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Research article

Almost periodic dynamics of a delayed differential neoclassical growth model with nonlinear depreciation rate

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Abstract: In this paper, we study a delayed almost periodic differential neoclassical growth model with nonlinear depreciation rate. Relying on the theory of almost periodic functions and by making use of the Lyapunov functional approach, some novel criteria for guaranteeing the existence and global exponential stability of positive almost periodic solutions of the studied model are established. Besides, the correctness of the theoretical results is validated by a numerical example. The established theoretical findings supplement and improve the conclusions in the literature.

Keywords: differential neoclassical growth model; almost periodic solution; nonlinear depreciation rate; global exponential stability

1. Introduction

The differential neoclassical growth model, as a kind of differential equation that discloses the intrinsic mechanisms and long-term trends of economic growth, has made abundant research achievements in recent years. For a better description of the behavior of economic activities changing over time under a simple economic structure, Matsumoto and Szidarovszky [1] first put forward the following differential neoclassical growth model:

$$p'(t) = -\gamma p(t) + \eta p^m(t-\delta)e^{-\beta p(t-\delta)}, \quad \gamma, \eta, \delta, \beta, m > 0, \tag{1.1}$$

among them, $p(t)$ means the per capita labor capital at time t, γ indicates the depreciation rate of capital, η stands for the maximum productivity, m can be regarded as an indicator to measure the return scale of the production function, δ represents the delay that occurs in the production process, and β reflects the intensity of the "negative effect" caused by the increase in capital concentration, which is mainly determined by the degree of damage of natural environment or energy resources. On account of its profound practical application background and theoretical value, model (1.1) and its generalized forms have drawn extensive attention and in-depth research from numerous economists and mathematicians [2–6].

For a long time, various factors (e.g., the retardation in information transmission and the time consuming of decision making process) can lead to the emergence of time varying delays in the actual economic operation. In addition, modifications in the natural environment and the interference of human policies can also bring about a non-negligible impact on economic growth. In order to reflect the dynamic characteristics of actual economic problems more truly, expressing the differential neoclassical growth model via delay differential equations is more in line with reality. It is well acknowledged that high-density capital may encounter higher maintenance costs or technological obsolescence risks, so the linear model with density-dependent depreciation rate is relatively precise for low-density capital [7]. Owing to the interactions among individuals within assets or resources (e.g., competition, synergy effects), the depreciation rate exhibits complex nonlinear characteristics with density changes. From this, it is more rational to introduce a new nonlinear depreciation rate term $\frac{\gamma p}{\alpha+p}(\gamma, \alpha > 0)$ in the differential neoclassical growth model. As a result, the study of delayed mathematical models with nonlinear depreciation rate has turned into a vital research subject. For example, Wang et al. [8] investigated the existence of positive periodic solutions for the delayed Nicholson's blowflies model with a nonlinear density-dependent mortality term; Garain et al. [9] analyzed the dynamic behaviors of the prey-predator model with nonlinear density-dependent mortality term and Allee effect; Wang et al. [10] established stability criteria for the hematopoiesis model with a nonlinear density-dependent mortality term. Hence, the study of the delayed mathematical model with nonlinear depreciation rate is of great significance.

On the flip side, it has been discovered that when considering the long-term dynamic behaviors of economic models, the periodic parameters of the model are often disrupted to some extent. This is due to the fact that the economy in reality is affected by multiple factors (e.g., technological advancement, policy adjustments, and international economic impacts). In contrast with the periodic effect, the almost periodic effect can depict the complex dynamic behavior of the economic system with higher accuracy. As a matter of fact, Bohr [11] has made detailed reports on such almost periodic phenomena and formulated the theory of almost periodic functions. In this sense, considering the almost periodicity of parameters in the model is reasonable and valuable. Numerous experiments have indicated that many key variables (e.g., savings rate, population growth rate) in economic growth are not fixed and immutable, but fluctuate considerably as time evolves. Additionally, for autonomous differential equations with constant coefficients and delays, the impact of economic cycles, policy shocks, or emergency events on the long-term growth path of the economy cannot be explained. Therefore, there is a need to construct a differential neoclassical growth model depicted by non-autonomous delay differential equations to contemplate economic growth phenomena, and numerous researchers have carried out profound studies on non-autonomous almost periodic differential neoclassical growth model [12–15]. Nonetheless, the reports on the research of the delayed almost periodic differential neoclassical growth model with nonlinear depreciation rate are scarce. Stimulated by the discussions stated earlier, we further investigate the non-autonomous delayed differential neoclassical growth model, which incorporates a nonlinear depreciation rate and almost periodic coefficients, as described by

$$p'(t) = -\frac{\gamma(t)p(t)}{\alpha(t) + p(t)} + \eta(t)p^m(t - \delta(t))e^{-\beta(t)p(t-\delta(t))}, \tag{1.2}$$

in which $\frac{\gamma(t)p(t)}{\alpha(t)+p(t)}$ denotes the reduction in per capita capital at time t. The economic implications of the leftover time dependent economic parameters are the same as what is detailed in (1.1). The principal objective of this paper is to deeply examine the existence and global exponential stability of positive almost periodic solutions for model (1.2).

With the aim of making the notations simpler, for a bounded continuous function u defined on $\mathbb{R}$, denote u^+ and u^- as

$$u^+ = \sup_{t\in\mathbb{R}} u(t), \quad u^- = \inf_{t\in\mathbb{R}} u(t).$$

During the paper, we invariably assume that

$$\gamma(t), \alpha(t), \eta(t), \beta(t), \delta(t) : \mathbb{R} \to (0, +\infty) \text{ are continuous almost periodic functions.}$$

Define $\delta^+ = \sup_{t\in\mathbb{R}} \delta(t)$ and $C = C([-\delta^+, 0], \mathbb{R})$ is the space of continuous functions supplemented with the supremum norm $\|\cdot\|$, and let $C_+ = C([-\delta^+, 0], \mathbb{R}_+)$ and $\mathbb{R}_+ = [0, +\infty)$. Provided that $p(t)$ is continuous and defined on $[t_0 - \delta^+, \zeta)$ with $t_0, \zeta \in \mathbb{R}$, then we define $p_t \in C$ for all $t \in [t_0, \zeta)$, in which $p_t(\rho) = p(t+\rho)$ for all $\rho \in [-\delta^+, 0]$.

The remainder of this paper has the following structure. In Section 2, we present some preliminaries. In Section 3, we establish novel results for guaranteeing the existence and global exponential stability of the almost periodic solutions for model (1.2). In Section 4, we illustrate the theoretical results by a numerical example.

2. Preliminaries

From the economic causes, only meaningful and thus admissible solutions of model (1.2) are the positive ones. Therefore, the initial conditions are presented as follows

$$p_{t_0} = \psi, \quad \psi \in C_+ \quad \text{and} \quad \psi(0) > 0. \tag{2.1}$$

We use $p_t(t_0, \psi)(p(t; t_0, \psi))$ to represent a solution of the admissible initial value problem (1.2)–(2.1), and $p_{t_0}(t_0, \psi) = \psi \in C_+$, $t_0 \in \mathbb{R}$. Additionally, the maximal right-interval of existence of $p_t(t_0, \psi)$ is indicated by $[t_0, \varsigma(\psi))$.

Let $0 < m < 1$, and it can be observed that the function $F(w) = e^{-w}w^{m-1}(m - w)$ is decreasing on the interval $(0, m + \sqrt{m})$ and increasing on the interval $(m + \sqrt{m}, +\infty)$. Then a unique $n_1 \in (0, m)$ can be found, such that

$$F(n_1) = \sup_{w\geq n_1} |F(w)| = |F(m + \sqrt{m})|.$$

Furthermore, one can see that $w^m e^{-w}$ increases on $[0, m]$ and decreases on $[m, +\infty)$. Then take n_2 as the unique number in $[m, +\infty)$, which satisfies

$$n_1^m e^{-n_1} = n_2^m e^{-n_2}.$$

Hereafter, we introduce several definitions and lemmas that are of great significance in the proof of the main results in Section 3.

Definition 2.1 (see [16]). *A continuous function $l(\cdot) : \mathbb{R} \to \mathbb{R}$ is called almost periodic if for any $\varepsilon > 0$, the set $A(l, \varepsilon) = \{\theta : |l(t + \theta) - l(t)| < \varepsilon$, for all $t \in \mathbb{R}\}$ is relatively dense, namely, for any $\varepsilon > 0$, there exists a real number $q = q(\varepsilon) > 0$, such that for any interval with length $q(\varepsilon)$, a number $\theta = \theta(\varepsilon)$ can be found in this interval satisfying $|l(t + \theta) - l(t)| < \varepsilon$, for all $t \in \mathbb{R}$.*

Definition 2.2 (see [17]). *A positive almost periodic solution $p^*(\cdot)$ of model* (1.2) *is called globally exponentially stable if for any solution $p(t;t_0,\psi)$ of model* (1.2)*, it is possible to find constants $\varepsilon > 0, K > 0$ and t^*, such that*

$$|p(t;t_0,\psi) - p^*(t)| < Ke^{-\varepsilon t}, \quad \textit{for all } t > t^*.$$

In light of the theory of almost periodic functions presented in [16], we have the ability to know that for any $\varepsilon > 0$, there exists a real number $q = q(\varepsilon) > 0$, such that for any interval with length $q(\varepsilon)$, a number $\theta = \theta(\varepsilon)$ can be found in this interval, satisfying

$$\begin{cases} |\gamma(t+\theta) - \gamma(t)| < \varepsilon, \quad |\alpha(t+\theta) - \alpha(t)| < \varepsilon, \quad |\eta(t+\theta) - \eta(t)| < \varepsilon, \\ |\beta(t+\theta) - \beta(t)| < \varepsilon, \quad |\delta(t+\theta) - \delta(t)| < \varepsilon, \end{cases} \tag{2.2}$$

for all $t \in \mathbb{R}$.

Lemma 2.1. *Given that* $0 < m < 1$. *Assume that it is possible to find a positive constant $N > n_1$ such that*

$$1 \le \beta^- \le \beta^+ \le \frac{n_2}{N}, \tag{2.3}$$

$$\sup_{t\in\mathbb{R}} \left\{ -\frac{\gamma(t)N}{\alpha(t)+N} + \frac{\eta(t)}{\beta^m(t)} m^m e^{-m} \right\} < 0, \tag{2.4}$$

and

$$\inf_{t\in\mathbb{R}, s\in[0,n_1]} \left\{ -\frac{\gamma(t)}{\alpha(t)+s} + \frac{\eta(t)}{\beta^m(t)} s^{m-1} e^{-s} \right\} > 0. \tag{2.5}$$

Then, each solution $p(t;t_0,\psi)$ of (1.2) *is positive and bounded on $[t_0, \varsigma(\psi))$, and $\varsigma(\psi) = +\infty$. Moreover, one can find $t^\psi > t_0$, satisfying*

$$n_1 < p(t;t_0,\psi) < N, \quad \textit{for all } t \ge t^\psi. \tag{2.6}$$

Proof. Because $\psi \in C_+$, with the use of invariant set theory [18], we possess $p_t(t_0,\psi) \in C_+$ for all $t \in [t_0, \varsigma(\varphi))$. Denote $p(t) = p(t;t_0,\psi)$. According to the model (1.2), one has

$$p'(t) \ge -\frac{\gamma(t)p(t)}{\alpha(t)} + \eta(t)p^m(t-\delta(t))e^{-\beta(t)p(t-\delta(t))}. \tag{2.7}$$

Due to the fact that $p(t_0) = \psi(0) > 0$, making the integration of the (2.7) from t_0 to t, it results that

$$\begin{aligned} p(t) &\ge e^{-\int_{t_0}^t \frac{\gamma(s)}{\alpha(s)}ds} p(t_0) + e^{-\int_{t_0}^t \frac{\gamma(s)}{\alpha(s)}ds} \int_{t_0}^t e^{\int_{t_0}^v \frac{\gamma(u)}{\alpha(u)}du} \eta(v)p^m(v-\delta(v))e^{-\beta(v)p(v-\delta(v))}dv \\ &> 0, \quad \text{for all } \ t \in [t_0, \varsigma(\psi)). \end{aligned}$$

With respect to every $t \in [t_0 - \delta^+, \varsigma(\psi))$, we define

$$\Lambda(t) = \max\{\sigma : \sigma \le t, p(\sigma) = \max_{t_0-\delta^+\le s\le t} p(s)\}.$$

We demonstrate that $p(t)$ is bounded on $[t_0, \varsigma(\psi))$. If the opposite is true, observe that $\Lambda(t) \to \varsigma(\psi)$ as $t \to \varsigma(\psi)$, we know

$$\lim_{t\to\varsigma(\psi)} p(\Lambda(t)) = +\infty.$$

Thus, we can select $t_* > t_0$, for which

$$\Lambda(t_*) > t_0 \quad \text{and} \quad p(\Lambda(t_*)) = \max_{t_0-\delta^+\le s\le t_*} p(s) > N. \tag{2.8}$$

It follows from (1.2), (2.4) and (2.8), in conjunction with the fact $\sup_{b\in\mathbb{R}_+} b^m e^{-b} = m^m e^{-m}$ that

$$\begin{aligned} 0 \le & p'(\Lambda(t_*)) \\ = & -\frac{\gamma(\Lambda(t_*))p(\Lambda(t_*))}{\alpha(\Lambda(t_*)) + p(\Lambda(t_*))} \\ & + \frac{\eta(\Lambda(t_*))}{\beta^m(\Lambda(t_*))}\beta^m(\Lambda(t_*))p^m(\Lambda(t_*) - \delta(\Lambda(t_*)))e^{-\beta(\Lambda(t_*))p(\Lambda(t_*)-\delta(\Lambda(t_*)))} \\ \le & -\frac{\gamma(\Lambda(t_*))N}{\alpha(\Lambda(t_*)) + N} + \frac{\eta(\Lambda(t_*))}{\beta^m(\Lambda(t_*))}m^m e^{-m} \\ \le & \sup_{t\in\mathbb{R}}\left\{ -\frac{\gamma(t)N}{\alpha(t)+N} + \frac{\eta(t)}{\beta^m(t)}m^m e^{-m}\right\} \\ < & 0, \end{aligned}$$

this presents a paradox and means that $p(t)$ is bounded on $[t_0, \varsigma(\psi))$. It follows directly from the continuation theorem that $\varsigma(\psi) = +\infty$.

Next, we prove that (2.6) is right. We first assert that one can find $t_1 \in [t_0, +\infty)$ satisfying

$$p(t_1) < N. \tag{2.9}$$

Otherwise,

$$p(t) \ge N, \quad \text{for all } t \in [t_0, +\infty). \tag{2.10}$$

With the aid of (1.2), (2.4) and (2.10), combined with the truth $\sup_{b\in\mathbb{R}_+} b^m e^{-b} = m^m e^{-m}$ that

$$\begin{aligned} p'(t) = & -\frac{\gamma(t)p(t)}{\alpha(t)+p(t)} + \frac{\eta(t)}{\beta^m(t)}\beta^m(t)p^m(t-\delta(t))e^{-\beta(t)p(t-\delta(t))} \\ \le & -\frac{\gamma(t)N}{\alpha(t)+N} + \frac{\eta(t)}{\beta^m(t)}m^m e^{-m} \\ \le & \sup_{t\in\mathbb{R}}\left\{ -\frac{\gamma(t)N}{\alpha(t)+N} + \frac{\eta(t)}{\beta^m(t)}m^m e^{-m}\right\} \\ < & 0, \quad \text{for all } t \ge t_0. \end{aligned}$$

This yields that

$$\begin{aligned} p(t) = & p(t_0) + \int_{t_0}^{t} p'(u)du \\ \le & p(t_0) + \sup_{t\in\mathbb{R}}\left\{ -\frac{\gamma(t)N}{\alpha(t)+N} + \frac{\eta(t)}{\beta^m(t)}m^m e^{-m}\right\}(t-t_0), \quad \text{for all } t \ge t_0. \end{aligned}$$

Hence,

$$\lim_{t\to+\infty} p(t) = -\infty,$$

this conflicts with the fact that $p(t)$ is positive and bounded on $[t_0, +\infty)$. Therefore, (2.9) is true. Thereafter, we verify that

$$p(t) < N, \quad \text{for all } t \in [t_1, +\infty). \tag{2.11}$$

With the reduction to absurdity, we presume that it is possible for one to find $t_2 \in (t_1, +\infty)$, such that

$$p(t_2) = N, \quad \text{and} \quad p(t) < N, \quad \text{for all } t \in [t_1, t_2). \tag{2.12}$$

In accordance with (1.2), (2.4) and (2.12), together with the statement $\sup\limits_{b\in\mathbb{R}_+} b^m e^{-b} = m^m e^{-m}$ that

$$\begin{aligned}
0 \le & p'(t_2) \\
= & -\frac{\gamma(t_2)p(t_2)}{\alpha(t_2)+p(t_2)} + \frac{\eta(t_2)}{\beta^m(t_2)}\beta^m(t_2)p^m(t_2-\delta(t_2))e^{-\beta(t_2)p(t_2-\delta(t_2))} \\
\le & -\frac{\gamma(t_2)N}{\alpha(t_2)+N} + \frac{\eta(t_2)}{\beta^m(t_2)}m^m e^{-m} \\
\le & \sup_{t\in\mathbb{R}}\left\{-\frac{\gamma(t)N}{\alpha(t)+N} + \frac{\eta(t)}{\beta^m(t)}m^m e^{-m}\right\} \\
< & 0.
\end{aligned}$$

This contradiction implies that (2.12) must be false, thereby confirming (2.11).

Hereafter, we demonstrate that there is a positive constant k, satisfying

$$\liminf_{t\to+\infty} p(t) = k. \tag{2.13}$$

In the contrary case, we have $\liminf\limits_{t\to+\infty} p(t) = 0$. With respect to every $t \ge t_0$, we define

$$\Gamma(t) = \max\{\sigma : \sigma \le t, p(\sigma) = \min_{t_0\le s\le t} p(s)\}.$$

Note that $\Gamma(t) \to +\infty$ as $t \to +\infty$ and

$$\lim_{t\to+\infty} p(\Gamma(t)) = 0. \tag{2.14}$$

In light of the definition of $\Gamma(t)$, we can obtain the results that $p(\Gamma(t)) = \min\limits_{t_0\le s\le t} p(s)$, as well as $p'(\Gamma(t)) \le 0$. Then, model (1.2) yields

$$\begin{aligned}
0 \ge & p'(\Gamma(t)) \\
= & -\frac{\gamma(\Gamma(t))p(\Gamma(t))}{\alpha(\Gamma(t))+p(\Gamma(t))} + \eta(\Gamma(t))p^m(\Gamma(t)-\delta(\Gamma(t)))e^{-\beta(\Gamma(t))p(\Gamma(t)-\delta(\Gamma(t)))} \\
\ge & -\frac{\gamma(\Gamma(t))p(\Gamma(t))}{\alpha(\Gamma(t))} + \eta(\Gamma(t))p^m(\Gamma(t)-\delta(\Gamma(t)))e^{-\beta(\Gamma(t))p(\Gamma(t)-\delta(\Gamma(t)))},
\end{aligned}$$

through a direct calculation, one could acquire

$$\frac{\gamma(\Gamma(t))p(\Gamma(t))}{\alpha(\Gamma(t))} \ge \eta(\Gamma(t))p^m(\Gamma(t)-\delta(\Gamma(t)))e^{-\beta(\Gamma(t))p(\Gamma(t)-\delta(\Gamma(t)))}, \tag{2.15}$$

which, along with (2.14), suggests that

$$\lim_{t\to+\infty} p(\Gamma(t)-\delta(\Gamma(t)))=0. \tag{2.16}$$

Considering the boundedness and continuity of the function $\gamma(t),\alpha(t),\eta(t)$ and $\beta(t)$, we can choose a sequence $\{t_i\}_{i=1}^{+\infty}$, satisfying

$$\begin{aligned}&\lim_{i\to+\infty} t_i=+\infty, \qquad \lim_{i\to+\infty}\gamma(\Gamma(t_i))=\gamma^{**}>0, \quad \lim_{i\to+\infty}\alpha(\Gamma(t_i))=\alpha^{**}>0,\\ &\lim_{i\to+\infty}\eta(\Gamma(t_i))=\eta^{**}, \quad \lim_{i\to+\infty}\beta(\Gamma(t_i))=\beta^{**}.\end{aligned} \tag{2.17}$$

On account of (2.15), it can be concluded that

$$\begin{aligned}\frac{\gamma(\Gamma(t_i))}{\alpha(\Gamma(t_i))} \geq&\eta(\Gamma(t_i))\frac{p^m(\Gamma(t_i)-\delta(\Gamma(t_i)))}{p(\Gamma(t_i))}e^{-\beta(\Gamma(t_i))p(\Gamma(t_i)-\delta(\Gamma(t_i)))}\\ \geq&\eta(\Gamma(t_i))\frac{p^m(\Gamma(t_i)-\delta(\Gamma(t_i)))}{p(\Gamma(t_i)-\delta(\Gamma(t_i)))}e^{-\beta(\Gamma(t_i))p(\Gamma(t_i)-\delta(\Gamma(t_i)))}\\ =&\eta(\Gamma(t_i))p^{m-1}(\Gamma(t_i)-\delta(\Gamma(t_i)))e^{-\beta(\Gamma(t_i))p(\Gamma(t_i)-\delta(\Gamma(t_i)))}.\end{aligned} \tag{2.18}$$

Setting $i\to+\infty$, (2.16), (2.17) and (2.18) signify

$$\frac{\gamma^{**}}{\alpha^{**}}\geq\eta^{**}\lim_{i\to+\infty}p^{m-1}(\Gamma(t_i)-\delta(\Gamma(t_i)))=+\infty,$$

this case evidently results in a contradiction, so (2.13) holds.

In the end, we confirm that $k>n_1$. For the sake of contradiction, we suppose that $0<k\leq n_1$. From the fluctuation lemma [19], one can find a sequence $\{t_l\}_{l=1}^{+\infty}$, fulfilling

$$t_l\to+\infty, \quad p(t_l)\to k, \ \text{ and } \ p'(t_l)\to 0, \ \text{ as } l\to+\infty. \tag{2.19}$$

Since $\{p_{t_l}\}$ is equicontinuous and bounded, the Arzela-Ascoli theorem implies the existence of a convergent subsequence (still represented as itself for brevity in notation) satisfying

$$p_{t_l}\to\psi^*(l\to+\infty), \quad \text{for } \psi^*\in C_+, \tag{2.20}$$

and

$$\psi^*(0)=k\leq\psi^*(\mu)\leq N, \quad \text{for } \mu\in[-\delta^+,0). \tag{2.21}$$

Because $\gamma(t),\alpha(t),\eta(t),\beta(t),\delta(t)$ are almost periodic functions, we can presume that $\gamma(t_l)$, $\alpha(t_l)$, $\eta(t_l)$, $\beta(t_l)$, and $\delta(t_l)$ are convergent to $\gamma^*,\alpha^*,\eta^*,\beta^*$, and δ^*, respectively. According to (2.3) and (2.21), we derive that

$$k\leq\beta^*\psi^*(-\delta^*)\leq\beta^+N\leq n_2,$$

and hence

$$(\beta^*\psi^*(-\delta^*))^m e^{-\beta^*\psi^*(-\delta^*)}\geq\min\{k^m e^{-k}, n_2^m e^{-n_2}\}=k^m e^{-k}. \tag{2.22}$$

By

$$p'(t_l)=-\frac{\gamma(t_l)p(t_l)}{\alpha(t_l)+p(t_l)}+\frac{\eta(t_l)}{\beta^m(t_l)}\beta^m(t_l)p^m(t_l-\delta(t_l))e^{-\beta(t_l)p(t_l-\delta(t_l))},$$

we pass to the limit in the above formula, along with (2.5), (2.19), (2.20), and (2.22), yielding

$$\begin{aligned}0 =& -\frac{\gamma^* k}{\alpha^* + k} + \frac{\eta^*}{(\beta^*)^m}(\beta^*)^m(\psi^*(-\delta^*))^m e^{-\beta^*\psi^*(-\delta^*)}\\ \geq& -\frac{\gamma^* k}{\alpha^* + k} + \frac{\eta^*}{(\beta^*)^m}k^m e^{-k}\\ =&k\Big(-\frac{\gamma^*}{\alpha^* + k} + \frac{\eta^*}{(\beta^*)^m}k^{m-1}e^{-k}\Big)\\ \geq& k\inf_{t\in\mathbb{R}, s\in[0,n_1]}\Big\{-\frac{\gamma(t)}{\alpha(t)+s} + \frac{\eta(t)}{\beta^m(t)}s^{m-1}e^{-s}\Big\}\\ >&0,\end{aligned}$$

which is a contradiction. The evidence shows that $k > n_1$. Therefore, by (2.11), one can find a $t^\psi > t_0$, satisfying

$$n_1 < p(t; t_0, \psi) < N, \quad \text{for all } t \geq t^\psi.$$

This completes the verification of Lemma 2.1. □

Lemma 2.2. *Apart from the assumptions in Lemma 2.1, and assume further that*

$$\sup_{t\in\mathbb{R}}\Big\{-\frac{\gamma(t)\alpha(t)}{(\alpha(t)+N)^2} + \eta(t)\beta^{1-m}(t)|F(m+\sqrt{m})|\Big\} < 0, \tag{2.23}$$

in which $F(w) = e^{-w}w^{m-1}(m-w)$. Furthermore, suppose that $p(t) = p(t; t_0, \psi)$ is a solution of (1.2) *with initial condition* (2.1) *as well as ψ' is bounded and continuous on* $[-\delta^+, 0]$. *Then, for any $\varepsilon > 0$, there exists $t^{**} > 0$ and $q = q(\varepsilon)$, such that for any $\xi \in R$, there exists a $\theta \in [\xi, \xi + q]$ satisfying*

$$|p(t+\theta) - p(t)| \leq \varepsilon, \quad \textit{for all } t > t^{**}.$$

Proof. Construct a continuous function $\Pi(v)$ given below

$$\Pi(v) = \sup_{t\in\mathbb{R}}\Big\{-\Big[\frac{\gamma(t)\alpha(t)}{(\alpha(t)+N)^2} - v\Big] + \eta(t)\beta^{1-m}(t)|F(m+\sqrt{m})|e^{v\delta^+}\Big\}, \quad v \geq 0.$$

By making use of (2.23), it can be effortlessly seen that

$$\Pi(0) = \sup_{t\in\mathbb{R}}\Big\{-\frac{\gamma(t)\alpha(t)}{(\alpha(t)+N)^2} + \eta(t)\beta^{1-m}(t)|F(m+\sqrt{m})|\Big\} < 0,$$

which shows that there are two constants $\varpi > 0$ and $\lambda \in (0, 1]$, fulfilling

$$\Pi(\lambda) = \sup_{t\in\mathbb{R}}\Big\{-\Big[\frac{\gamma(t)\alpha(t)}{(\alpha(t)+N)^2} - \lambda\Big] + \eta(t)\beta^{1-m}(t)|F(m+\sqrt{m})|e^{\lambda\delta^+}\Big\} < -\varpi < 0. \tag{2.24}$$

For all $t \in (-\infty, t_0 - \delta^+]$, we append the definition of $p(t)$ with $p(t) \equiv p(t_0 - \delta^+)$. Denote

$$\Phi(\theta, t) = -\Big[\frac{\gamma(t+\theta)p(t+\theta)}{\alpha(t+\theta)+p(t+\theta)} - \frac{\gamma(t)p(t+\theta)}{\alpha(t+\theta)+p(t+\theta)}\Big]$$

$$
\begin{aligned}
&-\left[\frac{\gamma(t)p(t+\theta)}{\alpha(t+\theta)+p(t+\theta)}-\frac{\gamma(t)p(t+\theta)}{\alpha(t)+p(t+\theta)}\right]\\
&+[\eta(t+\theta)-\eta(t)]p^m(t+\theta-\delta(t+\theta))e^{-\beta(t+\theta)p(t+\theta-\delta(t+\theta))}\\
&+\eta(t)\Big[p^m(t+\theta-\delta(t+\theta))e^{-\beta(t+\theta)p(t+\theta-\delta(t+\theta))}\\
&-p^m(t+\theta-\delta(t))e^{-\beta(t+\theta)p(t+\theta-\delta(t))}\Big]\\
&+\eta(t)\Big[p^m(t+\theta-\delta(t))e^{-\beta(t+\theta)p(t+\theta-\delta(t))}\\
&-p^m(t+\theta-\delta(t))e^{-\beta(t)p(t+\theta-\delta(t))}\Big],\qquad t\in\mathbb{R}.
\end{aligned}\tag{2.25}
$$

From Lemma 2.1, the solution $p(t)$ is bounded and

$$n_1 < p(t) < N, \quad \text{for all } t \geq t^{\psi},$$

which means that the right side of (1.2) is also bounded, and $p'(t)$ is a bounded function on $[t_0-\delta^+, +\infty)$. Therefore, in view of $p(t) \equiv p(t_0-\delta^+)$ for $t \in (-\infty, t_0-\delta^+]$, this yields that $p(t)$ is uniformly continuous on $\mathbb{R}$. Obverse that any finite set of almost periodic functions is a uniformly almost periodic family [16], and by virtue of the uniform continuity of $p(t)$, it is possible to select a constant θ for which every term in (2.25) could be arbitrarily small. Especially, with the aid of (2.2), for any $\varepsilon > 0$, there exists a real number $q = q(\varepsilon) > 0$, and $\theta \in [\xi, \xi+q], \xi \in \mathbb{R}$, such that

$$|\Phi(\theta,t)| \leq \frac{1}{2}\varpi\varepsilon, \quad \text{for all } t \in \mathbb{R}. \tag{2.26}$$

Denote $T \geq \max\{t_0, t_0-\theta, t^{\psi}+\delta^+, t^{\psi}+\delta^+-\theta\}$. For $t \in \mathbb{R}$, set

$$y(t) = p(t+\theta) - p(t).$$

Then, for all $t \geq T$, one yields

$$
\begin{aligned}
\frac{\mathrm{d}y(t)}{\mathrm{d}t} =& -\left[\frac{\gamma(t)p(t+\theta)}{\alpha(t)+p(t+\theta)}-\frac{\gamma(t)p(t)}{\alpha(t)+p(t)}\right]+\eta(t)\Big[p^m(t+\theta-\delta(t))e^{-\beta(t)p(t+\theta-\delta(t))}\\
&-p^m(t-\delta(t))e^{-\beta(t)p(t-\delta(t))}\Big]+\Phi(\theta,t).
\end{aligned}\tag{2.27}
$$

Let $F(w) = e^{-w}w^{m-1}(m-w)$, since

$$F(n_1) = \sup_{w\geq n_1}|F(w)| = |F(m+\sqrt{m})|,$$

then according to the mean value theorem, it can be inferred that

$$
\begin{aligned}
|x^m e^{-x} - z^m e^{-z}| =& |F(x+c(z-x))||x-z|\\
\leq& |F(m+\sqrt{m})||x-z|, \qquad x,z \in [n_1,+\infty),\ \ 0<c<1.
\end{aligned}\tag{2.28}
$$

It can be concluded from (2.3), (2.27), and (2.28) that

$$D^-(e^{\lambda s}|y(s)|)|_{s=t} = \lambda e^{\lambda t}|y(t)| + e^{\lambda t}\mathrm{sign}(y(t))\Bigg\{-\left[\frac{\gamma(t)p(t+\theta)}{\alpha(t)+p(t+\theta)}-\frac{\gamma(t)p(t)}{\alpha(t)+p(t)}\right]$$

$$
\begin{aligned}
&+\eta(t)\Big[p^m(t+\theta-\delta(t))e^{-\beta(t)p(t+\theta-\delta(t))}\\
&-p^m(t-\delta(t))e^{-\beta(t)p(t-\delta(t))}\Big]+\Phi(\theta,t)\Big\}\\
=&\lambda e^{\lambda t}|y(t)|+e^{\lambda t}\mathrm{sign}(y(t))\Big\{-\frac{\gamma(t)\alpha(t)y(t)}{(\alpha(t)+p(t+\theta))(\alpha(t)+p(t))}\\
&+\frac{\eta(t)}{\beta^m(t)}\Big[\beta^m(t)p^m(t+\theta-\delta(t))e^{-\beta(t)p(t+\theta-\delta(t))}\\
&-\beta^m(t)p^m(t-\delta(t))e^{-\beta(t)p(t-\delta(t))}\Big]+\Phi(\theta,t)\Big\}\\
\leq&\lambda e^{\lambda t}|y(t)|+e^{\lambda t}\Big[-\frac{\gamma(t)\alpha(t)|y(t)|}{(\alpha(t)+N)^2}\\
&+\frac{\eta(t)}{\beta^m(t)}\Big|\beta^m(t)p^m(t+\theta-\delta(t))e^{-\beta(t)p(t+\theta-\delta(t))}\\
&-\beta^m(t)p^m(t-\delta(t))e^{-\beta(t)p(t-\delta(t))}\Big|+|\Phi(\theta,t)|\Big]\\
\leq&\lambda e^{\lambda t}|y(t)|+e^{\lambda t}\Big[-\frac{\gamma(t)\alpha(t)}{(\alpha(t)+N)^2}|y(t)|\\
&+\frac{\eta(t)}{\beta^m(t)}|F(m+\sqrt{m})|\beta(t)|y(t-\delta(t))|+|\Phi(\theta,t)|\Big]\\
=&-\Big[\frac{\gamma(t)\alpha(t)}{(\alpha(t)+N)^2}-\lambda\Big]e^{\lambda t}|y(t)|\\
&+\eta(t)\beta^{1-m}(t)|F(m+\sqrt{m})|e^{\lambda\delta(t)}e^{\lambda(t-\delta(t))}|y(t-\delta(t))|\\
&+e^{\lambda t}|\Phi(\theta,t)|,\qquad \text{for all } t\geq T.
\end{aligned}\tag{2.29}
$$

Designate

$$\Omega(t)=\sup_{-\infty<s\leq t}\{e^{\lambda s}|y(s)|\}.$$

Clearly, $e^{\lambda t}|y(t)|\leq\Omega(t)$, and $\Omega(t)$ is non-decreasing. Thus, we can complete the subsequent verification in two cases.

Case one. Provided that $\Omega(t)>e^{\lambda t}|y(t)|$ for all $t\geq T$, we assert that

$$\Omega(t)\equiv\Omega(T)\quad\text{is a constant for all } t\geq T. \tag{2.30}$$

Employing the technique of contradiction, one can choose $t_3>T$, satisfying $\Omega(t_3)>\Omega(T)$. Owing to

$$e^{\lambda t}|y(t)|\leq\Omega(T),\quad \text{for all } t\leq T,$$

so there definitely exists a $\chi\in(T,t_3)$ for which

$$e^{\lambda\chi}|y(\chi)|=\Omega(t_3)\geq\Omega(\chi),$$

this contradicts the truth that $\Omega(\chi)>e^{\lambda\chi}|y(\chi)|$. Therefore, (2.30) is right. This indicates that it is possible to find a $t_4>T$ satisfying

$$|y(t)|<e^{-\lambda t}\Omega(t)=e^{-\lambda t}\Omega(T)<\varepsilon,\quad \text{for all } t\geq t_4.$$

Case two. Provided that there exists $t_5 \geq T$, fulfilling $\Omega(t_5) = e^{\lambda t_5}|y(t_5)|$. Based on (2.24), (2.26) and (2.29), it is easy to obtain

$$\begin{aligned}
0 \leq & D^-(e^{\lambda s}|y(s)|)|_{s=t_5} \\
\leq & -\left[\frac{\gamma(t_5)\alpha(t_5)}{(\alpha(t_5)+N)^2} - \lambda\right] e^{\lambda t_5}|y(t_5)| \\
& + \eta(t_5)\beta^{1-m}(t_5)|F(m+\sqrt{m})|e^{\lambda\delta(t_5)}e^{\lambda(t_5-\delta(t_5))}|y(t_5-\delta(t_5))| + e^{\lambda t_5}|\Phi(\theta, t_5)| \\
\leq & \left\{-\left[\frac{\gamma(t_5)\alpha(t_5)}{(\alpha(t_5)+N)^2} - \lambda\right] + \eta(t_5)\beta^{1-m}(t_5)|F(m+\sqrt{m})|e^{\lambda\delta^+}\right\}\Omega(t_5) + \frac{1}{2}e^{\lambda t_5}\varpi\varepsilon \\
< & -\varpi\Omega(t_5) + e^{\lambda t_5}\varpi\varepsilon,
\end{aligned}$$

which results in

$$e^{\lambda t_5}|y(t_5)| = \Omega(t_5) < e^{\lambda t_5}\varepsilon, \quad \text{and} \quad |y(t_5)| < \varepsilon. \tag{2.31}$$

For any $t > t_5$, if $\Omega(t) = e^{\lambda t}|y(t)|$, using the method as identical as the one in the derivation of (2.31), we have

$$e^{\lambda t}|y(t)| = \Omega(t) < e^{\lambda t}\varepsilon, \quad \text{and} \quad |y(t)| < \varepsilon. \tag{2.32}$$

On the flip side, if $\Omega(t) > e^{\lambda t}|y(t)|$ as well as $t > t_5$, we could select $t_6 \in [t_5, t)$, fulfilling

$$\Omega(t_6) = e^{\lambda t_6}|y(t_6)|, \quad \text{and} \quad \Omega(s) > e^{\lambda s}|y(s)|, \quad \text{for all } s \in (t_6, t],$$

which, in conjunction with (2.32), one has

$$e^{\lambda t_6}|y(t_6)| = \Omega(t_6) < e^{\lambda t_6}\varepsilon, \quad \text{and} \quad |y(t_6)| < \varepsilon.$$

By an argument analogous to that in Case one, it follows that

$$\Omega(s) \equiv \Omega(t_6) \quad \text{is a constant for all } s \in (t_6, t],$$

which indicates that

$$|y(t)| < e^{-\lambda t}\Omega(t) = e^{-\lambda t}\Omega(t_6) = |y(t_6)|e^{-\lambda(t-t_6)} < \varepsilon.$$

All in all, there exists a $t^{**} > \max\{t_4, t_5\}$ for which $|y(t)| \leq \varepsilon$ holds for all $t > t^{**}$. Hereby, we complete the proof of Lemma 2.2. □

3. Main results

The objective of this section is to derive some criteria that guarantee the existence and global exponential stability of almost periodic solutions for model (1.2).

Theorem 3.1. *Provided that the conditions of Lemma 2.2 are met, then model* (1.2) *admits at least one positive almost periodic solution $p^*(t)$. Furthermore, $p^*(t)$ is globally exponentially stable, namely, one can find constants $K > 0$ and t^*, fulfilling*

$$|p(t; t_0, \psi) - p^*(t)| < Ke^{-\lambda t}, \quad \textit{for all } t > t^*,$$

in which the definition of λ is given in (2.24).

Proof. Denote $X(t) = X(t; t_0, \psi^X)$ as a solution of model (1.2) with initial conditions satisfying the suppositions in Lemma 2.2. We additionally append the definition of $X(t)$ with $X(t) \equiv X(t_0 - \delta^+)$ for all $t \in (-\infty, t_0 - \delta^+]$. Designate

$$\begin{aligned}
\varphi(j,t) = &-\Big[\frac{\gamma(t+t_j)X(t+t_j)}{\alpha(t+t_j)+X(t+t_j)} - \frac{\gamma(t)X(t+t_j)}{\alpha(t+t_j)+X(t+t_j)}\Big] \\
&-\Big[\frac{\gamma(t)X(t+t_j)}{\alpha(t+t_j)+X(t+t_j)} - \frac{\gamma(t)X(t+t_j)}{\alpha(t)+X(t+t_j)}\Big] \\
&+[\eta(t+t_j)-\eta(t)]X^m(t+t_j-\delta(t+t_j))e^{-\beta(t+t_j)X(t+t_j-\delta(t+t_j))} \\
&+\eta(t)\Big[X^m(t+t_j-\delta(t+t_j))e^{-\beta(t+t_j)X(t+t_j-\delta(t+t_j))} \\
&-X^m(t+t_j-\delta(t))e^{-\beta(t+t_j)X(t+t_j-\delta(t))}\Big] \\
&+\eta(t)\Big[X^m(t+t_j-\delta(t))e^{-\beta(t+t_j)X(t+t_j-\delta(t))} \\
&-X^m(t+t_j-\delta(t))e^{-\beta(t)X(t+t_j-\delta(t))}\Big], \qquad t \in \mathbb{R},
\end{aligned}$$

in which $\{t_j\}_{j=1}^{+\infty}$ is an arbitrary sequence of real numbers. Then

$$\begin{aligned}
X'(t+t_j) = &-\frac{\gamma(t)X(t+t_j)}{\alpha(t)+X(t+t_j)} + \eta(t)X^m(t+t_j-\delta(t))e^{-\beta(t)X(t+t_j-\delta(t))} \\
&+\varphi(j,t), \qquad \text{for all } t+t_j \geq t_0.
\end{aligned} \tag{3.1}$$

As stated by Lemma 2.1, the solution $X(t)$ is bounded and

$$n_1 < X(t) < N, \quad \text{for all } t \geq t^{\psi^X},$$

which means that the right side of (1.2) is also bounded, and $X'(t)$ is a bounded function on $[t_0 - \delta^+, +\infty)$. Therefore, it follows from the fact that $X(t) \equiv X(t_0 - \delta^+)$ for $t \in (-\infty, t_0 - \delta^+]$ that $X(t)$ is uniformly continuous on $\mathbb{R}$. Then, in accordance with the almost periodicity of $\gamma(t), \alpha(t), \eta(t), \beta(t)$ and $\delta(t)$, we can choose a sequence $\{t_j\}_{j=1}^{+\infty}$, satisfying $\lim_{j\to+\infty} t_j = +\infty$ and

$$\begin{aligned}
&|\gamma(t+t_j)-\gamma(t)| < \frac{1}{j}, \quad |\alpha(t+t_j)-\alpha(t)| < \frac{1}{j}, \quad |\eta(t+t_j)-\eta(t)| < \frac{1}{j}, \\
&|\beta(t+t_j)-\beta(t)| < \frac{1}{j}, \quad |\delta(t+t_j)-\delta(t)| < \frac{1}{j}, \quad |\varphi(j,t)| \leq \frac{1}{j}, \quad \text{for all } t \in \mathbb{R}.
\end{aligned} \tag{3.2}$$

On account of the fact that $\{X(t+t_j)\}_{j=1}^{+\infty}$ is equiuniformly continuous and uniformly bounded, in view of the Arzela-Ascoli lemma, we can select a subsequence $\{t_{j_n}\}_{n=1}^{+\infty}$ of $\{t_j\}_{j=1}^{+\infty}$, for which $X(t+t_{j_n})$ (to facilitate this, we still represent it as $X(t+t_j)$) uniformly converges to a continuous function $p^*(t)$ on any compact set of $\mathbb{R}$, as well as

$$n_1 \leq p^*(t) \leq N, \quad \text{for all } t \in \mathbb{R}.$$

We now turn to demonstrate that $p^*(t)$ is a solution of model (1.2). Actually, for any $t \geq t_0$ and $t_7 \in \mathbb{R}$, (3.1) and (3.2) yield

$$p^*(t+t_7) - p^*(t) = \lim_{j\to+\infty}[X(t+t_7+t_j) - X(t+t_j)]$$

$$
\begin{aligned}
&= \lim_{j\to+\infty}\int_t^{t+t_7} X'(s+t_j)ds\\
&= \lim_{j\to+\infty}\int_t^{t+t_7}\Big[-\frac{\gamma(s)X(s+t_j)}{\alpha(s)+X(s+t_j)}\\
&\quad + \eta(s)X^m(s+t_j-\delta(s))e^{-\beta(s)X(s+t_j-\delta(s))}+\varphi(j,s)\Big]ds\\
&= \lim_{j\to+\infty}\int_t^{t+t_7}\Big[-\frac{\gamma(s)X(s+t_j)}{\alpha(s)+X(s+t_j)}\\
&\quad + \eta(s)X^m(s+t_j-\delta(s))e^{-\beta(s)X(s+t_j-\delta(s))}\Big]ds+\lim_{j\to+\infty}\int_t^{t+t_7}\varphi(j,s)ds\\
&= \int_t^{t+t_7}\Big[-\frac{\gamma(s)p^*(s)}{\alpha(s)+p^*(s)}+\eta(s)p^{*m}(s-\delta(s))e^{-\beta(s)p^*(s-\delta(s))}\Big]ds,
\end{aligned}
$$

in which $t+t_7\geq t_0$, then we can know that

$$
\frac{\mathrm{d}}{\mathrm{d}t}p^*(t) = -\frac{\gamma(t)p^*(t)}{\alpha(t)+p^*(t)}+\eta(t)p^{*m}(t-\delta(t))e^{-\beta(t)p^*(t-\delta(t))}.
$$

Hence, $p^*(t)$ is a solution of model (1.2).

Next, we are set to validate that $p^*(t)$ is an almost periodic solution of model (1.2). In light of Lemma 2.2, for any $\varepsilon>0$, there exists real numbers $q=q(\varepsilon)>0$ and $\theta\in[\xi,\xi+q]$ such that it is possible to find $t^{**}>0$ satisfying

$$
|X(t+\theta)-X(t)|\leq\varepsilon,\quad \text{for all } t>t^{**}.
$$

Then, for any fixed $s\in\mathbb{R}$, we can find a large enough positive integer $G>t^{**}$, such that for any $j>G$, we have

$$
s+t_j>t^{**}\quad\text{and}\quad |X(s+t_j+\theta)-X(s+t_j)|\leq\varepsilon.
$$

Letting $j\to+\infty$, it results in

$$
|p^*(s+\theta)-p^*(s)|\leq\varepsilon,
$$

this indicates that $p^*(t)$ is an almost periodic solution of model (1.2).

In the end, we verify that $p^*(t)$ is globally exponentially stable. Denote $p(t)=p(t;t_0,\psi)$ as well as $B(t)=p(t)-p^*(t)$, in which $t\in[t_0-\delta^+,+\infty)$. Then

$$
\begin{aligned}
B'(t) = &-\frac{\gamma(t)\alpha(t)}{(\alpha(t)+p(t))(\alpha(t)+p^*(t))}B(t)+\frac{\eta(t)}{\beta^m(t)}\Big[\beta^m(t)p^m(t-\delta(t))e^{-\beta(t)p(t-\delta(t))}\\
&-\beta^m(t)p^{*m}(t-\delta(t))e^{-\beta(t)p^*(t-\delta(t))}\Big].
\end{aligned}\tag{3.3}
$$

It can be deduced from Lemma 2.1 that one can find a $t^*>t_0$, satisfying

$$
n_1\leq p(t),p^*(t)\leq N,\quad \text{for all } t\in[t^*-\delta^+,+\infty).\tag{3.4}
$$

Consider the following Lyapunov function defined by

$$
I(t)=|B(t)|e^{\lambda t}.
$$

According to (3.3), one has

$$
\begin{aligned}
D^{-}(I(t)) =&\lambda e^{\lambda t}|B(t)| + e^{\lambda t}\text{sign}(B(t))B'(t)\\
=&\lambda e^{\lambda t}|B(t)| + e^{\lambda t}\text{sign}(B(t))\Big\{-\frac{\gamma(t)\alpha(t)}{(\alpha(t)+p(t))(\alpha(t)+p^{*}(t))}B(t)\\
&+\frac{\eta(t)}{\beta^{m}(t)}\Big[\beta^{m}(t)p^{m}(t-\delta(t))e^{-\beta(t)p(t-\delta(t))}-\beta^{m}(t)p^{*m}(t-\delta(t))e^{-\beta(t)p^{*}(t-\delta(t))}\Big]\Big\}\\
\leq&\lambda e^{\lambda t}|B(t)| - e^{\lambda t}\frac{\gamma(t)\alpha(t)}{(\alpha(t)+p(t))(\alpha(t)+p^{*}(t))}|B(t)|\\
&+\frac{\eta(t)}{\beta^{m}(t)}\Big|\beta^{m}(t)p^{m}(t-\delta(t))e^{-\beta(t)p(t-\delta(t))}-\beta^{m}(t)p^{*m}(t-\delta(t))e^{-\beta(t)p^{*}(t-\delta(t))}\Big|e^{\lambda t}\\
=&\Big[-\Big(\frac{\gamma(t)\alpha(t)}{(\alpha(t)+p(t))(\alpha(t)+p^{*}(t))}-\lambda\Big)|B(t)|\\
&+\frac{\eta(t)}{\beta^{m}(t)}\Big|\beta^{m}(t)p^{m}(t-\delta(t))e^{-\beta(t)p(t-\delta(t))}\\
&-\beta^{m}(t)p^{*m}(t-\delta(t))e^{-\beta(t)p^{*}(t-\delta(t))}\Big|\Big]e^{\lambda t}, \qquad \text{for all } t > t^{*}.
\end{aligned} \tag{3.5}
$$

We assert that

$$
\begin{aligned}
I(t) =&|B(t)|e^{\lambda t}\\
<&e^{\lambda t^{*}}\Big(\max_{t\in[t_0-\delta^{+},t^{*}]}|p(t)-p^{*}(t)|+1\Big)\\
:=&K, \quad \text{for all } t > t^{*}.
\end{aligned} \tag{3.6}
$$

For the purpose of achieving contradiction, there exists $t_8 > t^{*}$ satisfying

$$
I(t_8) = K \quad \text{and} \quad I(t) < K, \quad \text{for all } t \in [t_0-\delta^{+}, t_8). \tag{3.7}
$$

On the basis of (2.3), (2.24), (2.28), (3.4), (3.5), and (3.7), it is clear that

$$
\begin{aligned}
0 \leq&D^{-}(I(t_8))\\
\leq&\Big[-\Big(\frac{\gamma(t_8)\alpha(t_8)}{(\alpha(t_8)+p(t_8))(\alpha(t_8)+p^{*}(t_8))}-\lambda\Big)|B(t_8)|\\
&+\frac{\eta(t_8)}{\beta^{m}(t_8)}\Big|\beta^{m}(t_8)p^{m}(t_8-\delta(t_8))e^{-\beta(t_8)p(t_8-\delta(t_8))}\\
&-\beta^{m}(t_8)p^{*m}(t_8-\delta(t_8))e^{-\beta(t_8)p^{*}(t_8-\delta(t_8))}\Big|\Big]e^{\lambda t_8}\\
\leq&-\Big(\frac{\gamma(t_8)\alpha(t_8)}{(\alpha(t_8)+N)^{2}}-\lambda\Big)|B(t_8)|e^{\lambda t_8}\\
&+\frac{\eta(t_8)}{\beta^{m}(t_8)}|F(m+\sqrt{m})|\Big|\beta(t_8)p(t_8-\delta(t_8))-\beta(t_8)p^{*}(t_8-\delta(t_8))\Big|e^{\lambda t_8}\\
=&-\Big(\frac{\gamma(t_8)\alpha(t_8)}{(\alpha(t_8)+N)^{2}}-\lambda\Big)|B(t_8)|e^{\lambda t_8}\\
&+\eta(t_8)\beta^{1-m}(t_8)|F(m+\sqrt{m})||B(t_8-\delta(t_8))|e^{\lambda\delta(t_8)}e^{\lambda(t_8-\delta(t_8))}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big[-\Big(\frac{\gamma(t_8)\alpha(t_8)}{(\alpha(t_8)+N)^2}-\lambda\Big) + \eta(t_8)\beta^{1-m}(t_8)|F(m+\sqrt{m})|e^{\lambda\delta^+}\Big]K \\
&\leq \sup_{t\in\mathbb{R}}\Big\{ -\Big[\frac{\gamma(t)\alpha(t)}{(\alpha(t)+N)^2}-\lambda\Big] + \eta(t)\beta^{1-m}(t)|F(m+\sqrt{m})|e^{\lambda\delta^+}\Big\}K \\
&<0,
\end{aligned}
$$

this inconsistency makes (3.6) hold. It follows that

$$|B(t)| < Ke^{-\lambda t}, \quad \text{for all } t > t^*.$$

The proof of Theorem 3.1 is wrapped up. □

4. A numerical example

In this section, we present a numerical example to verify the theoretical results.

Example 4.1. Consider the following non-autonomous delayed differential neoclassical growth model with nonlinear depreciation rate:

$$p'(t) = -\frac{(8.3+0.1\sin\sqrt{2}t)p(t)}{(2.9+0.1\cos\sqrt{3}t)+p(t)} + (3+0.1\sin\sqrt{2}t)p^{\frac{1}{2}}(t-2e^{\sin^4 t})e^{-(1+0.1|\cos 4t|)p(t-2e^{\sin^4 t})}, \tag{4.1}$$

where $m = \frac{1}{2}$, $\gamma(t) = 8.3 + 0.1\sin\sqrt{2}t$, $\alpha(t) = 2.9 + 0.1\cos\sqrt{3}t$, $\eta(t) = 3 + 0.1\sin\sqrt{2}t$, $\beta(t) = 1 + 0.1|\cos 4t|$, and $\delta(t) = 2e^{\sin^4 t}$. Apparently, we have $n_1 = 0.341671$ and $n_2 = 0.701072$. Denote $N = 0.63$, a simple calculation reveals that

$$
\begin{aligned}
&1 = \beta^- < \beta^+ < \frac{n_2}{N} = 1.112717, \\
&\sup_{t\in\mathbb{R}}\Big\{ -\frac{\gamma(t)N}{\alpha(t)+N} + \frac{\eta(t)}{\beta^m(t)}m^m e^{-m}\Big\} = -0.093606 < 0, \\
&\inf_{t\in\mathbb{R}, s\in[0,n_1]}\Big\{ -\frac{\gamma(t)}{\alpha(t)+s} + \frac{\eta(t)}{\beta^m(t)}s^{m-1}e^{-s}\Big\} = 0.361335 > 0, \\
&\sup_{t\in\mathbb{R}}\Big\{ -\frac{\gamma(t)\alpha(t)}{(\alpha(t)+N)^2} + \eta(t)\beta^{1-m}(t)|F(m+\sqrt{m})|\Big\} = -1.116651 < 0,
\end{aligned}
$$

one can see that all conditions in Theorem 3.1 are met. Therefore, model (4.1) admits at least one positive almost periodic solution, which is globally exponentially stable(see Figure 1).

Remark 4.1. *Research on differential neoclassical growth models has predominantly focused on those with linear depreciation rates [3–5, 12–15], where the existence of almost periodic solutions is typically established using exponential dichotomy theory. In contrast, models incorporating nonlinear depreciation rates have received considerably less attention. In this paper, we aim to address this gap by investigating a differential neoclassical growth model with a nonlinear depreciation rate. To overcome the analytical challenges posed by the model's nonlinearity, we introduce novel methodological approaches. Our theoretical findings, therefore, represent a meaningful extension and supplement to the literature. Furthermore, throughout this paper, we only focus on the case where* $0 < m < 1$. *It is known that model* (1.2) *is the Lasota-Wazewska model and Nicholson's Blowflies model when* $m = 0$ *and* $m = 1$*, respectively. Numerous scholars have carried out extensive studies on the two aforementioned types of models [8, 20–24]. However, in contrast with the case of* $0 < m < 1$*, the dynamic behaviors of model* (1.2) *in the case of* $m > 1$ *are more challenging to handle. We will consider it as our future work.*

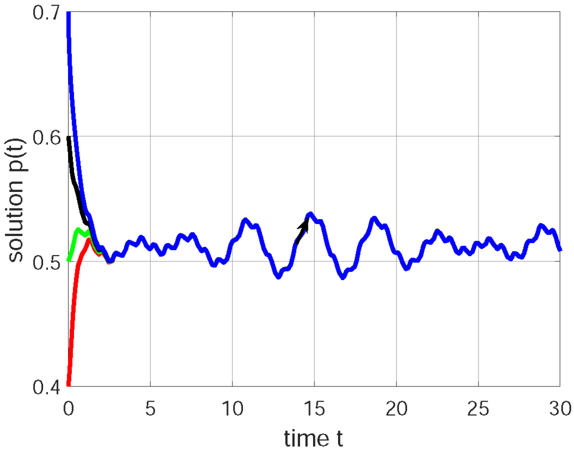

Figure 1. Numerical simulation of the model (4.1) with different initial values.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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