Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(11): 6805–6843.
DOI: 10.3934/era.2025301
Received: 06 April 2025
Revised: 28 September 2025
Accepted: 20 October 2025
Published: 17 November 2025

Research article

Equitable graphs of type II from groups: Studying and analyzing

Alaa Altassan[1], Anwar Saleh[2,*], Hanaa Alashwali[1], Marwa Hamed[1] and Najat Muthana[2,*]

[1] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

[2] Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 23218, Saudi Arabia

* **Correspondence:** Email: asaleh1@uj.edu.sa, nmuthana@uj.edu.sa.

Abstract: Algebraic graph theory explores the relationship between abstract algebra and graph theory. It uses algebraic concepts to define the structures of graphs and investigates how graph theory can characterize algebraic properties. There has been considerable scholarly interest in the connection between group-theoretic and graph-theoretic properties, especially concerning the role of symmetry in linking these two areas. This research contributes to the ongoing development of equitable graphs by introducing the concept of an equitable graph of Type II on finite groups which is defined on a group G, where two vertices x and y with different orders are adjacent if their orders are differ by at most the minimum of their orders ($|o(x) - o(y)| \leq \min\{o(x), o(y)\}$) or if one of them is the identity element. We investigate the properties of this graph for specific classes of groups, including cyclic, dihedral, and dicyclic groups. Furthermore, we derive general formulas for some degree-based indices of the equitable graph of Type II across various group families. Finally, we explore the relationship between the isomorphism of equitable graphs and the associated groups.

Keywords: finite group; equitable graph; entire topological indices; isomorphism

1. Introduction

Graphs and groups, two fundamental mathematical structures, exhibit a deep and intricate relationship, particularly when considering the concept of symmetry. This interplay offers valuable insights into the properties and applications of both. Cayley graphs provide a visual representation of groups, where the vertices correspond to the group elements, and the edges are represented by the group operation. Notably, the symmetries inherent in the Cayley graph directly reflect the underlying group structure. Highly symmetric groups often yield highly symmetric Cayley graphs. Graph automorphisms, permutations of vertices preserving the edge structure, encapsulate a graph's symmetries. The automorphism group, a group-theoretic object, fully characterizes a graph's symmetry. Graphs with high degrees of symmetry, such as complete graphs or cycle graphs, possess large automorphism groups.

The current scholarly research is concentrates on systematically exploring the connections between group-theoretic algebraic structures and graph-theoretic invariants. This investigation aims to uncover potential applications and synergies between these two areas of mathematics. Over the past decade, there has been a significant increase in scholarly contributions dedicated to examining graph-theoretic structures that are derived from group-theoretic algebraic systems. Cayley graphs are the initial concept introduced by Cayley [1]. Various other graphs were constructed on groups, such as the power graph [2, 3], the prime graph [4], the commuting graph [5], the intersection graph [6], the order divisor graph [7], and the recently defined equitable graph Type I [8]. For further information in this area, we refer to [9–11].

Investigating the parameters in graphs derived from groups such as domination, independence, and chromatic number enhances the understanding of their structural connectivity and complexity. These parameters are essential for classifying this type of graph, to explore their symmetries, and to optimize network-related applications, such as communication networks and security systems. Moreover, these parameters offer valuable insights into algebraic structures, aiding in solving problems associated with group theory, mathematical modeling, and combinatorial optimization.

The increasing knowledge of the importance of graphs associated with groups, as well as their role in classifying and characterizing both groups and graphs, has generated considerable academic interest in this area of research. Consequently, we aim to modify the definition of the equitable graph of Type I for groups [8], which is defined as a graph with a vertex set the set of elements of the group G in which any two distinct elements of G, a and b, are adjacent if and only if

$$\mid o(a) - o(b) \mid \leq \min\{o(a), o(b)\}.$$

Subsequently, we introduce and formalize the conceptual framework for a novel, secondary type of this graph-theoretic construct namely, the *equitable graph of Type II*. In this new version, we have taken different orders into account and discovered a method to make the graph connected, as the identity is adjacent to all other vertices.

For the readers, we explain the conventions and notations which are used in this research. Throughout this paper, G denotes a finite group and $|G|$ denotes the order of this group. The identity element is e and for any element $a \in G$, $o(a)$ denotes the order of a. For any prime number p, G is called a p-group if every element of G has an order power of p, whereas an elementary Abelian group (or elementary Abelian p-group) is an Abelian p-group in which each nonidentity elements has the order p. The dihedral group D_{2n} is a group of order $2n$, defined by

$$\langle a, b : a^n = b^2 = e, ab = ba^{-1}\rangle.$$

This group contains a cyclic subgroup $\langle a\rangle$ of order n, and all elements $a^i b$ have order 2. The dicyclic group is a group of order $4n$, denoted Q_{4n} and defined as

$$Q_{4n} = \langle a, b : a^{2n} = b^4 = e, a^n = b^2, ab = ba^{-1}\rangle.$$

This group contains a cyclic group $\langle a\rangle$ of order $2n$, and all elements $a^i b$ have order 4. If n is a power of 2, then this group is called the generalized quaternion group.

In a cyclic group G, if $m \mid |G|$, then at least one element $x \in G$ exists such that $o(x) = m$; and the number of the elements of G having order j is equal to $\phi(j)$, where ϕ is the *Euler's phi* function, given by $\phi(j) = |\{x \leq j : (x, j) = 1\}|$; and ϕ is a multiplicative function; that is $\phi(ab) = \phi(a) \cdot \phi(b)$, whenever a and b are relatively prime.

Let Γ be a simple graph with the vertex set V and the edge set E, where a simple graph is a graph having no loops or multiple edges. For any vertex v in V, $d(v)$ denotes the degree of v, which is equal to the number of edges incident to v, whereas the set of vertices adjacent to v is denoted by $N(v)$. Moreover, for any edge $e \in E$, the degree $d(e)$ of the edge e is the number of edges adjacent to e in the graph; that is, $d(e) = d(u) + d(v) - 2$, where u and v are the endpoints of e. The notation $\Delta(\Gamma)$ and $\delta(\Gamma)$ denote the maximum and minimum degree of Γ, respectively. The distance $d(u, v)$ between u and v is the length of the shortest u, v-path. The maximum distance between two vertices in the graph is called the diameter, diam(Γ), and the radius of the graph, $r(\Gamma)$, is equal to the minimum eccentricity of the graph, where the eccentricity of any vertex $v \in V$ is defined as $e(v) = \max\{d(v, u) : u \in V\}$. A connected graph is a graph in which any two vertices are connected by a path. It is said to be complete if and only if every pair of distinct vertices are adjacent. A complete graph of order n is denoted by K_n. A clique is a complete subgraph of Γ, and the clique number of Γ, $\omega(\Gamma)$, is the order of the maximum clique. The girth of the graph, $gr(\Gamma)$, is defined as the length of the shortest cycle in the graph. The domination number, $\gamma(\Gamma)$, is the cardinality of the minimum dominating set (γ-set) where the set S is said to be a dominating set if each vertex v in $V \setminus S$ is adjacent to at least one vertex from S. The chromatic number of Γ, denoted by $\chi(\Gamma)$, is defined to be the minimum number of colors required to label all vertices such that any two adjacent vertices have distinct colors, and the graph is perfect if and only if for each induced subgraph Λ of Γ, $\chi(\Lambda) = \omega(\Lambda)$. A set of vertices T of a connected graph is said to be a separating set (vertex cut) if $\Gamma(V - T)$ is disconnected, and the cardinality of the minimum separating set (minimum vertex cut) is called the vertex connectivity of Γ and is denoted by $\kappa(\Gamma)$. An independent set is a subset W of the vertex set V such that no two vertices in W are adjacent in Γ, and the number of vertices in the maximum independent set is called the independence number, $\alpha(\Gamma)$. A graph Γ is Eulerian if Γ has a closed spanning trail, while the graph is Hamiltonian if it has a spanning cycle. The graph is said to be planar if it can be drawn in a plane, such that its edges intersect only at the endpoints. The complement graph $\bar{\Gamma}$ of Γ is a graph with $V(\bar{\Gamma}) = V(\Gamma)$, and any two vertices u and v are adjacent in $\bar{\Gamma}$) if and only if they are nonadjacent in Γ. For more details, we refer to [12–14].

Through this research, we deal with finite groups and simple graphs. This study begins by formally introducing the definition of the second type of the equitable graph, denoted $\mathcal{E}_2(G)$, and subsequently provides a comprehensive characterization of the cases in which the graph $\mathcal{E}_2(G)$ is a star, tree, complete, path, or bipartite graph. We then explore some theoretical properties for equitable graphs of Type II and several classes of finite groups and find some relations between these properties. Next, the general formulae for several degree-based graph indices are derived, including the first and second entire indices, as well as the first and second Zagreb indices, for this graph of several group classes. Finally, we rigorously analyze and study the conditions under which it can be established that two arbitrary finite groups G and H are isomorphic if and only if their corresponding equitable graphs of Type I or of Type II, are also isomorphic. This analysis is accompanied by the presentation of relevant illustrative examples to substantiate the theoretical findings.

2. Definitions and general properties

In this section, we present the definition of equitable graphs of Type II for any finite group G. In our analysis, we provide a detailed characterization of the cases in which the graph $\mathcal{E}_2(G)$ takes the form of a star, tree, complete graph, path, or bipartite graph. We also investigate the theoretical properties of the equitable graph of Type II for specific classes of finite groups, including the chromatic number, independence number, and minimum degree. We establish connections among these properties. Additionally, we examine the planarity, Eulerian, and Hamiltonian properties of this graph for various classes of groups.

Definition 2.1. *Let G be a finite group. Then the equitable graph of Type II of G, denoted by $\mathcal{E}_2(G)$, is defined to be a graph with a vertex set is the group G and one in which any two distinct vertices a and b are adjacent if and only if either $o(a) \neq o(b)$ and $|o(a) - o(b)| \leq \min\{o(a),\ o(b)\}$ or one of them is the identity element.*

The fundamental difference between the two types of equitable graphs is that in the second type, the vertices are assigned distinct orders and each vertex is connected to the identity element. By ensuring connectivity, this construction makes the graph structurally richer and more suitable for detailed investigation than the first type.

Example 2.1. *Let G be the alternating group of degree* 4*, A_4. Then G has one element of order 1, three elements of order 2, and eight elements of order 3. The equitable graph of Type II of G can be shown in Figure 1. The description of the elements is as follows:*
v_1 denotes the identity permutation, $v_2 = (12)(34)$ $v_3 = (13)(24)$, $v_4 = (14)(23)$, $v_5 = (123)$, $v_6 = (132)$, $v_7 = (124)$, $v_8 = (142)$, $v_9 = (134)$, $v_{10} = (143)$, $v_{11} = (234)$, *and* $v_{12} = (243)$.

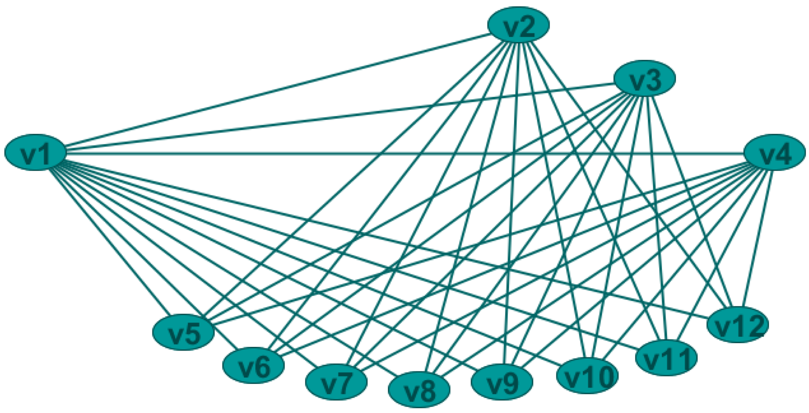


Figure 1. The equitable graph of Type II of the alternating group A_4.

For a comparison with the equitable graph Type I, Figure 2 illustrates the structure of $\mathcal{E}_1(A_4)$ with the same labeled vertices.

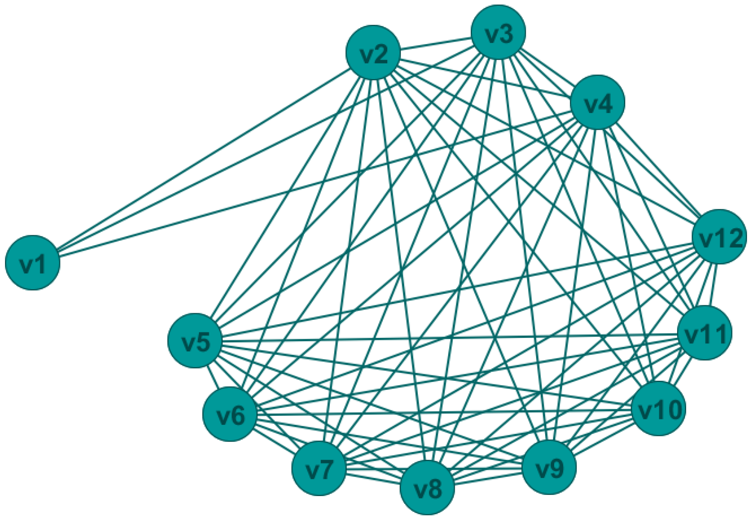


Figure 2. The equitable graph Type I of the alternating group A_4.

Note that the two conditions in Definition 2.1 are distinct vertices, with one vertex being the identity element to ensure connectivity while preserving the core combinatorial structure, as motivated by the topological requirements in [15–17].

Theorem 2.1. *Let G be a finite group. Then* $gr(\mathcal{E}_2(G)) = 3$ *or* ∞.

Proof. Let G be any finite group and assume that $\mathcal{E}_2(G)$ is not a star graph. Then, at least two adjacent vertices u and v exist which differ from the identity. Hence, from the definition of the graph, $v-u-e-v$ is a cycle of length 3. □

Theorem 2.2. *[13, Theorem 1.5.10] A graph is bipartite if and only if it contains no odd cycles.*

Theorem 2.3. *[13, Theorem 4.2.4] The number of edges in a tree on n vertices is* $n-1$. *Conversely, a connected graph on n vertices and* $n-1$ *edges is a tree.*

Observation 2.1. *Let G be a finite group of order greater than* 2. *In this case*

1) $\mathcal{E}_2(G)$ *is a connected graph.*

2) $\Delta(\mathcal{E}_2(G)) = n-1$.

3) $\operatorname{diam}(\mathcal{E}_2(G)) = 2$, *since* $|G| > 2$ *and it has at least two elements with the same order. As each vertex in this graph is adjacent to the identity, the shortest path between any nonadjacent vertices, say u and v, is* $u-e-v$, *which gives the maximum distance between all the vertices in the graph.*

4) $r(\mathcal{E}_2(G)) = 1$, *where* $r(\mathcal{E}_2(G))$ *is equal to the eccentricity of the identity element.*

5) $\gamma(\mathcal{E}_2(G)) = 1$ *and* $\{e\}$ *is a dominating set.*

6) Consider $|G| = n+1$. *Then* $\mathcal{E}_2(G)$ *is either of the form* $K_{1,n}$ *or not bipartite.*

7) The equitable graph of Type II of G, where $|G| = n \geq 3$, *is not a cycle graph . It is clear for* $n > 3$ *as* $d(e) = n-1 > 2$, *and for* $n = 3$, *that G has two elements of order* 3 *which are not adjacent.*

8) $\mathcal{E}_2(G)$ is a tree if and only if $\mathcal{E}_2(G)$ is a star graph.

9) If $\mathcal{E}_2(G)$ is a star graph, then $\delta(\mathcal{E}_2(G)) = 1 = \kappa(\mathcal{E}_2(G))$, and $\{e\}$ is the minimum separating set.

Theorem 2.4. *Let G be a finite group with more than two elements and let $\pi_e(G) = \{1, r_1, r_2, \dots, r_s\}$ be the orders of all nonidentity elements of G in increasing order (the spectrum of G). Then $\mathcal{E}_2(G)$ is a star graph if and only if $|r_{i+1} - r_i| > \min\{r_{i+1}, r_i\}$ for all $1 \leq i \leq s - 1$.*

Proof. Let G be any finite group such that $|G| > 2$. The proof then follows, by the definition of the graph, that $\mathcal{E}_1(G)$ is a star graph if and only if the orders of all elements of G are equal or $|r_{i+1} - r_i| > \min\{r_{i+1}, r_i\}$. □

Corollary 2.1. *Let G be a finite group and assume that G is isomorphic to one of the following groups:*

1) A p-group, where $p > 2$ is a prime number.

2) An elementary Abelian 2-group.

3) A dihedral group D_{2n}, where $n = 2$ or $n = p^k$, where $p > 3$ is a prime number and $k \geq 1$.

Then $\mathcal{E}_2(G)$ is a star graph.

Definition 2.2. *A connected graph Γ is called an equitable graph of of Type II if and only if a finite group exists G such that $\Gamma \cong \mathcal{E}_2(G)$.*

Proposition 2.1. *Let G be a finite group of order n. The following then hold:*

1) If Γ is an equitable graph of Type II, then $\Delta(\Gamma(G)) = n - 1$.

2) A complete graph of order $n \geq 2$ is an equitable graph of Type II if and only if $n = 2$.

3) A path graph of order $n \geq 2$ is an equitable graph of Type II if and only if $n = 2$ or $n = 3$.

Proof. Let G be a finite group of order n. In this case,

1) This is evident.

2) According to the definition of the equitable graph of Type II we find that any complete graph is an equitable graph of Type II if and only if every vertex is of full degree if and only if, for any order d of the elements of the group, G has at most one element of order d. Therefore, G contains no elements of order ≥ 3. Thus, $n \leq 2$, and we get $n = 2$. On the other hand, it is clear that in $n = 2$, $\Gamma \cong \mathcal{E}_2(\mathbb{Z}_2)$.

3) It is known that any path graph P_n is an equitable graph of Type II of the group G if and only if $d(v) \leq 2$ for each vertex $v \in G$, so by the first point, we have $n - 1 \leq 2$; that is, $n \leq 3$. However, $n \geq 2$, and we get the result. The other direction is obvious, since we have $\Gamma \cong \mathcal{E}_2(\mathbb{Z}_2)$ if $n = 2$, and $\Gamma \cong \mathcal{E}_2(\mathbb{Z}_3)$ if $n = 3$.

□

2.1. Minimum degree and vertex connectivity

In this subsection, we analyze the minimum degree of the equitable graph of Type II for certain finite groups and examine its relation to the vertex connectivity of this graph.

Theorem 2.5. *Let G be a cyclic group of order n. The following then hold:*

1) If n is odd, then $\delta(\mathcal{E}_2(G)) = 1$.

2) If $n = 2m$, m is odd, and $3 \nmid m$, then $\delta(\mathcal{E}_2(G)) = 1$.

3) If $n = 2 \cdot 3 \cdot m$, and m is odd, then $\delta(\mathcal{E}_2(G)) = 3$.

4) If $n = 2^2 \cdot m$, m is odd, $3 \nmid m$, $5 \nmid m$, and $7 \nmid m$, then $\delta(\mathcal{E}_2(G)) = 2$.

5) If $n = 2^2 \cdot m$, m is odd, $3 \nmid m$, and either $5|m$ or $7|m$, then $\delta(\mathcal{E}_2(G)) = 3$.

6) If $n = 2^2 \cdot 3 \cdot m$, m is odd, $3 \nmid m$, $5 \nmid m$, $7 \nmid m$, $11 \nmid m$, $13 \nmid m$, $17 \nmid m$, $19 \nmid m$, and $23 \nmid m$, then $\delta(\mathcal{E}_2(G)) = 3$.

7) If $n = 2^2 \cdot 3 \cdot m$, and there is a prime $p \in \{3, 5, 7, 11, 13, 17, 19, 23\}$ so that $p|m$, then $\delta(\mathcal{E}_2(G)) = 5$.

8) If $n = 2^3 \cdot m$ and $3 \nmid m$, then $\delta(\mathcal{E}_2(G)) = 3$.

9) If $n = 2^3 \cdot 3 \cdot m$, then $\delta(\mathcal{E}_2(G)) = 5$.

Proof. Let G be a cyclic group of order n. In this case,

1) Suppose that n is odd and let $x \in G$ be such that $o(x) = n$. Suppose that the nonidentity element $y \in G$ is adjacent to x. It follows that $o(x) \leq 2 \cdot o(y)$. Since we have $o(y)|o(x)$, we obtain $\frac{o(x)}{o(y)} \leq 2$. Now as $o(x)$ is odd, we must have $o(x) = o(y)$, which is a contradiction. It follows that $d(x) = 1$; hence, the result is obtained.

2) Let $n = 2m$, m is odd, $3 \nmid m$, and let $x \in G$ be such that $o(x) = 2$. Also let the nonidentity element $y \in G$ is adjacent to x. It follows that $o(y) \leq 2 \cdot o(x) = 4$. In our case, G has no element of order 2 other than x and no elements of order 4, so we have a contradiction. Hence, $d(x) = 1$ and $\delta(\mathcal{E}_2(G)) = 1$, as required.

3) Consider that $n = 2 \cdot 3 \cdot m$ and m is odd. Let $x \in G$ have order 2. As in the previous point, we can see that x is adjacent to all the elements of order 3, as well as e. Hence, $d(x) = 3$. Now suppose that $y \in G$ has order 3. We can see that y is adjacent to all elements of order 2 and of order 6, and also, to elements of order 5, if there are any. In any case, we have $d(y) \geq 1 + 1 + 2 = 4$. Now let $o(y) \geq 5$ for some $y \in G$. If $o(y)$ is odd, then there is an element $z \in G$ with $z^2 = y$. It follows that $o(z) = 2 \cdot o(y)$, and hence y is adjacent to z, and thus to all elements of G of order $o(z)$. Therefore, $d(y) \geq 1 + \phi(2o(y)) = 1 + \phi(o(y))$. Hence, either $o(y)$ is divisible by 9 or by some odd prime p, $p \geq 5$. It follows that $\phi(o(y)) \geq 4$, and hence $d(y) \geq 5$. Suppose that $o(y) = 2k$ with k being odd. Then, y^2 is adjacent to y. As in the previous case, it follows that $d(y) \geq 1 + \phi(k)$. Since $k \geq 3$, $\phi(k) \geq 2$. Thus, $d(y) \geq 3$ and we have $\delta(\mathcal{E}_2(G)) = 3$.

4) Assume that $n = 2^2 \cdot m,\ m$ is odd, $3 \nmid m,\ 5 \nmid m$, and $7 \nmid m$. Suppose that $y \in G$ with $o(y) = 2$. As before, we find that $d(y) = 3$, as y is adjacent to all the elements of order 4 and no other nonidentity element. Let $x \in G$ be an element of order 4. Thus, we also see that $d(x) = 1 + 1 = 2$. Now consider that $y \in G$ with $o(y) \geq 5$. Arguing as in the previous case, we can find an element $z \in G$ such that z is adjacent to y. Thus, $d(y) \geq 2$. Therefore, we get the desired outcome.

5) Suppose that $n = 2^2 \cdot m,\ m$ is odd, $3 \nmid m$, and either $5|m$ or $7|m$. Also suppose that $x \in G$ has order 2. As before, we get $d(x) = 1 + 2 = 3$, as x is adjacent to the identity and all elements of order 4. Now, let $y \in G$ have order 4. It follows that $d(y) \geq 1 + 1 + 4 = 6$ or $d(y) \geq 1 + 1 + 6 = 8$. Consider that $o(y) \geq 5$. By a similar argument, if $o(y)$ is odd, an element $z \in G$ exists such that $z^2 = y$. Then, $d(y) \geq 1 + \phi(o(y)) \geq 1 + 2 = 3$. Similarly, if $o(y)$ is even, y and y^2 are adjacent, and $o(y^2) \geq 3$ so $d(y) \geq 1 + \phi(o(y^2)) \geq 3$. It follows that $\delta(\mathcal{E}_2(G)) = 3$, as required.

6) Let $n = 2^2 \cdot 3 \cdot m,\ m$ is odd, and none of the primes 3, 5, 7, 11, 13, 17, 19, or 23 divide m. First, let $y \in G$ have order 2. We see that y is adjacent to the identity, all elements of order 3, and all elements of order 4. Hence, $d(y) = 1 + 2 + 2 = 5$. Now, if $o(y) = 3$, it is clear that y is adjacent to the identity and, at least, to all elements of order 2, 4, and 6. Note that, in any case, we have $d(y) \geq 1 + 1 + 2 + 2 = 6$. Moreover, if $o(y) = 4$, then $d(y) \geq 1 + 1 + 2 + 2 = 6$, and if $o(y) = 6$, then $d(y) \geq 1 + 2 + 2 + 4$. Moreover, if $o(y) = 12$, then $d(y) = 1 + 2 = 3$, as the only nonidentity elements adjacent to y are those that have order 6. Now consider that $o(y)$ is odd. Hence, we can assume that $o(y) \geq 13$. By a similar argument, there is an element $z \in G$ such that $z^2 = y$, and it follows that $d(y) \geq 1 + \phi(o(y))$, which is at least 5. Otherwise, if y has an even order, then y and y^2 are adjacent and $d(y) \geq 1 + \phi(o(y^2))$. Observe that $o(y^2) \geq 7$, and thus we find that $d(y) \geq 5$ and the result is obtained.

7) Let $n = 2^2 \cdot 3 \cdot m$ and assume that there is a prime $p \in \{3, 5, 7, 11, 13, 17, 19, 23\}$ such that $p|m$. The proof is similar to the previous point, except in the case in which $o(y) = 12$, where y is adjacent to all elements of order 6 and 9 if $p = 3$, and to all elements of order p if $p > 3$. Thus, we get $d(y) \geq 1 + 2 + 6$ or $d(y) \geq 1 + 2 + p - 1 = p + 2 > 5$. Therefore, by a similar argument, we get $\delta(\mathcal{E}_2(G)) = 5$.

8) Suppose that $n = 2^3 \cdot m$ and $3 \nmid m$. Let $o(x) = 2$ for some $x \in G$. We then have $d(x) = 3$ as x is adjacent only to the identity and the elements of order 4. Similarly, if there is an element $y \in G$ with $o(y) = 4$, then $d(y) = 1 + 1 + 4 = 6$ as y is adjacent to those elements of order 2 and 8. Moreover, let $o(y) \geq 5$, if $o(y)$ is odd, then there is $z \in G$ such that $z^2 = y$. Again, by a similar method, we find that $d(y) \geq 1 + \phi(o(y)) \geq 3$. Otherwise, if $o(y)$ is even, then y^2 is adjacent to y and we have $d(y) \geq 1 + \phi(o(y^2)) \geq 3$. We conclude that $\delta(\mathcal{E}_2(G)) = 3$.

9) Consider that $n = 2^3 \cdot 3 \cdot m$ and let $x \in G$ with $o(x) = 2$. It follows that $d(x) = 1 + 2 + 2 = 5$, as x is adjacent to the identity and all elements of order 3 and 4. It is clear that $d(y) \geq 5$ for any element $y \in G$ with $o(y) \leq 12$. For instance, assume that $o(y) = 12$, in which case, y is adjacent to the identity and all elements of order 6 and 8. Thus, $d(y) \geq 1 + 2 + 4 = 7$. Now, let $o(y) \geq 13$ be an odd number. In this case, $d(y) \geq 1 + \phi(o(y)) \geq 5$. Otherwise, if $o(y)$ is even, then $d(y) \geq 1 + \phi(o(y^2)) \geq 7$. Hence, we get $\delta(\mathcal{E}_2(G)) = 5$.

□

Theorem 2.6. *Let G be the dihedral group D_{2n}. The following then hold:*

1) If $n = 3$, then $\delta(\mathcal{E}_2(G)) = 3$.

2) If n is odd, and $n > 3$, then $\delta(\mathcal{E}_2(G)) = 1$.

3) If $n = 2m$, m is odd, and $3 \nmid m$, then $\delta(\mathcal{E}_2(G)) = 1$.

4) If $n = 2 \cdot 3 \cdot m$, and m is odd, then $\delta(\mathcal{E}_2(G)) = 3$.

5) If $n = 2^2 \cdot m$, and $3 \nmid m$, then $\delta(\mathcal{E}_2(G)) = 3$.

6) If $n = 2^2 \cdot 3 \cdot m$, then $\delta(\mathcal{E}_2(G)) = 5$.

Proof. Let G be the dihedral group D_{2n}. In this case,

1) Assume that $n = 3$. G has elements of order 1, 2, and 3. We have $d(e) = 5$. Moreover, for any element x of order 2 in G, then $d(x) = 1 + 2 = 3$, and for any element $y \in G$ of order 3, $d(y) = 1 + 3 = 4$. Hence, we conclude that $\delta(\mathcal{E}_2(G)) = 3$.

2) Let n be an odd number greater than 3, and let $x \in G$ be the element of order n. Suppose that $y \in G$ is a neighbor of x. If $o(y) = 2$, then n must be 3, which is a contradiction. Now $o(x) \leq 2o(y)$ and hence $\frac{o(x)}{o(y)} \leq 2$. Since n is odd, we get $o(x) = o(y)$, which is a contradiction. It follows that $d(x) = 1$, and thus we get the desired result.

3) Suppose that $n = 2m$, m is odd, and $3 \nmid m$. Let $x \in G$ with $o(x) = 2$. Then $d(x) = 1$, since there are no elements of order 3 or 4.

4) Let $n = 2 \cdot 3 \cdot m$, and m be odd. Let $x \in G$ be of order 2. We then see that $d(x) = 1 + 2 = 3$. Suppose that $y \in G$ is of order 3. Then $d(y) \geq 1 + (n + 1)$, since in this case, g has $n + 1$ elements of order 2. Now consider that $o(y) \geq 5$. If $o(y)$ is odd, then $z \in G$ exists such that $z^2 = y$. Hence, y is adjacent to elements that have an order $o(z) = 2o(y)$. Thus, $d(y) \geq 1 + \phi(o(y))$, which is at least 3, as $o(y)$ is odd. Otherwise, if $o(y) = 2k$, $k \geq 3$. Then y^2 has order k and is adjacent to y. We see that $d(y) \geq 1 + \phi(k) \geq 3$. Therefore, $\delta(\mathcal{E}_2(G)) = 3$.

5) Let $n = 2^2 \cdot m$, and $3 \nmid m$. Let $o(x) = 2$ for some $x \in G$. Then $d(x) = 1 + 2 = 3$, as x is adjacent to the identity and all elements of order 4. Suppose that $y \in G$ with $o(y) \geq 5$. Then, similarly to the previous case, we get $\delta(\mathcal{E}_2(G)) = 3$.

6) Let $n = 2^2 \cdot 3 \cdot m$, and let $x \in G$ with $o(x) = 2$. Then $d(x) = 1 + 2 + 2 = 5$, as x is adjacent to the identity and all elements of order 3 and 4. Let $y \in G$ with $o(y) = 3$ or 4. By a similar argument, we get $d(y) \geq 1+(n+1)$, as y is adjacent to elements of order 2. If $o(y)$ is 6, 7, or 8, then $d(y) \geq 1+2+2$, since y is adjacent to all elements of order 3 and 4 (or 4 and 6). Assuming that $o(y) \geq 9$, the result follows as in the previous two cases. Therefore, $\delta(\mathcal{E}_2(G)) = 5$.

□

Theorem 2.7. *Let G be the generalized quaternion group of order $4n$, $n = 2^k$ and $k \geq 1$. Then*

$$\delta(\mathcal{E}_2(G)) = \begin{cases} 2 & \text{if } k = 1; \\ 5 & \text{if } k = 3; \\ 6 & \text{otherwise.} \end{cases}$$

Proof. Let G be the generalized quaternion group of order $4n$, $n = 2^k$, and $k \geq 1$. Then for each $k \geq 1$, G has $2n + 2$ elements of order 4, a unique element of order 2, and $\phi(2^i)$ elements of order 2^i for each $3 \leq i \leq k + 1$. Thus, if $k = 1$, as $d(v_{(2)}) = d(e)$, since the order of the elements of the group in this case will be $1, 2, 4, 4, 4, 4, 4, 4$, which means that $d(v_{(2)}) = 4n - 1$; hence, we get the result. Now, since the elements of order 4 form the largest set of vertices, $\delta(\mathcal{E}_2(G)) \notin \{d(v_{(8)}), d(v_{(2)})\}$ and according to the adjacency method and the number of vertices, we get $d(v_{(4)}) = 6$ for each $k \geq 2$. If $k = 3$, $|2^4 - 2^2| > \min\{2^4, 2^2\}$, and hence $d(v_{(2^4)}) = \phi(2^3) + 1 = 5$, which gives the result. Otherwise, for $k = 2$, the minimum degree for the graph is obviously obtained. Moreover, for each $k > 3$, $d(v_{(2^{k+1})}) = \phi(2^k) + 1 > 6$, and $\phi(2^{i+1}) + \phi(2^{i-1}) + 1 > 6$ for each $3 \leq i \leq k$. Therefore, $\delta(\mathcal{E}_2(G)) = d(v_{(4)})$. □

Theorem 2.8. *Let G be the dicyclic group of order $4n$, $n = p^k$ where $p \geq 3$ is a prime number and $k \geq 1$. Then*

$$\delta(\mathcal{E}_2(G)) = \begin{cases} p & \text{if } p = 5 \text{ or } 7; \\ 6 & \text{if } p = 3; \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let G be the dicyclic group of order $4n$, $n = p^k$, where $p \geq 3$ is a prime number and $k \geq 1$. Then the group G has $2n$ elements of order 4 and a cyclic group of order $2p^k$. The orders of the elements in this group are arranged as follows:

$$1, 2, p, 2p, p^2, 2p^2, \ldots, p^k, 2p^k.$$

It is known that $|p - 4| \leq \min\{p, 4\}$ for each $p \in \{3, 5, 7\}$, and $|2p^i - p^{i+1}| > \min\{2p^i, p^{i+1}\}$ for each $p > 3$ and $1 \leq i \leq k - 1$. Consequently, $\delta(\mathcal{E}_2(G)) \notin \{d(v_{(2)}), d(v_{(p)}) : p = 3, 5 \text{ or } 7\}$. Thus, regarding the number of elements that have the same order and $\phi(2p^i) = \phi(p^i) = p - 1$, we get $d(v_{(2p)}) = \phi(p) + 1$, which gives the minimum degree for the graph when $p = 5$ or 7. Now, for all primes greater than 7, we have $|p - 4| > \min\{p, 4\}$, and hence the elements of order 4 are adjacent only to the identity and the unique element of order 2. Presently, consider the latter case where $p = 3$. If $k = 1$, then as $|6 - 4| \leq \min\{6, 4\}$, we conclude that $\delta(\mathcal{E}_2(G)) = d(v_{(4)}) = \phi(6) + \phi(3) + 2 = 6$. Otherwise, if $k > 1$, then $|2(3^i) - 3^{i+1}| \leq \min\{2(3^i), 3^{i+1}\}$ and $|3^2 - 4| > \min\{3^2, 4\}$. Thus, as $\phi(2(3^i)) = \phi(3^i) \geq 6$ for each $2 \leq i \leq k$, we get $\delta(\mathcal{E}_2(G)) = d(v_{(4)}) = 6$. □

Theorem 2.9. *Let G be a finite group of order at least 3. Then, one of the following is satisfied:*

1) *If all nonidentity elements form a connected component, then $\kappa(\mathcal{E}_2(G)) = \delta(\mathcal{E}_2(G))$ and $N(v)$ is a minimum separating set of $\mathcal{E}_2(G)$ where $\delta(\mathcal{E}_2(G)) = d(v)$ for some $v \in V(\mathcal{E}_2(G))$.*

2) *Otherwise, $\kappa(\mathcal{E}_2(G)) = 1$ and $\{e\}$ is a minimum separating set.*

Proof. Let G be a finite group. The second point is obvious. Now, assume that $\delta(\mathcal{E}_2(G)) = d(v)$ for some vertex v in G. Since $N(v) \neq G - v$, as the graph is not complete, this means that there is at least one element w of G with $v - w \notin E(\mathcal{E}_2(G))$. Hence v and w are not connected with any path in $\mathcal{E}_2(G - N(v))$. Therefore, $N(v)$ is a separating set. In a similar way, we see that for any nonidentity element $u \in G$, $N(u)$ is a separating set of the graph $\mathcal{E}_2(G)$. As all nonidentity elements form a component, the neighborhoods of these elements are the only separating sets. Thus, we conclude that $N(v)$ is a minimum separating set and hence $d(v) = |N(v)| = \kappa(\mathcal{E}_2(G)) = \delta(\mathcal{E}_2(G))$. □

2.2. The chromatic number

Within the ensuing section, the chromatic number of the graph $\mathcal{E}_2(G)$ will be ascertained for some finite group G. Moreover, an investigation into the property of perfectness is conducted for this graph.

Theorem 2.10. *Let G be any finite group. Then $\mathcal{E}_2(G)$ is perfect.*

To prove this theorem, we need some definitions and lemmas.

Definition 2.3. *[18] Let Γ be a graph. The following then hold:*

1) A hole of Γ is an induced subgraph of Γ which is a cycle of length at least 5.

2) An anti-hole of Γ is a hole of the complement graph $\bar{\Gamma}$.

3) An odd hole (respectively anti hole) is a hole (respectively anti hole) of odd length.

4) A graph Γ is Berge if it has no odd hole nor odd anti-hole.

Theorem 2.11. *[18] A graph is perfect if and only if it is Berge.*

Lemma 2.1. *Let G be a finite group, let x, y, and z be any distinct elements of G such that $o(x) = o(y)$. Then $xz \in E(\mathcal{E}_2(G))$ if and only if $yz \in E(\mathcal{E}_2(G))$.*

Lemma 2.2. *Let G be a finite group, and let H be a hole or anti-hole of $\mathcal{E}_2(G)$. Then the orders of the elements of H are all distinct.*

Proof. Let G be a finite group, and let H be a hole or anti hole of $\mathcal{E}_2(G)$. Assume for contradiction that H has two elements x and y that have the same order. We then have the following cases.
Case 1. If x and y are adjacent in H (anti-hole case).
Let w be the other neighbor of x in H. Then, by Lemma 2.1, $yw \in E(\mathcal{E}_2(G))$ which is a cord in H, which is a contradiction.
Case 2. If x and y are not adjacent in H.
Let $w \neq u$ be the neighbors of x in H. Then, by Lemma 2.1, we see that y is also adjacent to w and u. Therefore, H has a length 4, which is a contradiction. □

Lemma 2.3. *Let G be a finite group, and let x, y, and z be distinct elements of G such that $o(x) < o(y) < o(z)$ and $xz \in E(\mathcal{E}_2(G))$. Then $xy \in E(\mathcal{E}_2(G))$ and $yz \in E(\mathcal{E}_2(G))$.*

Lemma 2.4. *Let G be a finite group. Then $\mathcal{E}_2(G)$ contains neither a hole nor an anti hole.*

Proof. Let G be a finite group. Assume for contradiction that $\mathcal{E}_2(G)$ has either a hole or an anti hole H. Let z be the element of maximum order in H, and let x and y be its neighbors such that (without loss of generality) $o(x) < o(y)$; that is, $o(x) < o(y) < o(z)$. We then have the following cases.
Case 1. H is a hole.
This is a contradiction by Lemma 2.3, as $xy \in E(\mathcal{E}_2(G))$, forming a cord in H.
Case 2. H is an anti hole.
Let w be the other neighbor of y in H. By Lemma 2.2, we have either $o(w) < o(y)$ or $o(w) > o(y)$.
Case 2-1. If $o(w) < o(y)$.
As H is an anti hole and z and w are not adjacent in H, $wz \in E(\mathcal{E}_2(G))$ with $o(w) < o(y) < o(z)$. Thus, according to Lemma 2.3, $wy \in E(\mathcal{E}_2(G))$, which contradicts the assumption that w and y are adjacent

in H.

Case 2-2. If $o(w) > o(y)$**.**

By a similar argument, as x and w are not adjacent in H, $xw \in E(\mathcal{E}_2(G))$ with $o(x) < o(y) < o(w)$. Hence, by Lemma 2.3, $wy \in E(\mathcal{E}_2(G))$, which leads to the conclusion that w and y are not adjacent in H, which is a contradiction. □

Proof of Theorem 2.10. The proof follows from Theorem 2.11and Lemma 2.4. □

Theorem 2.12. *The chromatic number $\chi(\mathcal{E}_2)$ for any cyclic group of order 2^k, where $k > 1$, equals 3.*

Proof. Let G be any cyclic group of order 2^k, where $k > 1$. Observe that the identity has a unique color. Now, for each $1 \leq i \leq k-1$, we see that any element of order 2^i is adjacent only to each element of order 2^{i+1} and 2^{i-1}. Hence $\chi(\mathcal{E}_2(G)) \geq 3$. However, the elements that share the same order form an independent set, so we conclude that $\chi(\mathcal{E}_2(G)) \leq 3$. Therefore, we get the equality. □

Proposition 2.2. *Let G be a cyclic group of order $n = pq$, where $p < q$ are distinct primes. Then $\chi(\mathcal{E}_2(G)) = 3$, except when $p > 2$ and $|p - q| > \min\{p, q\}$, in which case $\chi(\mathcal{E}_2(G)) = 2$.*

Proof. Let G be a cyclic group of order $n = pq$, where p and q are distinct primes. If $|p-q| > \min\{p, q\}$ and $2 < p$, then the graph is a star graph and hence $\chi(\mathcal{E}_2(G)) = 2$. Otherwise, if n is odd, each element of order p is adjacent to the identity and all elements of order q. As there is no edge between elements of the same order, we conclude the result. Moreover, if n is even, observe that there is no edge between the element of order 2 and the elements of order n. Thus, the result is obtained. □

Theorem 2.13. *Let G be the dihedral group D_{2n} and let H be a cyclic group of order n. Then*

$$\chi(\mathcal{E}_2(G)) = \begin{cases} \chi(\mathcal{E}_2(H)) + 1 & 3 \textit{ divides } n \textit{ and } \mathcal{E}_2(H) \textit{ is a star graph;} \\ \chi(\mathcal{E}_2(H)) & \textit{otherwise.} \end{cases}$$

Proof. Let G be the dihedral group D_{2n} and let H be a cyclic group of order n. Then, the orders of the elements belong to the set $\pi_e = \{1,\ 2,\ d\ :\ d|n\ d > 1\}$. Observe that as $G \setminus B = H$, we have $\mathcal{E}_2(G - B) \cong \mathcal{E}_2(H)$, where $B = \{b, ab, a^2b, \ldots, a^{n-1}b\}$. Consider that n is divisible by 3 and $\mathcal{E}_2(H)$ is a star graph. Then the elements of order 2 are adjacent in $\mathcal{E}_2(G)$ to the elements of order 3 and the identity. Thus, an additional color is required, resulting in the outcome. Now assume the second case. There are then two possibilities: Either 3 divides n and $\mathcal{E}_2(H)$ is not a star graph or n is not divisible by 3. In both cases, the elements of order 2 in G will not affect the chromatic number of $\mathcal{E}_2(H)$. That is, if n is even, then the elements of order 2 in G will be colored in a similar was as in H, while if n is odd, then for such vertices, we can choose a color of any divisor d of n that is different from 1 and 3 (if it divides n) since for all $d > 3$, we have $|2 - d| > \min\{2, d\}$. □

Theorem 2.14. *Let G be the generalized quaternion group. Then*

$$\chi(\mathcal{E}_2(G)) = 3.$$

Proof. Let G be the generalized quaternion group. Then, the equitable graph of Type II of this group is defined as

$$K_1 \vee (W_{(2)} \vee W_{(4)} \vee \ldots \vee W_{(2^k)} \vee W_{(2^{k+1})}),$$

where K_1 has a unique vertex, which is the vertex associated with the identity element, and $W_{(2)} \vee W_{(4)} \vee \ldots \vee W_{(2^k)} \vee W_{(2^{k+1})}$ denote the sequential joining of the subgraphs $W_{(2)}, W_{(2^2)}, \ldots, W_{(2^k)}$, and $W_{(2^{k+1})}$, such that $W_{(2^i)} \vee W_{(2^{i+1})}$ for all $1 \leq i \leq k-1$, defined by adding an edge from each vertex in $W_{(2^i)}$ to each vertex in $W_{(2^{i+1})}$. Then proof is then similar to that of Theorem 2.12. □

Theorem 2.15. *Let G be the dicyclic group of order $4n$, $n = p^k$ where $p \geq 3$ is a prime number and $k \geq 1$. Then*

$$\chi(\mathcal{E}_2(G)) = \begin{cases} 4 & \text{if } p = 3; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Consider that $G \cong Q_{4n}$ with $n = p^k, p \geq 3$, and $k > 1$. First, assume that $p = 3$. According to the adjacency method in this graph, we get $\chi(\mathcal{E}_2(G)) \geq 4$. However, for each $2 \leq i \leq k$, $|2(3^i) - 3^{i+1}| \leq \min\{2(3^i), 3^{i+1}\}$ and $|2(3^i) - 2(3^{i+1})| > \min\{2(3^i), 2(3^{i+1})\}$ for all $1 \leq i \leq k-1$. Hence, $\mathcal{E}_2(G)$ contains the induced subgraph $K_1 \vee (W_{(6)} \vee W_{(3^2)} \vee W_{(2(3^2))} \vee \ldots \vee W_{(3^k)} \vee W_{(2(3^k))})$ and this implies that $\chi(\mathcal{E}_2(G)) \leq 4$. On the other hand, if $p > 3$, then $|p-2| > \min\{p, 2\}$, which implies $|p^{i+1} - 2p^i| > \min\{p^{i+1}, 2p^i\}$ for each $1 \leq i \leq k-1$. Thus, in this case, the equitable graph of Type II of G contains the induced subgraphs $K_1 \vee (W_{(p^i)} \vee W_{(2p^i)})$ for all $1 \leq i \leq k$ and hence $\chi(\mathcal{E}_2(G)) \geq 3$. However, since $|4 - 2p| > \min\{4, 2p\}$, we get $\chi(\mathcal{E}_2(G)) \leq 3$. □

2.3. Independence number

The independence number of any graph is the cardinality of the maximum independent set in that graph. The objective of this subsection is to study the independence number of the equitable graph of Type II of a certain finite group G.

Theorem 2.16. *Let G be a cyclic group of order $n = 2^k$, where $k > 1$ is a positive integer. In this case,*

$$\alpha(\mathcal{E}_2(G)) = \begin{cases} \frac{2n-2}{3} & \text{if } k \text{ is even;} \\[2ex] \frac{2n-1}{3} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let G be a cyclic group of order $n = 2^k$, where $k > 1$ is a positive integer and let W denote the maximum independent set of $\mathcal{E}_2(G)$. As the identity e is adjacent to all other vertices, e cannot be in any independent set. Now observe that each element of order 2^i is adjacent to all elements of orders 2^{i+1} and 2^{i-1} for all $2 \leq i \leq k-1$, and each divisor of n corresponds to an independent set. Let $W_{(j)}$ denote the independent set consisting of all elements of order j in G. The graph in this case is defined as

$$K_1 \vee (W_{(2)} \vee W_{(2^2)} \vee \ldots \vee W_{(2^k)}).$$

Thus, according to the fact that $\phi(2^k) > \phi(2^{k-1}) > \ldots > \phi(2) = 1$ and because of the adjacency method, we conclude that the independent set $W' = \{W_{(2^k)}, W_{(2^{k-2})}, \ldots, W_{(2^2)}\} \subseteq W$ if k is even (or $W' = \{W_{(2^k)}, W_{(2^{k-2})}, \ldots, W_{(2^3)}, W_{(2)}\} \subseteq W$ if k is odd) therefore, $W \subseteq W'$. Now consider the even case of k and let X be any maximal independent set and let $X = \{W_{(2^{i_1})}, W_{(2^{i+2})}, \ldots, W_{(2^{i_s})}\}$ in increasing order. As X is an independent set, for any $i_1 \leq j \leq i_s$, X cannot contain an element of order 2^j or 2^{j+1}. It follows that $s \leq \frac{k}{2}$, we also have $i_{z+1} - i_z \geq 2$ for all $1 \leq z \leq s$. Hence, $i_s \leq k,\ i_{s-1} \leq k-2,\ i_{s-2} \leq k-4,\ \ldots,\ i_1 \geq 2$ by the choice of n, and since the difference between any two consecutive orders is at least 2^2. Thus

$|X| \leq |W|$. For the odd case, the proof will follow a similar argument. Thus, if k is even, we have

$$\sum_{i=1}^{\frac{k}{2}} \phi(2^{2i}) = \sum_{i=1}^{\frac{k}{2}} 2^{2i-1} = \frac{1}{2}\sum_{i=1}^{\frac{k}{2}} 4^i = \frac{1}{2}\Big(\sum_{i=0}^{\frac{k}{2}} 4^i\Big) - 1 = \frac{1}{2}\Big(\frac{4^{\frac{k}{2}+1}-1}{4-1}\Big) - 1$$
$$= \frac{1}{2}\Big(\frac{4n-4}{3}\Big) = \frac{2n-2}{3}.$$

Similarly, if k is odd, we have

$$\sum_{i=0}^{\frac{k-1}{2}} \phi(2^{2i+1}) = \sum_{i=0}^{\frac{k-1}{2}} 2^{2i} = \sum_{i=0}^{\frac{k-1}{2}} 4^i = \frac{4^{\frac{k-1}{2}+1}-1}{4-1}$$
$$= \frac{2^{k-1+2}-1}{3} = \frac{2^{k+1}-1}{3} = \frac{2n-1}{3}.$$

□

Theorem 2.17. *Let G be a cyclic group of order $n = pq$, where $p < q$ are distinct primes. Then*

$$\alpha(\mathcal{E}_2(G)) = \begin{cases} q & \text{if } p = 2; \\ n-p & \text{if } p > 2 \text{ and } |p-q| < p; \\ n-1 & \text{otherwise.} \end{cases}$$

Proof. Let G be a cyclic group of order $n = pq$, where $p < q$ are distinct primes. According to the definition of the graph, the identity cannot be in any independent set. Consider that n is even. The graph $\mathcal{E}_2(G)$ is then defined as

$$K_1 \vee (W_{(2)} \vee W_{(q)} \vee W_{(n)}),$$

where $W_{(j)}$ is defined as in Theorem 2.16. Hence, if $|2-q| \leq \min\{2, q\}$ or $|2-q| > \min\{2, q\}$, then, similarly to Theorem 2.16, and since $|n-q| \leq \min\{n, q\}$, we find that $W_{(n)} \cup W_{(2)}$ is the maximum independent set, since $\phi(n) = \phi(2q) = \phi(2)\phi(q) = \phi(q)$.

Now assume that n is not even and $|p-q| > \min\{p, q\}$. Hence, the graph will be a star graph and the result is obvious. Alternatively, if $|p-q| \leq \min\{p, q\}$, then $\mathcal{E}_2(G)$ will be as follows:

$$(W_{(p)} \vee W_{(q)}) \vee K_1 \vee W_{(n)}.$$

Therefore, since $|q-n| > \min\{q, n\}$ and $p < q$, according to our analysis, we have $\phi(p) = p-1$, $\phi(q) = q-1$ and $\phi(n) = \phi(pq) = (p-1)(q-1)$, which implies that $\phi(p) < \phi(q) < \phi(n)$. Hence, it can be deduced that $W_{(n)} \cup W_{(q)}$ forms the maximum independent set for this graph. □

Theorem 2.18. *Let n be a positive integer, let G be the dihedral group D_{2n}, and consider the cyclic group of order n, say H, where $n = 2^k$; $k > 1$. In this case,*

$$\alpha(\mathcal{E}_2(G)) = \begin{cases} \alpha(\mathcal{E}_2(H)) + n - 1 & \text{if } k \text{ is even;} \\ \alpha(\mathcal{E}_2(H)) + n & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let G be the dihedral group D_{2n} and let H be the cyclic group of order n, where $n = 2^k$; $k > 1$. The graph $\mathcal{E}_2(G)$ is then defined as

$$K_1 \vee (W_{(2)} \vee W_{(2^2)} \vee \ldots \vee W_{(2^k)}),$$

where $W_{(j)}$ denotes the independent set consisting of all the elements of order j in the group G. It is known that the group $G = \langle a \rangle \cup \{b, ab, a^2b, \ldots, a^{2^k-1}b\}$ and the number of involutions is equal to $n + 1$, in this case, which forms an independent set with a large number of vertices, and hence $W_{(2)}$ is a subset of the maximum independent set, say W. By the same argument as in the proof of Theorem 2.16, we get $\{W_{(2)}, W_{(2^k)}, W_{(2^{k-2})}, \ldots, W_{(2^4)}[orW_{(2^3)}]\} \subset W$ where k is even [or odd].

Now, consider the even case, since the elements of orders 2 and 2^2 are adjacent and $W_{(2)} \subset W$. Hence $W_{(2^2)}$ cannot be in W. Moreover, any vertex of $G \setminus W$ is adjacent to at least four vertices in W. Hence, the equality holds in W. In comparison with $\mathcal{E}_2(H)$ and considering that each set $W_{(2^i)}$ contains $\phi(2^i) = 2^{i-1}$ vertices, we see that if k is odd, then $\alpha(\mathcal{E}_2(G)) = n+1+\sum_{i=1}^{\frac{k-1}{2}} \phi(2^{2i+1}) = n+1+(\alpha(\mathcal{E}_2(H))-1)$. While if k is even, we have $\alpha(\mathcal{E}_2(G)) = n + 1 + \sum_{i=2}^{\frac{k}{2}} \phi(2^{2i}) = n + 1 + (\alpha(\mathcal{E}_2(H)) - 2)$. □

Theorem 2.19. *Let G be the dihedral group D_{2n}, where $n = 3^k$ and $k > 1$. Then, $\alpha(\mathcal{E}_2(G)) = 2n - 3$.*

Proof. Let G be the dihedral group D_{2n}, where $n = 3^k$ and $k > 1$. Then all the vertices are adjacent only to the identity as $|3^{i+1} - 3^i| > \min\{3^{i+1}, 3^i\}$ for all $1 \leq i \leq k-1$, except the identity and the elements of orders 2 and 3 which form an induced subgraph $K_1 \vee (W_{(2)} \vee W_{(3)})$; that is, the elements of order 2 are adjacent to all the elements of order 3. Consequently, $W_{(3^i)}$ is a subset of the maximum independent set W for all $2 \leq i \leq k$ and one of the two sets $W_{(2)}$ and $W_{(3)}$. Considering the number of elements in each of them, we see that $W_{(2)} \subset W$ as it has n vertices in this case. Therefore, W consists of $n + \sum_{i=2}^{k} \phi(3^i) = n + (n-3) = 2n - 3$ vertices. □

Theorem 2.20. *Let G be the dihedral group D_{2n} where $n = pq$ with p and q are primes, such that $p < q$. Let H be the cyclic group of order n. Then*

$$\alpha(\mathcal{E}_2(G)) = \begin{cases} 2n - 3 & \text{if } p = 3 \text{ and } |p - q| > \min\{p, q\}; \\ \alpha(\mathcal{E}_2(H)) + n & \text{otherwise.} \end{cases}$$

Proof. Let G be the dihedral group D_{2n} and let H be the cyclic group of order n, where $n = pq$ such that $p < q$ are distinct primes. Then the group G consists of a cyclic group $\langle a \rangle$ of order n and the set $\{b, ab, a^2b, \ldots, a^kb\}$ of n involutions. Similar to the argument in the proof of Theorem 2.17, we can discuss the situation in each case. First, if $p = 2$, then $|n - q| \leq \min\{n, q\}$ and $|n - 2| > \min\{n, 2\}$. Since $W_{(2)}$ has $n + 1$ vertices, according to the adjacency, $W_{(2)} \cup W_{(n)}$ is a subset of W, which is the maximum independent set in $\mathcal{E}_2(G)$, since if X is any maximal independent set, then $W_{(2)}$ must be a subset of X. Moreover, by adjacency, in this case, we see that $W_{(q)}$ or $W_{(n)}$ is a subset of X. However, as $\phi(q) = \phi(2q)$, $|X| \leq |W|$ and $\alpha(\mathcal{E}_2(G)) = n + 1 + \phi(n) = q + n$. Now assume that $n = pq$, and $|p - q| \leq p$, $p > 2$, and thus there are two cases.

Case 1. If $p = 3$, then $|n - q| > \min\{n, q\}$ and $|q - 2| > \min\{q, 2\}$. Hence, $W_{(2)}$ which contains n vertices and $W_{(n)}$ are included in W. Moreover, since the elements of order q are adjacent only to the identity and the two elements of order 3, and as $p < q$, we get $W_{(q)} \subset W$.

Case 2. If $p > 3$, then $|p - 2| > \min\{p, 2\}$ and $|n - q| > \min\{n, q\}$. This implies that $W_{(2)} \cup W_{(n)} \subset W$. According to the choice of n and the number of elements in each order, we find that $W_{(q)}$ is contained in W. Furthermore, observe that in both cases, neither $W_{(p)}$ nor the identity can be in W. Therefore, $\alpha(\mathcal{E}_2(G)) = n+\phi(n)+\phi(q) = 2n-p$. Otherwise, the graph $\mathcal{E}_2(G)$ is a star graph except for $p = 3$. Then there is an edge from each element of order 2 to all elements of order 3, and hence $W_{(3)}$ is not included in W by considering the number of vertices in each of these two sets. Therefore, $W = \{W_{(2)}, W_{(q)}, W_{(n)}\}$ and $\alpha(\mathcal{E}_2(G)) = n + \phi(q) + \phi(n)$. □

Theorem 2.21. *Let G be the generalized quaternion group. Then* $\alpha(\mathcal{E}_2(G)) = 2\Big(\alpha(\mathcal{E}_2(D_{2n}))\Big)$.

Proof. Consider that $G = Q_{4n}, n = 2^k$, and $k > 1$. The proof of this theorem follows, since G has $2n+2$ elements of order 4, which must be included in the maximum independent set, say W, and a cyclic group with 2n elements. Then, by a similar argument as in the proof of Theorem 2.18, and as $\phi(2^i) = 2^{i-1}$, we find that if k is even,

$$\alpha(\mathcal{E}_2(G)) = 2n + 2 + \sum_{i=2}^{\frac{k}{2}} \phi(2^{2i+1}) = 2(n + 1 + \sum_{i=2}^{\frac{k}{2}} \phi(2^{2i})).$$

For the odd case of k, we have

$$\alpha(\mathcal{E}_2(G)) = 2n + 2 + \sum_{i=1}^{\frac{k-1}{2}} \phi(2^{2i+2}) = 2(n + 1 + \sum_{i=1}^{\frac{k-1}{2}} \phi(2^{2i+1})).$$

□

Theorem 2.22. *Let G be the dicyclic group of order* $4n$*,* $n = p^k$ *where* $p \geq 3$ *is a prime number, and* $k \geq 1$*. Then* $\alpha(\mathcal{E}_2(G)) = \alpha(\mathcal{E}_2(D_{2n})) + n$*.*

Proof. Let $G = Q_{4n}$, and $n = p^k$ where $p \geq 3, k \geq 1$, and consider that $p > 3$. According to the adjacency method mentioned for this group, since $\phi(p^i) = \phi(2p^i)$ for all $1 \leq i \leq k$, we see that $B = \{W_{(4)}, W_{(2p^i)} : 1 \leq i \leq k\} \subseteq W$, and no other vertex from $G \setminus B$ can be added to W because of the adjacency. Thus, $W = B$. Now, if any maximal independent set X exist, following the argument in the proof of Theorem 2.20, we see that $|X| \leq |W|$ and hence $\alpha(\mathcal{E}_2(G)) = 2n + \sum_{i=1}^{k} \phi(p^i)$. Considering that $\mathcal{E}_2(D_{2n})$ in this case is a star graph, we get the result. Now assume that $p = 3$, then $|4-6| \leq \min\{4, 6\}$, which implies that $B = \{W_{(4)}, W_{(2p^i)} : 2 \leq i \leq k\} \subseteq W$. Moreover, each vertex in $G \setminus B$ is adjacent to at least one vertex in B, and this shows that B is maximum, since if Y is any maximal independent set, then, again by similar criteria to those in the proof of Theorem 2.20, we get $|Y| \leq |W|$. Therefore, $\alpha(\mathcal{E}_2(G)) = 2n + \sum_{i=2}^{k} \phi(p^i)$, and a comparison with Theorem 2.19 leads to the announced result. □

Theorem 2.23. *Let G be the dicyclic group of order* $4n$*,* $n = 2p$*, and* $p \geq 3$ *is any prime. Then*

$$\alpha(\mathcal{E}_2(G)) = \begin{cases} 2n + 2 + \phi(4p) & \text{if } p \leq 7; \\ 2n + 2 + \phi(p) + \phi(4p) & \text{otherwise.} \end{cases}$$

Proof. Let G be the dicyclic group of order $4n$, $n = 2p$ and $p \geq 3$ is any prime. Then G has $2n + 2$ elements of order 4 and $\phi(d)$ elements for each order d. The orders of the elements of this group will be arranged as $1, 2, 4, p, 2p, 4p$. If $p \leq 7$, according to the adjacency criteria and the number of elements in each order, we find that $W = \{W_{(4)}, W_{(4p)}\}$. Hence, $\alpha(\mathcal{E}_2(G)) = 2n + 2 + \phi(2n)$. On the other hand, $W_{(4)}$ is separated from $W_{(d)}$ for all orders $d > 4$ and there is no edge between the elements of order p and order $2n$. Hence, in this case, $W = \{W_{(4)}, W_{(p)}, W_{(2n)}\}$, which leads to the result. □

Theorem 2.24. *Let G be the dicyclic group of order $4n$, $n = pq$, and $2 < p < q$ are prime numbers. Then*

$$\alpha(\mathcal{E}_2(G)) = \begin{cases} 2n + \phi(2n) + \phi(q) & \text{if } p = 3 \text{ or } 3 < p < 8 \text{ and } q < 2p; \\ 2n + \phi(2n) + \phi(p) + \phi(q) & \text{otherwise.} \end{cases}$$

Proof. Let G be the dicyclic group of order $4n$, $n = pq$, and let p, q be primes with $2 < p < q$. Then the elements of this group have the order $1, 2, 4, p, 2p, q, 2q, pq, 2pq$. It is clear that $|2n-2q| > \min\{2n, 2q\}$. Thus, there are two cases. First, assume that $p = 3$. Considering the adjacency method and the cardinality of the elements in each order, we get $W = \{W_{(4)}, W_{(2q)}, W_{(2n)}\}$. If we consider that $3 < p < 8$ and $2p < q$, then we see the elements of order q are adjacent to the elements of order p and order $2p$. Moreover, the elements of order $2p$ are adjacent to the elements of order $2q$. Thus, since $W_{(4)}$ and $W_{(2p)}$ are disconnected, we see that $W = \{W_{(4)}, W_{(2q)}, W_{(2n)}$. Otherwise, as $|4 - p| > \min\{4, p\}$, which implies that $W = \{W_{(4)}, W_{(p)}, W_{(2q)}, W_{(2n)}\}$. Therefore, the result is shown using the fact that $\phi(2p) = \phi(p)$ for any prime $p > 3$. □

2.4. Planar, Eulerian, and Hamiltonian

In this part of the research, the planarity, Eulerianess, and Hamiltonianess of equitable graph of Type II for some finite groups are studied. We have the following results.

Theorem 2.25. *[13, Theorem 8.4.3] $K_{3,3}$ is nonplanar.*

It is clear that any star graph is planar. The next theorem outlines the other cases in which the graph is planar.

Theorem 2.26. *Let G be any finite group such that $\mathcal{E}_2(G)$ is not a star graph. Then $\mathcal{E}_2(G)$ is planar if and only if G satisfies one of the following:*

1) G is a cyclic group of order 4.

2) G is isomorphic to the quaternion group Q_8.

Proof. Let G be any finite group. The proof will be by contradiction. Suppose that G does not satisfy any of the given conditions, and hence $|G| \geq 6$. If $|G| = 6$, according to the list of groups of small order, the only Abelian group of order 6 is $\mathbb{Z}_6$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$, and the only non Abelian group of this order is the dihedral group D_6, where $n = 3$. This group is isomorphic to the symmetric group S_3. As isomorphic groups have isomorphic graphs and since $\mathcal{E}_2(\mathbb{Z}_6$ and $\mathcal{E}_2(D_6)$ are isomorphic (as will be presented in the last section), we see that $\mathcal{E}_2(G)$ has $K_{3,3}$ as a subgraph. This implies that $\mathcal{E}_2(G)$ is nonplanar. This can be detected in Figure 3.

For $|G| = 8$, if we remove the case Q_8 and $\mathbb{Z}_{2^3}$ (as $\mathcal{E}_2(\mathbb{Z}_{2^3})$ is a star graph), the only Abelian groups are $\mathbb{Z}_8$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$. For the non Abelian case, there is only the dihedral group D_8 where $n = 4$. Considering that the equitable graph of Type II of D_8 and $\mathbb{Z}_8$ are isomorphic. This will be shown in the last section. It can be shown in Figure 4 that $\mathcal{E}_2(G)$ in each case has a subgraph isomorphic to $K_{3,3}$ and hence it is nonplanar.

Since the graph is not a star graph, there are nonidentity elements u_1, v_1 that are adjacent. It follows that u_1 and v_1 have distinct orders. Hence, for any finite group G of order greater than 8, as there exist at least three elements u_1, u_2, and u_3 share the same order, say d_1, and at least two elements v_1 and v_2 have the same order d_2, hence the subsets of vertices $U_1 = \{u_1, u_2, u_3\}$ and $U_2 = \{v_1, v_2, e\}$ form a subgraph isomorphic to $K_{3,3}$ by removing certain edges. Thus, we can always construct a subgraph of $\mathcal{E}_2(G)$ which is isomorphic to the graph $K_{3,3}$. Therefore, the equitable graph of Type II is nonplanar for any finite group G that does not satisfy the given conditions. On the other hand, in the case where G is a cyclic group of order 4 or $G \cong Q_8$, Figure 5 presents the planar graph of $\mathcal{E}_2(G)$ in these two cases.

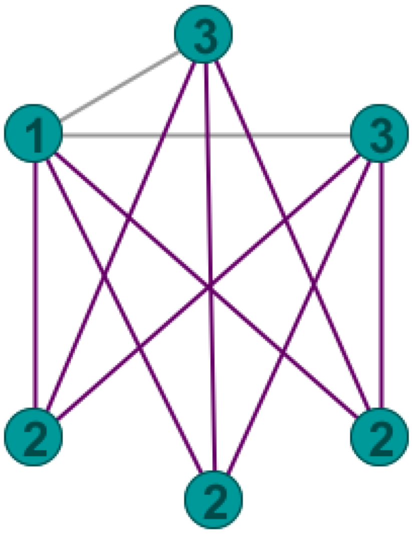
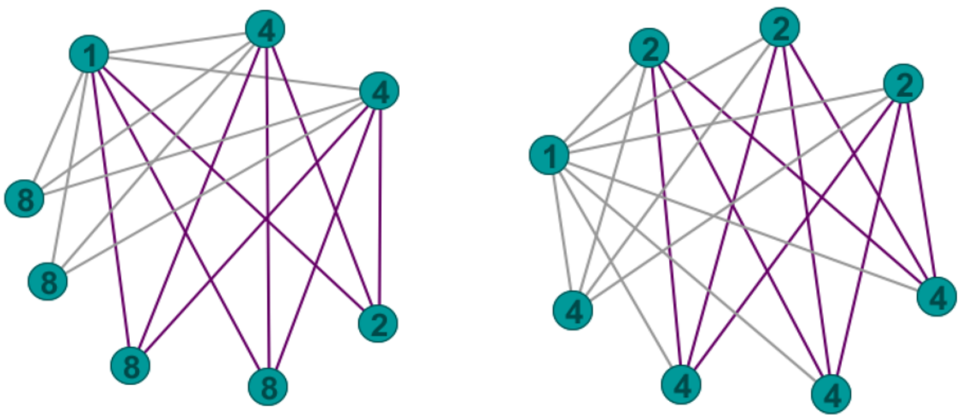


Figure 3. Nonplanar of the graph $\mathcal{E}_2(G)$ where $|G| = 6$.



(a) $\mathcal{E}_2(G); G \cong \mathbb{Z}_8$

(b) $\mathcal{E}_2(G); G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$

Figure 4. Nonplanar of the graph $\mathcal{E}_2(G)$ where $|G| = 8$.

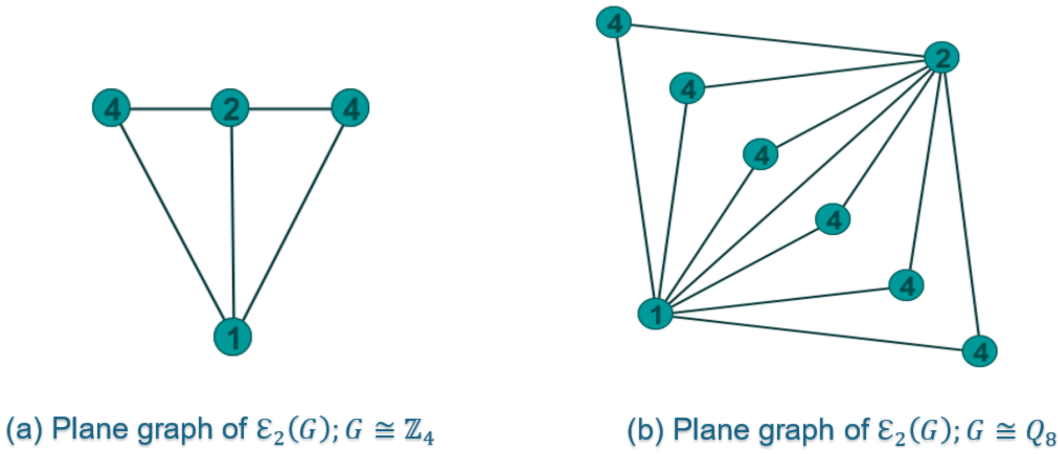


Figure 5. The planarity of $\mathcal{E}_2(G)$ where $G \cong \mathbb{Z}_4$ or Q_8.

□

Theorem 2.27. *[13, Theorem 6.2.2] For the nontrivial connected graph* Γ, *the following statements are equivalent:*

1) Γ *is Eulerian.*

2) The degree of each vertex of Γ *is an even positive integer.*

3) Γ *is an edge-disjoint union of cycles.*

Proposition 2.3. *Let G be a finite group, and assume that one of the following holds:*

1) $\mathcal{E}_2(G)$ *is a star graph.*

2) $|G|$ *is an even number.*

3) G is a cyclic group with an odd order.

Therefore, $\mathcal{E}_2(G)$ *is not Eulerian.*

Proof. Let G be a finite group of order n. The proof is then clear for the first and second points. Now let G be a cyclic group of odd order, say n, then $|n - \frac{n}{d}| > \min\{n.\frac{n}{d}\}$, where d is the smallest prime dividing n. Therefore, $d(v_{(n)}) = 1$, which proves the result. □

Proposition 2.4. *Let G be the dihedral group* D_{2n}. *Then* $\mathcal{E}_2(D_{2n})$ *is Hamiltonian if and only if* $n = 3$.

Proof. Let G be the dihedral group D_{2n}. If $n = 3$, then $\mathcal{E}_2(D_6)$ is Hamiltonian since $b - a - ab - a^2 - a^2b - e - b$ is a Hamiltonian cycle in $\mathcal{E}_2(D_6)$. Conversely, let C be a Hamiltonian cycle in $\mathcal{E}_2(D_6)$. We need to show that $n = 3$. Let $W_{(2)}$ be the set of elements of order 2 and let $NW_{(2)}$ be the set of the neighbors of the elements of $W_{(2)}$. As $|W_{(2)}| \geq n$ and $W_{(2)}$ and $NW_{(2)}$ are disjoint sets, we have $|NW_{(2)}| \leq |G| - |W_{(2)}| \leq n$. Moreover, $W_{(2)}$ is an independent set, so any two elements of $W_{(2)}$ in C must be separated by at least one element of their neighbors, so $|W_{(2)}| \leq |NW_{(2)}|$; that is, $n \leq |W_{(2)}| \leq |NW_{(2)}| \leq n$.

Thus we get $|W_{(2)}| = |NW_{(2)}| = n$. As G has n elements of order 2, n is odd, and hence any element of $G - W_{(2)}$ belongs to $NW_{(2)}$, which is the case of the elements of order n in particular. Therefore, $|n-2| \leq \min\{n, 2\}$, which implies that $n < 4$; therefore, $n = 1$ or $n = 3$. But n cannot be 1, since the group in this case has only two elements, and it has no cycle. Hence $n = 3$. □

Proposition 2.5. *Let G be the dicyclic group Q_{4n}, $n > 1$. The equitable graph of Type II of G is Hamiltonian if and only if $n = 3$.*

Proof. Let G be the dicyclic group Q_{4n}, $n > 1$. If $n = 3$, $\mathcal{E}_2(Q_{12})$ contains a Hamiltonian cycle $b - a - ab - a^5 - a^2b - a^2 - a^3b - a^4 - a^4b - a^3 - a^5b - e - b$, so it is Hamiltonian. On the other hand, let C be a Hamiltonian cycle in $\mathcal{E}_2(G)$. We need to show that $n = 3$. Let $W_{(4)}$ be the set of elements of order 4 and let $NW_{(4)}$ be the set of the neighbors of the elements of $W_{(4)}$. As $|W_{(4)}| \geq 2n$, and $W_{(4)}$ and $NW_{(4)}$ are disjoint sets, we have $|NW_{(4)}| \leq |G| - |W_{(4)}| \leq 2n$. Moreover, $W_{(4)}$ is an independent set, so any two elements of $W_{(4)}$ in C must be separated by at least one element of their neighbors, so $|W_{(4)}| \leq |NW_{(4)}|$; that is $2n \leq |W_{(4)}| \leq |NW_{(4)}| \leq 2n$. Thus we get $|W_{(4)}| = |NW_{(4)}| = 2n$. As G has $2n$ elements of order 4, $2n$ is not divisible by 4; therefore, n is odd, and hence any element of $G - W_{(4)}$ belongs to $NW_{(4)}$ which is the case of the elements of order $2n$ in particular. So $|2n - 4| \leq \min\{2n, 4\}$; this implies that $n < 4$; and therefore, $n = 3$ as $n > 1$. □

3. Topological indices

Topological indices play a crucial role in analyzing chemical compounds' physical and chemical characteristics. They include molecular structures based on degree and distance, as well as hybrid formulations. The identification of the physical properties, chemical reactivity, and biological activities of compounds can be achieved using these indices as the leading tools. In the context of algebraic graphs, topological indices are essential for understanding and exploring the graph's structure, which reflects the properties of the algebraic structure itself. Furthermore, algebraic graph invariants have an effective role in analyzing protein interaction networks and molecular symmetry, where Zagreb indices serve as predictors of molecular stability and complexity. Many researchers have conducted research in this field (see [19,20]). The most useful topological indices are the first and second Zagreb indices, which are defined as $M_1(\Gamma) = \sum_{u \in V} (d(u))^2$ and $M_2(\Gamma) = \sum_{uv \in E} d(u)d(v)$ for any graph Γ. In this section, we derive formulas for the first and second entire Zagreb indices as well as formulas for the first and second Zagreb indices for the equitable graph of Type II of various finite groups. Additionally, we provide some examples. For further information on the entire topological indices, we refer to [21,22].

Through this section, e denotes the vertex that is associated with the identity element of G, and $ev_{(j)}$ [or $v_{(i)}v_{(j)}$] means the edge which joins e with $v_{(j)}$ (or $v_{(i)}$ with $v_{(j)}$).

Definition 3.1. *[21] Let $\Gamma = (V, E)$ be a graph. The first and second entire Zagreb indices are defined by*

$$M_1^{\varepsilon}(\Gamma) = \sum_{x \in V(\Gamma) \cup E(\Gamma)} (d(x))^2,$$

and

$$M_2^{\varepsilon}(\Gamma) = \sum_{\substack{x \text{ is either} \\ \text{adjacent or incident to } y}} d(x)d(y).$$

Let G be any finite group such that $\mathcal{E}_2(G)$ is a star graph. The computation of the entire indices, along with the first and second Zagreb indices, can be derived from the subsequent results.

Proposition 3.1. *[21] For the star graph Γ of order n, we have*

1) $M_1^{\varepsilon}(\Gamma) = (n-1)[n^2 - 3n + 4]$.

2) $M_2^{\varepsilon}(\Gamma) = \frac{1}{2}(n-1)[n^3 - n(3n-4) - (n-3)^2 - 1]$.

Theorem 3.1. *[23] The first and second Zagreb indices of the star graph Γ of order n are*

$$M_1(\Gamma) = (n-1)^2 + (n-1) \quad \text{and} \quad M_2(\Gamma) = (n-1)^2.$$

Theorem 3.2. *Let G be a cyclic group of order 2^k, where $k \geq 4$. Then $\mathcal{E}_2(G)$ has the following properties:*

$$\begin{aligned}
1)\ M_1^{\varepsilon}(\mathcal{E}_2(G)) = \ & 2n^2 - 2n + 108 + (7)2^{3k-6} + 2^{k-1}\Big[(1+2^{k-2})^2 + (n-2+2^{k-2})^2\Big] \\
& + 2^{k-2}\Big[(1+(5)2^{k-3})^2 + (n+(5)2^{k-3}-2)^2\Big] \\
& + \sum_{i=2}^{k-1}\Big[2^{2i-1}\Big((15)2^{i-2}\Big)^2 + 2^{i-1}\Big((1+(5)2^{i-2})^2 + (n+(5)2^{i-2}-2)^2\Big)\Big].
\end{aligned}$$

$$\begin{aligned}
2)\ M_2^{\varepsilon}(\mathcal{E}_2(G)) = \ & (117)2^k + 7(2^{2k} - 2^{3k-5}) + (35)2^{4k-8} + (2^{k-1} + 2^{2k-3}) \\
& \times\Big[2^{k-2}(5+(7)2^{k-3}) - 2\Big] + 2^k\Big[52 + 5(2^k + 2^{2k-3}) + \sum_{i=4}^{k-1} 2^{i-1} d(v_e v_{(2^i)})\Big] \\
& + (2^k - 1)\Big[97 + (7)2^k + \sum_{i=4}^{k} 2^{i-1}(d(v_{(2^i)}) + d(v_e v_{(2^i)}))\Big] \\
& + (2^{k+2} + 32)\Big[30 + 8(d(v_{(2^3)}v_{(2^4)}) + d(v_e v_{(2^4)})) + (5)2^{2k-3} - 2^k + \sum_{i=5}^{k-1} 2^{i-1} d(v_e v_{(2^i)})\Big] \\
& + \sum_{i=4}^{k-1} 2^{i-1}\Bigg[d(v_{(2^i)})\Big(d(v_e v_{(2^i)}) + 2^{i-1} d(v_{(2^{i-1})}v_{(2^i)}) + 2^i(d(v_{(2^{i+1})}) + d(v_{(2^i)}v_{(2^{i+1})}))\Big) \\
& + d(v_e v_{(2^i)})\Big(2^{i-2} d(v_{(2^{i-1})}v_{(2^i)}) + 2^i d(v_{(2^i)}v_{(2^{i+1})}) + \sum_{j=i+1} 2^{j-1} d(v_e v_{(2^j)})\Big)\Bigg] \\
& + \sum_{i=3}^{k-2} 2^{2i-1} d(v_{(2^i)}v_{(2^{i+1})})\Big[2^{i+1} d(v_{(2^{i+1})}v_{(2^{i+2})})\Big] \\
& + \sum_{i=4}^{k}\Big[(2^{2i-3} - 2^{i-2})(d(v_e v_{(2^i)}))^2 + (4^{i-3}(3\cdot 2^i - 8))(d(v_{(2^{i-1})}v_{(2^i)}))^2\Big] \\
& + 44\Big[38 + 2^k + 8(d(v_{(2^4)}) + d(v_{(2^3)}v_{(2^4)}))\Big] + 960\, d(v_{(2^3)}v_{(2^4)}) + 6328.
\end{aligned}$$

Proof. Let G be a cyclic group of order 2^k and let $k \geq 4$ (the case of $k = 2$ or 3 is included in Example 3.1). Since the equitable graph of Type II of G is of the form

$$K_1 \vee (W_{(2)} \vee W_{(2^2)} \vee \ldots \vee W_{(2^k)}).$$

According to the definition of the entire indices, we get

$$M_1^{\varepsilon}(\mathcal{E}_2(G)) = (d(v_e))^2 + (d(v_{(2)}))^2 + (d(v_e v_{(2)}))^2 + \sum_{i=2}^{k} \varphi(2^i)\Big[(d(v_{(2^i)}))^2 + (d(v_e v_{(2^i)}))^2\Big]$$
$$+ \varphi(2)\varphi(2^2)(d(v_{(2)}v_{(2^2)}))^2 + \varphi(2^2)\varphi(2^3)(d(v_{(2^2)}v_{(2^3)}))^2 + \ldots$$
$$+ \varphi(2^{k-1})\varphi(2^k)(d(v_{(2^{k-1})}v_{(2^k)}))^2.$$

Now, using the fact that the cardinality of the elements that share the same order in a cyclic group is equal to the *Euler function* of that order and $\varphi(2^i) = 2^{i-1}$. In this graph, we have $d(v_{(2)}) = 1+\varphi(2^2) = 3$, $d(v_{(2^k)}) = 1 + \varphi(2^{k-1})$, $d(v_e v_{(2)}) = n - 2 + \varphi(2^2) = n$, $d(v_e v_{(2^k)}) = n - 2 + \varphi(2^{k-1})$, and for each $1 < i < k$, $d(v_{(2^i)}) = 1 + \varphi(2^{i-1}) + \varphi(2^{i+1})$, $d(v_e v_{(2^i)}) = n - 2 + \varphi(2^{i-1}) + \varphi(2^{i+1})$. Moreover, $d(v_{(2)}v_{(2^2)}) = 1+1+\varphi(2^2)-1+\varphi(2^3) = 7$, $d(v_{(2^{k-1})}v_{(2^k)}) = 1+1+\varphi(2^k)-1+\varphi(2^{k-1})-1+\varphi(2^{k-2}) = \varphi(2^k)+\varphi(2^{k-1})+\varphi(2^{k-2})$, and $d(v_{(2^i)}v_{(2^{i+1})}) = \varphi(2^{i-1}) + \varphi(2^i) + \varphi(2^{i+1}) + \varphi(2^{i+2})$. By making some mathematical operations, taking into consideration that symmetric vertices and edges have the same degree in this graph, the formula is obtained.

For the second entire index, according to the definition and the adjacency criteria, we get that

$$M_2^{\varepsilon}(\mathcal{E}_2(G)) = d(v_e)\Big[d(v_{(2)}) + d(v_e v_{(2)}) + \varphi(2^2)(d(v_{(2^2)}) + d(v_e v_{(2^2)})) + \ldots$$
$$+ \varphi(2^k)(d(v_{(2^k)}) + d(v_e v_{(2^k)}))\Big] + d(v_{(2)})\Big[d(v_e v_{(2)}) + \varphi(2^2)(d(v_{(2^2)}) + d(v_{(2)}v_{(2^2)}))\Big]$$
$$+ \varphi(2^2)d(v_{(2^2)})\Big[d(v_e v_{(2^2)}) + d(v_{(2)}v_{(2^2)}) + \varphi(2^3)(d(v_{(2^3)}) + d(v_{(2^2)}v_{(2^3)}))\Big] + \ldots$$
$$+ \varphi(2^i)d(v_{(2^i)})\Big[d(v_e v_{(2^i)}) + \varphi(2^{i-1})d(v_{(2^{i-1})}v_{(2^i)}) + \varphi(2^{i+1})(d(v_{(2^{i+1})}) + d(v_{(2^i)}v_{(2^{i+1})}))\Big] + \ldots$$
$$+ \varphi(2^k)d(v_{(2^k)})\Big[d(v_e v_{(2^k)}) + \varphi(2^{k-1})d(v_{(2^{k-1})}v_{(2^k)})\Big] + d(v_e v_{(2)})\Big[\varphi(2^2)d(v_e v_{(2^2)}) + \varphi(2^3)d(v_e v_{(2^3)}) + \ldots$$
$$+ \varphi(2^k)d(v_e v_{(2^k)}) + \varphi(2^2)d(v_{(2)}v_{(2^2)})\Big] + \ldots$$
$$+ \varphi(2^i)d(v_e v_{(2^i)})\Big[\varphi(2^{i+1})d(v_e v_{(2^{i+1})}) + \ldots$$
$$+ \varphi(2^k)d(v_e v_{(2^k)}) + \varphi(2^{i-1})d(v_{(2^{i-1})}v_{(2^i)}) + \varphi(2^{i+1})d(v_{(2^i)}v_{(2^{i+1})})\Big] + \ldots$$
$$+ \varphi(2^k)d(v_e v_{(2^k)})\Big[\varphi(2^{k-1})d(v_{(2^{k-1})}v_{(2^k)})\Big] + \varphi(2^2)d(v_{(2)}v_{(2^2)})\Big[\varphi(2^3)d(v_{(2^2)}v_{(2^3)})\Big] + \ldots$$
$$+ \varphi(2^{i-1})\varphi(2^i)d(v_{(2^{i-1})}v_{(2^i)})\Big[\varphi(2^{i+1})d(v_{(2^i)}v_{(2^{i+1})})\Big] + \ldots$$
$$+ \varphi(2^{k-2})\varphi(2^{k-1}) \times d(v_{(2^{k-2})}v_{(2^{k-1})})\Big[\varphi(2^k)d(v_{(2^{k-1})}v_{(2^k)})\Big] + (d(v_e v_{(2^2)}))^2 + \ldots$$
$$+ \sum_{j=1}^{\varphi(2^i)-1} j(d(v_e v_{(2^i)}))^2 + \ldots$$
$$+ \sum_{j=1}^{\varphi(2^k)-1} j(d(v_e v_{(2^k)}))^2 + (d(v_{(2)}v_{(2^2)}))^2 + \Big[2 \cdot \frac{4(4-1)}{2} + 4 \cdot \frac{2(2-1)}{2}\Big](d(v_{(2^2)}v_{(2^3)}))^2 + \ldots$$
$$+ \Big[2^{i-2} \cdot \frac{2^{i-1}(2^{i-1}-1)}{2} + 2^{i-1} \cdot \frac{2^{i-2}(2^{i-2}-1)}{2}\Big](d(v_{(2^{i-1})}v_{(2^i)}))^2 + \ldots$$
$$+ \Big[2^{k-2} \cdot \frac{2^{k-1}(2^{k-1}-1)}{2} + 2^{k-1} \cdot \frac{2^{k-2}(2^{k-2}-1)}{2}\Big](d(v_{(2^{k-1})}v_{(2^k)}))^2.$$

Observe that $\sum_{j=1}^{n} j = \dfrac{n(n+1)}{2}$. As the first entire index, use the degree values and make some computations to get the desired result. □

Theorem 3.3. *Let G be a cyclic group of order 2^k, where $k \geq 4$. The first and second Zagreb indices for $\mathcal{E}_2(G)$ are*

1) $M_1(\mathcal{E}_2(G)) = 9 + (n-1)^2 + 2^{k-1}[1 + 2^{k-4}(8+n)] + \sum_{i=2}^{k-1} 2^{i-1}(1 + (5)2^{i-2}).$

2) $M_2(\mathcal{E}_2(G)) = 36 + 2^{k-1}(3 + 2^{k-1} + 2^{2k-3}) + \sum_{i=2}^{k-1}(2^{i-1} + (5)2^{2i-3})[n - 1 + (5)2^{2i-1} + 2^i].$

Proof. Let G be a cyclic group of order 2^k, where $k \geq 4$. According to the definition of the first and second Zagreb indices and the adjacency method in this graph, we have

$$M_1(\mathcal{E}_2(G)) = (n-1)^2 + \sum_{i=1}^{k} \phi(2^i)(d(v_{(2^i)}))^2,$$

and

$$M_2(\mathcal{E}_2(G)) = (n-1)\Big[\sum_{i=1}^{k} \phi(2^i)d(v_{(2^i)})\Big] + \sum_{i=1}^{k-1} \phi(2^i)d(v_{(2^i)})[\phi(2^{i+1})d(v_{(2^{i+1})})].$$

Hence, by substituting the degrees as in Theorem 3.2 and performing some calculations, we obtain the required result. □

Theorem 3.4. *Let G be a cyclic group of order pq, where $p < q$ are distinct prime numbers. The equitable graph of Type II of G has two cases as follows:*

1) If $p = 2$ and $|2 - q| > \min\{2, q\}$, then

(a) $M_1^{\varepsilon}(\mathcal{E}_2(G)) = 4q^4 + 4q^3 - 24q^2 + 26q - 8.$

(b) $M_2^{\varepsilon}(\mathcal{E}_2(G)) = 4q^5 + 11q^4 - 53q^3 + 76q^2 - 49q + 12.$

2) If $p > 2$ and $|p - q| \leq \min\{p, q\}$, then

$$(a)\ M_1^{\varepsilon}(\mathcal{E}_2(G)) = (pq-1)^2 + (p-1)\Big[q^2 + (pq+q-3)^2\Big] + (q-1) \times\Big[p^2 + (pq+p-3)^2\Big] + (p-1)(q-1)\Big[1 + (pq-2)^2 + (p+q-2)^2\Big].$$

$$\begin{aligned}(b)\ M_2^{\varepsilon}(\mathcal{E}_2(G)) = \ & (pq-1)\Big[p^2q^2 + 2pq - 4p - 4q + 5\Big] + (pq-q)\Big[q^2 + 3pq - 2p - 2q - 1\Big] \\ & +(pq-p)\Big[p^2 + 2pq - 2p - q - 1\Big] \\ & +(p^2q - 3p - q + 3)\Big[p^2q^2 - p^2q + q^2 - 4q + 3\Big] \\ & +(pq^2 - 3q - p + 3)\Big[p^2q^2 - p^2q + p^2 - pq^2 - p + q\Big] \\ & +\Big[\tfrac{p^2-3p+2}{2}\Big] \times (pq+q-3)^2 + \Big[\tfrac{q^2-3q+2}{2}\Big](pq+p-3)^2 \\ & +\Big[\tfrac{(pq-p-q)(pq-p-q+1)}{2}\Big](pq-2)^2 + \Big[(p-1)\tfrac{q^2-3q+1}{2} + (q-1)\tfrac{p^2-3p+1}{2}\Big](p+q-2)^2 \\ & +p^2q^2 - p^2q - pq^2 - pq + 2p + 2q - 2.\end{aligned}$$

Proof. Let G be a cyclic group of order pq where $p < q$ are distinct primes and assume that $p = 2$ with $|2 - q| > \min\{2, q\}$ (the only case in which $|2 - q| \le \min\{2, q\}$ is when $q = 3$, which is included in Example 3.1). Then, in this case, we have $|q - n| \le \min\{q, n\}$ and the first entire index has the following form:

$$M_1^{\varepsilon}(\mathcal{E}_2(G)) = (d(v_e))^2 + (d(v_{(2)}))^2 + \varphi(q)(d(v_{(q)}))^2 + \varphi(2q)(d(v_{(2q)}))^2 + (d(v_e v_{(2)}))^2 + \varphi(q)(d(v_e v_{(q)}))^2 + \varphi(2q)(d(v_e v_{(2q)}))^2 + \varphi(q)\varphi(2q)(d(v_{(q)} v_{(2q)}))^2.$$

The second entire one will be determined via the following formula:

$$\begin{aligned} M_2^{\varepsilon}(\mathcal{E}_2(G)) = {} & d(v_e)\Big[d(v_{(2)}) + d(v_e v_{(2)}) + \varphi(q)(d(v_{(q)}) + d(v_e v_{(q)})) + \varphi(2q)(d(v_{(2q)}) + d(v_e v_{(2q)}))\Big] \\ & + d(v_{(2)})\Big[d(v_e v_{(2)})\Big] + \varphi(q)d(v_{(q)})[d(v_e v_{(q)}) + \varphi(2q)(d(v_{(2q)}) + d(v_{(q)} v_{(2q)}))\Big] \\ & + \varphi(2q)d(v_{(2q)})\Big[d(v_e v_{(2q)}) + \varphi(q)d(v_{(q)} v_{(2q)})\Big] + d(v_e v_{(2)})\Big[\varphi(q)d(v_e v_{(q)}) + \varphi(2q)d(v_e v_{(2q)})\Big] \\ & + \varphi(q)d(v_e v_{(q)})\Big[\varphi(2q)(d(v_e v_{(2q)}) + d(v_{(q)} v_{(2q)}))\Big] + \varphi(2q)d(v_e v_{(2q)})\Big[\varphi(q)d(v_{(q)} v_{(2q)})\Big] \\ & + \sum_{j=1}^{\varphi(q)-1} j(d(v_e v_{(q)}))^2 + \sum_{j=1}^{\varphi(2q)-1} j(d(v_e v_{(2q)}))^2 \\ & + \Big[(q-1)\frac{(q-1)(q-2}{2} + (q-1)\frac{(q-1)(q-2)}{2}\Big](d(v_{(q)} v_{(2q)}))^2. \end{aligned}$$

Since $\varphi(q) = \varphi(2q) = q - 1$, we have $d(v_{(2)}) = 1$, $d(v_{(q)}) = d(v_{(2q)}) = q$, $d(v_e v_{(2)}) = 2q - 2$, $d(v_e v_{(q)}) = d(v_e v_{(2q)}) = 3q - 3$, and $d(v_{(q)} v_{(2q)}) = 2q - 2$. Therefore, similarly to Theorem 3.2, we substitute these values. After performing the necessary calculations, we obtain the intended formula. On the other hand, if $p > 2$ and $|p - q| \le \min\{p, q\}$, then $|pq - q| > \min\{p, q\}$, and hence the first and second entire indices can be computed from the following equations:

$$M_1^{\varepsilon}(\mathcal{E}_2(G)) = (d(v_e))^2 + \varphi(p)\Big[(d(v_{(p)}))^2 + (d(v_e v_{(p)}))^2\Big] + \varphi(q)\Big[(d(v_{(q)}))^2 + (d(v_e v_{(q)}))^2\Big] + \varphi(pq)\Big[(d(v_{(pq)}))^2 + (d(v_e v_{(pq)}))^2\Big] + \varphi(p)\varphi(q)(d(v_{(p)} v_{(q)}))^2,$$

and

$$\begin{aligned} M_2^{\varepsilon}(\mathcal{E}_2(G)) = {} & d(v_e)\Big[\varphi(p)(d(v_{(p)}) + d(v_e v_{(p)})) + \varphi(q)(d(v_{(q)}) + d(v_e v_{(q)})) + \varphi(pq)(d(v_{(pq)}) + d(v_e v_{(pq)}))\Big] \\ & + \varphi(p)d(v_{(p)})\Big[d(v_e v_{(p)}) + \varphi(q)(d(v_{(q)}) + d(v_{(p)} v_{(q)}))\Big] + \varphi(q)d(v_{(q)})\Big[d(v_e v_{(q)}) + \varphi(p)d(v_{(p)} v_{(q)})\Big] \\ & + \varphi(pq)d(v_{(pq)})\Big[d(v_e v_{(pq)})\Big] + \varphi(p)d(v_e v_{(p)})\Big[\varphi(q)(d(v_e v_{(q)}) + d(v_{(p)} v_{(q)})) + \varphi(pq)d(v_e v_{(pq)})\Big] \\ & + \varphi(q)d(v_e v_{(q)})\Big[\varphi(pq)d(v_e v_{(pq)}) + \varphi(p)d(v_{(p)} v_{(q)})\Big] + \sum_{j=1}^{\varphi(p)-1} j(d(v_e v_{(p)}))^2 + \sum_{j=1}^{\varphi(q)-1} j(d(v_e v_{(q)}))^2 \\ & + \sum_{j=1}^{\varphi(pq)-1} j(d(v_e v_{(pq)}))^2 + \Big[(p-1)\frac{(q-1)(q-2)}{2} + (q-1)\frac{(p-1)(p-2)}{2}\Big](d(v_{(p)} v_{(q)}))^2. \end{aligned}$$

In this case, we have $\varphi(pq) = \varphi(p)\varphi(q) = (p - 1)(q - 1)$; hence $d(v_{(p)}) = q$, $d(v_{(q)}) = p$, $d(v_{(pq)}) = 1$, $d(v_e v_{(p)}) = pq + q - 3$, $d(v_e v_{(q)}) = pq + p - 3$, $d(v_e v_{(pq)}) = pq - 2$, and $d(v_{(p)} v_{(q)}) = p + q - 2$. Therefore, The final equation is revealed after the required computational operations are performed. □

Theorem 3.5. *Let G be a cyclic group of order pq, where $p < q$ are distinct prime numbers. The first and second Zagreb indices of the equitable graph of Type II of G are as follows:*

1) If $p = 2$ and $|2 - q| > \min\{2, q\}$, then

(a) $M_1(\mathcal{E}_2(G)) = 2q^3 + 2q^2 - 4q + 2.$

(b) $M_2(\mathcal{E}_2(G)) = q^4 + 2q^3 - 5q^2 + 4q - 1.$

2) If $p > 2$ and $|p - q| \leq \min\{p, q\}$, then

(a) $M_1(\mathcal{E}_2(G)) = p^2q^2 + p^2q + pq^2 - pq - (p^2 + q^2 + p + q) + 2.$

(b) $M_2(\mathcal{E}_2(G)) = 4p^2q^2 - 3(p^2q + pq^2) - pq + 2(p + q) - 1.$

Proof. Let G be a cyclic group of order pq, where $p < q$ are distinct prime numbers. Consider the first case in which $p = 2$ and $q > 3$. The first and second Zagreb indices can be obtained from the following equations and by similar arguments as in Theorem 3.4 as follows:

$$M_1(\mathcal{E}_2(G)) = d(e)^2 + d(v_{(2)})^2 + \phi(q)d(v_{(q)})^2 + \phi(n)d(v_{(n)})^2,$$

and

$$M_2(\mathcal{E}_2(G)) = d(e)\Big[d(v_{(2)}) + \phi(q)d(v_{(q)}) + \phi(n)d(v_{(n)})\Big] + \phi(q)d(v_{(q)})[\phi(n)d(v_{(n)})].$$

Now, assume that $p > 2$ and $|p - q| \leq \{p, q\}$. In this case, we have

$$M_1(\mathcal{E}_2(G)) = d(e)^2 + \phi(p)d(v_{(p)})^2 + \phi(q)d(v_{(q)})^2 + \phi(n)d(v_{(n)})^2,$$

and

$$M_2(\mathcal{E}_2(G)) = d(e)\Big[\phi(p)d(v_{(p)}) + \phi(q)d(v_{(q)}) + \phi(n)d(v_{(n)})\Big] + \phi(p)d(v_{(p)})[\phi(q)d(v_{(q)})].$$

Thus using the same degrees as in the previous theorem, the result will be obtained. □

Theorem 3.6. *Consider the dihedral group D_{2n}, where $n = 2^k$ and $k \geq 3$. Then $\mathcal{E}_2(D_{2n})$ has the following properties:*

$$1)\ M_1^{\varepsilon}(\mathcal{E}_2(D_{2n})) = 6n^3 + 58n^2 + 191n + 198 + 2^{k-1}\Big[(1 + 2^{k-2})^2 + (2n - 2 + 2^{k-2})^2\Big] + 2^{2k-3}\Big((7)2^{k-3}\Big)^2 + \sum_{i=3}^{k-1} 2^{i-1}\Big[(1 + (5)2^{i-2})^2 + (2n - 2 + (5)2^{i-2})^2\Big] + \sum_{i=2}^{k-2} 2^{2i-1}(t + (7)2^{i-1})^2,$$

where $t = \begin{cases} n + 1 & \text{if } i=2; \\ 2^{i-2} & \text{if } i > 2. \end{cases}$

$$2)\ M_2^{\varepsilon}(\mathcal{E}_2(D_{2n})) = 3n^4 + 18n^3 + 96n^2 + 208n + 127$$
$$+(2n-1)\Big[2n^2 + 13n + 21 + \sum_{i=3}^{k} 2^{i-1}(2n-1+(5)2^{i-1})\Big]$$
$$+(2n+12)\Big[n^2 + 14n + 13 + 4\big(d(v_{(2^3)}) + d(v_{(2^2)}v_{(2^3)})\big)\Big]$$
$$+(2n^2+2n)\Big[2n + 14 + \sum_{i=2}^{k} 2^{i-1}d(v_e v_{(2^i)})\Big]$$
$$+(6n+6)\Big[\sum_{i=3}^{k} 2^{i-1}d(v_e v_{(2^i)}) + n^2 + 8n + 7 + 4d(v_{(2^2)}v_{(2^3)})\Big]$$
$$+(2^{k-1} + 2^{2k-3})(2n - 2 + 2^{k-2} + (7)2^{2k-5}) + (7)2^{3k-6}(2n - 2 + 2^{k-2})$$
$$+\sum_{i=3}^{k-1} 2^{i-1}\Big[d(v_{(2^i)})\Big(d(v_e v_{(2^i)}) + 2^{i-2}d(v_{(2^{i-1})}v_{(2^i)}) + 2^i\big(d(v_{(2^{i+1})}) + d(v_{(2^i)}v_{(2^{i+1})})\big)\Big)$$
$$+d(v_e v_{(2^i)})\Big(2^{i-2}d(v_{(2^{i-1})}v_{(2^i)}) + 2^i d(v_{(2^i)}v_{(2^{i+1})}) + \sum_{j=i+1}^{k} 2^{j-1}d(v_e v_{(2^j)})\Big)\Big]$$
$$+\sum_{i=2}^{k}(2^{2i-3} - 2^{i-2})\big(d(v_e v_{(2^i)})\big)^2 + \sum_{i=2}^{k-1}(4^{i-3}(3\cdot 2^i - 8))\big(d(v_{(2^i)}v_{(2^{i+1})})\big)^2$$
$$+\sum_{i=2}^{k-2} 2^{3i}\Big[d(v_{(2^i)}v_{(2^{i+1})})d(v_{(2^{i+1})}v_{(2^{i+2})})\Big] + (8n^2 + 64n + 56)d(v_{(2^2)}v_{(2^3)}).$$

Proof. Let G be the dihedral group D_{2n} and let $n = 2^k$ where $k \geq 3$. Since $\pi_e(G) = \{1, 2, 2^2, \ldots, 2^k\}$, the proof is similar to Theorem 3.2 considering that the number of the elements of order 2 in this group is $n+1$ and $\varphi(2^i)$ elements of order 2^i for each $2 \leq i \leq k$. □

Theorem 3.7. *Consider the dihedral group D_{2n}, where $n = 2^k$ and $k \geq 3$. The first and second Zagreb indices of $\mathcal{E}_2(D_{2n})$ are*

$$1)\ M_1(\mathcal{E}_2(D_{2n})) = 6n^2 + 29n + 82 + 2^{k-1}(1+t)^2 + \sum_{i=3}^{k-1}(1+(5)2^{i-2})^2,$$

where, $t = \begin{cases} n+1 & \text{if } k=3; \\ 2^{k-2} & \text{if } k>3. \end{cases}$

$$2)\ M_2(\mathcal{E}_2(D_{2n})) = 16n^2 + 67n + 21 + (2n-1)\Big[2^{k-1} + 2^{2k-3} + \sum_{i=3}^{k-1}(2^{i-1} + (5)2^{2i-3})\Big]$$
$$+\sum_{i=2}^{k-1}\left[(2^{2i-1} + (5)2^{3i-3})(1+T)\right],$$

where, $T = \begin{cases} 2^{k-2} & \text{if } i = k-1; \\ 2^{i-1} + 2^{i+1} & \text{otherwise.} \end{cases}$

Proof. Consider the dihedral group D_{2n}, where $n = 2^k$ and $k \geq 3$. The proof is similar to Theorems 3.3 and 3.6. □

Theorem 3.8. *Let G be the dihedral group D_{2n}, where $n = 3^k$ and $k \geq 2$. Then $\mathcal{E}_2(G)$ has the following properties:*

1) $M_1^{\varepsilon}(\mathcal{E}_2(G)) = \; 6n^3 + 32n^2 - 7n + 11 + 2\sum_{i=2}^{k} 3^{i-1}[4n^2 - 8n + 5].$

2) $M_2^{\varepsilon}(\mathcal{E}_2(G)) = \; 3n^4 + 30n^3 + 71n^2 - 11n + 2 + 4\sum_{i=2}^{k-1} 3^{i-1}\Big[\sum_{j=i+1}^{k} 3^{j-1}(2n-2)^2\Big]$

$$+ \sum_{i=2}^{k}\left[(2)3^{i-1}\Big((2n-1)^2 + 4n^3 + 8n^2 - 18n + 6\Big) + \Big((2)3^{2i-2} - 3^{i-1}\Big)(2n-2)^2\right].$$

Proof. Consider the dihedral group D_{2n} where $n = 3^k$ and $k \geq 2$. Since we have $|2-3| \leq \min\{2,3\}$, $|3^i - 3^{i+1}| > \min\{3^i, 3^{i+1}\}$ for each $1 \leq i \leq k-1$, and considering that D_{2n}, in this case, has n elements of order 2, we find that the first and second entire indices of $\mathcal{E}_2(D_{2n})$ can be computed via the following equations:

$$\begin{aligned} M_1^{\varepsilon}(\mathcal{E}_2(D_{2n})) = {} & (d(v_e))^2 + n\Big[(d(v_{(2)}))^2 + (d(v_e v_{(2)}))^2\Big] + \varphi(3)\Big[(d(v_{(3)}))^2 + (d(v_e v_{(3)}))^2\Big] \\ & + \varphi(3^2)\Big[(d(v_{(3^2)}))^2 + (d(v_e v_{(3^2)}))^2\Big] + \ldots \\ & + \varphi(3^k)\Big[(d(v_{(3^k)}))^2 + (d(v_e v_{(3^k)}))^2\Big] + n\varphi(3)(d(v_{(2)}v_{(3)}))^2, \end{aligned}$$

and

$$\begin{aligned} M_2^{\varepsilon}(\mathcal{E}_2(D_{2n})) = {} & d(v_e)\Big[n(d(v_{(2)}) + d(v_e v_{(2)})) + \varphi(3)(d(v_{(3)}) + d(v_e v_{(3)})) + \ldots \\ & + \varphi(3^k)(d(v_{(3^k)}) + d(v_e v_{(3^k)}))\Big] + nd(v_{(2)})\Big[d(v_e v_{(2)}) + \varphi(3)(d(v_{(3)}) + d(v_{(2)}v_{(3)}))\Big] \\ & + \varphi(3)d(v_{(3)})\Big[d(v_e v_{(3)}) + nd(v_{(2)}v_{(3)})\Big] + \varphi(3^2)d(v_{(3^2)})\Big[d(v_e v_{(3^2)})\Big] + \ldots \\ & + \varphi(3^k)d(v_{(3^k)})\Big[d(v_e v_{(3^k)})\Big] + nd(v_e v_{(2)})\Big[\varphi(3)d(v_e v_{(3)}) + \varphi(3^2)d(v_e v_{(3^2)}) + \ldots \\ & + \varphi(3^k)d(v_e v_{(3^k)}) + \varphi(3)d(v_{(2)}v_{(3)})\Big] + \varphi(3)d(v_e v_{(3)})\Big[\varphi(3^2)d(v_e v_{(3^2)}) + \ldots \\ & + \varphi(3^k)d(v_e v_{(3^k)}) + nd(v_{(2)}v_{(3)})\Big] + \varphi(3^2)d(v_e v_{(3^2)})\Big[\varphi(3^3)d(v_e v_{(3^3)}) + \ldots \\ & + \varphi(3^k)d(v_e v_{(3^k)})\Big] + \ldots \\ & + \varphi(3^{k-1})d(v_e v_{(3^{k-1})})\Big[\varphi(3^k)d(v_e v_{(3^k)})\Big] + \sum_{j=1}^{n-1} j(d(v_e v_{(2)}))^2 + (d(v_e v_{(3)}))^2 + \sum_{j=1}^{\varphi(3^2)-1} j(d(v_e v_{(3^2)}))^2 + \ldots \\ & + \sum_{j=1}^{\varphi(3^k)-1} j(d(v_e v_{(3^k)}))^2 + \Big[n\frac{2(2-1)}{2} + 2\frac{n(n-1)}{2}\Big](d(v_{(2)}v_{(3)}))^2. \end{aligned}$$

The degree values of the vertices in $\mathcal{E}_2(D_{2n})$ are as follows:

$$d(v_{(2)}) = 3,\ d(v_{(3)}) = n+1,\ d(v_e v_{(2)}) = 2n,\ d(v_e v_{(3)}) = 3n-2,\ d(v_{(2)}v_{(3)}) = n+2,\ d(v_{(3^i)}) = 1,$$

and

$$d(v_e v_{(3^i)}) = 2n - 2 \text{ for all } 2 \leq i \leq k.$$

By substituting these values and making some calculations, we get the desired formula. □

Theorem 3.9. *Let G be the dihedral group D_{2n}, where $n = 3^k$ and $k \geq 2$. Then the first and second Zagreb indices for $\mathcal{E}_2(G)$ are*

1) $M_1(\mathcal{E}_2(G)) = 6n^2 + 9n + 3 + 2\sum_{i=2}^{k} 3^{i-1}$.

2) $M_2(\mathcal{E}_2(G)) = 16n^2 + 5n - 2 + (4n - 2)\sum_{i=2}^{k} 3^{i-1}$.

Proof. Let G be the dihedral group D_{2n}, where $n = 3^k$ and $k \geq 2$.The desired outcome can be determined by applying a similar argument to that used in Theorem 3.8 to the following equations:

$$M_1(\mathcal{E}_2(D_{2n})) = d(e)^2 + nd(v_{(2)})^2 + \sum_{i=1}^{k} \phi(3^i)d(v_{(3^i)})^2,$$

and

$$M_2(\mathcal{E}_2(D_{2n})) = d(e)\Big[nd(v_{(2)}) + \sum_{i=1}^{k} \phi(3^i)d(v_{(3^i)})\Big] + nd(v_{(2)})[\phi(3)d(v_{(3)})].$$

□

Theorem 3.10. *Consider the dihedral group D_{2n}, where $n = pq$; $p < q$ are distinct primes. The equitable graph of Type II of this group has the following cases:*

1) If $n = 2q$ and $q > 3$, then

(a) $M_1^{\varepsilon}(\mathcal{E}_2(D_{2n})) = 4q^4 + 68q^3 - 88q^2 + 48q - 8$.

(b) $M_2^{\varepsilon}(\mathcal{E}_2(D_{2n})) = 4q^5 + 106q^4 - 148q^3 + 75q^2 + 2q - 6$.

2) If $n = 3q$ and $q > 5$, then

(a) $M_1^{\varepsilon}(\mathcal{E}_2(D_{2n})) = 270q^3 + 108q^2 + 66q - 4$.

(b) $M_2^{\varepsilon}(\mathcal{E}_2(D_{2n})) = 729q^4 + 336q^3 + 563q^2 + 36q - 3$.

3) If $p > 3$ and $|p - q| \leq \min\{p, q\}$, then

$$\begin{aligned}(a)\ M_1^{\varepsilon}(\mathcal{E}_2(D_{2n})) = \quad & (2pq-1)^2 + pq\Big[4p^2q^2 - 8pq + 5\Big] \\ & +(pq - p - q + 1)\Big[1 + (2pq-2)^2 + (p+q-2)^2\Big] \\ & +(p-1)\Big[q^2 + (2pq + q - 3)^2\Big] + (q-1)\Big[p^2 + (2pq + p - 3)^2\Big]. \\ (b)\ M_2^{\varepsilon}(\mathcal{E}_2(D_{2n})) = \quad & (2n-1)\Big[4n^2 + n - pq^2 - 4p - 4q + 5\Big] + (n-q)\Big[4n - 2p - 2q + q^2 - 1\Big] \\ & +2n\Big[(n-1)^3 - 1\Big] + (n-p)\Big[3n + b^3 - 2p - q - 1\Big] \\ & +(2n^2 - 2n)\Big[2n^2 - 2n - 2p - 2q + 4\Big] \\ & +(q-1)(2n + q - 3)\Big[2n^2 - 2p^2q - 2pq^2 + p^2 + pq - p + q\Big] \\ & +\tfrac{1}{2}\Big[(p^2 - 3p + 2)(2n + q - 3)^2 + (q^2 - 3q + 2) \times (2n + p - 3)^2 \\ & +(n - p - q)(n - p - q + 1)(2n-2)^2\Big] \\ & +\Big[(p-1)\tfrac{q^2-3q+1}{2} + (q-1)\tfrac{p^2-3p+1}{2}\Big](p+q-2)^2 \\ & +4n^2 - 2p^2q - 2pq^2 + 2p + 2q - 2.\end{aligned}$$

Proof. Let G be the dihedral group D_{2n} and $n = pq$, where $p < q$ are distinct primes. If $n = 2q$ and $|2 - q| > \min\{2, q\}$, then the proof is similar to Theorem 3.4, considering that G in this case has $n + 1$ elements of order 2.

Now assume that $p = 3$ and $|3 - q| > \min\{3, q\}$. As $|n - q| > \min\{n, q\}$ and the cardinality of the elements that has order 2 is n, we get

$$M_1^{\varepsilon}(\mathcal{E}_2(G)) = (d(v_e))^2 + n\Big[(d(v_{(2)}))^2 + (d(v_e v_{(2)}))^2\Big] + \varphi(3)\Big[(d(v_{(3)}))^2 + (d(v_e v_{(3)}))^2\Big)^2\Big] + \varphi(q)\Big[(d(v_{(q)}))^2 + (d(v_e v_{(q)}))^2\Big] + \varphi(3q)\Big[(d(v_{(3q)}))^2 + (d(v_e v_{(3q)}))^2\Big] + n\varphi(3)(d(v_{(2)}v_{(3)}))^2,$$

and

$$\begin{aligned} M_2^{\varepsilon}(\mathcal{E}_2(G)) = d(v_e)\Big[&n(d(v_{(2)}) + d(v_e v_{(2)})) + \varphi(3)(d(v_{(3)}) + d(v_e v_{(3)})) + \varphi(q)(d(v_{(q)}) \\ &+ d(v_e v_{(q)})) + \varphi(3q)(d(v_{(3q)}) + d(v_e v_{(3q)}))\Big] \\ &+ nd(v_{(2)})\Big[d(v_e v_{(2)}) + \varphi(3)(d(v_{(3)}) + d(v_{(2)}v_{(3)}))\Big] + \varphi(3)d(v_{(3)})\Big[d(v_e v_{(3)}) + nd(v_{(2)}v_{(3)})\Big] \\ &+ \varphi(q)d(v_{(q)})\Big[d(v_e v_{(q)})\Big] + \varphi(3q)d(v_{(3q)})\Big[d(v_e v_{(3q)})\Big] \\ &+ nd(v_e v_{(2)})\Big[\varphi(3)(d(v_e v_{(3)}) + d(v_{(2)}v_{(3)})) + \varphi(q)d(v_e v_{(q)}) + \varphi(3q)d(v_e v_{(3q)})\Big] \\ &+ \varphi(3)d(v_e v_{(3)})\Big[nd(v_{(2)}v_{(3)}) + \varphi(q)d(v_e v_{(q)}) + \varphi(3q)d(v_e v_{(3q)})\Big] \\ &+ \varphi(q)d(v_e v_{(q)})\Big[\varphi(3q)d(v_e v_{(3q)})\Big] + \sum_{j=1}^{n-1} j(d(v_e v_{(2)}))^2 + \sum_{j=1}^{\varphi(q)-1} j(d(v_e v_{(q)}))^2 \\ &+ \sum_{j=1}^{\varphi(3q)-1} j(d(v_e v_{(3q)}))^2 + \Big[n\frac{2(2-1)}{2} + 2\frac{n(n-1)}{2}\Big](d(v_{(2)}v_{(3)}))^2. \end{aligned}$$

According to the adjacency method in this graph, we have $d(v_{(2)}) = 3$, $d(v_{(3)}) = 3q + 1$, $d(v_{(q)}) = d(v_{(3q)}) = 1$, $d(v_e v_{(2)}) = 2n$, $d(v_e v_{(3)}) = 3n - 2$, $d(v_e v_{(q)}) = d(v_e v_{(3q)}) = 2n - 2$, and $d(v_{(2)}v_{(3)}) = n + 2$. Therefore, by calculating these equations using the previous values, we obtain the required outcome.

Furthermore, consider that $n = pq$, where $p > 3$ and $|p - q| \leq \min\{p, q\}$. Then $|2 - p| > \min\{2, p\}$ and $|n - q| > \min\{n, q\}$. Hence, by a similar argument to the previous case, we get the result. □

Theorem 3.11. *Consider the dihedral group D_{2n}, where $n = pq$; $p < q$ are distinct primes. Then the first and second Zagreb indices of the the equitable graph of Type II of G are as follows:*

1) If $n = 2q$ and $q > 3$, then

- *(a)* $M_1(\mathcal{E}_2(D_{2n})) = 2q^3 + 14q^2 - 6q + 2.$
- *(b)* $M_2(\mathcal{E}_2(D_{2n})) = q^4 + 6q^3 - q^2 + 4q - 1.$

2) If $n = 3q$ and $q > 5$, then

- *(a)* $M_1(\mathcal{E}_2(D_{2n})) = 54q^2 + 30q.$
- *(b)* $M_2(\mathcal{E}_2(D_{2n})) = 162q^2 - 6q + 1.$

3) If $p > 3$ and $|p - q| \leq \min\{p, q\}$, then

(a) $M_1(\mathcal{E}_2(D_{2n})) = 4p^2q^2 - (p+q)^2 + p^2q + pq^2 - (p+q) + 2.$

(b) $M_2(\mathcal{E}_2(D_{2n})) = 9p^2q^2 - 5(p^2q + pq^2) - pq + 2(p+q) - 1.$

Proof. Let G be the dihedral group D_{2n}, where $n = pq$; $p < q$ are distinct primes. Assume that $n = 2q$, where $q > 3$. Since G in this case has $n + 1$ elements of order 2, we have

$$M_1(\mathcal{E}_2(G)) = d(e)^2 + (n+1)d(v_{(2)})^2 + \phi(q)d(v_{(q)})^2 + \phi(n)d(v_{(n)})^2,$$

and

$$M_2(\mathcal{E}_2(G)) = d(e)\Big[(n+1)d(v_{(2)}) + \phi(q)d(v_{(q)}) + \phi(n)d(v_{(n)})\Big] + \phi(q)d(v_{(q)})[\phi(n)d(v_{(n)})].$$

Thus, similarly to Theorem 3.4, the result will be achieved.

Now, if $p = 3$ and $|3 - q| > \min\{3, q\}$, we have

$$M_1(\mathcal{E}_2(G)) = d(e)^2 + \phi(3)d(v_{(3)})^2 + \phi(q)d(v_{(q)})^2 + \phi(n)d(v_{(n)})^2,$$

and

$$M_2(\mathcal{E}_2(G)) = d(e)\Big[\phi(3)d(v_{(3)}) + \phi(q)d(v_{(q)}) + \phi(n)d(v_{(n)})\Big] + nd(v_{(2)})[\phi(3)d(v_{(3)})].$$

Therefore, by substituting the degrees used in Theorem 3.10 with some calculations, the formula will be obtained.

Consequently, by applying an analogous methodology to that used in the previous case, the necessary result can be discerned. □

Theorem 3.12. *Let G be the generalized quaternion group Q_{4n}; $n = 2^k$ and $k > 3$. Then $\mathcal{E}_2(G)$ has the following properties:*

$$\begin{aligned} 1)\ M_1^{\varepsilon}(\mathcal{E}_2(G)) = \ & 72n^3 + 872n^2 + 3136n + 2738 + 2^k\Big[(1 + 2^{k-1})^2 + (4n - 2 + 2^{k-1})^2 \\ & + 2^{k-1}((7)2^{k-2})^2\Big] + \sum_{i=4}^{k} 2^{i-1}\Big[(1 + (5)2^{i-2})^2 + (4n - 2 + (5)2^{i-2})^2\Big] \\ & + \sum_{i=4}^{k-1} 2^{2i-1}((7)2^{i-1} + t)^2, \end{aligned}$$

where $t = \begin{cases} 2n+2 & \text{if } i = 4; \\ 2^{i-2} & \text{if } i > 4. \end{cases}$

$$
\begin{aligned}
2)\ M_2^{\varepsilon}(\mathcal{E}_2(G)) = \ & 72n^4 + 1092n^3 + 7108n^2 + 15458n + 32(2n+11)\Big[d(v_{(2^4)})\Big] \\
& +(128n^2 + 1344n + 1568)\Big[d(v_{(2^3)}v_{(2^4)})\Big] \\
& +2^k\Bigg[d(v_{(2^{k+1})})\Big(d(v_e v_{(2^{k+1})}) + 2^{k-1}d(v_{(2^k)}v_{(2^{k+1})})\Big) \\
& +d(v_e v_{(2^{k+1})})\Big(6n + 2^{k-1}d(v_{(2^k)}v_{(2^{k+1})})\Big)\Bigg] \\
& + \sum_{i=4}^{k} 2^{i-1}\Bigg[d(v_{(2^i)})\Bigg(d(v_e v_{(2^i)}) + 2^{i-2}d(v_{(2^{i-1})}v_{(2^i)}) + 2^i\big(d(v_{(2^{i+1})}) + d(v_{(2^i)}v_{(2^{i+1})})\big)\Bigg) \\
& +d(v_e v_{(2^i)})\Bigg(2^{i-2}d(v_{(2^{i-1})}v_{(2^i)}) + 2^i d(v_{(2^i)}v_{(2^{i+1})}) + \sum_{j=i+1}^{k+1} 2^{j-1}d(v_e v_{(2^j)})\Bigg) \\
& +2^{2i-2}\Big(d(v_{(2^{i-1})}v_{(2^i)})d(v_{(2^i)}v_{(2^{i+1})})\Big)\Bigg] + \sum_{i=4}^{k+1} 2^{i-1}\Bigg[(4n-1)\Big(d(v_{(2^i)}) + d(v_e v_{(2^i)})\Big) \\
& +\Big(8n^2 + 38n + 38\Big)d(v_e v_{(2^i)}) + \frac{2^{i-1}-1}{2}\big(d(v_e v_{(2^i)})\big)^2\Bigg] \\
& + \sum_{i=3}^{k}\Big[2^{i-1}(6n)d(v_e v_{(2^i)}) + (4^{i-3}(3\cdot 2^i - 8))\big(d(v_{(2^i)}v_{(2^{i+1})})\big)^2\Big] + 9265.
\end{aligned}
$$

Proof. Let G be the generalized quaternion group Q_{4n} where $n = 2^k$ and $k > 3$ (Example 3.1 includes the case of $k = 1, 2$, or 3). Observe that $\pi_e(G) = \{1, 2, 2^2, \ldots, 2^k, 2^{k+1}\}$ and G has $2n + 2$ elements of order 4, one of order 2, and $\varphi(2^i)$ elements of order 2^i for all $2 < i \leq k + 1$. Using the same method of proof mentioned in Theorem 3.2, we obtain the desired result. □

Theorem 3.13. *Let G be the generalized quaternion group Q_{4n}; $n = 2^k$ and $k > 3$. The first and second Zagreb indices of $\mathcal{E}_2(G)$ are*

1) $M_1(\mathcal{E}_2(G)) = 20n^2 + 76n + 82 + \sum_{i=3}^{k+1} 2^{i-1}(1+t)^2,$

where $t = \begin{cases} 2^{k-1} & \text{if } i = k+1; \\ (5)2^{i-2} & \text{otherwise.} \end{cases}$

2) $M_2(\mathcal{E}_2(G)) = 176n^2 + 730n + 549 + (4n-1)\Big[2^k + 2^{2k-1} + \sum_{i=3}^{k}(2^{i-1} + (5)2^{2i-3})\Big] + \sum_{i=3}^{k}(2^{2i-1} + (5)2^{3i-3})(1+T),$

where $T = \begin{cases} 2^{k-1} & \text{if } i = k; \\ (5)2^{i-2} & \text{otherwise.} \end{cases}$

Proof. Let G be the generalized quaternion group Q_{4n}; $n = 2^k$ and $k > 3$. The proof will be conducted using the same method outlined in Theorems 3.3 and 3.12. □

Theorem 3.14. *Consider the dicyclic group Q_{4n} where $n = p^k$ and $p > 3$; $k \geq 1$. We then have the following:*

1) If $p = 5$ or 7 and $k = 1$, then

(a) $M_1^{\varepsilon}(\mathcal{E}_2(Q_{4n})) = 48p^4 + 58p^3 - 66p^2 + 46p - 8$.

(b) $M_2^{\varepsilon}(\mathcal{E}_2(Q_{4n})) = 96p^5 + 197p^4 - 268p^3 + 246p^2 - 82p + 12$.

2) If $p = 5$ or 7 and $k \geq 2$, then

$$
\begin{aligned}
(a)\ M_1^{\varepsilon}(\mathcal{E}_2(Q_{4n})) = \ & 56n^2 - 28n + 6 + 2n\Big(4p^3 + 8np^2 - 5p^2 + 4n^2p + 3p + 16n^2 - 12n + 4\Big) \\
& +(p-1)\Big(4p^3 + 8np^2 - 8p^2 + 4n^2p + 8np + 52n^2 - 52n + 14\Big) \\
& +2\sum_{i=2}^{k}\varphi(p^i)\Big[(1+\varphi(p^i))^2 + (4n + \varphi(p^i) - 2)^2 + 2\Big(\varphi(p^i)\Big)^3\Big]. \\
(b)\ M_2^{\varepsilon}(\mathcal{E}_2(Q_{4n})) = \ & 8p^2n^3 + 16p^3n^2 - 4p^2n^2 + 24pn^2 + 8p^4n - 4p^3n + 12p^2n \\
& -14pn + 16n^2 + 6n + 4p^4 - 10p^3 + 4p^2 + 2p - 2 \\
& +(2n^2 - n)\Big[(4n + p - 2)^2 + ((2n + p)^2\Big] \\
& +\tfrac{p^2-3p+2}{2}\Big[(6n + p - 3)^2 + (4n + p - 3)^2\Big] \\
& +(3p^3 - 6p^2 + 3p) \times (2n + 2p - 1)^2 \\
& +(p^3 - 4p^2 + 5p - 2)(2n + 2p - 2)^2 \\
& +\Big(\varphi(p^k)\Big)^2(4n + \varphi(p^k) - 2) \times (4n + 5\varphi(p^k) - 2) \\
& +(4n-1)\Big[8n^2 + 16pn - 6n + 4p^2 - 10p + 5 + \beta\Big] \\
& +(6n-2)\Big[12n^2 - 14n + 14pn + 2p^2 - 8p + 6 + \beta\Big] \\
& +(8n^2 + 2pn - 4n) \times \Big[4p^2 - 10n + 12pn - 10p + 7 + \beta\Big] \\
& +(p-1)(6n + p - 3)\Big[3p^2 + 4n^2 - 8n + 10pn - 8p + 5 + \beta\Big] \\
& +(p^2 - 4p + 4np - 4n + 3)\Big[2p^2 - 2n + 2pn - 4p + 2 + \beta\Big] \\
& +\sum_{i=2}^{k-1}\varphi(p^i)(4n + \varphi(p^i) - 2)\Big[\varphi(p^i)(4n + 5\varphi(p^i) - 2) \\
& +\sum_{j=i+1}^{k}\varphi(p^j)(16n + 4\varphi(p^j) - 8)\Big] \\
& +\sum_{i=2}^{k}\varphi(p^i)\Big[(1 + \varphi(p^i))\Big(8n + 5(\varphi(p^i))^2 + 3\varphi(p^i) - 4\Big) \\
& +\Big((\varphi(p^i) - 1) \times (4n + \varphi(p^i) - 2)^2\Big) + \Big(4\varphi(p^i)(p^{2i-2}(p-1)^2(p^i - p^{i-1} - 1)\Big)\Big],
\end{aligned}
$$

where $\beta = \sum_{i=2}^{k}\varphi(p^i)\Big[8n + 2\varphi(p^i) - 4\Big]$.

3) If $p > 7$ and $k \geq 1$, then

$$
\begin{aligned}
(a)\ M_1^{\varepsilon}(\mathcal{E}_2(Q_{4n})) = \ & 40n^3 + 48n^2 - 16n + 6 \\
& +2\sum_{i=1}^{k}\varphi(p^i)\Big[(1+\varphi(p^i))^2 + (4n + \varphi(p^i) - 2)^2 + 4(\varphi(p^i))^4\Big].
\end{aligned}
$$

$$
\begin{aligned}
(b)\ M_2^{\varepsilon}(\mathcal{E}_2(Q_{4n})) = \ & 40n^4 + 96n^3 + 70n^2 - 10n - 1 \\
& + (\varphi(p^k))^2 (16n^2 + 24n\varphi(p^k) - 16n + 5\varphi(p^k)^2 - 12\varphi(p^k) + 4) \\
& + \sum_{i=1}^{k} \varphi(p^i) \Big[(4n-1)(8n + 4\varphi(p^i) - 2) \\
& + (1 + \varphi(p^i))(8n + 5(\varphi(p^i))^2 + 3\varphi(p^i) - 4) + (8n + 2\varphi(p^i) - 4)(6n^2 + 4n - 2) \\
& + 4\varphi(p^i)(p^{2i-2}(p-1)^2(p^i - p^{i-1} - 1) + (\varphi(p^i) - 1)(4n + \varphi(p^i) - 2)^2 \Big] \\
& + \sum_{i=1}^{k-1} \varphi(p^i)(4n + \varphi(p^i) - 2) \Big[\varphi(p^i)(4n + 5\varphi(p^i) - 2) \\
& + \sum_{j=i+1}^{k} \varphi(p^j)(16n + 4\varphi(p^j) - 8) \Big].
\end{aligned}
$$

Proof. Let G be the dicyclic group Q_{4n} and let $n = p^k$, where $k \geq 1$. In this case, G has $2n$ elements of order 4, one of order 2, and $\varphi(p^i)$ elements of order p^i for each $1 \leq i \leq k$. If $|4 - p| \leq \min\{4, p\}$ and $k = 1$, the equitable graph of Type II in this case has the following form:

$$K_1 \vee (W_{(2)} \vee W_{(4)} \vee W_{(p)} \vee W_{(2p)}).$$

Hence, the proof is a similar argument to that in Theorem 3.2.

Furthermore, assume that $k > 1$, then the entire indices can be computed from the following two equations:

$$
\begin{aligned}
M_1^{\varepsilon}(\mathcal{E}_2(G)) = \ & (d(v_e))^2 + (d(v_{(2)}))^2 + (d(v_e v_{(2)}))^2 + 2n\Big[(d(v_{(4)}))^2 + (d(v_e v_{(4)}))^2\Big] \\
& + \sum_{i=1}^{k} \varphi(p^i)\Big[(d(v_{(p^i)}))^2 + (d(v_e v_{(p^i)}))^2\Big] + \sum_{i=1}^{k} \varphi(2p^i)\Big[(d(v_{(2p^i)}))^2 + (d(v_e v_{(2p^i)}))^2\Big] \\
& + 2n(d(v_{(2)}v_{(4)}))^2 + 2n\varphi(p)(d(v_{(4)}v_{(p)}))^2 + \varphi(p)\varphi(2p)(d(v_{(p)}v_{(2p)}))^2,
\end{aligned}
$$

and

$$
\begin{aligned}
M_2^{\varepsilon}(\mathcal{E}_2(G)) = \ & d(v_e)\Big[d(v_{(2)}) + d(v_e v_{(2)}) + 2n(d(v_{(4)}) + d(v_e v_{(4)})) + \sum_{i=1}^{k} \varphi(p^i)(d(v_{(p^i)}) + d(v_e v_{(p^i)})) + \sum_{i=1}^{k} \varphi(2p^i)(d(v_{(2p^i)}) + d(v_e v_{(2p^i)}))\Big] \\
& + d(v_{(2)})\Big[d(v_e v_{(2)}) + 2n(d(v_{(4)}) + d(v_{(2)}v_{(4)}))\Big] + 2nd(v_{(4)})\Big[d(v_e v_{(4)}) + d(v_{(2)}v_{(4)}) + \varphi(p)(d(v_{(p)}) + d(v_{(4)}v_{(p)}))\Big] \\
& + \varphi(p)d(v_{(p)})\Big[d(v_e v_{(p)}) + 2nd(v_{(4)}v_{(p)}) + \varphi(2p)(d(v_{(2p)}) + d(v_{(p)}v_{(2p)}))\Big] + \varphi(2p)d(v_{(2p)})\Big[d(v_e v_{(2p)}) + \varphi(p)d(v_{(p)}v_{(2p)})\Big] \\
& + \sum_{i=2}^{k} \varphi(p^i)d(v_{(p^i)})\Big[d(v_e v_{(p^i)}) + \varphi(2p^i)(d(v_{(2p^i)}) + d(v_{(p^i)}v_{(2p^i)}))\Big] + \sum_{i=2}^{k} \varphi(2p^i)d(v_{(2p^i)})\Big[d(v_e v_{(2p^i)}) + \varphi(p^i)d(v_{(p^i)}v_{(2p^i)})\Big] \\
& + d(v_e v_{(2)})\Big[2n(d(v_e v_{(4)}) + d(v_{(2)}v_{(4)})) + \sum_{i=1}^{k} d(v_e v_{(p^i)}) + \sum_{i=1}^{k} \varphi(2p^i)d(v_e v_{(2p^i)})\Big] \\
& + 2nd(v_e v_{(4)})\Big[d(v_{(2)}v_{(4)}) + \varphi(p)(d(v_e v_{(p)}) + d(v_{(4)}v_{(p)})) + \varphi(2p)d(v_e v_{(2p)}) + \sum_{i=2}^{k} \varphi(p^i)d(v_e v_{(p^i)}) + \sum_{i=2}^{k} \varphi(2p^i)d(v_e v_{(2p^i)})\Big] \\
& + \varphi(p)d(v_e v_{(p)})\Big[2nd(v_{(4)}v_{(p)}) + \varphi(2p)(d(v_e v_{(2p)}) + d(v_{(p)}v_{(2p)})) + \sum_{i=2}^{k} \varphi(p^i)d(v_e v_{(p^i)}) + \sum_{i=2}^{k} \varphi(2p^i)d(v_e v_{(2p^i)})\Big] \\
& + \sum_{i=2}^{k-1} \varphi(p^i)d(v_e v_{(p^i)})\Big[\varphi(2p^i)(d(v_e v_{(2p^i)}) + d(v_{(p^i)}v_{(2p^i)})) + \sum_{j=i+1}^{k} \varphi(p^j)d(v_e v_{(p^j)}) + \sum_{j=i+1}^{k} \varphi(2p^j)d(v_e v_{(2p^j)})\Big] \\
& + \sum_{i=2}^{k-1} \varphi(2p^i)d(v_e v_{(2p^i)})\Big[\varphi(p^i)d(v_{(p^i)}v_{(2p^i)}) + \sum_{j=i+1}^{k} \varphi(p^j)d(v_e v_{(p^j)}) + \sum_{j=i+1}^{k} \varphi(2p^j)d(v_e v_{(2p^j)})\Big]
\end{aligned}
$$

$$+\varphi(p^k)d(v_ev_{(p^k)})\Big[\varphi(2p^k)(d(v_ev_{(2p^k)})+d(v_{(p^k)}v_{(2p^k)}))\Big]+\varphi(2p^k)d(v_ev_{(2p^k)})\Big[\varphi(p^k)d(v_{(p^k)}v_{(2p^k)})\Big]$$
$$+2n\,d(v_{(2)}v_{(4)})\Big[\varphi(p)d(v_{(4)}v_{(p)})\Big]+2n\,\varphi(p)d(v_{(4)}v_{(p)})\Big[\varphi(2p)d(v_{(p)}v_{(2p)})\Big]$$
$$+\sum_{j=1}^{2n-1}j(d(v_ev_{(4)}))^2+\sum_{i=1}^{k}\Big[\Big(\sum_{j=1}^{\varphi(p^i)-1}j(d(v_ev_{(p^i)}))^2\Big)+\Big(\sum_{j=1}^{\varphi(2p^i)-1}j(d(v_ev_{(2p^i)}))^2\Big)\Big]$$
$$+\sum_{j=1}^{2n-1}j(d(v_{(2)}v_{(4)}))^2+\Big(2n\frac{(p-1)(p-2)}{2}+(p-1)\frac{2n(2n-1)}{2}\Big)(d(v_{(4)}v_{(p)}))^2$$
$$+\sum_{i=1}^{k}\Big[\Big(\varphi(p^i)\frac{\varphi(2p^i)(\varphi(2p^i)-1)}{2}+\varphi(2p^i)\frac{\varphi(p^i)(\varphi(p^i)-1)}{2}\Big)(d(v_{(p^i)}v_{(2p^i)}))^2\Big].$$

Now, as $\varphi(p^i)=\varphi(2p^i)$ for each $1\le i\le k$, we have the following:

$$d(v_{(2)})=2n+1,\ d(v_{(4)})=p+1,\ d(v_{(p)})=2n+p,\ d(v_{(2p)})=p,\ d(v_ev_{(2)})=6n-2,$$
$$d(v_ev_{(4)})=4n+p-2,\ d(v_ev_{(p)})=6n+p-3,\ d(v_ev_{(2p)})=4n+p-3,\ d(v_{(2)}v_{(4)})=2n+p,$$
$$d(v_{(4)}v_{(p)})=2n+2p-1,\ d(v_{(p)}v_{(2p)})=2n+2p-2,\ d(v_{(p^i)})=d(v_{(2p^i)})=1+\varphi(p^i),$$
$$d(v_ev_{(p^i)})=d(v_ev_{(2p^i)})=4n+\varphi(p^i)-2,\text{ and } d(v_{(p^i)}v_{(2p^i)})=2\varphi(p^i)\text{ for all } 2\le i\le k.$$

Therefore, the result will be found using these degree values in the last two equations with some calculations.

For the latter case, if $p>7$ and $k\ge 1$, we have $|p-4|>\min\{p,4\}$, which means that this is the same as in the previous case, except that there is no edge between the vertices of order 4 and the vertices of order p. Hence, following a similar argument will yield the desired outcome. □

Theorem 3.15. *Consider the dicyclic group Q_{4n}, where $n=p^k$ and $p>3$; $k\ge 1$. The first and second Zagreb indices of $\mathcal{E}_2(Q_{4n})$ are as follows:*

1) If $p=5$ or 7 and $k=1$, then

- *(a)* $M_1(\mathcal{E}_2(Q_{4n}))=12p^3+14p^2-p+2.$
- *(b)* $M_2(\mathcal{E}_2(Q_{4n}))=9p^4+22p^3-3p^2+6p-1.$

2) If $p=5$ or 7 and $k\ge 2$, then

$$(a)\ M_1(\mathcal{E}_2(Q_{4n}))=\ 2p^2(p+3n-1)+4n^2(2+p)-2n+2+2\sum_{i=2}^{k}\phi(p^i)(1+\phi(p^i))^2.$$
$$(b)\ M_2(\mathcal{E}_2(Q_{4n}))=\ 8n^2+4n+(p^2+2np)^2+[2p^3+10np](2n-1)-(p-1)^2$$
$$+\sum_{i=2}^{k}\phi(p^i)(1+\phi(p^i))\Big[\phi(p^i)(1+\phi(p^i))+8n-2\Big].$$

3) If $p>7$ and $k\ge 1$, then

$$(a)\ M_1(\mathcal{E}_2(Q_{4n}))=20n^2+4n+2+2\sum_{i=1}^{k}\phi(p^i)(1+\phi(p^i))^2.$$
$$(b)\ M_2(\mathcal{E}_2(Q_{4n}))=32n^2+2n-1+\sum_{i=1}^{k}\phi(p^i)(1+\phi(p^i))\Big[\phi(p^i)(1+\phi(p^i))+8n-2\Big].$$

Proof. Let G be the dicyclic group Q_{4n} where $n = p^k$ and $p > 3$; $k \geq 1$. Consider that $n = 5$ or 7, according to the definition of the Zagreb indices, we have $M_1(\mathcal{E}_2(G)) = d(e)^2 + d(v_{(2)})^2 + 2nd(v_{(4)})^2 + \phi(n)d(v_{(n)})^2 + \phi(2n)d(v_{(2n)})^2$ and $M_2(\mathcal{E}_2(G)) = d(e)\Big[d(v_{(2)}) + 2nd(v_{(4)}) + \phi(n)d(v_{(n)}) + \phi(2n)d(v_{(2n)})\Big] + d(v_{(2)})[2nd(v_{(4)})] + 2nd(v_{(4)})[\phi(n)d(v_{(n)})] + \phi(n)d(v_{(n)})[\phi(2n)d(v_{(2n)})]$. Thus, using a similar argument as in Theorems 3.3 and 3.14, the result will be determined.

Now, assume that $n = p^k$ and $p \geq 5$. We then have two cases.

- If $|p - 4| \leq \min\{p, 4\}$, then the Zagreb indices can be found via the following equations:

$$1)\ M_1(\mathcal{E}_2(G)) = d(e)^2 + d(v_{(2)})^2 + 2nd(v_{(4)})^2 + \phi(p)d(v_{(p)})^2 + \phi(2p)d(v_{(2p)})^2 + \sum_{i=2}^{k}\Big[\phi(p^i)d(v_{(p^i)})^2 + \phi(2p^i)d(v_{(2p^i)})^2\Big].$$

$$2)\ M_2(\mathcal{E}_2(G)) = d(e)\Big[d(v_{(2)}) + 2nd(v_{(4)}) + \phi(p)d(v_{(p)}) + \phi(2p)d(v_{(2p)})\Big] + d(v_{(2)})[2nd(v_{(4)})] + 2nd(v_{(4)})[\phi(p)d(v_{(p)})] + \phi(p)d(v_{(p)})[\phi(2p)d(v_{(2p)})] + \sum_{i=2}^{k}\phi(p^i)d(v_{(p^i)})[\phi(2p^i)d(v_{(2p^i)})].$$

- If $|p - 4| > \min\{p, 4\}$, which is similar to the previous point except that the elements of orders 4 and p are not adjacent.

Consequently, the demonstration of the latter two cases utilizes the same approach as the proof presented in Theorem 3.14. □

Example 3.1. *Let G be a finite group. Table 1 presents the values of some topological indices for the equitable graph of of Type II of certain groups with orders not covered by the theories in this section.*

Table 1. The value of some topological indices of $\mathcal{E}_2(G)$ for some groups G.

G	$M_1^{\varepsilon}(\mathcal{E}_2(G))$	$M_2^{\varepsilon}(\mathcal{E}_2(G))$	$M_1(\mathcal{E}_2(G))$	$M_2(\mathcal{E}_2(G))$
$\mathbb{Z}_6$	440	1661	84	157
$\mathbb{Z}_8$	1218	5881	166	369
$\mathbb{Z}_{15}$	3408	24,449	290	540
D_6	440	1661	84	157
D_8	1218	5881	166	369
D_{12}	4598	31,529	402	1057
D_{30}	46,368	654,269	1820	5325
Q_8	845	3913	122	217
Q_{16}	18,930	175,065	1190	5025
Q_{32}	158,746	2,656,563	5086	26,561

4. Isomorphism of equitable graphs

This section discusses the isomorphism of any two finite groups and their associated equitable graph in its two types. It is known that two graphs Γ and Λ are isomorphic if and only if there is a bijective map, say $\Phi : V(\Gamma) \to V(\Lambda)$, such that Φ preserves adjacency from one graph to the other.

Observation 4.1. *Let G and H be any two finite groups and consider that G and H are isomorphic. Then $\mathcal{E}_i(G)$ and $\mathcal{E}_i(H)$ are isomorphic, where $i = \{1, 2\}$.*

Proof. The proof is straightforward by the definition of the graph. □

The other direction of this theorem for the equitable graph Type I of finite Abelian groups follows from Theorem 4.1.

Theorem 4.1. *[24] Two finite Abelian groups G and G′ are isomorphic if and only if $|G| = |G'|$ and for each positive integer k, k divides $|G|$, and $n_k = n'_k$.*

Here, n_k and n'_k denote the number of elements of order k in G and G', respectively. Thus, we have the following.

Theorem 4.2. *Let G and H be any two finite Abelian groups. Then $G \cong H$ if and only if $\mathcal{E}_1(G) \cong \mathcal{E}_1(H)$.*

The following counterexamples show that the reverse direction of Observation 4.1 does not hold for the first type of equitable graph for any finite non Abelian group in general.

Example 4.1. *1) Let $G \cong \mathbb{Z}_2 \times Q_8$ and $H \cong \langle x, y : x^4 = 1 = y^4, y^{-1}xy = x^3 \rangle$. Then each G and H has one identity element, three elements of order 2, and twelve elements of order 4. Hence $\mathcal{E}_1(G) \cong \mathcal{E}_1(H)$, while $G \ncong H$.*

2) Let $G \cong \langle x, y, z : x^4 = 1 = y^2 = z^2, xy = yx, xz = zx, y^{-1}zy = zx^2 \rangle$ and $H \cong \langle x, y, z : x^4 = 1 = y^2 = z^2, xz = zx, yz = zy, y^{-1}xy = xz \rangle$. Then each G and H has one identity element, seven elements of order 2, and eight elements of order 4. Thus $\mathcal{E}_1(G) \cong \mathcal{E}_1(H)$, while $G \ncong H$.

Now, as any finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$, the following counterexamples show that the reverse direction does not hold for the equitable graph of of Type II of any finite group.

Example 4.2. *$\mathcal{E}_2(D_6)$ and $\mathcal{E}_2(\mathbb{Z}_6)$ are isomorphic.*
To prove this, let $V(\mathcal{E}_2(D_6)) = \{v_{(1)} = e, v_{(2)} = b, v_{(2)} = ab, v_{(2)} = a^2b, v_{(3)} = a, v_{(3)} = a^2\}$ and $V(\mathcal{E}_2(\mathbb{Z}_6)) = \{u_{(1)} = 0, u_{(6)} = 1, u_{(2)} = 3.u_{(6)} = 5, u_{(3)} = 2, u_{(3)} = 4\}$, where $v_{(i)}$ and $u_{(i)}$ denote the element of order i. Let $W_{(i)}$ denote the set of elements of order i. The isomorphism then follows, since $\mathcal{E}_2(\mathbb{Z}_6 - \{e\})$ is a complete bipartite graph between the three elements of $W_{(2)} \cup W_{(6)}$ and the two elements of $W_{(3)}$, and $\mathcal{E}_2(D_6 - \{e\})$ is a complete bipartite graph between the three elements of $W_{(2)}$ and the two elements of $W_{(3)}$.

Example 4.3. *$\mathcal{E}_2(D_8)$ and $\mathcal{E}_2(\mathbb{Z}_8)$ are isomorphic.*
Let $V(\mathcal{E}_2(D_8)) = \{v_{(1)} = e, v_{(2)} = a^2, v_{(2)} = b, v_{(2)} = ab, v_{(2)} = a^2b, v_{(2)} = a^3b, v_{(4)} = a, v_{(4)} = a^3\}$ and $V(\mathcal{E}_2(\mathbb{Z}_8)) = \{u_{(1)} = 0, u_{(8)} = 1, u_{(8)} = 3, u_{(2)} = 4, u_{(8)} = 5, u_{(8)} = 7, u_{(4)} = 2, u_{(4)} = 6\}$. Then, the isomorphism follows since $\mathcal{E}_2(\mathbb{Z}_8 - \{e\})$ is a complete bipartite graph between the five elements of $W_{(2)} \cup W_{(8)}$ and the two elements of $W_{(4)}$ and $\mathcal{E}_2(D_8 - \{e\})$ is a complete bipartite graph between the five elements of $W_{(2)}$ and the two elements of $W_{(4)}$.

Example 4.4. *Consider the Abelian groups $G = \mathbb{Z}_9$ and $H = \mathbb{Z}_3 \times \mathbb{Z}_3$. These two groups are not isomorphic, since G has six elements of order 9 and two elements of order 3, but in H, all nonidentity elements have order 3. The equitable graphs of Type II of G and H are isomorphic, as both of them are star graphs of order 9.*

Remark 4.1. *According to the isomorphism established in Examples 4.2 and 4.3, these isomorphic graphs exhibit the same topological characteristics as elucidated in Example 3.1.*

Remark 4.2. *Note that the equitable graph Type I of the groups D_6 and $\mathbb{Z}_6$ (D_8 and $\mathbb{Z}_8$) are not isomorphic. The reason for this is that in Type I equitable graphs of any cyclic group of an even order, there is a pendant vertex (of degree equal to 1), while its existence is precluded in the resultant graph of the dihedral groups.*

5. Conclusions

This work has significantly contributed to the ongoing development of equitable graphs by introducing the concept of an equitable graph of Type II on finite groups. We conducted a thorough investigation into the theoretical properties of equitable graphs of Type II for various classes of groups, including cyclic, dihedral, and dicyclic groups. By deriving general formulas for degree-based graph indices such as the first and second entire indices and the first and second Zagreb indices, we provided robust mathematical frameworks for analyzing these indices within the context of specific group classes. These formulations not only enhance our theoretical understanding but also offer practical tools for computational analysis. Moreover, we explored the isomorphism property, and analyzed the conditions under which it can be established that two arbitrary finite groups are isomorphic if and only if their corresponding equitable graphs (of both types) are also isomorphic. This result was substantiated with illustrative examples, reinforcing the validity and applicability of our findings. This connection between group isomorphism and graph isomorphism opens new avenues for studying the structural similarities and differences between groups using graph-theoretic methods. As this represents the initial exploration of equitable graphs, numerous open problems and potential research directions remain to be addressed. Future studies could focus on the classification of groups derived from equitable graphs, which could lead to a deeper understanding of group structures and their graphical representations. Additionally, computing the spectral properties of equitable graphs, such as eigenvalues and eigenvectors, could provide insights into the graphs' connectivity, stability, and other intrinsic properties. Establishing connections between equitable graphs and other well-known graphs derived from groups, such as Cayley graphs or power graphs, could further enrich the interplay between group theory and graph theory. The potential applications of this research are vast. In theoretical mathematics, equitable graphs could serve as a bridge between group theory and graph theory, enabling researchers to leverage techniques from one field to solve problems in the other. For instance, the derived graph indices could be used to classify groups or predict their properties on the basis of graphical representations. In applied mathematics and computer science, equitable graphs could find applications in network analysis, where understanding the symmetries and structures of networks is crucial. For example, in social network analysis, equitable graphs could help identify isomorphic subgroups or communities within a larger network, providing insights into social dynamics and interactions. Furthermore, the spectral properties of equitable graphs could be utilized in machine learning and data science for dimensionality reduction or clustering tasks. By representing data points as nodes in a graph and leveraging the spectral decomposition of equitable graphs, one could develop more efficient algorithms for pattern recognition and data classification. In chemistry, equitable graphs could be applied to the analysis of molecular structure, where the symmetry and connectivity of molecules play a critical role in determining their properties and reactivity. In summary, this work lays the foundation for a rich and interdisciplinary research area with significant theoretical and practical implications. By addressing the open problems and exploring the potential applications outlined above, future research can advance our understanding of finite group theory, graph theory, and their interconnectedness, while also contributing to real-world applications in diverse fields such as network analysis, data science, and molecular chemistry.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. P. Cayley, Desiderata and suggestions: No. 2. The Theory of groups: Graphical representation, *Am. J. Math.*, **1** (1878), 174–176. https://doi.org/10.2307/2369306
2. P. J. Cameron, The power graph of a finite group, II, *J. Group Theory*, **13** (2010), 779–783. https://doi.org/10.1515/jgt.2010.023
3. P. J. Cameron, S. Ghosh, The power graph of a finite group, *Discrete Math.*, **311** (2011), 1220–1222. https://doi.org/10.1016/j.disc.2010.02.011
4. H. P. Tong-Viet, Finite groups whose prime graphs are regular, *J. Algebra*, **397** (2014), 18–31. https://doi.org/10.1016/j.jalgebra.2013.08.018
5. R. K. Nath, J. Dutta, Spectrum of commuting graphs of some classes of finite groups, *Mathematika*, **33** (2017), 87–95. https://doi.org/10.11113/matematika.v33.n1.812
6. S. Akhbari, F. Heydari, M. Maghasedi, The intersection graph of a group, *J. Algebra Appl.*, **14** (2015), 1550065. https://doi.org/10.1142/S0219498815500656
7. S. U. Rehman, A. Q. Baig, M. Imran, Z. U. Khan, Order divisor graphs of finite groups, *An. Stiint. Univ. Ovidius Constanta Ser. Mat.*, **26** (2018), 29–40. https://doi.org/10.2478/auom-2018-0031
8. A. Altassan, A. Saleh, M. Hamed, N. Muthana, Characterizing finite groups through equitable graphs: A graph-theoretical approach, *Mathematics*, **12** (2024), 2126. https://doi.org/10.3390/math12132126
9. F. Erdem, On the generating graphs of symmetric groups, *J. Group Theory*, **21** (2018), 629–649. https://doi.org/10.1515/jgth-2018-0004
10. R. K. Nath, W. N. T. Fasfous, K. C. Das, Y. Shang, Common neighborhood energy of commuting graphs of finite groups, *Symmetry*, **13** (2021), 1651. https://doi.org/10.3390/sym13091651
11. P. J. Cameron, Graphs defined on groups, *Int. J. Group Theory*, **11** (2022), 53–107. https://doi.org/10.22108/ijgt.2021.127679.1681

12. A. Kumar, L. Selvaganesh, P. J. Cameron, T. T. Chelvam, Recent developments on the power graph of finite groups-A survey, *AKCE Int. J. Graphs Comb.*, **18** (2021), 65–94. https://doi.org/10.1080/09728600.2021.1953359
13. R. Balakrishnan, K. Ranganathan, *A Textbook of Graph Theory*, 2^{nd} edition, Springer, 2012. https://doi.org/10.1007/978-1-4614-4529-6
14. T. W. Hungerford, *Algebra*, 1^{st} edition, Springer, 1974 https://doi.org/10.1007/978-1-4612-6101-8
15. X. Ma, L. Li, On the metric dimension of the reduced power graph of a finite group, *Taiwanese J. Math.*, **26** (2022), 1–15. https://doi.org/10.11650/tjm/210905
16. X. Ma, Forbidden subgraphs in intersection power graphs of finite groups, *Algebra Colloq.*, **32** (2025), 95–110. https://doi.org/10.1142/S1005386725000094
17. X. Ma, L. Li, G. Zhong, Perfect codes in proper intersection power graphs of finite groups, *Appl. Algebra Eng. Commun. Comput.*, **36** (2025), 543–555. https://doi.org/10.1007/s00200-023-00626-2
18. M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, *Ann. Math.*, **164** (2006), 51–229. https://doi.org/10.4007/annals.2006.164.51
19. A. Saleh, S. Alsulami, M. Alsulami, Entire irregularity indices: A comparative analysis and applications, *Mathematics*, **13** (2025), 146. https://doi.org/10.3390/math13010146
20. A. Saleh, N. Zeyada, M. S. Alharthi, Modified entire forgotten topological index of graphs: A theoretical and applied perspective, *Symmetry*, **17** (2025), 236. https://doi.org/10.3390/sym17020236
21. A. Alwardi, A. Alqesmah, R. Rangarajan, I. N. Cangul, Entire Zagreb indices of graphs, *Discrete Math. Algorithms Appl.*, **10** (2018), 1850037. https://doi.org/10.1142/S1793830918500374
22. A. Saleh, S. H. Alsulami, On the entire harmonic index and entire harmonic polynomial of graphs, *Symmetry*, **16** (2024), 208. https://doi.org/10.3390/sym16020208
23. I. N. Cangul, A. Yurttas, M. Togan, A. S. Cevik, Some formulae for the zagreb indices of graphs, in *AIP Conference Proceedings*, **1479** (2012), 365–367. https://doi.org/10.1063/1.4756139
24. R. McHaffey, Isomorphism of finite abelian groups, *Am. Math. Mon.*, **72** (1965), 48–50. https://doi.org/10.2307/2313001