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Research article

On 3-parameter generalized quaternions with higher order Leonardo numbers components

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Abstract: In this paper, a novel type of 3-parameter generalized quaternions (3-PGQs) is introduced, constructed from higher order Leonardo numbers and referred to as the higher order Leonardo 3-parameter generalized quaternions (shortly, higher order Leonardo 3-PGQs). Several fundamental properties of these quaternions are examined, including their recurrence relations, a Binet-type formula, and both generating and exponential generating functions. In addition, the obtained identities show that the higher order Leonardo 3-PGQs can be expressed in closed form in terms of Fibonacci numbers, Fibonacci generalized quaternions, and higher order Leonardo numbers.

Keywords: higher order Leonardo numbers; 3-parameter generalized quaternions; Binet-type formulas; generating functions

1. Introduction

Integer sequences have long attracted the interest of mathematicians due to their elegant recurrences and broad scientific applications. Among these, the Fibonacci sequence holds a prominent place as a result of its rich mathematical structure and wide relevance in various fields. Its regular patterns and symmetrical properties have inspired numerous applications in mathematics, physics, and engineering. The asymptotic behavior of the ratio of consecutive terms in the Fibonacci sequence is one of its fundamental characteristics. This ratio converges to the golden ratio, which is an irrational number of profound significance in both theory and nature. Because of these remarkable properties, the Fibonacci sequence has been widely studied, thus giving rise to a substantial body of literature on the subject [1,2].

For $n \geq 2$, the second-order homogeneous recurrence relations of Fibonacci and Lucas sequences are given as follows:

$$F_n = F_{n-1} + F_{n-2} \qquad F_0 = 0, F_1 = 1, \tag{1.1}$$

$$L_n = L_{n-1} + L_{n-2} \quad L_0 = 2, L_1 = 1, \tag{1.2}$$

respectively. The Binet formulas for the Fibonacci and Lucas numbers are given as follows:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
$$L_n = \alpha^n + \beta^n,$$

where α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$, with $\alpha + \beta = 1$ and $\alpha\beta = -1$.

The Leonardo sequence is defined by the following non-homogeneous recursive relation:

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \quad n \geq 2 \tag{1.3}$$

with the initial conditions $Le_0 = Le_1 = 1$ [3]. Additionally, there is the following homogeneous recursive relation:

$$Le_{n+1} = 2Le_n - Le_{n-2} \; n \geq 2. \tag{1.4}$$

Several researchers have studied Leonardo numbers and Leonardo quaternions (see [4–7]). Recent studies have expanded these ideas by developing higher order generalizations and embedding them into algebraic structures such as quaternions. The higher order Fibonacci numbers were defined by Özvatan [8]. Kızılateş and Kone [9] investigated these numbers within the framework of quaternion algebra. Cook et al. [10] derived various identities involving Jacobsthal and Jacobsthal-Lucas numbers that satisfied higher order recurrence relations. In [11], Gül studied quaternions derived from higher order Leonardo number structures. Additionally, Özimamoğlu [12] examined hypercomplex numbers whose components were based on higher order Pell numbers.

The quaternions were introduced by Hamilton [13] as an extension of the complex numbers. A quaternion is defined as follows:

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \tag{1.5}$$

where q_0, q_1, q_2, and q_3 are real numbers, and the basis vectors e_0, e_1, e_2, and e_3 form the standard orthonormal basis of $\mathbb{R}^4$, which satisfy the following quaternion multiplication rules:

$$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1, \tag{1.6}$$
$$e_1 e_2 = e_3 = -e_2 e_1, \; e_2 e_3 = e_1 = -e_3 e_2, \; e_3 e_1 = e_2 = -e_1 e_3. \tag{1.7}$$

These equations demonstrate that the multiplication of real quaternions is non-commutative. This observation provides an important insight into quaternion algebra. Moreover, by altering the defining conditions of the quaternionic units, one can construct new types of quaternionic algebras, some of which are commutative while others remain non-commutative.

Quaternions are generally divided into two classes: non-commutative types (real, split, semi, split-semi, quasi, hyperbolic, elliptical, hyperbolic-split, 2-parameter generalized, and 3-parameter generalized quaternions) and commutative types (generalized Segre and dual quaternions).

Table 1. Conditions of the multiplication rules.

2-parameter generalized quaternion, for all λ_1, $\lambda_2 \in \mathbb{R}$ [14–17]	$e_1^2 = -\lambda_1$, $e_2^2 = -\lambda_2$, $e_3^2 = -\lambda_1\lambda_2$, $e_1e_2 = -e_2e_1 = e_3$, $e_2e_3 = -e_3e_2 = \lambda_2 e_1$, $e_3e_1 = -e_1e_3 = \lambda_1 e_2$
3-parameter generalized quaternion, for all λ_1, λ_2, $\lambda_3 \in \mathbb{R}$ [18]	$e_1^2 = -\lambda_1\lambda_2$, $e_2^2 = -\lambda_1\lambda_3$, $e_3^2 = -\lambda_2\lambda_3$ $e_1e_2 = -e_2e_1 = \lambda_1 e_3$, $e_2e_3 = -e_3e_2 = \lambda_3 e_1$, $e_3e_1 = -e_1e_3 = \lambda_2 e_2$.

The generalization of quaternions whose components are based on integer sequences, such as Fibonacci, Lucas, and Leonardo, has been extensively investigated in the literature. For instance, Horadam defined the nth Fibonacci and Lucas quaternions as follows:

$$\begin{aligned} Q_n &= F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3, \\ K_n &= L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3, \end{aligned}$$

respectively [19]. Thereafter, Halici [20] derived generating functions and numerous significant identities for these quaternions. Halici [21] studied complex Fibonacci quaternions by deriving their generating function, Binet-type formula, and matrix representations. Horadam [22] provided a structural basis for quaternion recurrence relations, which subsequently supported the development of various Fibonacci-type quaternion sequences. Iyer [23] obtained identities that connected Fibonacci and Lucas quaternions and outlined the fundamental relations between these two quaternion families. In [24], the authors presented a generalization of quaternions by introducing the Fibonacci generalized quaternions and derived several relations for them using well-known Fibonacci and Lucas identities. In [25], the author proposed a further generalization of Fibonacci and Lucas quaternions. Moreover, in various studies such as [26–29], many authors have investigated the quaternions associated with different types of sequences, thereby enriching the algebraic and structural properties of quaternionic number systems. In several previous studies, various 3-PGQs derived from different integer sequences were introduced. Among these are quaternion types constructed from Fibonacci numbers [30], higher order generalized Fibonacci numbers [31], Gaussian Leonardo numbers [32], k-Fibonacci and k-Lucas numbers [33], third-order Jacobsthal numbers [34], generalized Tribonacci numbers [35], and Horadam numbers [36]. These studies demonstrated that a wide range of recurrence sequences can generate significant and structurally meaningful quaternionic forms within the 3-parameter generalized quaternion framework.

Motivated by recent studies on the generalization of number sequences and their applications in quaternion algebra, in this paper, we introduce a new family of 3-PGQs constructed from higher order generalized Leonardo numbers. We refer to these structures as higher order Leonardo 3-PGQs. Although Fibonacci-like quaternions have been extensively studied, the quaternionic generalization of higher order Leonardo numbers is addressed here for the first time. We present key algebraic and analytical properties of these generalized quaternions, including recurrence relations, the Binet-type formula, generating and exponential generating functions, and fundamental identities. These structures are applicable in combinatorics, coding theory, and mathematical modeling, and enable the

Table 2. Multiplication rule of 3-PGQs [18].

.	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	$-\lambda_1\lambda_2$	$\lambda_1 e_3$	$-\lambda_2 e_2$
e_2	e_2	$-\lambda_1 e_3$	$-\lambda_1\lambda_3$	$\lambda_3 e_1$
e_3	e_3	$\lambda_2 e_2$	$-\lambda_3 e_1$	$-\lambda_2\lambda_3$

discovery of new algebraic structures.

2. Preliminaries

In this section, we recall some terminology that is used throughout this paper with respect to both 3-PGQs and Leonardo numbers.

The set of 3-PGQs is denoted by $H_{\lambda_1\lambda_2\lambda_3}$ and is defined as follows:

$$H_{\lambda_1\lambda_2\lambda_3} = \{q = q_0 + q_1e_1 + q_2e_2 + q_3e_3, q_0, q_1, q_2, q_3, \lambda_1\lambda_2, \lambda_3 \in \mathbb{R}\}, \tag{2.1}$$

where the quaternionic units satisfy the multiplication rules given in Table 2.

For $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$, $p = p_0 + p_1e_1 + p_2e_2 + p_3e_3$, taking the rules in Table 2 into account, some basic algebraic properties are listed below.

Equality: $q = p \Leftrightarrow q_0 = p_0, q_1 = p_1, q_2 = p_2, q_3 = p_3$

Addition and subtraction: $q \pm p = q_0 \pm p_0 + (q_1 \pm p_1)e_1 + (q_2 \pm p_2)e_2 + (q_3 \pm p_3)e_3$

Multiplication by a scalar: $cq = cq_0 + cq_1e_1 + cq_2e_2 + cq_3e_3, c \in \mathbb{R}$

Scalar and vector part: $q = S_q + V_q$, where $S_q = q_0$ is scalar part and $V_q = q_1e_1 + q_2e_2 + q_3e_3$ is vector part.

Multiplication: $qp = S_qS_p - f(V_q, V_p) + S_qV_q + S_pV_p + V_q \wedge V_p$,

where $f(V_q, V_p) = \lambda_1\lambda_2q_1p_1 + \lambda_1\lambda_3q_2p_2 + \lambda_2\lambda_3q_3p_3$ and

$$V_q \wedge V_p = \begin{vmatrix} \lambda_3e_1 & \lambda_2e_2 & \lambda_1e_3 \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = \lambda_3(q_2p_3 - q_3p_2)e_1 + \lambda_2(q_3p_1 - q_1p_3)e_2 + \lambda_1(q_1p_2 - q_2p_1)e_3.$$

Conjugation: $\bar{q} = q_0 - q_1e_1 - q_2e_2 - q_3e_3$.

Inverse: $q^{-1} = \frac{\bar{q}}{N_q} = \frac{q_0 - q_1e_1 - q_2e_2 - q_3e_3}{(q_0)^2 + \lambda_1\lambda_2(q_1)^2 + \lambda_1\lambda_3(q_2)^2 + \lambda_2\lambda_3(q_3)^2}$, $N_q \neq 0$.

Inner product: $\langle q, p\rangle = q_0p_0 + \lambda_1\lambda_2q_1p_1 + \lambda_1\lambda_3q_2p_2 + \lambda_2\lambda_3q_3p_3$.

Norm: $N_q = qq^* = q^*q = (q_0)^2 + \lambda_1\lambda_2(q_1)^2 + \lambda_1\lambda_3(q_2)^2 + \lambda_2\lambda_3(q_3)^2$.

More detailed terminology for 3-PGQs can be referred to the studies [18, 30].

Additionally, we give some identities between Leonardo, Fibonacci, and Lucas numbers as follows:

$$F_{m+n} = F_mF_{n+1} + F_{m-1}F_n, \tag{2.2}$$

$$F_mF_{n+1} - F_{m+1}F_n = (-1)^nF_{m-n}, \tag{2.3}$$

$$F_{n+m} + (-1)^mF_{n-m} = L_mF_n, \tag{2.4}$$

$$F_{n+m} - (-1)^mF_{n-m} = F_mL_n, \tag{2.5}$$

$$L_{n+m} + (-1)^mL_{n-m} = L_mL_n, \tag{2.6}$$

$$\begin{aligned} L_{n+m} - (-1)^m L_{n-m} &= 5F_m F_n, & (2.7)\\ Le_n &= 2F_{n+1} - 1, & (2.8)\\ Le_{n+m} + (-1)^m Le_{n-m} &= L_m(Le_n + 1) - 1 - (-1)^m, & (2.9)\\ Le_{n+m} - (-1)^m Le_{n-m} &= L_{n+1}(Le_{m-1} + 1) - 1 + (-1)^m. & (2.10) \end{aligned}$$

More information about Leonardo, Fibonacci, and Lucas numbers can be found in [1,3,6,37,38].

For a positive integer s, the higher order Leonardo numbers $\{L_n^{(s)}\}$ are defined by the following: (see [39])

$$L_n^{(s)} = \frac{Le_{sn}}{Le_s}, n = 0, 1, 2 \ldots . \tag{2.11}$$

Since $Le_s \geq 1$ for all $s \geq 0$, the denominator Le_s is never equal to zero. This ensures that $L_n^{(s)}$ is well-defined. When $s = 1$, the higher order Leonardo numbers $L_n^{(1)}$ coincide with the ordinary Leonardo numbers. The Binet's formula for the higher order Leonardo numbers $L_n^{(s)}$ can be written as follows:

$$L_n^{(s)} = \frac{2\alpha^{sn+1} - 2\beta^{sn+1} - \alpha + \beta}{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta}. \tag{2.12}$$

3. On higher order Leonardo 3-parameter generalized quaternions

In this section, we introduce the higher order Leonardo 3-PGQs and derive several new identities related to these newly defined quaternions.

Definition 1. *The nth higher order Leonardo 3-PGQ is defined by the following:*

$$\mathbb{Q}L_n^{(s)} = L_n^{(s)} + L_{n+1}^{(s)} e_1 + L_{n+2}^{(s)} e_2 + L_{n+3}^{(s)} e_3 \tag{3.1}$$

where $L_n^{(s)}$ is the nth higher order Leonardo number, and e_1, e_2, and e_3 satisfy the rules given in Table 2.

For $s = 1$, Equation (3.1) becomes the nth Leonardo 3-PGQ, denoted by $\mathbb{Q}Le_n$. The higher order Leonardo 3-PGQs can be written as folows:

$$\mathbb{Q}L_n^{(s)} = L_n^{(s)} + u$$

where $u = L_{n+1}^{(s)} e_1 + L_{n+2}^{(s)} e_2 + L_{n+3}^{(s)} e_3$.
The conjugate of the higher order Leonardo 3-PGQ is denoted by $(\mathbb{Q}L_n^{(s)})^*$ and is given by the following:

$$(\mathbb{Q}L_n^{(s)})^* = L_n^{(s)} - u. \tag{3.2}$$

For the higher order Leonardo 3-PGQs, we have

$$\mathbb{Q}L_n^{(s)} + (\mathbb{Q}L_n^{(s)})^* = 2L_n^{(s)}.$$

Proposition 2. *The higher order Leonardo 3-PGQs satisfy the following identity:*

$$\mathbb{Q}L_n^{(s)} (\mathbb{Q}L_n^{(s)})^* = (L_n^{(s)})^2 + \lambda_1\lambda_2 (L_{n+1}^{(s)})^2 + \lambda_1\lambda_3 (L_{n+2}^{(s)})^2 + \lambda_2\lambda_3 (L_{n+3}^{(s)})^2.$$

Proof. Using (3.1) and (3.2), we obtain the following:

$$\begin{aligned}
\mathbb{Q}L_n^{(s)}(\mathbb{Q}L_n^{(s)})^* &= (L_n^{(s)} + L_{n+1}^{(s)}e_1 + L_{n+2}^{(s)}e_2 + L_{n+3}^{(s)}e_3)(L_n^{(s)} - L_{n+1}^{(s)}e_1 - L_{n+2}^{(s)}e_2 - L_{n+3}^{(s)}e_3) \\
&= (L_n^{(s)})^2 - L_n^{(s)}L_{n+1}^{(s)}e_1 - L_n^{(s)}L_{n+2}^{(s)}e_2 - L_n^{(s)}L_{n+3}^{(s)}e_3 + L_{n+1}^{(s)}L_n^{(s)}e_1 - L_{n+1}^{(s)}L_{n+1}^{(s)}e_1e_1 \\
&\quad -L_{n+1}^{(s)}L_{n+2}^{(s)}e_1e_2 - L_{n+1}^{(s)}L_{n+3}^{(s)}e_1e_3 + L_{n+2}^{(s)}L_n^{(s)}e_2 - L_{n+2}^{(s)}L_{n+1}^{(s)}e_2e_1 - L_{n+2}^{(s)}L_{n+2}^{(s)}e_2e_2 \\
&\quad -L_{n+2}^{(s)}L_{n+3}^{(s)}e_2e_3 + L_{n+3}^{(s)}L_n^{(s)}e_3 - L_{n+3}^{(s)}L_{n+1}^{(s)}e_3e_1 - L_{n+3}^{(s)}L_{n+2}^{(s)}e_3e_2 - L_{n+3}^{(s)}L_{n+3}^{(s)}e_3e_3 \\
&= (L_n^{(s)})^2 + \lambda_1\lambda_2(L_{n+1}^{(s)})^2 + \lambda_1\lambda_3(L_{n+2}^{(s)})^2 + \lambda_2\lambda_3(L_{n+3}^{(s)})^2.
\end{aligned}$$

□

Proposition 3. *The higher order Leonardo* 3*-PGQs satisfy the following identity:*

$$(\mathbb{Q}L_n^{(s)})^2 = 2L_n^{(s)}\mathbb{Q}L_n^{(s)} - \mathbb{Q}L_n^{(s)}(\mathbb{Q}L_n^{(s)})^*.$$

Proof. Using (3.1) and the multiplication rules presented in Table 2, we have the following:

$$\begin{aligned}
(\mathbb{Q}L_n^{(s)})^2 &= (L_n^{(s)} + L_{n+1}^{(s)}e_1 + L_{n+2}^{(s)}e_2 + L_{n+3}^{(s)}e_3)(L_n^{(s)} + L_{n+1}^{(s)}e_1 + L_{n+2}^{(s)}e_2 + L_{n+3}^{(s)}e_3) \\
&= (L_n^{(s)})^2 + L_n^{(s)}L_{n+1}^{(s)}e_1 + L_n^{(s)}L_{n+2}^{(s)}e_2 + L_n^{(s)}L_{n+3}^{(s)}e_3 + L_{n+1}^{(s)}L_n^{(s)}e_1 + L_{n+1}^{(s)}L_{n+1}^{(s)}e_1e_1 \\
&\quad +L_{n+1}^{(s)}L_{n+2}^{(s)}e_1e_2 + L_{n+1}^{(s)}L_{n+3}^{(s)}e_1e_3 + L_{n+2}^{(s)}L_n^{(s)}e_2 + L_{n+2}^{(s)}L_{n+1}^{(s)}e_2e_1 + L_{n+2}^{(s)}L_{n+2}^{(s)}e_2e_2 \\
&\quad +L_{n+2}^{(s)}L_{n+3}^{(s)}e_2e_3 + L_{n+3}^{(s)}L_n^{(s)}e_3 + L_{n+3}^{(s)}L_{n+1}^{(s)}e_3e_1 + L_{n+3}^{(s)}L_{n+2}^{(s)}e_3e_2 + L_{n+3}^{(s)}L_{n+3}^{(s)}e_3e_3 \\
&= (L_n^{(s)})^2 - \lambda_1\lambda_2(L_{n+1}^{(s)})^2 - \lambda_1\lambda_3(L_{n+2}^{(s)})^2 - \lambda_2\lambda_3(L_{n+3}^{(s)})^2 + 2L_n^{(s)}(L_{n+1}^{(s)}e_1 + L_{n+2}^{(s)}e_2 + L_{n+3}^{(s)}e_3) \\
&= 2L_n^{(s)}\mathbb{Q}L_n^{(s)} - \mathbb{Q}L_n^{(s)}(\mathbb{Q}L_n^{(s)})^*.
\end{aligned}$$

□

Theorem 4. *The higher order Leonardo* 3*-PGQs satisfy the following identity:*

$$\mathbb{Q}L_{n+1}^{(s)} + (-1)^s\mathbb{Q}L_{n-1}^{(s)} = L_s\mathbb{Q}L_n^{(s)} + (L_2^{(s)} + (-1)^sL_0^{(s)} - L_sL_1^{(s)})(1 + e_1 + e_2 + e_3).$$

Proof. From the Equations (2.3), (2.8), and (2.11), we find that

$$\begin{aligned}
L_{n+1}^{(s)} + (-1)^sL_{n-1}^{(s)} &= \frac{Le_{sn+s} + (-1)^sLe_{sn-s}}{Le_s} \\
&= \frac{2F_{sn+s+1} + 2(-1)^sF_{sn-s+1} - 1 - (-1)^s}{Le_s} \\
&= \frac{2L_sF_{sn+1} - 1 - (-1)^s}{Le_s} \\
&= \frac{L_sLe_{sn} + L_s - 1 - (-1)^s}{Le_s} \\
&= L_sL_n^{(s)} + (Le_s)^{-1}(L_s - 1 - (-1)^s).
\end{aligned}$$

From (2.9), we can write the following:

$$L_s + (-1)^{s+1} - 1 = Le_{2s} + (-1)^sLe_0 - L_sLe_s.$$

Therefore, we have the following:

$$
\begin{aligned}
\mathbb{Q}L_{n+1}^{(s)} + (-1)^s \mathbb{Q}L_{n-1}^{(s)} &= L_{n+1}^{(s)} + (-1)^s L_{n-1}^{(s)} + (L_{n+2}^{(s)} + (-1)^s L_n^{(s)})e_1 \\
&\quad + (L_{n+3}^{(s)} + (-1)^s L_{n+1}^{(s)})e_2 + (L_{n+4}^{(s)} + (-1)^s L_{n+2}^{(s)})e_3 \\
&= L_s L_n^{(s)} + (Le_s)^{-1}(L_s - 1 - (-1)^s) + (L_s L_{n+1}^{(s)} + (Le_s)^{-1}(L_s - 1 - (-1)^s))e_1 \\
&\quad + (L_s L_{n+2}^{(s)} + (Le_s)^{-1}(L_s - 1 - (-1)^s))e_2 \\
&\quad + (L_s L_{n+3}^{(s)} + (Le_s)^{-1}(L_s - 1 - (-1)^s))e_3 \\
&= L_s \mathbb{Q}L_n^{(s)} + (L_2^{(s)} + (-1)^s L_0^{(s)} - L_s L_1^{(s)})(1 + e_1 + e_2 + e_3).
\end{aligned}
$$

□

Lemma 5. *For $s, n \geq 1$, we have the following summation formula:*

$$\sum_{i=0}^{s} Le_{sn+i-2} = Le_{sn+s} - Le_{sn-1} - s - 1.$$

Proof. Using (1.4), we have the following:

$$
\begin{aligned}
Le_{sn+s} &= 2Le_{sn+s-1} - Le_{sn+s-3}, \\
Le_{sn+s-1} &= 2Le_{sn+s-2} - Le_{sn+s-4}, \\
Le_{sn+s-2} &= 2Le_{sn+s-3} - Le_{sn+s-5}, \\
&\vdots \\
Le_{sn+1} &= 2Le_{sn} - Le_{sn-2}, \\
Le_{sn} &= 2Le_{sn-1} - Le_{sn-3}.
\end{aligned}
$$

By adding the equations, we derive the following result:

$$
\begin{aligned}
Le_{sn+s} &= (Le_{sn+s-1} - Le_{sn+s-3}) + (Le_{sn+s-2} - Le_{sn+s-4}) + \cdots + (Le_{sn} - Le_{sn-2}) + (2Le_{sn-1} - Le_{sn-3}) \\
&= Le_{sn+s-2} + Le_{sn+s-3} + \cdots + Le_{sn-1} + Le_{sn-2} + Le_{sn-1} + s + 1.
\end{aligned}
$$

Therefore, we have $\sum_{i=0}^{s} Le_{sn+i-2} = Le_{sn+s} - Le_{sn-1} - s - 1$. □

Theorem 6. *For $s, n \geq 1$, we have the following summation formula:*

$$\sum_{i=0}^{s} \mathbb{Q}Le_{sn+i-2} = \mathbb{Q}Le_{sn+s} - \mathbb{Q}Le_{sn-1} - (s+1)(1 + e_1 + e_2 + e_3),$$

where $\mathbb{Q}Le_n$ is the nth Leonardo 3-PGQ.

Proof. From Lemma 5 and the definition of Leonardo 3-PGQs, we have the following:

$$
\begin{aligned}
\sum_{i=0}^{s} \mathbb{Q}Le_{sn+i-2} &= \sum_{i=0}^{s} Le_{sn+i-2} + e_1 \sum_{i=0}^{s} Le_{sn+i-1} + e_2 \sum_{i=0}^{s} Le_{sn+i} + e_3 \sum_{i=0}^{s} Le_{sn+i+1} \\
&= \mathbb{Q}Le_{sn+s} - \mathbb{Q}Le_{sn-1} - (s+1)(1 + e_1 + e_2 + e_3).
\end{aligned}
$$

□

Theorem 7. *For the higher order Leonardo numbers, we have the following:*

$$L_{n+1}^{(s)} = L_s L_n^{(s)} + (-1)^{s-1}(L_{n-1}^{(s)} + L_0^{(s)}) + \frac{1}{2}(L_1^{(s)} + L_0^{(s)}).$$

Proof. Using (1.3), (1.4), (2.3), (2.8), and (2.11), we obtain the following:

$$\begin{aligned}
L_{n+1}^{(s)} - F_{s-1}L_n^{(s)} &= \frac{Le_{sn+s}}{Le_s} - F_{s-1}\frac{Le_{sn}}{Le_s} \\
&= \frac{2F_{sn+s+1} - 2F_{s-1}F_{sn+1}}{Le_s} \\
&= \frac{2F_{sn+1}F_{s+1} + 2F_{sn}F_s - 2F_{s-1}F_{sn+1}}{Le_s} \\
&= \frac{F_{s+1}(2F_{sn+1} - 1) + F_{s+1} + 2(-1)^{s-1}F_{sn-s+1}}{Le_s} \\
&= \frac{F_{s+1}Le_{sn} + F_{s+1} + (-1)^{s-1}Le_{sn-s} + (-1)^{s-1}}{Le_s} \\
&= \frac{\frac{1}{2}Le_{sn}(2F_{s+1} - 1) + \frac{1}{2}Le_{sn} + F_{s+1} + (-1)^{s-1}Le_{sn-s} + (-1)^{s-1}}{Le_s} \\
&= \frac{\frac{1}{2}Le_sLe_{sn} + \frac{1}{2}Le_{sn} + F_{s+1} + (-1)^{s-1}Le_{sn-s} + (-1)^{s-1}}{Le_s} \\
&= \frac{1}{2}L_n^{(s)}(Le_s + 1) + (-1)^{s-1}L_{n-1}^{(s)} + \frac{\frac{1}{2}Le_s + \frac{1}{2}Le_0 + Le_0(-1)^{s-1}}{Le_s} \\
&= \frac{1}{2}L_n^{(s)}(Le_s + 1) + (-1)^{s-1}L_{n-1}^{(s)} + \frac{1}{2}L_1^{(s)} + \frac{1}{2}L_0^{(s)} + L_0^{(s)}(-1)^{s-1}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
L_{n+1}^{(s)} &= \frac{1}{2}L_n^{(s)}(Le_s + 1 + 2F_{s-1}) + (-1)^{s-1}L_{n-1}^{(s)} + \frac{1}{2}L_1^{(s)} + \frac{1}{2}L_0^{(s)} + L_0^{(s)}(-1)^{s-1} \\
&= L_sL_n^{(s)} + (-1)^{s-1}(L_{n-1}^{(s)} + L_0^{(s)}) + \frac{1}{2}(L_1^{(s)} + L_0^{(s)}).
\end{aligned}$$

□

Theorem 8. *The Binet-type formula for the higher order Leonardo 3-PGQs is given by*

$$\mathbb{Q}L_n^{(s)} = \frac{2\alpha^{sn+1}\widetilde{\alpha} - 2\beta^{sn+1}\widetilde{\beta} - \alpha\gamma + \beta\gamma}{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta}, \tag{3.3}$$

where $\widetilde{\alpha} = 1 + \alpha^s e_1 + \alpha^{2s}e_2 + \alpha^{3s}e_3$, $\widetilde{\beta} = 1 + \beta^s e_1 + \beta^{2s}e_2 + \beta^{3s}e_3$ *and* $\gamma = 1 + e_1 + e_2 + e_3$.

Proof. Using the identity of the higher order Leonardo 3-PGQs in (3.1) and the Binet's formula of the higher order Leonardo numbers (2.12), we have the following:

$$\begin{aligned}
\mathbb{Q}L_n^{(s)} &= L_n^{(s)} + L_{n+1}^{(s)}e_1 + L_{n+2}^{(s)}e_2 + L_{n+3}^{(s)}e_3 \\
&= \frac{2\alpha^{sn+1} - 2\beta^{sn+1} - \alpha + \beta}{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta} + \frac{2\alpha^{s(n+1)+1} - 2\beta^{s(n+1)+1} - \alpha + \beta}{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta}e_1
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\alpha^{s(n+2)+1}-2\beta^{s(n+2)+1}-\alpha+\beta}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}e_2+\frac{2\alpha^{s(n+3)+1}-2\beta^{s(n+3)+1}-\alpha+\beta}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}e_3\\
=&\ \frac{2\alpha^{sn+1}(1+\alpha^s e_1+\alpha^{2s}e_2+\alpha^{3s}e_3)-2\beta^{sn+1}(1+\beta^s e_1+\beta^{2s}e_2+\beta^{3s}e_3)+(-\alpha+\beta)(1+e_1+e_2+e_3)}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}\\
=&\ \frac{2\alpha^{sn+1}\widetilde{\alpha}-2\beta^{sn+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}.
\end{aligned}
$$

Hence, the proof is completed. □

Theorem 9. *For all integers $n \geq 1$, the higher order Leonardo 3-PGQs satisfy the following recurrence relation:*

$$\mathbb{Q}L_{n+1}^{(s)}=L_s\mathbb{Q}L_n^{(s)}+(-1)^{s+1}\mathbb{Q}L_{n-1}^{(s)}+\gamma(L_2^{(s)}+(-1)^sL_0^{(s)}-L_sL_1^{(s)}).$$

Proof. From the Binet-type formula for the higher order Leonardo 3-PGQs, we have the following:

$$
\begin{aligned}
\mathbb{Q}L_{n+1}^{(s)} =&\ \frac{2\alpha^{sn+s+1}\widetilde{\alpha}-2\beta^{sn+s+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}\\
=&\ \frac{1}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}\alpha^s(2\alpha^{sn+1}\widetilde{\alpha}-2\beta^{sn+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma)-2\beta^s\alpha^{sn+1}\widetilde{\alpha}+2\alpha^s\beta^{sn+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma\\
&-(\alpha^s+\beta^s)(-\alpha\gamma+\beta\gamma)+\beta^s(2\alpha^{sn+1}\widetilde{\alpha}-2\beta^{sn+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma)\\
=&\ \frac{1}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}((\alpha^s+\beta^s)\left(2\alpha^{sn+1}\widetilde{\alpha}-2\beta^{sn+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma\right)\\
&+(-1)^{s+1}(2\alpha^{sn-s+1}\widetilde{\alpha}-2\beta^{sn-s+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma)\\
&+(-\alpha^s-\beta^s-(-1)^{s+1}+1)(-\alpha+\beta)\gamma).
\end{aligned}
$$

From (2.11) and (2.12), we have

$$\mathbb{Q}L_{n+1}^{(s)}=L_s\mathbb{Q}L_n^{(s)}+(-1)^{s+1}\mathbb{Q}L_{n-1}^{(s)}+(Le_s)^{-1}\gamma(L_s+(-1)^{s+1}-1).$$

Using (2.9), we derive the following result:

$$\mathbb{Q}L_{n+1}^{(s)}=L_s\mathbb{Q}L_n^{(s)}+(-1)^{s+1}\mathbb{Q}L_{n-1}^{(s)}+\gamma(L_2^{(s)}+(-1)^sL_0^{(s)}-L_sL_1^{(s)}).$$

□

Theorem 10. *For any integer s and n, the following equations hold:*

$$
\begin{aligned}
\mathbb{Q}L_{-n}^{(s)} =&\ (-1)^{sn-1}\frac{-2(\alpha^{sn-1}-\beta^{sn-1})(\widetilde{\alpha}+\widetilde{\beta})+2(\alpha^{sn-1}\widetilde{\alpha}-\beta^{sn-1}\widetilde{\beta})+(-1)^{sn-1}\gamma(-\alpha+\beta)}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta},\\
\mathbb{Q}L_{-n}^{(-s)} =&\ \mathbb{Q}L_n^{(s)}L_{-1}^{(-s)},\\
\mathbb{Q}L_{n}^{(-s)} =&\ \mathbb{Q}L_{-n}^{(s)}L_{-1}^{(-s)}.
\end{aligned}
$$

Proof. Using the Binet-type formula for the higher order Leonardo 3-PGQs, we obtain the following:

$$\mathbb{Q}L_{-n}^{(s)} = \frac{2\alpha^{-sn+1}\widetilde{\alpha}-2\beta^{-sn+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}$$

$$= \frac{1}{(\alpha\beta)^{sn-1}} \frac{2\beta^{sn-1}\widetilde{\alpha} - 2\alpha^{sn-1}\widetilde{\beta} - (-1)^{sn-1}\alpha\gamma + (-1)^{sn-1}\beta\gamma}{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta}$$

$$= (-1)^{sn-1} \frac{-2(\alpha^{sn-1} - \beta^{sn-1})(\widetilde{\alpha} + \widetilde{\beta}) + 2(\alpha^{sn-1}\widetilde{\alpha} - \beta^{sn-1}\widetilde{\beta}) + (-1)^{sn-1}\gamma(-\alpha + \beta)}{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta}.$$

$$\begin{aligned}
\mathbb{Q}L_{-n}^{(-s)} &= \frac{2\alpha^{sn+1}\widetilde{\alpha} - 2\beta^{sn+1}\widetilde{\beta} - \alpha\gamma + \beta\gamma}{2\alpha^{-s+1} - 2\beta^{-s+1} - \alpha + \beta} \\
&= -(-1)^{s-1} \frac{2\alpha^{sn+1}\widetilde{\alpha} - 2\beta^{sn+1}\widetilde{\beta} - \alpha\gamma + \beta\gamma}{2\alpha^{s-1} - 2\beta^{s-1} + (-1)^{s-1}\alpha - \beta} \\
&= (-1)^{s}\mathbb{Q}L_{n}^{(s)} Le_s (2F_{s-1} + (-1)^{s-1})^{-1} \\
&= (-1)^{s}\mathbb{Q}L_{n}^{(s)} Le_s (2(-1)^{-s}F_{-s+1} + (-1)^{s-1})^{-1} \\
&= \mathbb{Q}L_{n}^{(s)} Le_s (2F_{-s+1} - 1)^{-1} \\
&= \mathbb{Q}L_{n}^{(s)} L_{-1}^{(-s)}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{Q}L_{n}^{(-s)} &= \frac{2\alpha^{-sn+1}\widetilde{\alpha} - 2\beta^{-sn+1}\widetilde{\beta} - \alpha\gamma + \beta\gamma}{2\alpha^{-s+1} - 2\beta^{-s+1} - \alpha + \beta} \\
&= \frac{(\alpha\beta)^{s-1}}{(\alpha\beta)^{sn-1}} \frac{2\beta^{sn-1}\widetilde{\alpha} - 2\alpha^{sn-1}\widetilde{\beta} + (-1)^{sn-1}(-\alpha\gamma + \beta\gamma)}{2\beta^{s-1} - 2\alpha^{s-1} + (-1)^{s-1}(-\alpha + \beta)} \\
&= (\alpha\beta)^{s-1} \frac{1}{(\alpha\beta)^{sn-1}} \frac{2\beta^{sn-1}\widetilde{\alpha} - 2\alpha^{sn-1}\widetilde{\beta} - (-1)^{sn-1}\alpha\gamma + (-1)^{sn-1}\beta\gamma}{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta} \\
&\quad \frac{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta}{2\beta^{s-1} - 2\alpha^{s-1} + (-1)^{s-1}(-\alpha + \beta)} \\
&= \mathbb{Q}L_{-n}^{(s)} L_{-1}^{(-s)}.
\end{aligned}$$

□

Theorem 11. *The generating function of the higher order Leonardo* 3*-PGQs is given by*

$$G(x, s) = \frac{\mathbb{Q}L_0^{(s)} - ((1 + L_s)\mathbb{Q}L_0^{(s)} + L_0^{(s)})x + (-1)^s(\mathbb{Q}L_{-1}^{(s)})x^2}{(1 - (1 + L_s)x + (L_s + (-1)^s)x^2 - (-1)^s x^3)}.$$

Proof. If the generating function for higher order Leonardo 3-PGQs is $G_h(x) = \sum_{n=0}^{\infty} \mathbb{Q}L_n^{(s)} x^n$, then, using the Binet-type formula (3.3), we can write the following:

$$\begin{aligned}
G_h(x) &= \frac{1}{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta} \left(2\widetilde{\alpha} \sum_{n=0}^{\infty} \alpha^{sn+1} x^n - 2\widetilde{\beta} \sum_{n=0}^{\infty} \beta^{sn+1} x^n + (-\alpha + \beta)\gamma \sum_{n=0}^{\infty} x^n \right) \\
&= \frac{1}{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta} \left(2\alpha\widetilde{\alpha} \sum_{n=0}^{\infty} (\alpha^s x)^n - 2\beta\widetilde{\beta} \sum_{n=0}^{\infty} (\beta^s x)^n + (-\alpha + \beta)\gamma \sum_{n=0}^{\infty} x^n \right) \\
&= \frac{1}{2\alpha^{s+1} - 2\beta^{s+1} - \alpha + \beta} \left(\frac{2\alpha\widetilde{\alpha}}{1 - \alpha^s x} - \frac{2\beta\widetilde{\beta}}{1 - \beta^s x} + (-\alpha + \beta)\gamma \frac{1}{1 - x} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta} \\
&\quad (\frac{2\alpha\widetilde{\alpha}-2\beta\widetilde{\beta}+(-\alpha+\beta)\gamma-(2\alpha\widetilde{\alpha}-2\beta\widetilde{\beta}+(\alpha^s+\beta^s)(2\alpha\widetilde{\alpha}-2\beta\widetilde{\beta})+(-\alpha+\beta)(\alpha^s+\beta^s)\gamma)x}{1-(1+\alpha^s+\beta^s)x+(\alpha^s+\beta^s+(\alpha\beta)^s)x^2-(\alpha\beta)^s x^3} \\
&\quad +\frac{(2\alpha\widetilde{\alpha}\beta^s-2\beta\widetilde{\beta}\alpha^s+(-\alpha+\beta)(\alpha\beta)^s\gamma)x^2}{1-(1+\alpha^s+\beta^s)x+(\alpha^s+\beta^s+(\alpha\beta)^s)x^2-(\alpha\beta)^s x^3}) \\
&= \frac{\mathbb{Q}L_0^{(s)}-((1+L_s)\mathbb{Q}L_0^{(s)}+L_0^{(s)})x+(-1)^s(\mathbb{Q}L_{-1}^{(s)})x^2}{(1-(1+L_s)x+(L_s+(-1)^s)x^2-(-1)^s x^3)}.
\end{aligned}
$$

□

Theorem 12. *The exponential generating function of higher-order Leonardo 3-PGQs is given by*

$$\sum_{n=0}^{\infty}\mathbb{Q}L_n^{(s)}\frac{x^n}{n!}=\frac{2\alpha\widetilde{\alpha}e^{\alpha^s x}-2\beta\widetilde{\beta}e^{\beta^s x}+\gamma(-\alpha+\beta)e^x}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}.$$

Proof. Let $G^{(s)}(x)=\sum_{n=0}^{\infty}\mathbb{Q}L_n^{(s)}\frac{x^n}{n!}$ be the exponential generating function of $\mathbb{Q}L_n^{(s)}$. Then, from (3.3), we have the following:

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathbb{Q}L_n^{(s)}\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{2\alpha^{sn+1}\widetilde{\alpha}-2\beta^{sn+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}\right)\frac{x^n}{n!} \\
&= \frac{1}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}2\alpha\widetilde{\alpha}\sum_{n=0}^{\infty}\frac{(\alpha^s x)^n}{n!}-2\beta\widetilde{\beta}\sum_{n=0}^{\infty}\frac{(\beta^s x)^n}{n!}+\gamma(-\alpha+\beta)\sum_{n=0}^{\infty}\frac{x^n}{n!} \\
&= \frac{2\alpha\widetilde{\alpha}e^{\alpha^s x}-2\beta\widetilde{\beta}e^{\beta^s x}+\gamma(-\alpha+\beta)e^x}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}.
\end{aligned}
$$

□

Theorem 13. *The following identities are satisfied:*

$$\widetilde{\alpha}\widetilde{\beta}=A+(-1)^{s+1}\left(B+\sqrt{5}F_s C\right) \tag{3.4}$$

and

$$\widetilde{\beta}\widetilde{\alpha}=A+(-1)^{s+1}(B-\sqrt{5}F_s C) \tag{3.5}$$

where $A=1+L_s e_1+L_{2s}e_2+L_{3s}e_3$*,* $B=\lambda_1\lambda_2+(-1)^s\lambda_1\lambda_3+\lambda_2\lambda_3$*,* $C=\lambda_1 e_3-\lambda_2 e_2 L_s+(-1)^s\lambda_3 e_1$*, and* L_s *and* F_s *represent the sth Lucas and Fibonacci numbers, respectively.*

Proof. By applying the multiplication rule of 3-PGQs in Table 2, we have the following:

$$
\begin{aligned}
\widetilde{\alpha}\widetilde{\beta} &= (1+\alpha^s e_1+\alpha^{2s}e_2+\alpha^{3s}e_3)(1+\beta^s e_1+\beta^{2s}e_2+\beta^{3s}e_3) \\
&= \widetilde{\beta}+\alpha^s e_1+\alpha^s\beta^s(e_1)^2+\alpha^s\beta^{2s}e_1e_2+\alpha^s\beta^{3s}e_1e_3+\alpha^{2s}e_2+\alpha^{2s}\beta^s e_2e_1+\alpha^{2s}\beta^{2s}(e_2)^2+\alpha^{2s}\beta^{3s}e_2e_3 \\
&\quad +\alpha^{3s}e_3+\alpha^{3s}\beta^s e_3e_1+\alpha^{3s}\beta^{2s}e_3e_2+\alpha^{3s}\beta^{3s}(e_3)^2 \\
&= \widetilde{\beta}+\widetilde{\alpha}-1+(-1)^s((e_1)^2+(-1)^s(e_2)^2+(e_3)^2)+e_1e_2(\alpha^s\beta^{2s}-\alpha^{2s}\beta^s)
\end{aligned}
$$

$$
\begin{aligned}
& +e_1e_3(\alpha^s\beta^{3s}-\alpha^{3s}\beta^s)+e_2e_3(\alpha^{2s}\beta^{3s}-\alpha^{3s}\beta^{2s}) \\
= \;& 1+L_se_1+L_{2s}e_2+L_{3s}e_3+(-1)^{s+1}(\lambda_1\lambda_2+(-1)^s\lambda_1\lambda_3+\lambda_2\lambda_3) \\
& +(-1)^{s+1}\sqrt{5}F_s(\lambda_1e_3-\lambda_2e_2L_s+(-1)^s\lambda_3e_1) \\
= \;& A+(-1)^{s+1}\left(B+\sqrt{5}F_sC\right).
\end{aligned}
$$

Moreover, the other identity can be obtained in a similar manner. □

Theorem 14. *The following identities are satisfied:*

$$
\begin{aligned}
\widetilde{\alpha}-\widetilde{\beta} &= \sqrt{5}F_s(e_1+L_se_2+L_{2s}e_3+(-1)^se_3), \\
\widetilde{\alpha}\beta^s-\widetilde{\beta}\alpha^s &= (-1)^s\sqrt{5}F_s(-1+e_2+L_se_3).
\end{aligned}
$$

Proof. Considering the definitions of $\widetilde{\alpha}$ and $\widetilde{\beta}$, we have the following:

$$
\begin{aligned}
\widetilde{\alpha}-\widetilde{\beta} &= 1+\alpha^se_1+\alpha^{2s}e_2+\alpha^{3s}e_3-1-\beta^se_1-\beta^{2s}e_2-\beta^{3s}e_3 \\
&= (\alpha^s-\beta^s)(e_1+(\alpha^s+\beta^s)e_2+(\alpha^{2s}+\beta^{2s})e_3+(-1)^se_3) \\
&= \sqrt{5}F_s(e_1+L_se_2+L_{2s}e_3+(-1)^se_3).
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\alpha}\beta^s-\widetilde{\beta}\alpha^s &= \beta^s+\alpha^s\beta^se_1+\alpha^{2s}\beta^se_2+\alpha^{3s}\beta^se_3-\alpha^s-\alpha^s\beta^se_1-\alpha^s\beta^{2s}e_2-\alpha^s\beta^{3s}e_3 \\
&= -(\alpha^s-\beta^s)+(\alpha\beta)^s(\alpha^s-\beta^s)e_2+(\alpha\beta)^s(\alpha^{2s}-\beta^{2s})e_3 \\
&= (-1)^s\sqrt{5}F_s(-1+e_2+L_se_3).
\end{aligned}
$$

□

Theorem 15 (Vajda's identity)**.** *For any integer n, m, and r, we have the following:*

$$
\begin{aligned}
\mathbb{Q}L^{(s)}_{n+r}\mathbb{Q}L^{(s)}_{n+m}-\mathbb{Q}L^{(s)}_{n}\mathbb{Q}L^{(s)}_{n+m+r} = \;& (L^s_0)^2[4((-1)^{sn+1}(A+((-1)^{s+1}B)(F_{sm}F_{sr}))+(-1)^{s+1}F_sC(-F_{sr}L_{sm})) \\
& -2\gamma(\mathbb{Q}_{s(n+r)+1}-\mathbb{Q}_{sn+1}+\mathbb{Q}_{s(n+m)+1}-\mathbb{Q}_{s(n+m+r)+1})]
\end{aligned}
$$

where $\mathbb{Q}_n$ is the nth Fibonacci 3*-PGQ.*

Proof. In order to simplify the notation, the left-hand side of the equation will be expressed by A_n throughout the proof, that is,

$$
A_n := \mathbb{Q}L^{(s)}_{n+r}\mathbb{Q}L^{(s)}_{n+m}-\mathbb{Q}L^{(s)}_{n}\mathbb{Q}L^{(s)}_{n+m+r}.
$$

Using the Binet-type formula, we find that

$$
\begin{aligned}
A_n = \;& \frac{2\alpha^{sn+sr+1}\widetilde{\alpha}-2\beta^{sn+sr+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}\,\frac{2\alpha^{sn+sm+1}\widetilde{\alpha}-2\beta^{sn+sm+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta} \\
& -\frac{2\alpha^{sn+1}\widetilde{\alpha}-2\beta^{sn+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}\,\frac{2\alpha^{sn+sm+sr+1}\widetilde{\alpha}-2\beta^{sn+sm+sr+1}\widetilde{\beta}-\alpha\gamma+\beta\gamma}{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}.
\end{aligned}
$$

After performing the required mathematical operations, the expression becomes the following:

$$
A_n = \frac{1}{(2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta)^2}(4\widetilde{\alpha}\widetilde{\beta}(-1)^{sn+1}(-(-1)^{sr}\beta^{sm-sr}+\beta^{sm+sr})+2\gamma\widetilde{\alpha}\alpha^{sn}(-\alpha^{sr}+1)(\alpha^2+1)
$$

$$+4\widetilde{\beta}\widetilde{\alpha}(-1)^{sn+1}\left(-(-1)^{sr}\alpha^{sm-sr}+\alpha^{sm+sr}\right)+2\gamma\widetilde{\beta}\beta^{sn}(-\beta^{sr}+1)(\beta^{2}+1)$$
$$+(-\alpha+\beta)\gamma(2\widetilde{\alpha}\alpha^{sn+sm+1}(1-\alpha^{sr})-2\widetilde{\beta}\beta^{sn+sm+1}(1-\beta^{sr}))).$$

Taking the identities (2.6), (2.7), (3.4), and (3.5) into account, along with the previously established equality, the following result is obtained:

$$\begin{aligned}
A_n &= \frac{1}{(2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta)^2}4(-1)^{sn+1}((A+(-1)^{s+1}\left(B+\sqrt{5}F_sC\right)(-(-1)^{sr}\beta^{sm-sr}+\beta^{sm+sr})\\
&\quad+(A+(-1)^{s+1}\left(B-\sqrt{5}F_sC\right)(-(-1)^{sr}\alpha^{sm-sr}+\alpha^{sm+sr}))\\
&\quad+2\gamma(\widetilde{\alpha}\alpha^{sn}(-\alpha^{sr}+1)(\alpha^2+1)+\widetilde{\beta}\beta^{sn}(-\beta^{sr}+1)(\beta^2+1))\\
&\quad+(-\alpha+\beta)\gamma(2\widetilde{\alpha}\alpha^{sn+sm+1}-2\widetilde{\beta}\beta^{sn+sm+1}-2\widetilde{\alpha}\alpha^{sn+sm+sr+1}+2\widetilde{\beta}\beta^{sn+sm+sr+1})\\
&= \frac{1}{(2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta)^2}4(-1)^{sn+1}((A+(-1)^{s+1}\left(B+\sqrt{5}F_sC\right)(-(-1)^{sr}\beta^{sm-sr}+\beta^{sm+sr})\\
&\quad+(A+(-1)^{s+1}\left(B-\sqrt{5}F_sC\right)(-(-1)^{sr}\alpha^{sm-sr}+\alpha^{sm+sr}))\\
&\quad+2\gamma(\sqrt{5}\widetilde{\alpha}\alpha^{sn}(-\alpha^{sr+1}+\alpha)-\sqrt{5}\widetilde{\beta}\beta^{sn}(-\beta^{sr+1}+\beta))\\
&\quad+(-\alpha+\beta)\gamma(2\widetilde{\alpha}\alpha^{sn+sm+1}-2\widetilde{\beta}\beta^{sn+sm+1}-2\widetilde{\alpha}\alpha^{sn+sm+sr+1}+2\widetilde{\beta}\beta^{sn+sm+sr+1})\\
&= \frac{1}{(2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta)^2}4(-1)^{sn+1}(A+((-1)^{s+1}B(-(-1)^{sr}L_{sm-sr}+L_{sm+sr}))\\
&\quad+5(-1)^{s+1}F_sC((-1)^{sr}F_{sm-sr}-F_{sm+sr}))\\
&\quad-10\gamma(\mathbb{Q}_{s(n+r)+1}-\mathbb{Q}_{sn+1}+\mathbb{Q}_{s(n+m)+1}-\mathbb{Q}_{s(n+m+r)+1})\\
&= (L_0^s)^2[4((-1)^{sn+1}(A+((-1)^{s+1}B)(F_{sm}F_{sr}))+(-1)^{s+1}F_sC(-F_{sr}L_{sm}))\\
&\quad-2\gamma(\mathbb{Q}_{s(n+r)+1}-\mathbb{Q}_{sn+1}+\mathbb{Q}_{s(n+m)+1}-\mathbb{Q}_{s(n+m+r)+1})].
\end{aligned}$$

□

Now, we obtain the following special cases from the Vajda's identity:

Corollary 16. *For $r=-m=1$, the Cassini's identity is as follows:*

$$\begin{aligned}
\mathbb{Q}L_{n+1}^{(s)}\mathbb{Q}L_{n-1}^{(s)}-(\mathbb{Q}L_n^{(s)})^2 &= (L_0^s)^2[4((-1)^{sn+1}(A+((-1)^{s+1}B)((-1)^{s+1}(F_s)^2))\\
&\quad+(-1)^{s+1}F_sC((-1)^{s+1}F_sL_s))-2\gamma(\mathbb{Q}_{sn+s+1}-2\mathbb{Q}_{sn+1}+\mathbb{Q}_{sn-s+1})].
\end{aligned}$$

Corollary 17. *For $m=-r$, the Catalan's identity is as follows:*

$$\begin{aligned}
\mathbb{Q}L_{n+r}^{(s)}\mathbb{Q}L_{n-r}^{(s)}-(\mathbb{Q}L_n^{(s)})^2 &= (L_0^s)^2[4((-1)^{sn+1}(A+((-1)^{s+1}B)((-1)^{sr+1}(F_{sr})^2))\\
&\quad+(-1)^{s+1}F_sC((-1)^{sr+1}F_{sr}L_{sr}))-2\gamma(\mathbb{Q}_{sn+sr+1}-2\mathbb{Q}_{sn+1}+\mathbb{Q}_{sn-sr+1})].
\end{aligned}$$

Corollary 18. *For $m=1$, $r=j-n$, the d'Ocagne's identity is as follows:*

$$\begin{aligned}
\mathbb{Q}L_j^{(s)}\mathbb{Q}L_{n+1}^{(s)}-\mathbb{Q}L_n^{(s)}\mathbb{Q}L_{j+1}^{(s)} &= =(L_0^s)^2[4((-1)^{sn+1}(A+((-1)^{s+1}B)(F_sF_{sj-sn}))+(-1)^{s+1}F_sC(-F_{sj-sn}L_s))\\
&\quad-2\gamma(\mathbb{Q}_{sj+1}-\mathbb{Q}_{sn+1}+\mathbb{Q}_{sn+s+1}-\mathbb{Q}_{sj+s+1})].
\end{aligned}$$

4. Conclusions

In this study, we introduced and examined the higher order Leonardo 3-PGQs, whose components were defined using higher order Leonardo numbers. We systematically investigated their structural and functional properties, including recurrence relations, Binet-type formulas, generating functions, exponential generating functions, and several fundamental identities. We expect that these findings will inspire further research on this particular family of quaternions and open up new possibilities for potential applications in coding theory, cryptography, combinatorial designs, and other areas of the mathematical sciences.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

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