has the form $H = H_1 \times \cdots \times H_n$ if and only if $\theta_i$ is the trivial homomorphism for $1 \leq i \leq n-1$, where $\theta_i : \prod_i(H) \twoheadrightarrow \overline{H}_{i+1}/H(i+1|1,\ldots,i)$. In fact, if $H = H_1 \times \cdots \times H_n$, then $\prod_i(H) = H_1 \times \cdots \times H_i$ and $\overline{H}_{i+1} = H(i+1|1,\ldots,i) = H_{i+1}$. Thus, $\theta_i$ is trivial homomorphism. Conversely, by induction, we can obtain the assertion.

**Theorem 2.13** ( [8]). *The subgroup $H$ of $G_1 \times \cdots \times G_n$ has the form of $H = \widetilde{H} \times H_n$ if and only if $\widetilde{H} = \ker(\theta_{n-1}) = \prod_{n-1}(H)$ and $H_n = H(n|1,2,\ldots,n-1)$, where $\ker(\theta_{n-1}) \lhd G_1 \times \cdots \times G_{n-1}$, $H(n|1,\ldots,n-1) \lhd \overline{H}_n$, and the map*

$$\theta_{n-1} : \prod\nolimits_{n-1}(H) \twoheadrightarrow \overline{H}_n/H(n|1,\ldots,n-1)$$

*is a surjective homomorphism, $\prod_{n-1} : G_1 \times \cdots \times G_n \twoheadrightarrow G_1 \times \cdots \times G_{n-1}$ is the standard projection.*

Applying Theorem 2.13 and by induction, we can obtain the following Corollary.

**Corollary 2.14** ( [8]). *The subgroup $H$ of $G_1 \times \cdots \times G_n$ has the form of $H = H_1 \times \cdots \times H_n$ if and only if $\prod_i(H) = \ker(\theta_i) = H_1 \times \cdots \times H_i$ and $H_{i+1} = H(i+1|1,\ldots,i)$, where the map*

$$\theta_i : \prod\nolimits_i(H) \twoheadrightarrow \overline{H}_{i+1}/H(i+1|1,\ldots,i)$$

*is a surjective homomorphism for $1 \leq i \leq n-1$.*

## 3. Goursat's lemma for $R$-modules and $R$-algebras

A $R$-module $M$ is also an Aabelian group. A $R$-algebra is not only a $R$-module but also satisfies a scalar multiplication. In this section, we introduce the symmetry and asymmetric version of Goursat's lemma for $R$-modules and $R$-algebras, respectively. We also give the answer to the question: What submodules (resp. subalgebras) of $M_1 \times \cdots \times M_n$ (resp. $A_1 \times \cdots \times A_n$) can be written as $\widetilde{N} \times N_n$ (resp. $\widetilde{B} \times B_n$) and $N_1 \times \cdots \times N_n$ (resp. $B_1 \times \cdots \times B_n$), which we give in Theorem 3.9 (resp. Theorem 3.20) and Corollary 3.10 (resp. Corollary 3.21), respectively.

### *3.1. Symmetric and asymmetric version of Goursat's lemma for R-modules*

When the ring $R$ is commutative with identity and $M$ is a left $R$-module, we can make $M$ into a right $R$-module by defining $mr = rm$ for $m \in M$ and $r \in R$.

Let $R$ be commutative ring with identity, and $M_a$ and $M_b$ be submodules of $R$-modules $M_1$ and $M_2$, respectively. It is obvious that $M_a \times M_b$ is a submodule of $M_1 \times M_2$. However, the reverse is not necessarily true. Let $M_1$ and $M_2$ be $R$-modules and $M$ be a submodule of $M_1 \times M_2$. We write

$$\begin{aligned} M_{11} &= \{a \in M_1 | (a,0) \in M\}, \\ M_{12} &= \{a \in M_1 | (a,b) \in M \text{ for some } b \in M_2\}, \\ M_{21} &= \{b \in M_2 | (0,b) \in M\}, \\ M_{22} &= \{b \in M_2 | (a,b) \in M \text{ for some } a \in M_1\}. \end{aligned}$$

In [10], we know that every submodule $N$ of an $R$-module $M$ is 'normal' in the sense that we always have the quotient module $M/N$.

**Theorem 3.1** (**Symmetric version of Goursat's lemma for** 2 $R$**-modules, [4]**)**.** *Let $R$ be a commutative ring with identity, and $M_1, M_2$ are $R$-modules.*

1) *Let $M$ ba a submodule of $M_1 \times M_2$, then $M_{i1}$ and $M_{i2}$ are submodules of $M_i$ with $M_{i1} \subseteq M_{i2}$ for $i = 1, 2$, and the map*

$$f_M : M_{12}/M_{11} \to M_{22}/M_{21}$$

*given by $f_M(a + M_{11}) = b + M_{21}$ is a $R$-module isomorphism, where $(a, b) \in M$.*

2) Suppose that $M_{i1}$ and $M_{i2}$ are submodules of $M_i$ with $M_{i1} \subseteq M_{i2}$ for $i = 1, 2$, and the map

$$f : M_{12}/M_{11} \to M_{22}/M_{21}$$

is a $R$-module isomorphism, then

$$M = \{(a, b) \in M_{12} \times M_{22} | f(a + M_{11}) = b + M_{21}\}$$

is a submodule of $M_1 \times M_2$.

3) The constructions given in 1) and 2) are inverse to each other.

**Lemma 3.2** (**Symmetric version of Goursat's lemma for** 2 $R$**-modules**, [7, Theorem 3])**.** *Let $R$ be a commutative ring with identity. There is a bijective correspondence between submodule $M$ of $R$-module $M_1 \times M_2$ and quintuples $\{M_{11}, M_{12}, M_{21}, M_{22}, f_M\}$, where $M_{i1}$ is a submodule of $M_i$, $M_{i2}$ is a submodule of $M_{i1}$ for $i = 1, 2$, respectively, and the map $f_M : M_{12}/M_{11} \to M_{22}/M_{21}$ is a $R$-module isomorphism.*

Subsequently, for the case $M_1 \times M_2 \times M_3$, $M$ is a submodule of $M_1 \times M_2 \times M_3$, we should consider 12 submodules of $M_1, M_2, M_3$ as follows:

$$\begin{aligned}
M_{111} &= \{a \in M_1 | (a, b, c) \in M \text{ for some } b \in M_2, c \in M_3\},\\
M_{112} &= \{a \in M_1 | (a, b, 0) \in M \text{ for some } b \in M_2\},\\
M_{121} &= \{a \in M_1 | (a, 0, c) \in M \text{ for some } c \in M_3\},\\
M_{122} &= \{a \in M_1 | (a, 0, 0) \in M\};\\
M_{211} &= \{b \in M_2 | (a, b, c) \in M \text{ for some } a \in M_1, c \in M_3\},\\
M_{212} &= \{b \in M_2 | (a, b, 0) \in M \text{ for some } a \in M_1\},\\
M_{221} &= \{b \in M_2 | (0, b, c) \in M \text{ for some } c \in M_3\},\\
M_{222} &= \{b \in M_2 | (0, b, 0) \in M\};\\
M_{311} &= \{c \in M_3 | (a, b, c) \in M \text{ for some } a \in M_1, b \in M_2\},\\
M_{312} &= \{c \in M_3 | (a, 0, c) \in M \text{ for some } a \in M_1\},\\
M_{321} &= \{c \in M_3 | (0, b, c) \in M \text{ for some } b \in M_2\},\\
M_{322} &= \{c \in M_3 | (0, 0, c) \in M\}.
\end{aligned}$$

Note that if $M_3 = \{0\}$, then $M_{111} = M_{112}$, $M_{121} = M_{122}$, $M_{211} = M_{212}$, $M_{221} = M_{222}$, $M_{311} = M_{312}$, and $M_{321} = M_{322}$.

Similar as the case $G_1 \times G_2 \times G_3$, it is also most likely impossible to give a symmetric version of Goursat's lemma for $M_1 \times M_2 \times M_3$. In the following, we use additional condition to obtain some related results similar to those in Theorems 2.4–2.6.

Let $\pi_i : M_1 \times M_2 \times M_3 \twoheadrightarrow M_i$ be the standard projection onto the $i$th factor and a $R$-submodule $M \subseteq M_1 \times M_2 \times M_3$ such that $\pi_i(M) = M_i$ for $1 \leq i \leq 3$. Define the homomorphism $\eta_1 : M \to M_2 \times M_3, (m_1, m_2, m_3) \mapsto (m_2, m_3)$. If $\eta_1 : M \to M_2 \times M_3$ is injective, then for any $(a, b, c), (a', b, c) \in M$, we have $a = a'$. This is similar to homomorphisms $\eta_2 : M \to M_1 \times M_3$ and $\eta_3 : M \to M_1 \times M_2$.

**Theorem 3.3.** *Let $M \subseteq M_1 \times M_2 \times M_3$ be a $R$-submodule satisfying $\pi_i(M) = M_i$ for $1 \leq i \leq 3$. Then*

$$M = \{(a, b, c) \in M_1 \times M_2 \times M_3 | (a + M_{122}, b + M_{222}, c + M_{322}) \in M/(M_{122} \times M_{222} \times M_{322})\}.$$

*Proof.* Let $\Omega_M := \{(a, b, c) \in M_1 \times M_2 \times M_3 | (a + M_{122}, b + M_{222}, c + M_{322}) \in M/(M_{122} \times M_{222} \times M_{322})\}$ and $N_M := M_{122} \times M_{222} \times M_{322}$. Since $\pi_i(M) = M_i$, we have $M_{i22}$ is a submodule of $M_i$ for $1 \leq i \leq 3$. Also, since $N_M \subseteq M$, so $N_M$ is a submodule of $M$.

For any $(m_1, m_2, m_3) \in M$, we have $(m_1 + M_{122}, m_2 + M_{222}, m_3 + M_{322}) \in M/N_M$. Thus, $(m_1, m_2, m_3) \in \Omega_M$. Conversely, for any $(a, b, c) \in \Omega_M$, i.e., $(a + M_{122}, b + M_{222}, c + M_{322}) \in M/N_M$, there exists $(a', b', c') \in M$, such that $(a + M_{122}, b + M_{222}, c + M_{322}) = (a' + M_{122}, b' + M_{222}, c' + M_{322})$. This means that $(a, b, c) = (a' + x, b' + y, b' + z)$ for some $(x, y, z) \in N_M$. Since $N_M$ is a submodule of $M$, we have $(x, y, z) \in M$ and thus $(a, b, c) \in M$. Therefore, $M = \Omega_M$.

**Theorem 3.4.** *Let $M \subseteq M_1 \times M_2 \times M_3$ be a $R$-submodule satisfying $\pi_i(M) = M_i$, and $\eta_i$ is injective for $1 \leq i \leq 3$. Let*

$$\begin{aligned} f_{12} &: M_{112} \to M_{212}, a \mapsto b, (a, b, 0) \in M, \\ f_{13} &: M_{121} \to M_{312}, a \mapsto c, (a, 0, c) \in M, \\ f_{23} &: M_{221} \to M_{321}, b \mapsto c, (0, b, c) \in M, \end{aligned}$$

*then*

$$\begin{aligned} f_{12}|_{M_{112} \cap M_{121}} &: M_{112} \cap M_{121} \to M_{212} \cap M_{221}, \\ f_{23}|_{M_{212} \cap M_{221}} &: M_{212} \cap M_{221} \to M_{312} \cap M_{321}, \\ f_{13}|_{M_{112} \cap M_{121}} &: M_{112} \cap M_{121} \to M_{312} \cap M_{321}, \end{aligned}$$

*are isomorphisms and $f_{13}|_{M_{112} \cap M_{121}} = f_{23}|_{M_{212} \cap M_{221}} \circ f_{12}|_{M_{112} \cap M_{121}}$.*

*Proof.* We show only that $f_{12}|_{M_{112} \cap M_{121}} : M_{112} \cap M_{121} \to M_{212} \cap M_{221}$ is an isomorphism. This is similar to $f_{13}|_{M_{112} \cap M_{121}}, f_{23}|_{M_{212} \cap M_{221}}$. Since $\pi_i(M) = M_i$ and $\eta_i$ is injective, then $f_{12}|_{M_{112} \cap M_{121}}$ is well-defined. In fact, for any $x \in M_{112} \cap M_{121}$, there exist unique $y \in M_{212}, z \in M_{312}$, such that $(x, y, 0), (x, 0, z) \in M$ following the injectivity $\eta_i$ and then $f_{12}(x) = y, f_{13}(x) = z$. Since $(x, y, 0) + (-x, 0, -z) = (0, y, -z) \in M$, we have $y \in M_{221}, z \in M_{321}$. Thus, $y \in M_{212} \cap M_{221}$ and $z \in M_{312} \cap M_{321}$.

Suppose that for any $x_1, x_2 \in M_{112} \cap M_{121}$, $f_{12}(x_1) = y_1, f_{12}(x_2) = y_2$, then $(x_1, y_1, 0), (x_2, y_2, 0) \in M$. Since $M$ is a submodule of $M_1 \times M_2 \times M_3$, we have $(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0) \in M$. This means that $f_{12}(x_1 + x_2) = y_1 + y_2 = f_{12}(x_1) + f_{12}(x_2)$. Additionally, for any $r \in R$ and $x \in M_{112} \cap M_{121}$, $f_{12}(x) = y$, then $(x, y, 0) \in M$ and $r \cdot (x, y, 0) = (rx, ry, 0) \in M$. Thus, $f_{12}(rx) = ry = rf_{12}(x)$. It follows that $f_{12}|_{M_{112} \cap M_{121}}$ is a homomorphism.

Suppose that $f_{12}(x_1) = f_{12}(x_2) = y$ for any $x_1, x_2 \in M_{112} \cap M_{121}$, then $(x_1, y, 0), (x_2, y, 0) \in M$. Since $\eta_1$ is injective, we have $x_1 = x_2$. Thus, $f_{12}|_{M_{112} \cap M_{121}}$ is injective. Further, for any $y \in M_{212} \cap M_{221}$, we have $(x, y, 0) \in M$ for some $x \in M_{112}$ and $(0, y, z) \in M$ for some $z \in M_{321}$. Thus, $(x, y, 0) + (0, -y, -z) = (x, 0, -z) \in M$, which means that $x \in M_{121}$, and then $x \in M_{112} \cap M_{121}$ and $f_{12}(x) = y$. It tells that $f_{12}|_{M_{112} \cap M_{121}}$ is an isomorphism.

Suppose that $f_{12}(x) = y$ and $f_{13}(x) = z$ for any $x \in M_{112} \cap M_{121}$, then $(x, y, 0), (x, 0, z) \in M$ and $(x, y, 0) + (-x, 0, -z) = (0, y, -z) \in M$. It follows that $f_{23}(y) = -z$. Additionally, following the definition $f_{23}$, if $f_{23}(y) = z$, then $(0, y, z) \in M$. With the injectivity $\eta_i$, we have $z = -z$. Thus, $f_{13}(x) = f_{23}(f_{12}(x))$ for any $x \in M_{112} \cap M_{121}$. Therefore, $f_{13}|_{M_{112} \cap M_{121}} = f_{23}|_{M_{212} \cap M_{221}} \circ f_{12}|_{M_{112} \cap M_{121}}$.

**Theorem 3.5.** *Let $M \subseteq M_1 \times M_2 \times M_3$, satisfying $\pi_i(M) = M_i$ and $\eta_i$ is injective for $1 \leq i \leq 3$. Suppose $M_{i12} \cap M_{i21} = M_{i22}$ for $1 \leq i \leq 3$, then*

$$f_{12} : M_{111}/M_{121} \to M_{211}/M_{221}, m_1 + M_{121} \mapsto m_2 + M_{221}, (m_1, m_2, m_3) \in M \text{ for some } m_3 \in M_3,$$
$$f_{23} : M_{211}/M_{212} \to M_{311}/M_{312}, m_2 + M_{212} \mapsto m_3 + M_{312}, (m_1, m_2, m_3) \in M \text{ for some } m_1 \in M_1,$$
$$f_{13} : M_{111}/M_{112} \to M_{311}/M_{321}, m_1 + M_{112} \mapsto m_3 + M_{321}, (m_1, m_2, m_3) \in M \text{ for some } m_2 \in M_2,$$

*are isomorphisms and*

$$\begin{aligned} M =& \{(m_1, m_2, m_3) \in M_{111} \times M_{211} \times M_{311} | f_{12}(m_1 + M_{121}) = m_2 + M_{221}, \\ & f_{23}(m_2 + M_{212}) = m_3 + M_{312}, f_{13}(m_1 + M_{112}) = m_3 + M_{321}\}. \end{aligned}$$

*Proof.* Since $M_{i12} \cap M_{i21} = M_{i22}$ and $\pi_i(M) = M_i$ for $1 \leq i \leq 3$, we have $M_{i21}$ and $M_{i12}$ are submodules of $M_{i11}$ for $1 \leq i \leq 3$. We only prove that $f_{12}$ is an isomorphism. Similar for $f_{23}, f_{13}$. Suppose that $m_1 + M_{121} = m_1' + M_{121}$ (i.e., $-m_1 + m_1' \in M_{121}$) for $(m_1, m_2, m_3), (m_1', m_2', m_3') \in M$, then there exists $(a, 0, c) \in M$, such that $-m_1 + m_1' = a$, i.e., $m_1' = m_1 + a$. Thus,

$$(m_1, m_2, m_3) + (a, e, c) + (-m_1', -m_2', -m_3') = (0, m_2 - m_2', m_3 + c - m_3') \in M.$$

It follows that $m_2 M_{221} = m_2' M_{221}$ and then $f_{12}$ is well-defined. Additionally, for any $r \in R$ and $a, a' \in M_{111}$ with $(a, b, c), (a', b', c') \in M$, we have

$$f_{12}((a + M_{121}) + (a' + M_{121})) = (b + M_{221}) + (b' + M_{221}) = f_{12}(a + M_{121}) + f_{12}(a' + M_{121}),$$
$$f_{12}(r(a + M_{121})) = f_{12}(ra + M_{121}) = rb + M_{221} = r(b + M_{221}) = r f_{12}(a + M_{121}),$$

which means that $f_{12}$ is a homomorphism.

For any $b + M_{221} \in M_{211}/M_{221}$, there exists $(a, b, c) \in M$ by the fact $\pi_i(M) = M_i$. Thus, $f_{12}(a + M_{121}) = b + M_{221}$ and then $f_{12}$ is surjective. Consequently, suppose that $f_{12}(m_1 + M_{121}) = M_{221}$ for $(m_1, m_2, m_3) \in M$, it suffices to prove $m_1 \in M_{121}$ for the injective $f_{12}$. Since $m_2 \in M_{221}$, we have $(0, m_2, c') \in M$. Thus, $(m_1, m_2, m_3) + (0, -m_2, -c') = (m_1, 0, m_3 - c') \in M$. This means that $m_1 \in M_{121}$ by the injectivity $\eta_i$ and $M_{i12} \cap M_{i21} = M_{i22}$. Therefore, $f_{12}$ is an isomorphism.

Let $\Gamma_M := \{(m_1, m_2, m_3) \in M_{111} \times M_{211} \times M_{311} | f_{12}(m_1 + M_{121}) = m_2 + M_{221}, f_{23}(m_2 + M_{212}) = m_3 + M_{312}, f_{13}(m_1 + M_{112}) = m_3 + M_{321}\}$. For any $(m_1, m_2, m_3) \in M$, we have $f_{12}(m_1 + M_{121}) = m_2 + M_{221}, f_{23}(m_2 + M_{212}) = m_3 + M_{312}, f_{13}(m_1 + M_{112}) = m_3 + M_{321}$, that is, $(m_1, m_2, m_3) \in \Gamma_M$. On the other hand, suppose $(m_1', m_2', m_3') \in \Gamma_M$, i.e., $f_{12}(m_1' + M_{121}) = m_2' + M_{221}, f_{23}(m_2' + M_{212}) = m_3' + M_{312}, f_{13}(m_1' + M_{112}) = m_3' + M_{321}$, since $\pi_i(M) = M_i$, there exists $(m_1', b, c) \in M$, such that $b + M_{221} = m_2' + M_{221}, c + M_{321} = m_3' + M_{321}$. Thus, by the injectivity of $\eta_i$ and $M_{i12} \cap M_{i21} = M_{i22}$, $b = m_2'$ implies $c = m_3'$. We obtain the assertion.

**Lemma 3.6** (**Asymmetric version of Goursat's lemma for 2 $R$-modules**, [9, Lemma 3.3])**.** *Let $R$ be a commutative ring with identity. There is a bijective correspondence between submodule $M$ of $R$-module $M_1 \times M_2$ and quadruples $\{M_{11}, M_{21}, M_{22}, f_M\}$, where $M_{i1}$ is a submodule of $M_i$ for $i = 1, 2$, respectively, $M_{22}$ is a submodule of $M_{21}$ and the map $f_M : M_{11} \to M_{21}/M_{22}$ is a surjective homomorphism.*

Before we extend an asymmetric version of Goursat's lemma for a finite numbers of $R$-modules, we introduce some convenient notation for any submodule $M$ of $R$-module $M_1 \times \cdots \times M_n$.

**Definition 3.7** ( [9, Definition 3.9], [11])**.** Let $R$ be a commutative ring with identity and $M$ be a submodule of $M_1 \times \cdots \times M_n$, where $M_i$ is a $R$-module, $1 \leq i \leq n$. Let $S \subsetneq \{1, 2, \ldots, n\} =: [n]$, and $i \in [n] \backslash S$. Then

$$M(i|S) := \{x_i \in M_i | (x_1, \ldots, x_i, \ldots, x_n) \in M \text{ for some } x_j \in M_j,\ 1 \leq j \leq n,\ j \neq i, \text{with } x_j = 0 \text{ if } j \in S\}.$$

From [11], we know that $M(i|S)$ is a submodule of $M_i$ and $M(i|S)$ is a submodule of $M(i|T)$ if $T \subseteq S$. Let $\overline{M}_i := M(i|\emptyset)$, i.e.,

$$\overline{M}_i := \{x_i \in M_i | (x_1, \ldots, x_i, \ldots, x_n) \in M \text{ for some } x_j \in M_j,\ 1 \leq j \leq n,\ j \neq i\}.$$

**Lemma 3.8** (**Asymmetric version of Goursat's lemma for $n$ $R$-modules with $n \geq 2$**, [9, Lemma 3.10])**.** *Let $R$ be a commutative ring with identity. There is a bijective correspondence between the submodule $M$ of $R$-module $M_1 \times \cdots \times M_n$ and $3(n-2)$-tuples*

$$Q_n(M) := \{\overline{M}_1, \overline{M}_2, M(2|1), \lambda_1, \ldots, \overline{M}_n, M(n|1, \ldots, n-1), \lambda_{n-1}\},$$

*where $M(i+1|1, \ldots, i)$ and $\overline{M}_{i+1}$ are submodules of $M_{i+1}$ with $M(i+1|1, \ldots, i) \subseteq \overline{M}_{i+1}$, and $\lambda_i : \Omega_i \twoheadrightarrow \overline{M}_{i+1}/M(i+1|1, \ldots, i)$ is a $R$-module homomorphism. Here $\Omega_i$ is a submodule of $M_1 \times \cdots \times M_i$ is defined recursively, $1 \leq i \leq n-1$, by setting $\Omega_1 := \overline{M}_1$ and*

$$\Omega_{i+1} := \Gamma_2(\{\Omega_i, \overline{M}_{i+1}, M(i+1|1, \ldots, i), \lambda_i\})$$

*is a submodule of $(M_1 \times \cdots \times M_i) \times M_{i+1}$, with*

$$\Gamma_2(\{\Omega_i, \overline{M}_{i+1}, M(i+1|1, \ldots, i), \lambda_i\}) := p_i^{-1}(\phi_{\lambda_i}),$$

*where $\phi_{\lambda_i} \subseteq \Omega_i \times (\overline{M}_{i+1}/M(i+1|1, \ldots, i))$ is the graph of $\lambda_i$ and $p_i : \Omega_i \times \overline{M}_{i+1} \to \Omega_i \times (\overline{M}_{i+1}/M(i+1|1, \ldots, i))$ is the natural surjection.*

**Theorem 3.9.** *Let $R$ be a commutative ring with identity. The submodule $M$ of $M_1 \times \cdots \times M_n$ can be expressed as $M = \widetilde{N} \times N_n$ if and only if $\prod_{n-1}(M) = \ker(\lambda_{n-1}) = \widetilde{N}, M(n|1, \ldots, n-1) = N_n$, where $\ker(\lambda_{n-1}) \subseteq \prod_{n-1}(M)$, $M(n|1, \ldots, n-1) \subseteq \overline{M}_n$, and the map:*

$$\lambda_{n-1} : \prod\nolimits_{n-1}(M) \twoheadrightarrow \overline{M}_n/M(n|1, \ldots, n-1)$$

*is a surjective homomorphism.*

*Proof.* Suppose that the sumodule $M$ of $M_1 \times \cdots \times M_n$ can be expressed as $M = \widetilde{N} \times N_n$, then $\widetilde{N} = \prod_{n-1}(M)$ and $N_n = M(n|1\ldots, n-1)$. For any $k_n \in \overline{M}_n$, there exist $(k_1, \ldots, k_{n-1}) \in M_1 \times \cdots \times M_{n-1}$ such that $(k_1, \ldots, k_{n-1}, k_n) \in M = \widetilde{N} \times N_n$, which implies that $k_n \in N_n$. Thus, we have $\overline{M}_n \subseteq N_n$. It follows that $\overline{M}_n/M(n|1, \ldots, n-1)$ is trivial. Since $\lambda_{n-1}$ is surjective, we have $\ker(\lambda_{n-1}) = \prod_{n-1}(M)$.

Conversely, suppose that $\prod_{n-1}(M) = \ker(\lambda_{n-1}) = \widetilde{N}$, $M(n|1, \ldots, n-1) = N_n$, then for any $(k, \ell) \in M, k \in M_1 \times \cdots \times M_{n-1}, \ell \in M_n$, we have $k \in \widetilde{N}$ following the fact $\prod_{n-1}(M) = \widetilde{N}$. Since $\lambda_{n-1}$ is surjective, we have $\ell \in N_n$. Thus, $M = \widetilde{N} \times N_n$.

Following Theorem 3.9, and by induction, we have the following corollary.

**Corollary 3.10.** *The submodule $M$ of $M_1 \times \cdots \times M_n$ can be expressed as $M = N_1 \times \cdots \times N_n$ if and only if $\prod_i(M) = \ker(\lambda_i) = N_1 \times \cdots \times N_i$ and $M(i+1|1, \ldots, i) = N_{i+1}$, where* $\ker(\lambda_i) \subseteq \prod_i(M)$, *$M(i+1|1, \ldots, i) \subseteq \overline{M}_{i+1}$, and the map*

$$\lambda_i : \prod\nolimits_i(M) \twoheadrightarrow \overline{M}_{i+1}/M(i+1|1, \ldots, i)$$

*is a surjective homomorphism for* $1 \leq i \leq n-1$.

As the statement for groups in [8, Section 4], we can give that the submodule $M \subseteq M_1 \times \cdots \times M_n$ has the form $M = N_1 \times \cdots \times N_n$ if and only if $\lambda_i$ is the trivial homomorphism for $1 \leq i \leq n-1$, where $\lambda_i : \prod_i(M) \twoheadrightarrow \overline{M}_{i+1}/M(i+1|1, \ldots, i)$.

### *3.2. Symmetric and asymmetric version of Goursat's lemma for R-algebra*

A $R$-algebra $A$ has the ring structure and $R$-module structure concurrently, and the operations of these two structures are compatible, that is, $r(xy) = (rx)y = x(ry)$ for any $x, y \in A$ and $r \in R$.

**Definition 3.11** ( [12, Definition 7.1])**.** Let $R$ be a commutative ring with identity. A $R$-algebra is a ring $A$ together with
1) $(A, +)$ is a $R$-module,
2) $r(ab) = (ra)b = a(rb)$ for all $r \in R$, $a$, $b \in A$.

From [12], we know that an algebra (left, right, two-side) ideal of $R$-algebra $A$ is a (left, right, two-side) ideal of the ring $A$ that is also a submodule of $A$. If $A$ is a $R$-algebra, an ideal of the ring $A$ need not be an algebra ideal of $A$. However, if $A$ has an identity, every ideal is also an algebra ideal.

If $A_1$ and $A_2$ are $R$-algebras, and $A_a$ and $A_b$ are subalgebras of $A_1$ and $A_2$, respectively, then $A_a \times A_b$ is a subalgebra of $A_1 \times A_2$. However, the reverse is not necessarily true. Let $A_1$ and $A_2$ be $R$-algebras and $A$ be a subalgebra of $A_1 \times A_2$. We write

$$\begin{aligned}
A_{11} &= \{a \in A_1 | (a, 0) \in A\}, \\
A_{12} &= \{a \in A_1 | (a, b) \in A \text{ for some } b \in A_2\}, \\
A_{21} &= \{b \in A_2 | (0, b) \in A\}, \\
A_{22} &= \{b \in A_2 | (a, b) \in A \text{ for some } a \in A_1\}.
\end{aligned}$$

**Theorem 3.12** (**Symmetric version of Goursat's lemma for** 2 ***R*-algebras**, [7])**.** *Let $R$ be a commutative ring with identity and $A_1$ and $A_2$ be $R$-algebras.*

1) *Let $A$ be a subalgebra of $A_1 \times A_2$. Then $A_{i1}$ and $A_{i2}$ are subalgebras of $A_i$ such that $A_{i1}$ is an algebraic ideal of $A_{i2}$ for $i = 1, 2$, and the map*

$$f_A : A_{12}/A_{11} \to A_{22}/A_{21}, a + A_{11} \mapsto b + A_{21}$$

*is a R-algebra isomorphism, where $(a, b) \in A$.*

2) *Suppose that $A_{i1}$ and $A_{i2}$ are subalgebras of $A_i$ such that $A_{i1}$ is an algebraic ideal of $A_{i2}$ for $i = 1, 2$, and $f : A_{12}/A_{11} \to A_{22}/A_{21}$ is a R-algebra isomorphism, then*

$$A = \{(a, b) \in A_{12} \times A_{22} | f(a + A_{11}) = b + A_{21}\}$$

*is a subalgebra of $A_1 \times A_2$.*

3) *The construction given in 1) and 2) is inverse to each other.*

**Lemma 3.13** (**Symmetric version of Goursat's lemma for** 2 ***R*-algebras** [7, Corollary 1]). *Let $R$ be a commutative ring with identity. There is a bijective correspondence between subalgebra $A$ of R-algebra $A_1 \times A_2$ and quintuples $\{A_{11}, A_{12}, A_{21}, A_{22}, f_A\}$, where $A_{i1}$, $A_{i2}$ are subalgebras of $A_i$, such that $A_{i1}$ is a R-algebra ideal of $A_{i2}$ for $i = 1, 2$, and the map $f_A : A_{12}/A_{11} \to A_{22}/A_{21}$ is a R-algebra isomorphism.*

Subsequently, for the case $A_1 \times A_2 \times A_3$, $A$ is a $R$-subalgebra of $A_1 \times A_2 \times A_3$, and we should consider 12 sualgebras of $A_1, A_2, A_3$ as follows:

$$\begin{aligned}
A_{111} &= \{a \in A_1 | (a, b, c) \in A \text{ for some } b \in A_2, c \in A_3\},\\
A_{112} &= \{a \in A_1 | (a, b, 0) \in A \text{ for some } b \in A_2\},\\
A_{121} &= \{a \in A_1 | (a, 0, c) \in A \text{ for some } c \in A_3\},\\
A_{122} &= \{a \in A_1 | (a, 0, 0) \in A\};\\
A_{211} &= \{b \in A_2 | (a, b, c) \in A \text{ for some } a \in A_1, c \in A_3\},\\
A_{212} &= \{b \in A_2 | (a, b, 0) \in A \text{ for some } a \in A_1\},\\
A_{221} &= \{b \in A_2 | (0, b, c) \in A \text{ for some } c \in A_3\},\\
A_{222} &= \{b \in A_2 | (0, b, 0) \in A\};\\
A_{311} &= \{c \in A_3 | (a, b, c) \in A \text{ for some } a \in A_1, b \in A_2\},\\
A_{312} &= \{c \in A_3 | (a, 0, c) \in A \text{ for some } a \in A_1\},\\
A_{321} &= \{c \in A_3 | (0, b, c) \in A \text{ for some } b \in A_2\}.\\
A_{322} &= \{c \in A_3 | (0, 0, c) \in A\}.
\end{aligned}$$

Note that if $A_3 = \{0\}$, then $A_{111} = A_{112}, A_{121} = A_{122}$, $A_{211} = A_{212}, A_{221} = A_{222}$, $A_{311} = A_{312}$, and $A_{321} = A_{322}$.

Similar as the case $G_1 \times G_2 \times G_3$, it is also most likely impossible to give a symmetric version of Goursat's lemma for $A_1 \times A_2 \times A_3$. In the following, we use additional conditions to obtain some related results as those in Theorems 3.3–3.5.

Let $\pi_i : A_1 \times A_2 \times A_3 \twoheadrightarrow A_i$ be the standard projection onto the $i$th factor and a $R$-subalgebra $A \subseteq A_1 \times A_2 \times A_3$, such that $\pi_i(A) = A_i$ for $1 \leq i \leq 3$. Define the homomorphism $\tau_1 : A \to A_2 \times A_3, (a_1, a_2, a_3) \mapsto (a_2, a_3)$. If $\tau_1 : A \to A_2 \times A_3$ is injective, then for any $(a, b, c), (a', b, c) \in A$, we have $a = a'$. Similar for the homomorphisms $\tau_2 : A \to A_1 \times A_3$ and $\tau_3 : A \to A_1 \times A_2$. Similar as the proofs of Theorems 3.3–3.5, we can obtain the following results easily.

**Theorem 3.14.** *Let $A \subseteq A_1 \times A_2 \times A_3$ be a R-subalgebra satisfying $\tau_i(A) = A_i$ for $1 \leq i \leq 3$. Then*

$$A = \{(a, b, c) \in A_1 \times A_2 \times A_3 | (a + A_{122}, b + A_{222}, c + A_{322}) \in A/(A_{122} \times A_{222} \times A_{322})\}.$$

**Theorem 3.15.** *Let $A \subseteq A_1 \times A_2 \times A_3$ be a R-subalgebra satisfying $\tau_i(A) = A_i$, and $\tau_i$ is injective for $1 \leq i \leq 3$. Let*

$$f_{12} : A_{112} \to A_{212}, a \mapsto b, (a, b, e) \in A,$$
$$f_{13} : A_{121} \to A_{312}, a \mapsto c, (a, e, c) \in A,$$
$$f_{23} : A_{221} \to A_{321}, b \mapsto c, (e, b, c) \in A,$$

*then*

$$f_{12}|_{A_{112} \cap A_{121}} : A_{112} \cap A_{121} \to A_{212} \cap A_{221},$$
$$f_{23}|_{A_{212} \cap A_{221}} : A_{212} \cap A_{221} \to A_{312} \cap A_{321},$$
$$f_{13}|_{A_{112} \cap A_{121}} : A_{112} \cap A_{121} \to A_{312} \cap A_{321},$$

*are isomorphisms and $f_{13}|_{A_{112} \cap A_{121}} = f_{23}|_{A_{212} \cap A_{221}} \circ f_{12}|_{A_{112} \cap A_{121}}$.*

**Theorem 3.16.** *Let $A \subseteq A_1 \times A_2 \times A_3$ satisfying $\tau_i(A) = A_i$ and $\tau_i$ is injective for $1 \leq i \leq 3$. Suppose $A_{i12} \cap A_{i21} = A_{i22}$ for $1 \leq i \leq 3$, then*

$$f_{12} : A_{111}/A_{121} \to A_{211}/A_{221}, a_1 A_{121} \mapsto a_2 A_{221}, (a_1, a_2, a_3) \in A \text{ for some } a_3 \in A_3,$$
$$f_{23} : A_{211}/A_{212} \to A_{311}/A_{312}, a_2 A_{212} \mapsto a_3 A_{312}, (a_1, a_2, a_3) \in A \text{ for some } a_1 \in A_1,$$
$$f_{13} : A_{111}/A_{112} \to A_{311}/A_{321}, a_1 A_{112} \mapsto a_3 A_{321}, (a_1, a_2, a_3) \in A \text{ for some } a_2 \in A_2,$$

*are isomorphisms and*

$$A = \{(a_1, a_2, a_3) \in A_{111} \times A_{211} \times A_{311} | f_{12}(a_1 A_{121}) = a_2 A_{221}, f_{23}(a_2 A_{212}) = a_3 A_{312}, f_{13}(a_1 A_{112}) = a_3 A_{321}\}.$$

**Lemma 3.17** (**Asymmetric version of Goursat's lemma for 2 *R*-algebras**, [11, Corollary 2])**.** *Let $R$ be a commutative ring with identity. There is a bijective correspondence between subalgebra $A$ of R-algebra $A_1 \times A_2$ and quadruples $\{A_{12},\ A_{21},\ A_{22},\ f_A\}$, where $A_{i1}$, $A_{i2}$ are subalgebras of $A_i$ for $i = 1, 2$, respectively, $A_{21}$ is a R-algebra ideal of $A_{22}$ and the map $f_A : A_{12} \to A_{22}/A_{21}$ is a surjective homomorphism.*

In the following, we introduce some convenient notation for any subalgebra $A$ of $R$-algebra $A_1 \times \cdots \times A_n$.

**Definition 3.18.** Let $R$ be a commutative ring with identity. Let $A$ be a subalgebra of $A_1 \times \cdots \times A_n$, where $A_i$ is a $R$-algebra, $1 \leq i \leq n$. Let $S \subsetneq \{1, 2, \ldots, n\} = [n]$, and $i \in [n] \backslash S$. Then

$$A(i|S) := \{x_i \in A_i | (x_1, \ldots, x_i, \ldots, x_n) \in A \text{ for some } x_j \in A_j,\ 1 \leq j \leq n,\ j \neq i, \text{with } x_j = 0 \text{ if } j \in S\}.$$

From [13], we know that $A(i|S)$ is a subalgebra of $A_i$ and $A(i|S)$ is a subalgebra of $A(i|T)$ if $T \subseteq S$. Let $\overline{A}_i := A(i|\emptyset)$, i.e.,

$$\overline{A}_i := \{x_i \in A_i | (x_1, \ldots, x_i, \ldots, x_n) \in A \text{ for some } x_j \in A_j,\ 1 \leq j \leq n,\ j \neq i\}.$$

**Theorem 3.19** (**Asymmetric version of Goursat's lemma for** $n$ $R$**-algebras with** $n \geq 2$, [11, Corallary 4])**.** *Let $R$ be a commutative ring with identity. There is a bijective correspondence between the subalgebra $A$ of $R$-algebra $A_1 \times \cdots \times A_n$ and $(3n-2)$-tuples*

$$Q_n(A) := \left\{\overline{A}_1, \overline{A}_2, A(2|1), \delta_1, \ldots, \overline{A}_n, A(n|1, \ldots, n-1), \delta_{n-1}\right\},$$

*where $A(i|1, \ldots, i-1)$ and $\overline{A}_i$ are subalgebras of $A_i$, $A(i|1, \ldots, i-1)$ is a $R$-algebra ideal of $\overline{A}_i$, and the map*

$$\delta_i : \Theta_i \twoheadrightarrow \overline{A}_{i+1}/A(i+1|1, \ldots, i)$$

*is a $R$-algebra epimorphism. Here $\Theta_i$ is an algebra ideal of $A_1 \times \cdots \times A_i$ is defined recursively, $1 \leq i \leq n-1$, by setting $\Theta_1 := \overline{A}_1$ and*

$$\Theta_{i+1} := \Gamma_2(\{\Theta_i, \overline{A}_{i+1}, A(i+1|1, \ldots, i), \delta_i\})$$

*is a subalgebra of $A_1 \times \cdots \times A_i$ with*

$$\Gamma_2(\{\Theta_i, \overline{A}_{i+1}, A(i+1|1, \ldots, i), \delta_i\}) := p_i^{-1}(\mathfrak{g}_{\delta_i}),$$

*where $\mathfrak{g}_{\delta_i} \subseteq \Theta_i \times (\overline{A}_{i+1}/A(i+1|1, \ldots, i))$ which is the graph of $\delta_i$ and $p_i : \Theta_i \times \overline{A}_{i+1} \to \Theta_i \times (\overline{A}_{i+1}/A(i+1|1, \ldots, i))$ is the natural surjection.*

Using Theorem 3.9, we can easily obtain the following Theorem.

**Theorem 3.20.** *Let $R$ be a commutative ring with identity. The subalgebra $A$ of $A_1 \times \cdots \times A_n$ can be expressed as $A = \widetilde{B} \times B_n$ if and only if $\prod_{n-1}(A) = \ker(\delta_{n-1}) = \widetilde{B}$, $A(n|1, \ldots, n-1) = B_n$, where $\ker(\delta_{n-1})$ and $A(n|1, \ldots, n-1)$ are algebraic ideals of $\prod_{n-1}(A)$ and $\overline{A}_n$, respectively, and the map:*

$$\delta_{n-1} : \textstyle\prod_{n-1}(A) \twoheadrightarrow \overline{A}_n/A(n|1, \ldots, n-1)$$

*is a surjective homomorphism.*

Following Theorem 3.20, and by induction, we have the following corollary.

**Corollary 3.21.** *The subalgebra $A$ of $A_1 \times \cdots \times A_n$ can be expressed as $A = B_1 \times \cdots \times B_n$ if and only if $\prod_i(A) = \ker(\delta_i) = B_1 \times \cdots \times B_i$, $A(i+1|1, \ldots, i) = B_{i+1}$, where $\ker(\delta_{n-1})$ and $A(n|1, \ldots, n-1)$ are algebra ideals of $\prod_{n-1}(A)$ and $\overline{A}_n$, respectively, and the map:*

$$\delta_i : \textstyle\prod_i(A) \twoheadrightarrow \overline{A}_{i+1}/A(i+1|1, \ldots, i),$$

*is a surjective homomorphism for $1 \leq i \leq n-1$.*

As the statement for groups in [8, Section 4], we can give that the subalgebra $A \subseteq A_1 \times \cdots \times A_n$ has the form $A = B_1 \times \cdots \times B_n$ if and only if $\delta_i$ is the trivial homomorphism for $1 \leq i \leq n-1$, where $\delta_i : \prod_i(A) \twoheadrightarrow \overline{A}_{i+1}/A(i+1|1, \ldots, i)$.

Since rings are the special case of $R$-algebras, we can obtain the following results about rings according to Theorems 3.14–3.16, and Theorem 3.20 and Corollary 3.21. The notations $T_{ijk}$ are similar as $A_{ijk}$ in Section 3.2.

**Corollary 3.22.** *Let $T \subseteq R_1 \times R_2 \times R_3$ be a subring satisfying $\pi_i(T) = R_i$ for $1 \leq i \leq 3$. Then*

$$T = \{(a, b, c) \in R_1 \times R_2 \times R_3 | (a + T_{122}, b + T_{222}, c + T_{322}) \in T/(T_{122} \times T_{222} \times T_{322})\}.$$

**Corollary 3.23.** *Let $T \subseteq R_1 \times R_2 \times R_3$ be a subring satisfying $\pi_i(T) = R_i$, and $\eta_i$ is injective for $1 \leq i \leq 3$. Let*

$$\begin{aligned} f_{12} &: T_{112} \to T_{212}, a \mapsto b, (a, b, 0) \in T, \\ f_{13} &: T_{121} \to T_{312}, a \mapsto c, (a, 0, c) \in T, \\ f_{23} &: T_{221} \to T_{321}, b \mapsto c, (0, b, c) \in T, \end{aligned}$$

*then*

$$\begin{aligned} f_{12}|_{T_{112} \cap T_{121}} &: T_{112} \cap T_{121} \to T_{212} \cap T_{221}, \\ f_{23}|_{T_{212} \cap T_{221}} &: T_{212} \cap T_{221} \to T_{312} \cap T_{321}, \\ f_{13}|_{T_{112} \cap T_{121}} &: T_{112} \cap T_{121} \to T_{312} \cap T_{321}, \end{aligned}$$

*are isomorphisms and $f_{13}|_{T_{112} \cap T_{121}} = f_{23}|_{T_{212} \cap T_{221}} \circ f_{12}|_{T_{112} \cap T_{121}}$.*

**Corollary 3.24.** *Let $T \subseteq R_1 \times R_2 \times R_3$ satisfying $\pi_i(T) = R_i$ and $\eta_i$ is injective for $1 \leq i \leq 3$. Suppose $T_{i12} \cap T_{i21} = T_{i22}$ for $1 \leq i \leq 3$, then*

$$\begin{aligned} f_{12} &: T_{111}/T_{121} \to T_{211}/T_{221}, r_1 + T_{121} \mapsto r_2 + T_{221}, (r_1, r_2, r_3) \in T \text{ for some } r_3 \in T_3, \\ f_{23} &: T_{211}/T_{212} \to T_{311}/T_{312}, r_2 + T_{212} \mapsto r_3 + T_{312}, (r_1, r_2, r_3) \in T \text{ for some } r_1 \in T_1, \\ f_{13} &: T_{111}/T_{112} \to T_{311}/T_{321}, r_1 + T_{112} \mapsto r_3 + T_{321}, (r_1, r_2, r_3) \in T \text{ for some } r_2 \in T_2, \end{aligned}$$

*are isomorphisms and*

$$\begin{aligned} T = \{&(r_1, r_2, r_3) \in T_{111} \times T_{211} \times T_{311} | f_{12}(r_1 + T_{121}) = r_2 + T_{221}, \\ &f_{23}(r_2 + T_{212}) = r_3 + T_{312}, f_{13}(r_1 + T_{112}) = r_3 + T_{321}\}. \end{aligned}$$

**Corollary 3.25.** *The subring $T$ of $R_1 \times \cdots \times R_n$ can be expressed as $T = \widetilde{T} \times T_n$ if and only if $\prod_{n-1}(T) = \ker(f_{n-1}) = \widetilde{T}$, $T(n|1, \ldots, n-1) = T_n$, where $\ker(f_{n-1})$ and $T(n|1, \ldots, n-1)$ are ideals of $\prod_{n-1}(T)$ and $\overline{T}_n$, respectively, and the map:*

$$f_{n-1} : \textstyle\prod_{n-1}(T) \twoheadrightarrow \overline{T}_n / T(n|1, \ldots, n-1)$$

*is a surjective homomorphism.*

**Corollary 3.26** ( [14])**.** *The subring $T$ of $R_1 \times \cdots \times R_n$ can be expressed as $T = T_1 \times \cdots \times T_n$ if and only if $\prod_i(T) = \ker(f_i) = T_1 \times \cdots \times T_i$, $T(i+1|1, \ldots, i) = T_{i+1}$, where $\ker(f_i)$ and $T(i+1|1, \ldots, i)$ are ideals of $\prod_i(T)$ and $\overline{T}_{i+1}$, respectively, and the map:*

$$f_i : \textstyle\prod_i(T) \twoheadrightarrow \overline{T}_{i+1} / T(i+1|1, \ldots, i)$$

*is a surjective homomorphism for $1 \leq i \leq n-1$.*

As the statement for groups in [8, Section 4], we can give that the subring $T \subseteq R_1 \times \cdots \times R_n$ has the form $T = T_1 \times \cdots \times T_n$ if and only if $f_i$ is the trivial homomorphism for $1 \leq i \leq n-1$, where $f_i : \prod_i(T) \twoheadrightarrow \overline{T}_{i+1}/T(i+1|1, \ldots, i)$.

## 4. Conclusions

The symmetric version of Goursat's lemma is useful to deal with the expressions of subgroups of the direct product $G_1 \times G_2$ of two groups $G_1, G_2$ (even for rings, $R$-modules, $R$-algebras). However, it is most likely impossible to give the symmetric version of Goursat's lemma for $G_1 \times \cdots \times G_n$ ($n \geq 3$), even for rings, $R$-modules, $R$-algebras. With the help of the additional hypothesis given by [8, Section 5], we use additional conditions to obtain some related results for groups (also $R$-modules, $R$-algebras (rings as corollary)). We subsequently use the asymmetric version of Goursat's lemma to give the subproduct of a $R$-submodule of $M_1 \times \cdots \times M_n$ ($n \geq 3$), and $R$-algebras (rings as corollary).

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China (11701111; 12031003) and Ministry of Science and Technology of China (CSN-XXZ-20120700649).

The authors would like to sincerely express our thanks for the useful comments from the kind referees.

## Conflict of interest

The authors declare there are no conflicts of interest.

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