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Research article

B_σ-grand Morrey spaces

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Abstract: In this paper, we define the B_σ-type grand Morrey spaces and establish the extrapolation theorem on the B_σ-type grand Morrey spaces. In the process of proving the theorem, we find that the predual spaces of these spaces are H_σ-block spaces and obtain the boundedness of the Hardy–Littlewood maximal operator on the predual spaces. By the extrapolation theory, the boundedness of the Calderón–Zygmund operator and commutators on the nonhomogeneous B_σ-type grand Morrey space is also obtained. In particular, the classical bounded mean oscillation (BMO) spaces are characterized by establishing the John–Nirenberg inequality on the nonhomogeneous B_σ type grand Morrey space.

Keywords: grand Lebesgue space; B_σ type space; extrapolation theory; BMO space

1. Introduction

The grand Lebesgue spaces were introduced by Iwaniec and Sbordone [1] in 1992 on the integrability of Jacobian functions. Since its introduction, the grand Lebesgue space has been widely studied and generalized. Fiorenza [2] introduced the small Lebesgue space in this study and proved that it is the dual space of the grand Lebesgue space. Anatriello and Fiorenza [3] introduced the fully measurable grand Lebesgue spaces and proved the boundedness of Hardy–Littlewood maximal operator through Hardy's inequality. Generalized grand Lebesgue spaces were introduced by Capone et al. [4], and Formica et al. [5] obtained control over the spectral radius of some linear operators and proved the boundedness for two Hardy operators. For the study of the related spatial properties, the reader can refer to [2–7].

Morrey spaces are regarded as a localized generalization of classical Lebesgue spaces, which were defined and used to study the local properties of solutions of second-order elliptic equations by Morrey [8]. Grand Morrey spaces, as a generalization of grand Lebesgue spaces and classical Morrey spaces, were introduced by Meskhi [9] in 2010. The study of grand Morrey spaces were further promoted due to the advantages of Morrey-type spaces in the study of local properties. Generalized grand Morrey spaces [10], weighted grand Morrey spaces [11], grand Morrey spaces on Euclidean spaces [12], and grand variable exponent Morrey spaces [13], etc., as generalizations of the more general form of the grand Morrey spaces have been proposed and studied in the course of the study of solvability problems of nonlinear partial differential equations in mathematical models. In this process, the duality of the corresponding spaces, the operators' boundedness, and other excellent properties have also been systematically studied, and related work can be found in the relevant references [11, 13–15].

In the research and development of space theory, centralized-type function spaces are also important research objects. In 2011, Matsuoka and Nakai [16] completed the unification of central Morrey spaces and general Morrey–Campanato spaces by introducing function spaces $B^{p,\lambda}(\mathbb{R}^n)$ with the Morrey–Campanato norm. Subsequently, Komori-Furuya et al. [17] introduced B_σ-type function spaces in 2013. Due to the unification of B_σ-type function spaces, the boundedness of various operators such as Calderón–Zygmund operators, fractional integral operators, and commutators, etc. on B_σ-type spaces were proved, and the corresponding work can be referred to in [18–21]. As a profound application of weight theory, the extrapolation theorem can be used to solve the problem of boundedness of operators on the spaces. However, its proof relies on the duality of the corresponding spaces and the boundedness of Hardy–Littlewood maximal operators on dual or predual spaces. The classical extrapolation theorem was first formulated by de Francia and Luis [22]. In subsequent studies, the extrapolation theorem was generalized to more spaces such as grand Morrey spaces [23], quasi-Banach function spaces [24], weighted product Morrey spaces [25], B_σ-type mixed Morrey space [26], grand Morrey spaces on Euclidean spaces [27], etc.

Inspired by the results above, the main purpose of this paper is to consider the theory of operators on B_σ-type grand Morrey spaces. To solve this problem, we first define the grand B_σ-type grand Morrey spaces which unify the grand Lebesgue space, the grand Morrey space, and the grand central Morrey space. At the same time, the embedding property, dual theory, and the boundedness of the Hardy–Littlewood maximal operator in predual spaces are established. On this basis, the extrapolation theory of the spaces is also solved. By the extrapolation theory, we establish some boundedness results for some classical operators. In particular, the extrapolation theory can obtain a new characterization of bounded mean oscillation (BMO) space.

We establish some conventions regarding notation. For any $p \in [1, \infty]$, we denote its conjugate index p' such that $1/p + 1/p' = 1$. In the following, let $Q(x, r)$ represent an open cube with x as the center and a side length of $2r$, or an open sphere with x as the center and a radius of r. Specifically, when x is the origin, we use Q_r to represent it. We also use $\mathcal{Q}$ to denote the set of all cubes whose edges are parallel to the coordinate axes. We always denote a positive constant by C, independent of the main parameters, but it may vary from line to line. The notation $A \lesssim B$ means that $A \leq CB$ with some positive constant C that is independent of the appropriate quantities. Additionally, if $A \lesssim B \lesssim A$, we write $A \sim B$. For a measurable set E, we use χ_E to denote the characteristic function of E, and $|E|$ signifies its n-dimensional Lebesgue-measure.

2. B_σ-type grand Morrey spaces

In this section, we define the B_σ-type grand Morrey spaces. Initially, we recall and review some properties of the grand Lebesgue spaces and grand Morrey spaces.

Definition 1. Let $Q \in \mathcal{Q}$, for any Lebesgue-measurable function f and $s > 0$,write

$$d_{f,Q}(s) := \frac{1}{|Q|}|\{x \in Q : |f(x)| > s\}|$$

and

$$f_Q^*(t) := \inf\left\{s > 0 : d_{f,Q}(s) \le t\right\} \quad t > 0.$$

Definition 2. Let $p \in (0, \infty)$ and $Q \in \mathcal{Q}$. The grand Lebesgue space $L^{p)}(Q)$ then consists of all Lebesgue-measurable functions f satisfying

$$\|f\|_{L^{p)}(Q)} = \sup_{0<t<1}(1 - \ln t)^{-\frac{1}{p}}\left(\int_t^1 (f_Q^*(s))^p ds\right)^{\frac{1}{p}} < \infty.$$

The small Lebesgue space $L^{(p}(Q)$ consists of all Lebesgue-measurable functions f satisfying

$$\|f\|_{L^{(p}(Q)} = \int_0^1 (1 - \ln t)^{-\frac{1}{p}}\left(\int_0^t (f_Q^*(s))^p ds\right)^{\frac{1}{p}} \frac{dt}{t} < \infty.$$

Lemma 1. [28] When $p \in (1, \infty)$, the grand Lebesgue spaces and the small Lebesgue spaces are originally defined in terms of the following norms:

$$\|f\|^*_{L^{p)}(Q)} = \sup_{0<\epsilon<p-1}\left(\frac{\epsilon}{|Q|}\int_Q |f(x)|^{p-\epsilon}dx\right)^{\frac{1}{p-\epsilon}} < \infty,$$

$$\|g\|^*_{L^{(p}(Q)} = \inf_{g=\sum g_k}\sum_{k=1}^{\infty}\inf_{0<\epsilon<p-1}\epsilon^{-\frac{1}{p-\epsilon}}\left(\frac{1}{|Q|}\int_Q |g_k(x)|^{(p-\epsilon)'}dx\right)^{\frac{1}{(p-\epsilon)'}}.$$

$\|\cdot\|^*_{L^{p)}(Q)}$ and $\|\cdot\|^*_{L^{(p}(Q)}$ are equivalent norms of $\|\cdot\|_{L^{p)}(Q)}$ and $\|\cdot\|_{L^{(p}(Q)}$, respectively.

Lemma 2. [2] The associated space of $L^{p)}(Q)$ is $L^{(p'}(Q)$ and vice versa, and the Hölder-type inequality holds

$$\frac{1}{|Q|}\int_Q |f(x)g(x)|dx \le C\|f\|_{L^{p)}(Q)}\|g\|_{L^{(p'}(Q)}$$

and the norm conjugation formula is

$$\|f\|_{L^{p)}(Q)} = \sup_{\|g\|_{L^{(p'}(Q)}\neq 0}\frac{\frac{1}{|Q|}\int_Q |f(x)g(x)|dx}{\|g\|_{L^{(p'}(Q)}}.$$

Definition 3. For any function $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|dy.$$

Lemma 3. [23] Let $1 < p < \infty$. For any $Q \in \mathcal{Q}$ and $f \in L^{(p}(Q)$, a constant $C > 0$ exists such that

$$\|M(\chi_Q f)\|_{L^{(p}(Q)} \le C\|f\|_{L^{(p}(Q)}.$$

Definition 4. [23] (Grand Morrey space) Let $p \in (0,\infty)$ and let $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue-measurable function. In this case, the grand Morrey space $M_u^{p)}(\mathbb{R}^n)$ consists of all Lebesgue-measurable functions f satisfying

$$\|f\|_{M_u^{p)}} := \sup_{Q(x,r)\in\mathcal{Q}} \frac{1}{u(x,r)} \|f\chi_{Q(x,r)}\|_{L^{p)}(Q(x,r))}.$$

For any $Q = Q(x,r) \in \mathcal{Q}$, we write $u(Q) = u(x,r)$.

Next, we introduce the B_σ-type grand Morrey spaces and study its related properties.

Definition 5. Let $\sigma \geq 0$, $p \in (0,\infty)$, and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue-measurable function. The homogeneous B_σ-type grand Morrey space $\dot{B}_\sigma(M_u^{p)})(\mathbb{R}^n)$ is defined to be the set of all measurable functions f with

$$\|f\|_{\dot{B}_\sigma(M_u^{p)})} := \sup_{r>0} \frac{1}{r^\sigma} \|f\chi_{Q_r}\|_{M_u^{p)}} = \sup_{r>0} \sup_{Q\in\mathcal{Q}} \frac{1}{r^\sigma u(Q)} \|f\chi_{Q\cap Q_r}\|_{L^{p)}(Q)} < \infty.$$

Similarly, the nonhomogeneous B_σ-type grand Morrey space $B_\sigma(M_u^{p)})(\mathbb{R}^n)$ consists of all measurable functions f with

$$\|f\|_{B_\sigma(M_u^{p)})} := \sup_{r\geq 1} \frac{1}{r^\sigma} \|f\chi_{Q_r}\|_{M_u^{p)}} = \sup_{r\geq 1} \sup_{Q\in\mathcal{Q}} \frac{1}{r^\sigma u(Q)} \|f\chi_{Q\cap Q_r}\|_{L^{p)}(Q)} < \infty.$$

Remark 1.

1) When $0 \leq \sigma_1 \leq \sigma_2 < \infty$, it is obvious that $B_{\sigma_1}(M_u^{p)})(\mathbb{R}^n) \hookrightarrow B_{\sigma_2}(M_u^{p)})(\mathbb{R}^n)$.

2) Let $q > 0$. By the q-convexification of grand Lebesgue space and Definition 2, we conclude that

$$\begin{aligned} \|f^q\|^{\frac{1}{q}}_{B_{\sigma q}(M_{u^q}^{\frac{p}{q})})} &= \sup_{r\geq 1} \sup_{Q\in\mathcal{Q}} \left(\frac{1}{r^{\sigma q} u^q(Q)} \|f^q \chi_{Q\cap Q_r}\|_{L^{\frac{p}{q})}(Q)} \right)^{\frac{1}{q}} \\ &= \sup_{r\geq 1} \sup_{Q\in\mathcal{Q}} \frac{1}{r^\sigma u(Q)} \|f\chi_{Q\cap Q_r}\|_{L^{p)}(Q)} \\ &= \|f\|_{B_\sigma(M_u^{p)})}. \end{aligned}$$

That is, the q-convexification of $B_\sigma(M_u^{p)})(\mathbb{R}^n)$ is $B_{\sigma q}(M_{u^q}^{\frac{p}{q})})(\mathbb{R}^n)$.

Proposition 1. Let $\sigma \geq 0$, $1 < p < \infty$, and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue-measurable function. If u satisfies

$$Cr'^{-\frac{n}{p}} < u(x,r'), \quad r' > 1, \tag{2.1}$$

$$C \leq u(x,r'), \quad r' \leq 1, \tag{2.2}$$

then for any $Q \in \mathcal{Q}$, we obtain $\chi_Q \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$ and $\|\chi_Q\|_{B_\sigma(M_u^{p)})} \leq C(1 + |Q|^{\frac{1}{p}})$.

Proof. By Lemma 1 and Inequality (2.1) for any $Q(x,r') \in \mathcal{Q}$ with $r' > 1$

$$\begin{aligned}\frac{1}{r^{\sigma}u(x,r')}\|\chi_{Q_r\cap Q(x,r')\cap Q}\|_{L^{p)}(Q(x,r'))} &\le \frac{C}{r^{\sigma}}\cdot\frac{1}{r'^{\frac{n}{p}}u(x,r')}|Q_r\cap Q(x,r')\cap Q|^{\frac{1}{p}}\\ &\le \frac{C}{r^{\sigma}}\cdot\frac{1}{r'^{\frac{n}{p}}u(x,r')}|Q|^{\frac{1}{p}}\\ &\le \frac{C}{r^{\sigma}}|Q|^{\frac{1}{p}}.\end{aligned}$$

Thus,

$$\|\chi_Q\|_{B_\sigma(M_u^{p)})} \le C|Q|^{\frac{1}{p}}.$$

By Lemma 1 and Inequality (2.2) for any $Q(x,r') \in \mathcal{Q}$ with $r' \le 1$,

$$\begin{aligned}\frac{1}{r^{\sigma}u(x,r')}\|\chi_{Q_r\cap Q(x,r')\cap Q}\|_{L^{p)}(Q(x,r'))} &\le \frac{C}{r^{\sigma}}\cdot\frac{1}{r'^{\frac{n}{p}}u(x,r')}|Q_r\cap Q(x,r')\cap Q|^{\frac{1}{p}}\\ &\le \frac{C}{r^{\sigma}}\cdot\frac{1}{r'^{\frac{n}{p}}u(x,r')}|Q(x,r')|^{\frac{1}{p}}\\ &\le Cr^{-\sigma}.\end{aligned}$$

Thus,

$$\|\chi_Q\|_{B_\sigma(M_u^{p)})} \le C.$$

Definition 6. Let $\sigma \ge 0$, $1 < p < \infty$, and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue-measurable function.

1) If a function $A \in L^{(p'}(Q)$ satisfies $supp(A) \subset Q \cap Q_r$ and $\|A\|_{L^{(p'}(Q)} \le \frac{1}{r^{\sigma}u(Q)|Q|}$ for some $r > 0$ and $Q \in \mathcal{Q}$, the function A is called a (p', u, σ, r)-block.

2) Let $\dot{\mathscr{A}}_\sigma\left(\mathscr{H}_u^{(p'}\right)$ be the collection of all sequences $\{(A_i, r_i, Q_i)\}_{i=1}^{\infty}$ for which each A_i is a (p', u, σ, r_i)-block. The homogeneous H_σ-block space $\dot{H}_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$ is defined by all Lebesgue-measurable functions f such that

$$f = \sum_{i=1}^{\infty}\lambda_i A_i \quad a.e. x \in \mathbb{R}^n,$$

for some $\{\lambda_i\}_{i=1}^{\infty} \in \ell^1$ and $\{(A_i, r_i, Q_i)\}_{i=1}^{\infty} \in \dot{\mathscr{A}}_\sigma\left(\mathscr{H}_u^{(p'}\right)$. The norm $\|f\|_{\dot{H}_\sigma(\mathscr{H}_u^{(p'})}$ can be defined by

$$\|f\|_{\dot{H}_\sigma(\mathscr{H}_u^{(p'})} := \inf\left\{\sum_{i=1}^{\infty}|\lambda_i| : f = \sum_{i=1}^{\infty}\lambda_i A_i \quad a.e., \{(A_i, r_i, Q_i)\}_{i=1}^{\infty} \in \dot{\mathscr{A}}_\sigma\left(\mathscr{H}_u^{(p'}\right)\right\} < \infty.$$

3) Let $\mathscr{A}_\sigma\left(\mathscr{H}_u^{(p'}\right)$ be the collection of all sequences $\{(A_i, r_i, Q_i)\}_{i=1}^{\infty}$ for which each A_i is a (p', u, σ, r_i)-block, where $r_i \ge 1$. The nonhomogeneous H_σ-block space $H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$ is defined by all Lebesgue-measurable functions f such that

$$f = \sum_{i=1}^{\infty}\lambda_i A_i \quad a.e. x \in \mathbb{R}^n,$$

for some $\{\lambda_i\}_{i=1}^{\infty} \in \ell^1$ and $\{(A_i, r_i, Q_i)\}_{i=1}^{\infty} \in \mathscr{A}_\sigma\left(\mathscr{H}_u^{(p')}\right)$. The norm $\|f\|_{H_\sigma(\mathscr{H}_u^{(p')})}$ can be defined by

$$\|f\|_{H_\sigma(\mathscr{H}_u^{(p')})} := \inf\left\{\sum_{i=1}^{\infty}|\lambda_i| : f = \sum_{i=1}^{\infty}\lambda_i A_i \quad a.e., \{(A_i, r_i, Q_i)\}_{i=1}^{\infty} \in \mathscr{A}_\sigma\left(\mathscr{H}_u^{(p')}\right)\right\} < \infty.$$

3. The duality of B_σ-type function spaces

In this section, we primarily establish the predual spaces of B_σ-type grand Morrey spaces. Prior to presenting the key theorems, we conduct some preliminary lemmas related to $\dot{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ and $H_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$. Given their similar properties, we prove only some of the conclusions related to $H_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ in this section, and the proof of the conclusions $\dot{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ are analogous.

Lemma 4. Let $\sigma \geq 0$, $p \in (1, \infty)$, and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue-measurable function. If function $g \in L^{(p'}$, $Q \in \mathcal{Q}$, then, for any $r > 0$ and $r \geq 1$, the following inequalities hold, respectively:

$$\left\|\frac{\chi_{Q_r\cap Q}g}{r^\sigma u(Q)|Q|}\right\|_{\dot{H}_\sigma(\mathscr{H}_u^{(p')})} \leq \|g\|_{L^{(p'}(Q)},$$

and

$$\left\|\frac{\chi_{Q_r\cap Q}g}{r^\sigma u(Q)|Q|}\right\|_{H_\sigma(\mathscr{H}_u^{(p')})} \leq \|g\|_{L^{(p'}(Q)}.$$

Proof. When $Q \in \mathcal{Q}$ and $r \geq 1$, let

$$B := \frac{\chi_{Q_r\cap Q}g}{r^\sigma u(Q)|Q|\|g\|_{L^{(p'}(Q)}}.$$

Obviously, B satisfies $supp(B) \subset Q_r \cap Q$ and

$$\|B\|_{L^{(p'}(Q)} \leq \frac{1}{r^\sigma u(Q)|Q|} \cdot \frac{\|\chi_{Q_r\cap Q}g\|_{L^{(p'}(Q)}}{\|g\|_{L^{(p'}(Q)}} \leq \frac{1}{r^\sigma u(Q)|Q|}.$$

This implies that B is a (p', u, σ, r)-block. Thus,

$$\left\|\frac{\chi_{Q_r\cap Q}g}{r^\sigma u(Q)|Q|}\right\|_{H_\sigma(\mathscr{H}_u^{(p')})} \leq \|g\|_{L^{(p'}(Q)}.$$

Next, we introduce $\mathscr{H}_\sigma$-block spaces and prove its density in H_σ-spaces.

Definition 7. Let $\sigma \geq 0$, $p \in (1, \infty)$, and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue-measurable function.

1) The $\mathscr{H}_\sigma$-block space $\dot{\mathscr{H}}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$, which is the linear subspace of $\dot{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$, is defined to be the set of all functions f satisfying $supp(f) \subset Q_R \backslash Q_{R^{-1}}$ for some $R > 0$.

2) The $\mathscr{H}_\sigma$-block space $\mathscr{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$, which is the linear subspace of $H_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$, is defined to be the set of all functions f satisfying $supp(f) \subset Q_R$ for some $R \geq 1$.

Lemma 5. Let $\sigma \geq 0$, $p \in (1, \infty)$, and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue-measurable function. The spaces $\dot{\mathscr{H}}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ and $\mathscr{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ are dense in $\dot{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ and $H_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$, respectively.

Proof. For any $f \in H_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$, there are sequences $\{(A_i, r_i, Q_i)\}_{i=1}^\infty$ in which A_i is a (p', u, σ, r_i)-block satisfying

$$f = \sum_{i=1}^{\infty} \lambda_i A_i \quad a.e. x \in \mathbb{R}^n \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i| < \infty.$$

Consequently, it is evident that

$$\sum_{i=1}^{k} \lambda_i A_i \in \mathscr{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n).$$

Then, for any $k \in \mathbb{N}$,

$$\left\| f - \sum_{i=1}^{k} \lambda_i A_i \right\|_{H_\sigma(\mathscr{H}_u^{(p')})} = \left\| \sum_{i=k+1}^{\infty} \lambda_i A_i \right\|_{H_\sigma(\mathscr{H}_u^{(p')})} \leq \sum_{i=k+1}^{\infty} |\lambda_i|,$$

so it will suffice to show that $\|f - \sum_{i=1}^{k} \lambda_i A_i\|_{H_\sigma(\mathscr{H}_u^{(p')})} \to 0$ when $k \to \infty$.

Therefore, we complete the proof that the space $\mathscr{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ is dense in $H_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$.

For any $f \in \dot{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$, there are sequences $\{(A_i, r_i, Q_i)\}_{i=1}^\infty$ in which A_i is a (p', u, σ, r_i)-block satisfying

$$f = \sum_{i=1}^{\infty} \lambda_i A_i \quad a.e. x \in \mathbb{R}^n \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i| < \infty.$$

From this, we have $supp(\chi_{Q_R \backslash Q_{R^{-1}}} A_i) \subset Q_R \backslash Q_{R^{-1}}$ and $supp(A_i - \chi_{Q_R \backslash Q_{R^{-1}}} A_i) \subset Q \cap Q_i$ for any block A_i and some $Q \in \mathcal{Q}$.

Let $g_k = \sum_{i=1}^{k} \lambda_i \chi_{Q_R \backslash Q_{R^{-1}}} A_i$. It will suffice to show that $g_k \in \dot{\mathscr{H}}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$. By definition of the space $\dot{\mathscr{H}}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ and Lemma 4, we have

$$\|A_i - \chi_{Q_R \backslash Q_{R^{-1}}} A_i\|_{\dot{H}_\sigma(\mathscr{H}_u^{(p')})} \leq r^\sigma u(Q)|Q| \cdot \|A_i - \chi_{Q_R \backslash Q_{R^{-1}}} A_i\|_{L^{(p')}(Q)}. \tag{3.1}$$

Combining (3.1), we can obtain the following inequality:

$$\begin{aligned} \|f - g_k\|_{\dot{H}_\sigma(\mathscr{H}_u^{(p')})} &\leq \sum_{i=1}^{k} \lambda_i \|A_i - \chi_{Q_R \backslash Q_{R^{-1}}} A_i\|_{\dot{H}_\sigma(\mathscr{H}_u^{(p')})} + \left\| \sum_{i=k+1}^{\infty} \lambda_i A_i \right\|_{\dot{H}_\sigma(\mathscr{H}_u^{(p')})} \\ &\leq \sum_{i=1}^{k} \lambda_i r^\sigma u(Q)|Q| \cdot \|A_i - \chi_{Q_R \backslash Q_{R^{-1}}} A_i\|_{L^{(p')}(Q)} + \sum_{i=k+1}^{\infty} |\lambda_i|. \end{aligned}$$

Therefore, we can deduce that $\|f - g_k\|_{\dot{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)} \to 0$ when $R \to \infty$ and $k \to \infty$.

It implies that the space $\dot{\mathscr{H}}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ is dense in $\dot{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$.

Theorem 1. Let $\sigma \geq 0$, $p \in (1, \infty)$, and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue-measurable function. The spaces $\dot{H}_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ and $H_\sigma(\mathscr{H}_u^{(p')})(\mathbb{R}^n)$ are the predual spaces of $\dot{B}_\sigma(M_u^{p)})(\mathbb{R}^n)$ and $B_\sigma(M_u^{p)})(\mathbb{R}^n)$, respectively.

Proof. Let $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$ and $g \in H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$. By definition of the space $H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$, for any $\epsilon > 0$, a decomposition $g = \sum_{i=1}^\infty \lambda_i A_i$ exists such that

$$\sum_{i=1}^\infty |\lambda_i| \le (1+\epsilon)\|g\|_{H_\sigma(\mathscr{H}_u^{(p'})}. \tag{3.2}$$

In addition, by the Hölder inequality on grand Lebesgue spaces and small Lebesgue spaces and (3.2),

$$\begin{aligned}
\|fg\|_{L^1} &\le \sum_{i=1}^\infty |\lambda_i| \int_{\mathbb{R}^n} |f\chi_{Q_i\cap Q_{r_i}} A_i| dx \\
&\lesssim \sum_{i=1}^\infty |\lambda_i| \cdot |Q_i| \cdot \|f\chi_{Q_i\cap Q_{r_i}}\|_{L^{p)}(Q_i)} \|A_i\|_{L^{(p'}(Q_i)} \\
&\lesssim \sum_{i=1}^\infty |\lambda_i| \cdot \|f\chi_{Q_i\cap Q_{r_i}}\|_{L^{p)}(Q_i)} \frac{1}{r_i^\sigma u(Q_i)} \\
&\lesssim \left(\sum_{i=1}^\infty |\lambda_i|\right) \|f\|_{B_\sigma(M_u^{p)})} \\
&\lesssim (1+\epsilon)\|g\|_{H_\sigma(\mathscr{H}_u^{(p'})} \|f\|_{B_\sigma(M_u^{p)})}.
\end{aligned}$$

Hence, we get

$$\|fg\|_{L^1} \lesssim \|g\|_{H_\sigma(\mathscr{H}_u^{(p'})} \|f\|_{B_\sigma(M_u^{p)})}. \tag{3.3}$$

Moreover, we define the functional $L_f : H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n) \to \mathbb{C}$ as

$$L_f(g) = \int_{\mathbb{R}^n} f(x)g(x)dx, \quad g \in H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n).$$

By the definition of the norm of the operator L_f on $H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$,

$$\|L_f\|_{(H_\sigma(\mathscr{H}_u^{(p'}))^*} := \sup_{g\neq 0} \frac{1}{\|g\|_{H_\sigma(\mathscr{H}_u^{(p'})}} \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right|.$$

Thus, it can be inferred that

$$\|L_f\|_{(H_\sigma(\mathscr{H}_u^{(p'}))^*} \lesssim \sup_{g\neq 0} \frac{\|g\|_{H_\sigma(\mathscr{H}_u^{(p'})} \|f\|_{B_\sigma(M_u^{p)})}}{\|g\|_{H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)}} \lesssim \|f\|_{B_\sigma(M_u^{p)})}. \tag{3.4}$$

Let $L : H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n) \to \mathbb{C}$ is a bounded linear functional. Furthermore, for $r \ge 1$ and $Q \in \mathcal{Q}$, the functional $L_{r,Q} : L^{(p'}(Q) \to \mathbb{C}$ is defined by

$$L_{r,Q}(g) = L\left(\frac{\chi_{Q\cap Q_r} g}{r^\sigma u(Q)|Q|}\right), \quad g \in L^{(p'}(Q).$$

According to the definition of the norm of the operator $L_{r,Q}$ on $L^{(p'}(Q)$, it can be inferred that

$$\|L_{r,Q}\|_{(L^{(p'}(Q))^*} := \sup_{g\neq 0} \frac{|L\left(\frac{\chi_{Q\cap Q_r} g}{r^\sigma u(Q)|Q|}\right)|}{\|g\|_{L^{(p'}(Q)}}.$$

From Lemma 2 and the Riesz representation theorem, for each cube Q and $r \geq 1$, there is a $f_{r,Q} \in L^{p)}(Q)$ such that

$$L_{r,Q}(g) = \int_{\mathbb{R}^n} f_{r,Q}(x)g(x)dx,$$

for all $g \in L^{(p'}(Q)$ and that $\|f_{r,Q}\|_{L^{p)}(Q)} = \|L_{r,Q}\|_{(L^{(p'}(Q))^*}$.

By using the boundedness of the functional L and Lemma 4

$$\begin{aligned} |L_{r,Q}(g)| &\leq \|L\|_{(H_\sigma(\mathscr{H}_u^{(p'}))^*} \left\| \frac{\chi_{Q\cap Q_r} g}{r^\sigma u(Q)|Q|} \right\|_{H_\sigma(\mathscr{H}_u^{(p'})} \\ &\leq \|L\|_{(H_\sigma(\mathscr{H}_u^{(p'}))^*} \|g\|_{L^{(p'}(Q)}. \end{aligned}$$

Hence,

$$\|f_{r,Q}\|_{L^{p)}(Q)} = \|L_{r,Q}\|_{(L^{(p'}(Q))^*} \leq \|L\|_{(H_\sigma(\mathscr{H}_u^{(p'}))^*}.$$

By the definition of the functional L, for any $r \geq 1$ and $Q \in \mathcal{Q}$, we have

$$L(\chi_{Q\cap Q_r} g) = \int_{\mathbb{R}^n} r^\sigma |Q| u(Q) f_{r,Q}(x) g(x) dx.$$

Thus, when $1 \leq r_1 \leq r_2$ and $Q_1 \subset Q_2$, we have

$$L\left(\chi_{Q_1\cap Q_{r_1}} g\right) = L\left(\chi_{Q_1\cap Q_{r_1}} \chi_{Q_1\cap Q_{r_1}} g\right) = L\left(\chi_{Q_2\cap Q_{r_2}} \chi_{Q_1\cap Q_{r_1}} g\right).$$

Furthermore, it can be concluded that

$$r_1^\sigma |Q| u(Q) f_{r_1,Q_1} = r_2^\sigma |Q| u(Q) f_{r_2,Q_2} \quad a.e. x \in Q_1 \cap Q_{r_1}.$$

That implies the definition of the function f independent of r and Q as follows:

$$f = r^\sigma u(Q)|Q| \cdot f_{r,Q} \quad a.e. x \in Q \cap Q_r.$$

It can then be concluded that

$$\frac{1}{r^\sigma |Q| u(Q)} \|f \chi_{Q\cap Q_r}\|_{L^{p)}(Q)} \leq \|f_{r,Q}\|_{L^{p)}(Q)} \leq \|L\|_{(H_\sigma(\mathscr{H}_u^{(p'}))^*},$$

and

$$\|f\|_{B_\sigma(M_u^{p)})} \leq \|L\|_{(H_\sigma(\mathscr{H}_u^{(p'}))^*}. \tag{3.5}$$

Thus, for all $g \in L^{(p}(Q)$, it can be concluded that

$$\int_{\mathbb{R}^n} f_{r,Q}(x)g(x)dx = \frac{1}{r^\sigma |Q| u(Q)} L_f\left(\chi_{Q\cap Q_r} g\right)$$

and

$$\int_{\mathbb{R}^n} f_{r,Q}(x)g(x)dx = L\left(\frac{1}{r^\sigma |Q| u(Q)} \chi_{Q\cap Q_r} g\right) = \frac{1}{r^\sigma |Q| u(Q)} L\left(\chi_{Q\cap Q_r} g\right).$$

Therefore, L is identical on the (p', u, σ, r_i)-block. The set of finite linear combinations of the (p', u, σ, r_i)-block is $\mathscr{H}_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$, and through Lemma 5, (3.4), and (3.5), we get $L = L_f$ and

$$\|f\|_{B_\sigma(M_u^{p)})} \sim \|L_f\|_{(H_\sigma(\mathscr{H}_u^{(p'}))^*}.$$

Corollary 1. If $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$ for all $g \in H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |f(x)g(x)|dx < \infty$, then for all $g \in H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$, the following equivalence holds:

$$\|f\|_{B_\sigma(M_u^{p)})} \sim \sup\left\{\int_{\mathbb{R}^n} |f(x)g(x)|dx : \|g\|_{H_\sigma(\mathscr{H}_u^{(p'})} \leq 1\right\}.$$

4. Extrapolation

In this section, we extend the extrapolation theory which is initially introduced by Rubio de Francia in [22] to B_σ-type grand Morrey spaces. The explanation of extrapolation theory is based on Muckenhoupt weight functions. Thus, we start by revisiting the definition of Muckenhoupt weight functions.

Definition 8.

1) Let $1 < p < \infty$. We say that a locally integrable function $\omega : \mathbb{R}^n \to (0, \infty)$ belongs to A_p weight if

$$[\omega]_{A_p} := \sup_{Q\in\mathcal{Q}}\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty.$$

2) We say that a locally integrable function $\omega : \mathbb{R}^n \to (0, \infty)$ belongs to the A_1 weight if for any $Q \in \mathcal{Q}$

$$\frac{1}{|Q|}\int_Q \omega(y)dy \leq C\omega(x) \quad a.e. x \in Q,$$

for some constants $C > 0$. The infimum of all such C is denoted by $[\omega]_{A_1}$.

3) We define $A_\infty := \bigcup_{p\geq 1} A_p$.

Lemma 6. Let $\sigma \geq 0$, $p \in (1, \infty)$, and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue-measurable function. If for any $Q \in \mathcal{Q}$, u satisfies

$$\sum_{i=1}^{\infty} u(2^i Q) \leq Cu(Q). \tag{4.1}$$

Then, Hardy–Littlewood maximal operator M is then bounded on $\dot{H}_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$ and $H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$.

Proof. Let $f \in H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$. Then the function f can be represented as

$$f = \sum_{i=1}^{\infty} \lambda_i A_i \quad a.e. x \in \mathbb{R}^n.$$

Thus, it is sufficient to prove that for any (p', u, σ, r_i)-block A satisfies $\|MA\|_{H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)} \leq C$.

Let A be a (p', u, σ, r)-block. Then, perform the following decomposition:

$$MA = \chi_{2Q}MA + \sum_{i=1}^{\infty} \chi_{2^{i+1}Q\setminus 2^i Q}MA.$$

Write $A_0 := \chi_{2Q}MA$ and $A_i := \chi_{2^{i+1}Q\setminus 2^i Q}MA$.

By Lemma 3, a constant C which is independent of Q exists such that

$$\begin{aligned}\|A_0\|_{L^{(p'}(2Q)} &\le C\|A\|_{L^{(p'}(2Q)} \le C\|A\|_{L^{(p'}(Q)}\\ &\le C\frac{1}{u(Q)|Q|r^{\sigma}} = C\frac{1}{u(2Q)|2Q|r^{\sigma}}\frac{u(2Q)}{u(Q)}.\end{aligned}$$

Thus, $\frac{u(Q)}{Cu(2Q)}A_0$ is a (p', u, σ, r)-block and

$$\|A_0\|_{H_\sigma(\mathscr{H}_u^{(p'})} \le C\frac{u(2Q)}{u(Q)}. \tag{4.2}$$

Further, for any $i \in \mathbb{N}^+$ and $x \in 2^{i+1}Q\backslash 2^iQ$, we have

$$\begin{aligned}A_i &\le \frac{1}{|2^iQ|}\int_{Q\cap Q_r}|A(y)|dy\\ &\le C\frac{|Q|}{|2^iQ|}\|A\|_{L^{(p'}(Q)}\|\chi_Q\|_{L^{p)}(Q)}\\ &\le C\frac{1}{r^{\sigma}u(Q)}\frac{1}{|2^iQ|}.\end{aligned}$$

Applying the norm $\|\cdot\|_{L^{(p}(2^{i+1}Q)}$ on both sides of the inequality above, we have

$$\begin{aligned}\|A_i\|_{L^{(p}(2^{i+1}Q)} &\le C\frac{1}{r^{\sigma}u(Q)}\frac{1}{|2^iQ|}\|\chi_{2^{i+1}Q\backslash 2^iQ}\|_{L^{(p}(2^{i+1}Q)}\\ &\le C\frac{1}{r^{\sigma}u(Q)|Q|}\frac{|Q|}{|2^{i+1}Q|}\\ &= C\frac{1}{r^{\sigma}u(2^{i+1}Q)|2^{i+1}Q|}\cdot\frac{u(2^{i+1}Q)}{u(Q)},\end{aligned}$$

where the constant C is independent of Q. Thus, $\frac{u(Q)}{Cu(2^{i+1}Q)}A_i$ is a (p', u, σ, r)-block and

$$\|A_i\|_{H_\sigma(\mathscr{H}_u^{(p'})} \le C\frac{u(2^{i+1}Q)}{u(Q)}. \tag{4.3}$$

Then, by Inequality (4.1)–(4.3), for any (p', u, σ, r_i)-block A, a constant C exists such that

$$\|MA\|_{H_\sigma(\mathscr{H}_u^{(p'})} \le C\frac{1}{u(Q)}\sum_{i=1}^{\infty}u(2^iQ) \le C.$$

Hence, we deduce that $MA \in H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$. Similarly, it can ultimately be concluded that M is bounded on $\dot{H}_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$ and $H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$.

Drawing on the boundedness of the Hardy–Littlewood maximal operator M, we establish the subsequent iteration algorithm generated by this operator.

Definition 9. Let $\sigma \geq 0$, $p \in (1, \infty)$, $q \in (0, p)$, and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue-measurable function. If u satisfies

$$\sum_{i=1}^{\infty} u^q(2^i Q) \leq C u^q(Q) \tag{4.4}$$

for any $Q \in \mathcal{Q}$. Define B as the operator norm of M on $H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})(\mathbb{R}^n)$, i.e.,

$$B = \|M\|_{H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})(\mathbb{R}^n) \to H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})(\mathbb{R}^n)}.$$

For any non-negative locally integral function h, the iteration algorithm $\mathscr{R}$ is defined by

$$\mathscr{R}h := \sum_{k=0}^{\infty} \frac{M^k h}{2^k B^k},$$

where M^k is the k-th iterations of M and we write $M^0 h = h$.

Proposition 2. Let $\sigma \geq 0$, $p \in (1, \infty)$, $q \in (0, p)$, and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgu-measurable function. If u satisfies (4.4) for $Q \in \mathcal{Q}$, for any $h \in H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})(\mathbb{R}^n)$, the operator $\mathscr{R}$ has the following properties:

$$h(x) \leq \mathscr{R}h(x),$$
$$\|\mathscr{R}h\|_{H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})} \leq 2\|h\|_{H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})},$$
$$[\mathscr{R}h]_{A_1} \leq 2B.$$

These properties can be deduced from the definition of the operator $\mathscr{R}$ and the boundedness of the Hardy–Littlewood maximal operator M on the space $H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n)$.

Theorem 2. Let $\sigma \geq 0$, $p \in (1, \infty)$, $q \in (0, p)$, and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue-measurable function. If u satisfies (4.4) for any $Q \in \mathcal{Q}$, and if $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$ and g is a non-negative Lebesgue-measurable function such that for every

$$\omega \in \left\{ \mathscr{R}h : h \in H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})(\mathbb{R}^n) \quad \text{and} \quad \|h\|_{H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})} \leq 1 \right\},$$

that satisfy

$$\int_{\mathbb{R}^n} |g(x)|^q \omega(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^q \omega(x) dx < \infty,$$

this implies that $g \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$ and

$$\|g\|_{B_\sigma(M_u^{p)})} \leq C\|f\|_{B_\sigma(M_u^{p)})}.$$

Proof. From Corollary 1, we can obtain

$$\begin{aligned} \|g\|_{B_\sigma(M_u^{p)})}^q &= \|g^q\|_{B_{\sigma q}(M_{u^q}^{\frac{p}{q})})} \\ &\sim \sup\left\{ \left| \int_{\mathbb{R}^n} g(x)^q h(x) dx \right| : \|h\|_{H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})} \leq 1 \right\}. \end{aligned}$$

By Proposition 2,

$$
\begin{aligned}
|\int_{\mathbb{R}^n} g(x)^q h(x)dx| &\le \int_{\mathbb{R}^n} |g(x)^q| \mathscr{R}h(x)dx \\
&\le C\int_{\mathbb{R}^n} |f(x)^q| \mathscr{R}h(x)dx \\
&\le C\|f^q\|_{B_{\sigma q}(M_{u^q}^{\frac{p}{q})})} \|h\|_{H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})} \\
&\le C\|f\|^q_{B_\sigma(M_u^{p)})}.
\end{aligned}
$$

Further, it implies that

$$
\|g\|_{B_\sigma(M_u^{p)})} \le C\|f\|_{B_\sigma(M_u^{p)})}.
$$

Corollary 2. Let $\sigma \ge 0$, $p \in (1,\infty)$, $q \in (0,p)$, and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue-measurable function. If u satisfies (4.4) for any $Q \in \mathcal{Q}$, we assume that for every

$$
\omega \in \left\{ \mathscr{R}h : h \in H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})(\mathbb{R}^n) \quad \text{and} \quad \|h\|_{H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})} \le 1 \right\},
$$

the operators $T : L^q_\omega(\mathbb{R}^n) \to L^q_\omega(\mathbb{R}^n)$ and $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$ satisfy

$$
\int_{\mathbb{R}^n} |Tf(x)|^q \omega(x)dx \le C\int_{\mathbb{R}^n} |f(x)|^q \omega(x)dx.
$$

We then have

$$
\|Tf\|_{B_\sigma(M_u^{p)})} \le C\|f\|_{B_\sigma(M_u^{p)})}.
$$

Remark 2. In this section, although we only proved the extrapolation theory on the space $B_\sigma(M_u^{p)})(\mathbb{R}^n)$, the extrapolation theory on the space $\dot{B}_\sigma(M_u^{p)})(\mathbb{R}^n)$ can be established because the proof is similar.

5. Operators on the $B_\sigma(M_u^{p)})(\mathbb{R}^n)$

In this section, we will use the extrapolation theorem to prove the boundedness of the Calderón–Zygmund operator and the commutators in $B_\sigma(M_u^{p)})(\mathbb{R}^n)$ space.

5.1. The Calderón–Zygmund operator

The Calderón–Zygmund singular integral operator, a classic operator in harmonic analysis, was introduced by Calderón and Zygmund [29]. Since its introduction, the Calderón–Zygmund singular integral operator has been extensively studied due to its wide application in partial differential equations, mathematical physics, signal processing, and other fields.

First, we review the definition of the Calderón–Zygmund operator from [30] and the related lemma.

Definition 10. A continuous linear operator T is a Calderón–Zygmund-type singular operator if T is bounded on $L^2(\mathbb{R}^n)$ and T has the kernel function $K : \mathbb{R}^{2n} \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ such that

$$
Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \quad x \notin supp(f)
$$

for all $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, and there is a constant $C > 0$ such that the kernel function K satisfies

$$|K(x,y)| \le \frac{C}{|x-y|^n}, \quad x \ne y,$$

$$|K(x,y) - K(x,z)| \le C\frac{|y-z|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0 \quad if \quad |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \le C\frac{|x-\xi|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0 \quad if \quad |x-y| > 2|x-\xi|.$$

Lemma 7. [31] Let $p \in (1,\infty)$ and T be a Calderón–Zygmund-type singular operator. If $\omega \in A_p$, then for any $f \in L^p_\omega$, we have

$$\int_{\mathbb{R}^n} |Tf|^p \omega(x)dx \le C \int_{\mathbb{R}^n} |f|^p \omega(x)dx.$$

We then prove the boundedness of the Calderón–Zygmund operator in $B_\sigma(M_u^{p)})(\mathbb{R}^n)$ through the extrapolation theorem.

Theorem 3. Let $\sigma \ge 0$, $p \in (1,\infty)$, $q \in (1,p)$, and T be a Calderón–Zygmund-type singular operator and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue-measurable function. If u satisfies (2.1), (2.2), and (4.4) for any $Q \in \mathcal{Q}$, then for any $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$

$$\|Tf\|_{B_\sigma(M_u^{p)})} \le C\|f\|_{B_\sigma(M_u^{p)})}.$$

Proof. Let $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$ and any $h \in H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})(\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^q \mathscr{R}h(x)dx &\le C\|f^q\|_{B_{\sigma q}(M_{u^q}^{\frac{p}{q})})}\|\mathscr{R}h\|_{H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})} \\ &\le C\|f\|^q_{B_\sigma(M_u^{p)})}\|h\|_{H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})} < \infty.\end{aligned}$$

By Proposition 2 and Lemma 7, we get $f(x) \in L^q_{\mathscr{R}h}$ and

$$\int_{\mathbb{R}^n} |Tf(x)|^q \mathscr{R}h(x)dx \le C \int_{\mathbb{R}^n} |f(x)|^q \mathscr{R}h(x)dx.$$

By Corollary 2, we have

$$\|Tf\|_{B_\sigma(M_u^{p)})} \le C\|f\|_{B_\sigma(M_u^{p)})}.$$

5.2. Commutators

In the research framework of harmonic analysis, the commutators generated by the Calderón–Zygmund operator play an important role. These commutators play a crucial role in promoting the regularity analysis of solutions to elliptic, parabolic, and hyper parabolic partial differential equations [32–34]. The Coifman–Rochberg–Weiss theorem [35] pioneered the study of commutators, constructed a profound operator description for BMO spaces, and established an essential connection between the abstract definition of function spaces and the specific boundedness conditions of singular integral operators. It is worth noting that scholars such as Rochberg and Weiss [36] have proposed and deeply explored a class of nonlinear commutators with more complex structures. Subsequent studies have further extended and applied their boundedness theory to a broader range of function space systems on this basis. For instance, it has been extended to new functional frameworks such as Morrey–Banach spaces [37], and a systematic theoretical generalization has been formed.

We review the commutators generated by the Calderón–Zygmund operator in this section through the lens of [38]. They involve the higher-order commutators and the nonlinear commutators.

Definition 11. [38] Let $b \in \mathrm{BMO}$, $k \in \mathbb{N}$, and T be a Calderón–Zygmund type singular operator. For any $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$, the higher-order commutators T_b^k are defined as

$$T_b^k f(x) := \int_{\mathbb{R}^n} (b(x) - b(y))^k K(x, y) f(y) dy.$$

Lemma 8. [38] Let $p \in (0, \infty)$, $\omega \in A_\infty$, and $b \in \mathrm{BMO}$. We obtain

$$\int_{\mathbb{R}^n} |T_b^k f(x)|^p \omega(x) dx \le C \|b\|_{BMO}^{kp} [\omega]_{A_\infty}^{(k+1)p} \int_{\mathbb{R}^n} M^{k+1} f(x)^p \omega(x) dx,$$

where M is the Hardy–Littlewood maximal operator.

Theorem 4. Let $\sigma \ge 0$, $p \in (1, \infty)$, $q \in (1, p)$, T be a Calderón–Zygmund type singular operator, and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue-measurable function. If u satisfies (2.1), (2.2), and (4.4) for any $Q \in \mathcal{Q}$, $k \in \mathbb{N}$ and $b \in \mathrm{BMO}$, then for any $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$, we have

$$\|T_b^k f\|_{B_\sigma(M_u^{p)})} \le C \|f\|_{B_\sigma(M_u^{p)})}.$$

Proof. Let $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$, for any $h \in H_{\sigma q}(\mathscr{H}_{u^q}^{((\frac{p}{q})'})(\mathbb{R}^n)$ and $\omega \in A_\infty$. When $q > 1$, the M is bounded in $L^q_{\mathscr{R}h}$-space. Therefore, we can deduce from Proposition 2 and Lemma 8 that

$$\begin{aligned}\int_{\mathbb{R}^n} |T_b^k f(x)|^q \omega(x) dx &\le C \|b\|_{BMO}^{kq} [\omega]_{A_\infty}^{(k+1)q} \int_{\mathbb{R}^n} M^{k+1} f(x)^q \mathscr{R}h(x) dx \\ &\le C \int_{\mathbb{R}^n} |f(x)|^q \mathscr{R}h(x) dx.\end{aligned}$$

By Corollary 2,

$$\|T_b^k f\|_{B_\sigma(M_u^{p)})} \le C \|f\|_{B_\sigma(M_u^{p)})}.$$

Definition 12. [38] Let T be a Calderón–Zygmund type singular operator. For any $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$, the nonlinear order commutator N is defined as

$$Nf := T(f \ln |f|) - Tf \ln(|Tf|).$$

Lemma 9. [38] Let $p \in (1, \infty)$ and $\omega \in A_p$. Then a constant $C > 0$ exists such that

$$\int_{\mathbb{R}^n} |Nf(x)|^p \omega(x) dx \le C [\omega]_{A_p}^{3p} \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx.$$

Theorem 5. Let $\sigma \geq 0$, $p \in (1, \infty)$, $q \in (1, p)$, and T be a Calderón–Zygmund type singular operator and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ is a Lebesgue-measurable function. If u satisfies (2.1), (2.2), and (4.4) for any $Q \in \mathcal{Q}$, then for any $f \in B_\sigma(M_u^{p)})(\mathbb{R}^n)$, we have

$$\|Nf\|_{B_\sigma(M_u^{p)})} \leq C\|f\|_{B_\sigma(M_u^{p)})}.$$

Proof. By direct observation, the conditions for Corollary 2 are satisfied. Therefore, the conclusion $\|Nf\|_{B_\sigma(M_u^{p)})} \leq C\|f\|_{B_\sigma(M_u^{p)})}$ is valid.

6. A new characterization of BMO

The BMO space introduced by John and Nirenberge [39] can be seen as a natural generalization of the essentially bounded function space $L^\infty(\mathbb{R}^n)$. This generalization can be used to solve the endpoint estimation of classical singular integral operators and commutators. Therefore, BMO plays a crucial role in harmonic analysis and has been studied deeply by researchers, with the characterization of BMO becoming an important topic.

In this section, we will provide another characterization of BMO space through the extrapolation theorem. For the convenience of description, we define $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$. Before defining the new BMO space, we first perform some preliminary work.

Lemma 10. (John–Nirenberg inequality) For any $Q \in \mathcal{Q}$, $f \in$ BMO and $\|f\|_{BMO} \neq 0$, we have

$$|\{x \in Q : |f(x) - f_Q| > t\}| \leq Ce^{-\frac{C_1 t}{\|f\|_{BMO}}}|Q|.$$

Theorem 6. Let $\sigma \geq 0$, $p \in (1, \infty)$, $q \in (1, p)$, and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue-measurable function. If u satisfies (2.1), (2.2), and (4.4) for any $Q \in \mathcal{Q}$, such that $C, C_1 > 0$ for any $t > 0$, $f \in$ BMO, and $\|f\|_{BMO} \neq 0$, we have

$$\|\chi_{\{x\in Q:|f(x)-f_Q|>t\}}\|_{B_\sigma(M_u^{p)})} \leq Ce^{-\frac{C_1 t}{\|f\|_{BMO}}}\|\chi_Q\|_{B_\sigma(M_u^{p)})}.$$

Proof. For any $\omega \in A_1 \subset A_\infty$, $\epsilon > 0$ exists such that for any Lebesgue-measurable set $E \subset Q$, we can obtain

$$\frac{\omega(E)}{\omega(Q)} \leq C(\frac{|E|}{|Q|})^\epsilon, \tag{6.1}$$

which, by the definition of $\omega(E)$, is $\omega(E) := \int_E \omega(x)dx$.

By combining the John–Nirenberg inequality and (6.1)

$$\int_{\mathbb{R}^n} \chi_{\{x\in Q:|f(x)-f_Q|>t\}}(x)\omega(x)dx \leq Ce^{-\frac{C_1 t}{\|f\|_{BMO}}}\int_{\mathbb{R}^n}\chi_Q(x)\omega(x)dx.$$

Therefore, the conditions of Theorem 2 are all satisfied, so we have

$$\|\chi_{\{x\in Q:|f(x)-f_Q|>t\}}\|_{B_\sigma(M_u^{p)})} \leq Ce^{-\frac{C_1 t}{\|f\|_{BMO}}}\|\chi_Q\|_{B_\sigma(M_u^{p)})}.$$

Next, we define a new type of BMO space based on the $B_\sigma(M_u^{p)})(\mathbb{R}^n)$ space.

Definition 13. Let $\sigma \geq 0$, $p \in (1,\infty)$, $q \in (1,p)$, and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue-measurable function. If u satisfies (2.1), (2.2), and (4.4) for any $Q \in \mathcal{Q}$, the $BMO_{B_\sigma(M_u^{p)})}(\mathbb{R}^n)$ is defined by

$$BMO_{B_\sigma(M_u^{p)})}(\mathbb{R}^n) := \left\{ f \in L_{loc} : \|f\|_{BMO_{B_\sigma(M_u^{p)})}} := \sup_{Q\in\mathcal{Q}} \frac{\|(f - f_Q)\chi_Q\|_{B_\sigma(M_u^{p)})}}{\|\chi_Q\|_{B_\sigma(M_u^{p)})}} < \infty \right\}.$$

Lemma 11. Let $\sigma \geq 0$, $p \in (1,\infty)$, $q \in (1,p)$, and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue-measurable function. If u satisfies (2.1), (2.2), and (4.4) for any $Q \in \mathcal{Q}$, then we have

$$C^{-1}|Q| \leq \|\chi_Q\|_{B_\sigma(M_u^{p)})} \cdot \|\chi_Q\|_{H_\sigma(\mathscr{H}_u^{(p'})} \leq C|Q|.$$

Proof. According to Theorem 1, it can be concluded that

$$|Q| = \int_{\mathbb{R}^n} \chi_Q^2(x)dx \leq C\|\chi_Q\|_{B_\sigma(M_u^{p)})} \cdot \|\chi_Q\|_{H_\sigma(\mathscr{H}_u^{(p'})}.$$

According to Corollaries 1 and 2, we find that

$$\begin{aligned}
&\|\chi_Q\|_{B_\sigma(M_u^{p)})} \cdot \|\chi_Q\|_{H_\sigma(\mathscr{H}_u^{(p'})} \\
&\quad =C \sup\left\{ |\int_Q g(x)dx| \cdot \|\chi_Q\|_{H_\sigma(\mathscr{H}_u^{(p'})} : g \in H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n), \|g\|_{H_\sigma(\mathscr{H}_u^{(p'})} \leq 1 \right\} \\
&\quad \leq C|Q| \sup\left\{ \|Mg\|_{H_\sigma(\mathscr{H}_u^{(p'})} : g \in H_\sigma(\mathscr{H}_u^{(p'})(\mathbb{R}^n), \|g\|_{H_\sigma(\mathscr{H}_u^{(p'})} \leq 1 \right\} \\
&\quad \leq C|Q|
\end{aligned}$$

for some $C > 0$.

Theorem 7. Let $\sigma \geq 0$, $p \in (1,\infty)$, $q \in (1,p)$, and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue-measurable function. If u satisfies (2.1), (2.2), and (4.4) for any $Q \in \mathcal{Q}$, then $\|\cdot\|_{BMO}$ and $\|\cdot\|_{BMO_{B_\sigma(M_u^{p)})}}$ are equivalent norms.

Proof. By Theorem 1 and Lemma 11, for any $f \in$ BMO,

$$\begin{aligned}
\int_{\mathbb{R}^n} |(f(x) - f_Q)|\chi_Q dx &\leq C\|(f - f_Q)\chi_Q\|_{B_\sigma(M_u^{p)})} \cdot \|\chi_Q\|_{H_\sigma(\mathscr{H}_u^{(p'})} \\
&\leq C|Q| \frac{\|(f - f_Q)\chi_Q\|_{B_\sigma(M_u^{p)})}}{\|\chi_Q\|_{B_\sigma(M_u^{p)})}}.
\end{aligned} \tag{6.2}$$

By (6.2), we have $\|f\|_{BMO} \leq C\|f\|_{BMO_{B_\sigma(M_u^{p)})}}$.

According to Theorem 6, for any $i \in \mathbb{N}$, we have

$$\begin{aligned}
\|\chi_{\{x\in Q:2^i<|f(x)-f_Q|\leq 2^{i+1}\}}\|_{B_\sigma(M_u^{p)})} &\leq \|\chi_{\{x\in Q:|f(x)-f_Q|>2^i\}}\|_{B_\sigma(M_u^{p)})} \\
&\leq Ce^{-\frac{C_1 2^i}{\|f\|_{BMO}}}\|\chi_Q\|_{B_\sigma(M_u^{p)})}.
\end{aligned}$$

Multiplying 2^{i+1} on both sides and summing over i, we have

$$\|(f - f_Q)\chi_Q\|_{B_\sigma(M_u^{p)})} \leq C\|f\|_{BMO}\|\chi_Q\|_{B_\sigma(M_u^{p)})}.$$

Hence, we obtain

$$\|f\|_{BMO_{B_\sigma(M_u^{p)})}} \leq C\|f\|_{BMO}.$$

Use of AI tools declaration

The authors declare that artificial intelligence (AI) tools played no part in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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