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http://www.aimspress.com/journal/era

ERA, 33(10): 5937–5964.
DOI: 10.3934/era.2025264
Received: 10 April 2025
Revised: 06 August 2025
Accepted: 06 August 2025
Published: 11 October 2025

Research article

Controllability of second-order evolution systems in Banach spaces

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Abstract: This paper examines the controllability of the second-order evolution problems in Banach spaces in the presence of a nonlinear term depending on both the solution and its first derivative, and of a time-dependent linear term. The first goal of the paper is to introduce a definition of controllability for such problems that considers not only the solution but also its derivative at the final point using the same control function. Subsequently, the paper addresses the second goal: Finding conditions that guarantee the solvability of the controllability problem. The main statements of the paper are proved by the combination of Schauder fixed point theorem, approximate solvability, and weak topology. Using the method of approximate solvability allows us to avoid any compactness restrictions, and the results are therefore proved under non restrictive conditions imposed on the fundamental operator as well as on the right-hand side.

Keywords: Banach space; second-order system; controllability; mild solution; approximation solvability method; weak topology

1. Introduction

Consider the semilinear second-order control problem

$$\left.\begin{array}{rl} \ddot{x}(t) = A(t)x(t) + f(t, x(t), \dot{x}(t)) + Bu(t), & \text{for a.a. } t \in [0, T], \\ x(0) = x_0, \dot{x}(0) = \overline{x}_0, & \end{array}\right\} \tag{1.1}$$

and let

(*i*) X be a Banach space that is reflexive and has a Schauder basis $\{e_k\}_k$;

(*ii*) $\{A(t)\}_t$, with $A(t) : D(A) \subset X \to X$, for every $t \in [0, T]$, be a family of closed, linear, and densely defined operators that generates an evolutionary fundamental system $\{S(t, s)\}_{t,s}$;

(iii) $f\colon [0,T] \times X \times X \to X$;
(iv) $x_0, \overline{x}_0 \in X$;
(v) B be a linear and bounded operator that maps a uniformly convex Banach space U into X;
(vi) the control function $u \in L^p([0,T], U)$, with $p \in (1, +\infty)$.

Nonlinear partial differential equations, and specifically second-order partial differential equations evolving in time, play a crucial role in describing a range of problems in physics, biology, and many other fields (see, e.g., [1–3] and the references cited therein). Being motivated by different applications, many authors have been interested in investigating the existence of solutions to partial differential equations. Recently, a new approach to studying partial differential equations has appeared in the literature. It involves transforming the partial differential equation that describes the model into the corresponding ordinary differential equation in a convenient infinite-dimensional space. A significant amount of theoretical analysis on evolution equations based on this technique has emerged recently. We refer, e.g., to [4], where the existence of solutions for a class of nonlinear partial differential equations of the parabolic type with nonlinearities with superlinear growth is considered; [5], which studied non-autonomous semilinear neutral second-order differential inclusions; [6], which investigated state-dependent abstract retarded functional differential equations with infinite delay; and [7] which deal with nonlocal transport and diffusion equations in Banach spaces. We also recall [8], which provides a numerical analysis for a two-dimensional variable coefficient evolution equation.

The equation in (1.1) is the abstract representation of, e.g., a wave equation describing a vibrating string. In this scenario, the linear term $A(t)$ is a time-dependent perturbation of the spatial second derivative operator, representing properties that may vary with time, while the dependence of the nonlinear term f by the derivative function $\dot{x}$ describes nonlinear effects such as damping.

Whereas the research dealing with second-order problem in Banach spaces with a right-hand side (r.h.s.) independent of the first derivative is quite wide, only a few authors have considered the dependence on the first derivative. See, e.g., [9–11] for Dirichlet or Floquet problems (without $A(t)$, with $A(t)$ being bounded, or with A independent of t), [12–14] for the study of integro-differential second-order equations, or [15] for second-order impulsive Cauchy problems. Furthermore, in [16], the existence and properties of mild solutions for semilinear evolution equations and their relevance to second-order hyperbolic partial differential equations has been studied in the case when the r.h.s. depends on the first derivative.

The concept of controllability is a fundamental pillar of modern control theory because corresponding problems arise in a wide variety of practical situations in engineering, physics, economics, biology, and other fields (see, e.g., [17, 18], and the references therein). Mathematically, controllability is related to feedback controls, optimization, and stability analysis, which are prerequisites for the implementation of effective control strategies in real-world applications.

The notion of controllability consists of finding a control function u that steers the state variable x from any fixed initial configuration to any fixed final configuration. While there is unanimous agreement on the definition of controllability for first-order semilinear systems in Banach spaces, the appropriate theory for second-order semilinear differential problems in Banach spaces is still being developed. Although the transfer of second-order evolution systems to the associated first-order systems is one of the possibilities for solving the problem, it is not always the most effective, since it is sometimes possible to obtain better existence and localization results when dealing directly with the second-order problem. This is the case when using topological methods, like fixed point theorems or degree theory. For example, the lower and upper solution method for second-order scalar equations is based on the natural ordering structure, which loses meaning for a first-order vectorial system; the bounding function technique (in Mawhin's coincidence degree framework) for second-order equations deeply exploits the Hartman–Nagumo a priori bound on the first derivative, while it is harder to achieve the transversality condition in the first-order case. As a consequence, many authors have decided not to transform the second-order problem into the first-order one, and we will also achieve results in line with this strategy.

Most authors have assumed a definition of controllability for second-order equations that only ensures steering the state function to the target value. As pointed out in [19], unfortunately, in these papers the authors did not consider the damping term in the definition of the exact controllability of the corresponding systems. This violates the controllability definition because $\dot{x}$ is also a state variable for second-order problems. Only a few authors have tried to extend the definition of exact controllability in the direction of prescribing the existence of a control function such that not only x but also $\dot{x}$ reaches the prescribed target value. However, in these papers, the controllability is obtained by requiring the surjectivity of a certain operator taking values in the Banach space X. This condition, as pointed out in [20], implies the linear part A to be bounded on the whole Banach space. In the recent paper [20], a definition of controllability in second-order semilinear problems in Banach spaces has been introduced, and sufficient conditions for such controllability of the Cauchy problem have been discussed. The controllability definition in [20] allows steering, using a unique control, not only the solution but also its derivative in the final point to the prescribed values. The results in [20] have been obtained under a weaker surjectivity assumption, required only on a linear subspace of X, that has not yielded to the boundedness of A. The results in [20] have been proven without the reduction of the second-order problem to a first-order problem, for nonlinear terms depending on the first derivative and the linear operators A in (1.1) not depending on t. The paper provides an application of the theoretical result to the Klein–Gordon equation, where the weaker surjectivity is shown to hold.

As far as we know, the literature concerning controllability when the linear term is time-dependent is quite rare (see [21–24]). Although impulsive equations with delay and nonlocal conditions are considered therein, the definition of controllability does not take the derivative into account in the target point, and the strong surjectivity condition is assumed in certain cases, together with the compactness of the fundamental system generated by $\{A(t)\}_t$, which, according to the well-known contradiction identified by Triggiani (see [25,26]), yields the fact that the Banach space X must be finite-dimensional. In the applications, the problem whether the surjectivity condition is verified or not is not addressed and/or the surjectivity is not proved.

Therefore, motivated by previous results, the aim of this paper is to investigate the controllability of the Cauchy problem in a Banach space with the r.h.s. also being dependent on the first derivative, and with the operator A being dependent on t, namely studying the problem (1.1) above. More concretely, the generalization of the definition introduced in [20] to second-order problems with linear terms depending on time will be introduced, and relevant sufficient conditions for such controllability will be subsequently obtained.

The definition of the solution will be considered in a mild sense in this paper. For this purpose, let us define the linear, bounded operator

$$C(t, s) = -\frac{\partial}{\partial s} S(t, s),$$

for each $t, s \in [0, T]$. Furthermore, let $x_0 \in Y$, where

$$Y = \left\{ x \in X \mid \exists \frac{\partial}{\partial t} C(t, s)x \, \forall t, s \in [0, T] \text{ and } (t, s) \to \frac{\partial}{\partial t} C(t, s)x \text{ is continuous in } [0, T] \times [0, T] \right\}.$$

By a *mild solution* of (1.1), we mean a C^1-function $x\colon [0, T] \to X$ satisfying, for all $t \in [0, T]$,

$$x(t) = C(t, 0)\, x_0 + S(t, 0)\overline{x}_0 + \int_0^t S(t, s) f(s, x(s), \dot{x}(s))\, ds + \int_0^t S(t, s) Bu(s)\, ds, \tag{1.2}$$

where $u \in L^p([0, T], U)$.
Notice that the function specified in Eq (1.2) is continuously differentiable (according to [27, Remark 2.3] and since $x_0 \in Y$). Furthermore, for all $t \in [0, T]$, it holds that

$$\dot{x}(t) = \frac{\partial}{\partial t} C(t, 0)x_0 + \frac{\partial}{\partial t} S(t, 0)\overline{x}_0 + \int_0^t \frac{\partial}{\partial t} S(t, s) f(s, x(s), \dot{x}(s))\, ds + \int_0^t \frac{\partial}{\partial t} S(t, s) Bu(s)\, ds.$$

Definition 1.1. We say that the problem (1.1) is *controllable on* $[0, T]$ if, for every $x_0, x_1 \in Y$, $\overline{x}_0, \overline{x}_1 \in X$, there is a control $u \in L^p([0, T], U)$ and a mild solution x of (1.1) satisfying

$$x(T) = x_1 \quad \text{and} \quad \dot{x}(T) = \overline{x}_1. \tag{1.3}$$

Remark 1.1. *In [28, 29], the author defines the notion of controllability for the semilinear wave equation*

$$\begin{cases} y_{tt} - y_{xx} + f(y) = h\chi_{(l_1, l_2)}, & t \in (0, T), x \in (0, 1) \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x),\ y_t(0, x) = \overline{y_0}(x), & x \in (0, 1). \end{cases} \tag{1.4}$$

with $0 \leq l_1 < l_2 \leq 1$, *requiring that, for every* $y_0, y_1 \in H_0^1(0, 1)$, $\overline{y}_0, \overline{y}_1 \in L^2(0, 1)$, *a control* $h \in L^2((l_1, l_2) \times (0, T))$ *and a mild solution* y *of (1.4) exist such that*

$$y(T, x) = y_1(x), \quad y_t(T, x) = \overline{y}_1(x), \quad x \in (0, 1)$$

(see also the survey [30] for a variety of results on the controllability of hyperbolic equations).
Our definition of controllability extends the previous one to a general non–autonomous semilinear second-order problem in a Banach space with the nonlinearity also depending on the first derivative (see also Remark 3.1).

Definition 1.2. An ordered pair (x, u) that consists of a mild solution x of (1.1) fulfilling (1.3) and of the matching control $u \in L^p([0, T], U)$ is called a *solution to the controllability problem* (1.1).

The proofs of our main statements will be established by combining Schauder fixed point theorem with the method of approximation solvability and weak topology, thereby avoiding any compactness requirements and achieving solution localization in a suitable bounded set.

The paper has the following structure. In Section 2, we illustrate the basic facts about the Schauder basis and the evolutionary fundamental systems, as well as key theorems on fundamental systems which are applied in the proofs of our main results, the latter being presented in Section 3.

2. Materials and methods

Let X be an infinite dimensional real Banach space endowed with the norm $\|\cdot\|$ and let X^* be its dual. By X^ω, the space X equipped with the weak topology will be indicated throughout the paper, and the symbols $\|\cdot\|_C$ and $\|\cdot\|_{C^1}$ will mean the $C([0,T],X)$-norm and the $C^1([0,T],X)$-norm, respectively. They will be defined in a standard manner as

$$\|x\|_C = \max_{t\in[0,T]} \|x(t)\|, \text{ for all } x \in C([0,T],X),$$

$$\|x\|_{C^1} = \max\{\|x\|_C, \|\dot{x}\|_C\}, \text{ for all } x \in C^1([0,T],X).$$

The notation $\mathcal{L}(X)$ will mean the Banach space of bounded and linear operators from X into itself.

In the proofs of main results, the approximation solvability method will be used as one of the key tools; we therefore recall basic facts about natural projections and the Schauder basis.

Definition 2.1. A *Schauder basis* for X is a sequence $\{e_k\}_k$ of vectors belonging to X such that, for every $x \in X$, there is a unique sequence of real numbers $\alpha_k = \alpha_k(x)$, $k \in \mathbb{N}$, with

$$\left\| x - \sum_{m=1}^{k} \alpha_m e_m \right\| \to 0, \quad \text{as } k \to \infty.$$

It is possible to prove that $\alpha_k \in X^*$, for every $k \in \mathbb{N}$, (see, e.g., [31, pp 18–20]).
Let $\{e_k\}_k$ be a Schauder basis for X and let $X_k = \text{span}\{e_1, \ldots, e_k\}$ be the k-dimensional Banach space generated by the first k vectors of this basis. The *natural projection* $P_k : X \to X_k$ of X onto X_k is then defined by

$$P_k\Big(\sum_{m=1}^{\infty} \alpha_m e_m\Big) = \sum_{m=1}^{k} \alpha_m e_m.$$

Every sequence $\{\|P_k\|\}_k$ defined in this way is bounded, i.e., it holds that $\|P_k(x)\| \le K^*\|x\|$, for all $k \in \mathbb{N}$ and $x \in X$, for some $K^* \ge 1$ (see [32, Proposition 1.a.2]). The Schauder basis $\{e_k\}_k$ is called *monotone* if $K^* = 1$, i.e., if $\|P_k\| = 1$, for all $k \in \mathbb{N}$.

Remark 2.1. If $\Omega \subset \mathbb{R}^k$ is bounded and $p \in (1;\infty)$, then the space $L^p(\Omega,\mathbb{R})$ has a Schauder basis that is monotone (see, e.g., [33, Chapter 1.3 and 1.4]). Namely, when $p = 2$, every orthonormal system creates a monotone Schauder basis.

The main properties of the natural projections P_k that will be subsequently applied in the paper are summarized in the following lemma (see, e.g., [34, Lemma 6], [35, Lemma 2.2], and [36, Proposition 7]).

Lemma 2.1. *The natural projection $P_k : X \to X_k$ fulfills the following properties:*

(a) $P_k : X^\omega \to X_k$ is continuous;
(b) if $x_k \to x$, then $P_k(x_k) \to x$;
(c) if $x_k \rightharpoonup x$, then $P_k x_k \rightharpoonup x$;
(d) if $f_k \rightharpoonup f$ in $L^1([0,T],X)$, then $P_k f_k \rightharpoonup f$ in $L^1([0,T],X)$;
(e) for every $x \in X$, $\|P_k(x) - x\| \to 0$;

In this paper, the controllability problem (1.1) is investigated under the assumption that the family $\{A(t)\}_t$ generates a fundamental system. This is a concept which was introduced in [37, 38]. We will also consider the domain $D(A)$ of $A(t)$ to be a closed and dense subset of X and independent of t. For this purpose, main properties of fundamental systems will be summarized now.

A set of bounded linear operators $S(t,s) : X \to X$, $t, s \in [0,T]$, is called a *fundamental system* if

(i) $S(t,t) = 0$, for each $t \in [0,T]$;

(ii) the mapping $(t,s) \to S(t,s)x$ is of class C^1, for each $x \in X$;

(iii)
$$\frac{\partial}{\partial t}S(t,s)\Big|_{t=s} x = x, \quad \frac{\partial}{\partial s}S(t,s)\Big|_{t=s} x = -x,$$
for all $t, s \in [0,T]$ and each $x \in X$‘

(iv) it holds, for all $t, s \in [0,T]$, that if $x \in D(A)$, then $S(t,s)x \in D(A)$, the map $(t,s) \to S(t,s)x$ is of class C^2 and
$$\frac{\partial^2}{\partial t^2}S(t,s)x = A(t)S(t,s)x,$$
$$\frac{\partial^2}{\partial s \partial t}S(t,s)\Big|_{t=s} x = 0;$$
$$\frac{\partial^2}{\partial s^2}S(t,s)x = S(t,s)A(s)x,$$

(v) it holds, for all $t, s \in [0,T]$, that if $x \in D(A)$, then $\frac{\partial}{\partial s}S(t,s)x \in D(A)$, the map $(t,s) \to A(t)\frac{\partial}{\partial s}S(t,s)x$ is continuous, and
$$\frac{\partial^3}{\partial t^2 \partial s}S(t,s)x = A(t)\frac{\partial}{\partial s}S(t,s)x$$
$$\frac{\partial^3}{\partial s^2 \partial t}S(t,s)x = \frac{\partial}{\partial t}S(t,s)A(s)x.$$

The map $S : [0,T] \times [0,T] \to \mathcal{L}(X)$ is called a *fundamental operator* if $\{S(t,s)\}_{t,s}$ is a fundamental system.

We can obtain, by applying the Banach–Steinhaus theorem, that a fundamental system fulfills other useful properties referred to in the next lemma.

Lemma 2.2. *(see [27, Lemma 2.2]) If $\{S(t,s)\}_{t,s}$ is a fundamental system, then there is a constant $K_T > 0$ satisfying, for all $t, s \in [0,T]$,*

(i) $\|C(t,s)\| \le K_T$;

(ii) $\|S(t,s)\| \le K_T$;

(iii) $\|\frac{\partial}{\partial t}S(t,s)\| \le K_T$.

For studying the controllability, we require the fundamental system $\{S(t,s)\}_{t,s}$ generated by $\{A(t)\}_t$ to be evolutionary, i.e., that

$$C(t,r)S(r,s)x + S(t,r)\frac{\partial}{\partial r}S(r,s)x = S(t,s)x, \tag{2.1}$$

for every $t, r, s \in [0, T]$, and $x \in X$ (see [39, Definition 3.1]).
In the following, we will also need the linear subspace Y_T defined by

$$Y_T = \{x \in X \,|\, C(\cdot, T)x \text{ is continuously differentiable}\}. \tag{2.2}$$

We will make use of the following lemma.

Lemma 2.3. *If $\{S(t, s)\}_{t,s}$ is an evolutionary system, then*

$$C(t, r)C(r, s)x + S(t, r)\frac{\partial}{\partial r}C(r, s)x = C(t, s)x, \tag{2.3}$$

for every $t, r, s \in [0, T]$ and $x \in Y$.
Moreover, $C(T, 0)x, S(T, s)y \in Y_T$, for every $x \in Y$, $s \in [0, T]$, and $y \in X$.

Proof. Since $x \in Y$, according to the definition of $C(t, s)$, the map

$$(t, s) \to \frac{\partial^2}{\partial t \partial s}S(t, s)x$$

is continuous in $[0, T] \times [0, T]$, and hence the Schwarz theorem implies that

$$\exists \frac{\partial^2}{\partial s \partial t}S(t, s)x = \frac{\partial^2}{\partial t \partial s}S(t, s)x = -\frac{\partial}{\partial t}C(t, s)x,$$

for every $t, s \in [0, T]$.
Since $S(\cdot, \cdot)x$ is continuously differentiable in $[0, T] \times [0, T]$, for every $x \in X$, and $C(t, r)$ and $S(t, r)$ are linear and bounded, we find from (2.1), that for every $t, r, s \in [0, T]$, $x \in Y$,

$$\begin{aligned} C(t, s)x = & -\frac{\partial}{\partial s}S(t, s)x = -\frac{\partial}{\partial s}\Big[C(t, r)S(r, s)x + S(t, r)\frac{\partial}{\partial r}S(r, s)x\Big] \\ = & -C(t, r)\frac{\partial}{\partial s}S(r, s)x - S(t, r)\frac{\partial^2}{\partial s \partial r}S(r, s)x \\ = & \ C(t, r)C(r, s)x + S(t, r)\frac{\partial}{\partial r}C(r, s)x. \end{aligned}$$

Taking $r = T$ and $s = 0$ in (2.3), we subsequently find that

$$C(t, T)C(T, 0)x + S(t, T)\frac{\partial}{\partial r}C(r, 0)\Big|_{r=T} x = C(t, 0)x,$$

for every $t \in [0, T]$ and $x \in Y$. From the continuous differentiability of $S(\cdot, \cdot)x$ we then see that $C(T, 0)x \in Y_T$ because

$$\exists \frac{\partial}{\partial t}C(t, T)C(T, 0)x = \frac{\partial}{\partial t}C(t, 0)x - \frac{\partial}{\partial t}S(t, T)\frac{\partial}{\partial r}C(r, 0)\Big|_{r=T} x.$$

To prove the last part of lemma, we again use (2.1) and the continuous differentiability of $S(\cdot, \cdot)y$ in $[0, T] \times [0, T]$, for every $y \in X$, and obtain

$$\exists \frac{\partial}{\partial t}C(t, r)S(r, s)y = \frac{\partial}{\partial t}S(t, s)y - \frac{\partial}{\partial t}S(t, r)\frac{\partial}{\partial r}S(r, s)y,$$

for every $t, r, s \in [0, T], y \in X$. The required conclusion then follows, taking $r = T$ in the previous formula.

Remark 2.2. *Very simple and easily verifiable conditions guaranteeing that $\{A(t)\}_t$ generates a fundamental system were obtained when $A(t)$ is the perturbation of a strongly continuous cosine family generator A, i.e., when there is a one-parameter family $\{C(t)\}_t$ of linear, bounded operators that map X into itself, such that*

- $C(0) = I$;
- $C(t+s) + C(s-t) = 2C(s)C(t)$, *for every* $t,\ s \in [0,T]$;
- *the mapping* $t \to C(t)x$ *is continuous, for every fixed* $x \in X$;

and $A : D(A) \subset X \to X$ *is the linear closed operator given by the formula*

$$Ax = \frac{d^2}{dt^2}\Big[C(t)x\Big]_{t=0} = 2\lim_{t\to 0^+}\frac{C(t)x - x}{t^2}$$

with

$$D(A) = \left\{x \in X : \exists \lim_{t\to 0^+}\frac{C(t)x - x}{t^2}\right\}.$$

In this case, the mild solution to a Cauchy problem associated with the second-order equation

$$\ddot{x} = Ax$$

is continuously differentiable if and only if the initial data x_0 *belongs to the set*

$$\tilde{Y} = \{x \in X : C(\cdot)x \text{ is continuously differentiable}\}.$$

The first result of this kind was obtained in [40] for an additive perturbation $A(t) = A + B(t)$, *with* $B : [0,T] \to \mathcal{L}(\tilde{Y}, X)$ *being strongly continuously differentiable. The explicit formula for the fundamental system in this case is*

$$S(t,s)z = \tilde{S}(t-s)z + \int_s^t \tilde{S}(t-\xi)B(\xi)S(\xi,s)z\,d\xi, \quad s,t \in [0,\infty).$$

with

$$\tilde{S}(t)x = \int_0^t \tilde{C}(s)x\,ds.$$

The second related result was obtained in [41] for a multiplicative perturbation $A(t) = a(t)A$, *with* $a : [0,T] \to (0,+\infty)$ *being continuously differentiable. The explicit formula for the fundamental system this time is*

$$S(t,s)y := \frac{1}{\sqrt{a(s)}}\left[\tilde{S}\Big(\int_s^t \sqrt{a(r)}\,dr\Big)y - \frac{1}{2}\int_s^t \frac{a'(r)}{a(r)}\tilde{S}\Big(\int_r^t \sqrt{a(\tau)}\,d\tau\Big)\Big(\frac{\partial S(r,s)}{\partial r}y\Big)dr\right].$$

In both of these cases, it is possible to prove, reasoning as in [27] and in [41], respectively, that

$$Y = Y_T = \tilde{Y}.$$

In what follows, we also need the following result.

Proposition 2.1. *(see [36, Proposition 1]) Let E be a uniformly convex Banach space, let F be a normed vector space, and let $V : E \to F$ be a linear, bounded, and surjective operator. Then*

(i) the mapping $\mathcal{V} : E/\ker V \to F$ defined, for every $u \in E$, by $\mathcal{V}([u]) = V(u)$ is bounded, linear, one to one, and onto;
(ii) a continuous mapping $\Pi : E/\ker V \to E$ exists that

$$(\Pi([u])) = V(u) \quad \textit{and} \quad \|\Pi([u])\| = \min\{\|v\| : V(u) = V(v)\};$$

(iii) the mapping $\tilde{V}^{-1} = \Pi \circ \mathcal{V}^{-1}$ is a continuous right inverse of V and

$$\|\tilde{V}^{-1}(w)\| = \min\left\{\|u\| : u \in V^{-1}(w)\right\}.$$

Furthermore, if E is a Hilbert space, then

(iv) $\tilde{V}^{-1}$ is linear.

Proposition 2.1 was proven in [36] for F being a Banach space, but it is also true in this more general case with F being a normed vector space.

In what follows, we will also make use of the compactness result stated in Theorem 2.1 below, which was proved in [42, Corollary 5.1.1] in the case of a C_0 semigroup instead of the fundamental system. Before its formulating, let us recall the notion of semicompactness.

Definition 2.2. *We say that a sequence $\{f_n\}_n \subset L^1([0,T],X)$ is* semicompact *if it is integrably bounded, i.e., $\varphi \in L^1([0,T],X)$ exists that*

$$\|f_n(t)\| \le \varphi(t) \quad \textit{for all } n \in \mathbb{N}, \textit{ and a.a. } t \in [0,T],$$

and the set $\{f_n(t)\}_n$ is relatively compact for a.a. $t \in [0,T]$.

Theorem 2.1. *Let $\{S(t,s)\}_{t,s\in[0,T]}$ be a fundamental system. Then the linear operator $F : L^1([0,T],X) \to C^1([0,T],X)$ given by the formula*

$$F(f)(t) = \int_0^t S(t,s)f(s)\,ds, \quad f \in L^1([0,T],X), t \in [0,T]$$

is well defined. Moreover, for every semicompact sequence $\{f_n\}_n \subset L^1([0,T],X)$, the sequence $\{F(f_n)\}_n$ is relatively compact in $C^1([0,T],X)$. Furthermore, if $f_n \rightharpoonup f_0$ in $L^1([0,T],X)$, then $F(f_n) \to F(f_0)$ in $C^1([0,T],X)$.

Proof. Since $\{S(t,s)\}_{t,s\in[0,T]}$ is a fundamental system, it holds, for all $f \in L^1([0,T],X)$ and $s \in [0,T]$, that the mappings $t \to S(t,s)f(s)$ and $t \to \frac{\partial}{\partial t}S(t,s)f(s)$ are continuous. Moreover, according to Lemma 2.2,

$$\|S(t,s)f(s)\| \le K_T\|f(s)\| \tag{2.4}$$

and

$$\left\|\frac{\partial}{\partial t}S(t,s)f(s)\right\| \le K_T\|f(s)\|, \tag{2.5}$$

for all $t, s \in [0, T]$. Recalling that $S(t,t)x = 0$, for every $x \in X$, we then see that $F(f) \in C^1([0\,,T]\,,X)$, for every $f \in L^1([0\,,T]\,,X)$ with

$$F(f)'(t) = \int_0^t \frac{\partial}{\partial t} S(t,s) f(s)\, ds.$$

Now take a semicompact sequence $\{f_n\}_n$. According to [42, Corollary 4.2.3], for every $\delta > 0$, a sequence $\{g_n\}_n \subset L^1([0\,,T]\,,X)$ and a compact set G exist such that

$$g_n(t) \in G \tag{2.6}$$

and

$$\|f_n(t) - g_n(t)\| \le \frac{\delta}{2K_T T}, \tag{2.7}$$

for every $t \in [0, T]$ and $n \in \mathbb{N}$.
Now consider the set

$$Q_1 = \{(S(t,s)x : t, s \in [0, T], x \in G\}.$$

Notice that the function $h_1 : [0, T] \times [0, T] \times G \to X$ defined as

$$h_1(t, s, x) = S(t, s)x$$

is continuous on the compact set $[0, T] \times [0, T] \times G$. In fact, given a sequence $\{(t_n, s_n, x_n)\}_n$ converging to $(t_0, s_0, x_0) \in [0, T] \times [0, T] \times G$, we find according to Lemma 2.2 and the continuity of the map $(t, s) \to S(t, s)x_0$, that

$$\begin{aligned} \|S(t_n, s_n)x_n - S(t_0, s_0)x_0\| \le & \ \|S(t_n, s_n)x_n - S(t_n, s_n)x_0\| + \|S(t_n, s_n)x_0 - S(t_0, s_0)x_0\| \\ \le & \ K_T\|x_n - x_0\| + \|S(t_n, s_n)x_0 - S(t_0, s_0)x_0\| \to 0. \end{aligned}$$

Hence $Q_1 = \operatorname{Im} h_1$ is compact and h_1 is uniformly continuous. If we take

$$q = \max_{x \in G} \|x\|,$$

(2.6) implies that

$$\|S(t, s)g_n(s)\| \le K_T q,$$

for every $(t, s) \in [0, T] \times [0, T]$ and $n \in \mathbb{N}$. Therefore, $\{F(g_n)(t)\}_n$ is relatively compact, for every $t \in [0, T]$, because $\{S(t, s)g_n(s)\}_n \subset Q_1$, and

$$\|F(g_n)(t)\| \le T K_T q,$$

for every $n \in \mathbb{N}$ and $t \in [0, T]$, i.e., the sequence $\{F(g_n)\}_n$ is bounded in $C([0\,,T]\,,X)$.
Let us now prove that $\{F(g_n)\}_n$ is also equicountinuous. For this purpose, given $\epsilon > 0$, the uniform continuity of h_1 yields the existence of $\rho > 0$ such that

$$\|S(t_2, s)x - S(t_1, s)x\| \le \frac{\epsilon}{2T}$$

for every $x \in G, t_1, t_2, s \in [0, T]$ with $|t_1 - t_2| \leq \rho$. Without loss of generality, we can consider that

$$\rho \leq \frac{\epsilon}{2K_T q}.$$

It then follows that, for every $t_1, t_2 \in [0, T]$ with $|t_1 - t_2| \leq \rho$, and every $n \in \mathbb{N}$,

$$\begin{aligned} \|F(g_n)(t_2) - F(g_n)(t_1)\| &\leq \int_0^{t_1} \|[S(t_2, s)g_n(s) - S(t_1, s)g_n(s)]\| \, ds + \left| \int_{t_1}^{t_2} \|S(t_2, s)g_n(s)\| \, ds \right| \\ &\leq \frac{\epsilon}{2T} t_1 + K_T q|t_2 - t_1| \leq \epsilon. \end{aligned}$$

Similarly, defining

$$Q_2 = \left\{ \frac{\partial}{\partial t} S(t, s)x : t, s \in [0, T], x \in G \right\}$$

and $h_2 : [0, T] \times [0, T] \times G \to X$ as

$$h_2(t, s, x) = \frac{\partial}{\partial t} S(t, s)x,$$

it is possible to show that $\{F(g_n)'\}_n$ is bounded in $C([0\,, T]\,, X)$ and equicontinuous and that $\{F(g_n)'(t)\}_n$ is relatively compact, for every $t \in [0, T]$. The Ascoli–Arzelá Theorem then implies that $\{F(g_n)\}_n$ is relatively compact in $C^1([0\,, T]\,, X)$. Hence, for every $\delta > 0$, a finite set $\{p_1, \cdots p_m\} \in C^1(0, T], X)$ exists such that, for every $n \in \mathbb{N}$

$$\|F(g_n) - p_i\|_C \leq \frac{\delta}{2} \quad \text{ and } \|F(g_n)' - p_i'\|_C \leq \frac{\delta}{2},$$

for some $i = 1, \ldots, m$. Consequently, according to (2.7) and Lemma 2.2

$$\begin{aligned} \|F(f_n) - p_i\|_C &\leq \|F(f_n) - F(g_n)\|_C + \|F(g_n) - p_i\|_C \\ &\leq \max_{t \in [0,T]} \int_0^T \|S(t, s)f_n(s) - S(t, s)g_n(s)\| ds + \frac{\delta}{2} \\ &\leq K_T T \frac{\delta}{2K_T T} + \frac{\delta}{2} \leq \delta. \end{aligned}$$

Similarly, we can obtain

$$\|F(f_n) - p_i\|_C \leq \delta.$$

Therefore, $\{F(f_n)\}_n$ is relatively compact in $C^1([0\,, T]\,, X)$.

Let us now suppose that $f_n \rightharpoonup f_0$. According to (2.4) and (2.5), F is a bounded and linear operator, and therefore

$$F(f_n) \rightharpoonup F(f_0)$$

in $C^1([0\,, T]\,, X)$. Since $\{F(f_n)\}_n$ is relatively compact in $C^1([0\,, T]\,, X)$, by the uniqueness of the weak limit, we see that every subsequence of $\{F(f_n)\}_n$ admits a subsequence which converges to $F(f_0)$ in $C^1([0\,, T]\,, X)$. Therefore, the whole sequence must converges to $F(f_0)$ in $C^1([0\,, T]\,, X)$.

3. Results

In this part of the paper, we discuss the existence of a mild solution to the controllability problem (1.1). First, by using the concept of the Schauder basis and the natural projections of X, the sequence of fixed points of the operators defined by formula (3.1) below will be found by Schauder fixed point theorem. Subsequently, we apply a limiting procedure and the concept of weak topology for proving that the sequence of fixed points admits a subsequence weakly pointwise converging to a solution to the controllability problem (1.1). Thanks to the approach used, we avoid any compactness requirements in the assumptions, and the results will be proven without any restrictive assumptions on the r.h.s or the fundamental system.

Theorem 3.1. *Let us consider the control problem* (1.1)*, where* $x_0 \in Y$*,* $x_1 \in Y_T$*, with* Y_T *as defined in* (2.2)*,* $\overline{x}_0, \overline{x}_1 \in X$*, and let* $f : [0,T] \times X \times X \to X$ *fulfill the following conditions:*

($f1$) *For every* $(x,y) \in X \times X$*,* $f(\cdot, x, y) : [0,T] \to X$ *is measurable with respect to (w.r.t.) the Lebesgue measure on* $[0,T]$ *and the Borel measure on X;*

($f2$) *For a.a.* $t \in [0,T]$*,* $f(t,\cdot,\cdot) : X^w \times X^w \to X^w$ *is sequentially continuous, i.e., it holds that*

$$\text{if } x_k \rightharpoonup x \text{ and } y_k \rightharpoonup y, \text{ then } f(t, x_k, y_k) \rightharpoonup f(t,x,y), \text{ for a.a. } t \in [0,T];$$

($f3$) *For every* $n \in \mathbb{N}$*, there is a function* $\varphi_n \in L^1([0,T],\mathbb{R})$*, with*

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_{L^1}}{n} = 0,$$

such that

$$\|f(t,x,y)\| \le \varphi_n(t),$$

for a.a. $t \in [0,T]$ *and every* $(x,y) \in nB \times nB$*, where* $B = \{x \in X : \|x\| \le 1\}$*.*

Moreover, let the following additional assumptions hold:

(SB) $\{e_k\}_k \subset Y_T$;

(WZ) *The linear and bounded operator* $N : L^p([0,T],U) \to Y_T \times X$ *given by formula*

$$Nu = \left(\int_0^T S(T,s)Bu(s)\,ds, \int_0^T \frac{\partial}{\partial t} S(t,s)\Big|_{t=T} Bu(s)\,ds \right),$$

is surjective.

Then the problem (1.1) *is controllable on the interval* $[0,T]$*.*

Proof. Let us consider arbitrary $x_0 \in Y$, $x_1 \in Y_T, \overline{x}_0, \overline{x}_1 \in X$, and let us show that a control $u \in L^p([0,T],U)$ and a mild solution q of (1.1) exist such that $q(T) = x_1$ and $\dot{q}(T) = \overline{x}_1$.

In what follows, we will suppose, for the sake of simplicity, that a Schauder basis of the space X is monotone, i.e., that, for all $k \in \mathbb{N}$, $\|P_k\| = 1$. We note that in the case of a non–monotone Schauder basis, only minor modifications of the proof will be necessary.

N otice first of all that, thanks to (SB) and since Y_T is a linear subspace, $P_k x \in Y_T$ for every $x \in Y_T, k \in \mathbb{N}$.

In the following, given $v = (v_1, v_2) \in X \times X$, we will use the notation

$$P_k(v) = (P_k(v_1), P_k(v_2)).$$

Recall that Y_T is a linear subspace, and therefore is also a normed space endowed with the norm induced by the norm defined in X. Hence, $Y_T \times X$ is a normed space with the norm defined as

$$\|v\| = \|v_1\| + \|v_2\|.$$

We stress that, according to Lemma 2.3, $S(T, s)Bu(s)$ belongs to the linear subspace Y_T of X for every $s \in [0, T]$. Thus, it is known that

$$\int_0^T S(T, s)Bu(s)\, ds \in Y_T,$$

for every $u \in L^p([0, T], U)$, i.e., N is well defined. Moreover, since Y_T is a linear subspace, $x_1 \in Y_T$, $x_0 \in Y$, it follows from Lemma 2.3 that

$$x_1 - C(T, 0)\, x_0 - S(T, 0)\overline{x}_0 - \int_0^T S(T, s)f(s)\, ds \in Y_T,$$

for all $f \in L^p([0, T], X)$.

For proving the existence of a control $u \in L^p([0, T], U)$ and a mild solution to the problem (1.1), the approximation solvability method will be applied. For this purpose, for each $k \in \mathbb{N}$, let us consider the map $g_k : [0, T] \times X_k \times X_k \to X_k$ given by the formula $g_k = P_k \circ f$, where X_k is the linear subspace generated by the first k elements of the Schauder basis of X, and the operator $\Sigma_k : C^1([0, T], X_k) \to C^1([0, T], X_k)$, defined as follows:

$$\begin{aligned}\Sigma_k(q)(t) = \quad & P_k C(t, 0)\, x_0 + P_k S(t, 0)\, \overline{x}_0 + \int_0^t P_k S(t, s) g_k(s, q(s), \dot{q}(s))\, ds \\ & + \int_0^t P_k S(t, s) B(\tilde{N}^{-1}(P_k(p_q))(s))\, ds,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}p_q = \quad & \Big(x_1 - C(T, 0)\, x_0 - S(T, 0)\overline{x}_0 - \int_0^T S(T, s)f(s, q(s), \dot{q}(s))\, ds, \\ & \overline{x}_1 - \frac{\partial}{\partial t}C(t, 0)\Big|_{t=T} x_0 - \frac{\partial}{\partial t}S(t, 0)\Big|_{t=T} \overline{x}_0 - \int_0^T \frac{\partial}{\partial t}S(t, s)\Big|_{t=T} f(s, q(s), \dot{q}(s))\, ds\Big).\end{aligned}$$

$\tilde{N}^{-1}$ is a continuous mapping defined according to Proposition 2.1 and satisfying $N \circ \tilde{N}^{-1} = id_{Y_T \times X}$, whose existence is guaranteed by (WZ) and the fact that $L^p([0, T], U)$ inherits the uniform convexity of U.

Notice that

$$\begin{aligned}\dot{\Sigma}_k(q)(t) = \quad & P_k \frac{\partial}{\partial t} C(t, 0)x_0 + P_k \frac{\partial}{\partial t} S(t, 0)\overline{x}_0 + \int_0^t P_k \frac{\partial}{\partial t} S(t, s) g_k(s, q(s), \dot{q}(s))ds \\ & + \int_0^t P_k \frac{\partial}{\partial t} S(t, s) B(\tilde{N}^{-1}(P_k(p_q))(s))\, ds.\end{aligned}$$

The proof will be divided into two steps:

Step 1 Showing that, for all $k \in \mathbb{N}$, Σ_k has a fixed point q_k;
Step 2 Proving that the sequence $\{q_k\}_k$ from **Step 1** has a subsequence that weakly converges pointwise to a solution q to the controllability problem (1.1), and showing that q satisfies $q(T) = x_1$ and $\dot{q}(T) = \overline{x}_1$.

To prove in **Step 1** that Σ_k has a fixed point, we will show that this map fulfills all assumptions of the Schauder fixed point theorem. For this end, given $n \in \mathbb{N}$, we utilize the following notation: nB_k will represented the closed, convex, and bounded subset of the Banach space $C^1([0, T], X_k)$ given by

$$nB_k = \{q \in C^1([0, T], X_k) : \|q(t)\|, \|\dot{q}(t)\| \leq n, \text{ for every } t \in [0, T]\}.$$

Moreover, $K_T > 0$ denotes the constant from Lemma 2.2 such that

$$\|C(t, s)\| \leq K_T, \quad \|S(t, s)\| \leq K_T, \text{ and } \left\|\frac{\partial}{\partial t} S(t, s)\right\| \leq K_T, \qquad \text{for all } s, t \in [0, T].$$

Furthermore, since $x_0 \in Y$, the following exists

$$M = \max_{t \in [0,T]} \left\|\frac{\partial}{\partial t} C(t, 0)x_0\right\|.$$

To implement the Schauder fixed point theorem in **Step 1**, we will show that for every $k \in \mathbb{N}$, the following hold:

(a) $\Sigma_k(nB_k)$ is a relatively compact subset of $C^1([0, T], X_k)$ for all $n \in \mathbb{N}$,
(b) $\Sigma_k : nB_k \to C^1([0, T], X_k)$ is continuous, for every $n \in \mathbb{N}$,
(c) $N_0 \in \mathbb{N}$ (not dependent on k) exists such that $\Sigma_k(N_0 B_k) \subset N_0 B_k$.

Step 1 (a) *Proving that, for every $k \in \mathbb{N}$, $\Sigma_k(nB_k)$ is a relatively compact subset of $C^1([0, T], X_k)$, for all $n \in \mathbb{N}$.*
Let k, $n \in \mathbb{N}$ be fixed and let us prove that each sequence $\{\Sigma_k(q_m)\}_m$, $q_m \in nB_k$, for all $m \in \mathbb{N}$, admits a subsequence that is convergent in $C^1([0, T], X_k)$. For this purpose, let us denote $f_m(\cdot) = f(\cdot, q_m(\cdot), \dot{q_m}(\cdot))$. By ($f3$), we can see that the sequence $\{f_m\}_m \subset L^1([0, T], X)$ is bounded and uniformly integrable, and that for a.a. $s \in [0, T]$, the sequence $\{f_m(s)\}_m$ is bounded in X. Since X is reflexive, the Dunford–Pettis theorem can be applied and the existence of a subsequence (denoted for the sake of simplicity as the sequence) and of a function f_0 such that $f_m \rightharpoonup f_0$ in $L^1([0, T], X)$ can be obtained. By Lemma 2.1 (d), it is also possible to see that $P_k f_m \rightharpoonup P_k f_0$ in $L^1([0, T], X_k)$. Therefore, Theorem 2.1 implies that

$$\int_0^t P_k S(t, s) P_k f_m(s)\, ds \to \int_0^t P_k S(t, s) P_k f_0(s)\, ds \tag{3.2}$$

and

$$\int_0^t P_k \frac{\partial}{\partial t} S(t, s) P_k f_m(s)\, ds \to \int_0^t P_k \frac{\partial}{\partial t} S(t, s) P_k f_0(s)\, ds, \tag{3.3}$$

uniformly in $[0, T]$ and

$$\int_0^T S(T, s) f_m(s)\, ds \to \int_0^T S(T, s) f_0(s)\, ds$$

and

$$\int_0^T \frac{\partial}{\partial t} S(t,s)\bigg|_{t=T} f_m(s)\,ds \to \int_0^T \frac{\partial}{\partial t} S(t,s)\bigg|_{t=T} f_0(s)\,ds.$$

Therefore,

$$\begin{aligned} p_{q_m} \to p_0 = \ & \Big(x_1 - C(T,0)\,x_0 - S(T,0)\overline{x}_0 - \int_0^T S(T,s) f_0(s)\,ds, \\ & \overline{x}_1 - \frac{\partial}{\partial t} C(t,0)\bigg|_{t=T} x_0 - \frac{\partial}{\partial t} S(t,0)\bigg|_{t=T} \overline{x}_0 - \int_0^T \frac{\partial}{\partial t} S(t,s)\bigg|_{t=T} f_0(s)\,ds\Big) \text{ in } Y_T \times X. \end{aligned}$$

Through Lemma 2.1 (a), we can then obtain $P_k p_{q_m} \to P_k p_0$ in $Y_T \times X$.
The continuity of $\tilde{N}^{-1}$ subsequently ensures that

$$\alpha_m = \tilde{N}^{-1}(P_k(p_{q_m})) \to \tilde{N}^{-1}(P_k(p_0)) = \alpha_0$$

in $L^p([0,T],U)$. Thus, by the Hölder inequality, we obtain

$$\begin{aligned} & \left\| \int_0^t P_k S(t,s) B(\tilde{N}^{-1}(P_k(p_{q_m}))(s)) ds - \int_0^t P_k S(t,s) B(\tilde{N}^{-1}(P_k(p_0))(s)) ds \right\| \\ & = \left\| \int_0^t P_k S(t,s) B(\alpha_m(s) - \alpha_0(s))\,ds \right\| \\ & \le \int_0^t \| P_k S(t,s) B(\alpha_m(s) - \alpha_0(s)) \|\,ds \\ & \le \int_0^t K_T \|B\| \|\alpha_m(s) - \alpha_0(s)\|_U\,ds \le K_T \|B\| T^{1-\frac{1}{p}} \|\alpha_m - \alpha_0\|_{L_p}, \end{aligned}$$

for every $t \in [0,T]$.
In a similar way, it is possible to prove that for every $t \in [0,T]$

$$\begin{aligned} & \left\| \int_0^t P_k \frac{\partial}{\partial t} S(t,s) B(\tilde{N}^{-1}(P_k(p_{q_m}))(s)) ds - \int_0^t P_k \frac{\partial}{\partial t} S(t,s) B(\tilde{N}^{-1}(P_k(p_0))(s)) ds \right\| \\ & \le\ K_T\, \|B\|\, T^{1-\frac{1}{p}}\, \|\alpha_m - \alpha_0\|_{L_p}. \end{aligned}$$

Hence

$$\int_0^t P_k S(t,s) B(\tilde{N}^{-1}(P_k(p_{q_m}))(s))\,ds \to \int_0^t P_k S(t,s) B(\tilde{N}^{-1}(P_k(p_0))(s))\,ds \tag{3.4}$$

and

$$\int_0^t P_k \frac{\partial}{\partial t} S(t,s) B(\tilde{N}^{-1}(P_k(p_{q_m}))(s))\,ds \to \int_0^t P_k \frac{\partial}{\partial t} S(t,s) B(\tilde{N}^{-1}(P_k(p_0))(s))\,ds \tag{3.5}$$

uniformly in $[0,T]$.
The relations (3.2)–(3.5) then ensure that $\{\Sigma_k(q_m)\}_m$, $q_m \in nB_k$, admits a subsequence that is convergent in $C^1([0,T],X_k)$.

Step 1 (b) *Proving that, for every* $k \in \mathbb{N}$, $\Sigma_k : nB_k \to C^1([0,T],X_k)$ *is continuous, for every* $n \in \mathbb{N}$.
Let k, $n \in \mathbb{N}$ be arbitrary and let $\{q_m\}_m$ be a sequence in nB_k that converges to q in $C^1([0,T],X_k)$. We show that $\Sigma_k(q_m) \to \Sigma_k(q)$ as $m \to \infty$ in $C^1([0,T],X_k)$. To prove what is required, it is sufficient, according to **Step 1** (a), to verify that $\Sigma_k(q_m)(t) \to \Sigma_k(q)(t)$ and $\dot{\Sigma}_k(q_m)(t) \to \dot{\Sigma}_k(q)(t)$ as $m \to \infty$ for every $t \in [0,T]$.

Let $t \in [0, T]$ be fixed. Since $q_m \to q$ in $C^1([0, T], X_k)$, $q_m \rightharpoonup q$ in $C^1([0, T], X)$, we can find, according to [43, Theorem 4.3], that $q_m(s) \rightharpoonup q(s)$ and $\dot{q}_m(s) \rightharpoonup \dot{q}(s)$, for every $s \in [0, t]$. Thus, when defining $f_m(\cdot) = f(\cdot, q_m(\cdot), \dot{q_m}(\cdot))$ and $f_0(\cdot) = f(\cdot, q(\cdot), \dot{q}(\cdot))$, by $(f2)$ we get

$$f_m(s) \rightharpoonup f_0(s), \text{ for a.a. } s \in [0, t].$$

Since P_k is a bounded operator with values in the finite-dimensional space X_k, we subsequently obtain

$$P_k f_m(s) \to P_k f_0(s), \text{ for a.a. } s \in [0, t].$$

Furthermore, the boundedness of $S(t, s)$ and $\frac{\partial}{\partial t} S(t, s)$ then implies that

$$P_k S(t, s) P_k f_m(s) \to P_k S(t, s) P_k f_0(s), \text{ for a.a. } s \in [0, t]$$

and

$$P_k \frac{\partial}{\partial t} S(t, s) P_k f_m(s) \to P_k \frac{\partial}{\partial t} S(t, s) P_k f_0(s), \text{ for a.a. } s \in [0, t].$$

The properties of $S(t, s)$ and $\frac{\partial}{\partial t} S(t, s)$ together with $(f3)$ lead to the following estimates:

$$\|P_k S(t, s) P_k f_m(s)\| \leq K_T \varphi_n(s)$$

and

$$\left\| P_k \frac{\partial}{\partial t} S(t, s) P_k f_m(s) \right\| \leq K_T \varphi_n(s),$$

for a.a. $s \in [0, t]$, which ensure, by applying the dominated convergence theorem, that

$$\int_0^t P_k S(t, s) P_k f_m(s)\, ds \to \int_0^t P_k S(t, s) P_k f_0(s)\, ds \tag{3.6}$$

and

$$\int_0^t P_k \frac{\partial}{\partial t} S(t, s) P_k f_m(s)\, ds \to \int_0^t P_k \frac{\partial}{\partial t} S(t, s) P_k f_0(s)\, ds \tag{3.7}$$

as $m \to \infty$.

Similarly,

$$\int_0^T P_k S(T, s) f_m(s)\, ds \to \int_0^T P_k S(T, s) f_0(s)\, ds$$

and

$$\int_0^T P_k \frac{\partial}{\partial t} S(t, s)\Big|_{t=T} f_m(s)\, ds \to \int_0^T P_k \frac{\partial}{\partial t} S(t, s)\Big|_{t=T} f_0(s)\, ds$$

as $m \to \infty$. Therefore, $P_k(p_{q_m}) \to P_k(p_q)$ as $m \to \infty$.

Hence, by reasoning as in **Step 1** (a), by the continuity of $\tilde{N}^{-1}$ and the Hölder inequality, we get

$$\int_0^t P_k S(t, s) B(\tilde{N}^{-1}(P_k(p_{q_m}))(s))\, ds \to \int_0^t P_k S(t, s) B(\tilde{N}^{-1}(P_k(p_q))(s))\, ds$$

and

$$\int_0^t P_k \frac{\partial}{\partial t} S(t, s) B(\tilde{N}^{-1}(P_k(p_{q_m}))(s))\, ds \to \int_0^t P_k \frac{\partial}{\partial t} S(t, s) B(\tilde{N}^{-1}(P_k(p_q))(s))\, ds,$$

which, together with (3.6) and (3.7), gives the convergence $\Sigma_k(q_m)(t) \to \Sigma_k(q)(t)$ and $\dot{\Sigma}_k(q_m)(t) \to \dot{\Sigma}_k(q)(t)$ as $m \to \infty$, for every $t \in [0, T]$, as required.

Step 1 (c) *Proving that $N_0 \in \mathbb{N}$ (not dependent on k) exists such that $\Sigma_k(N_0 B_k) \subset N_0 B_k$.*
Let $k,\ n \in \mathbb{N}$ be arbitrary and let $q \in nB_k$. Then

$$\left\|P_k p_q\right\| \leq \|x_1\| + \|\overline{x}_1\| + M + K_T\left(\|x_0\| + 2\|\overline{x}_0\|\right) + 2K_T\|\varphi_n\|_{L^1}.$$

According to Proposition 2.1

$$\begin{aligned} \|\tilde{N}^{-1}(P_k(p_q))\|_{L^p} &= \left\|\Pi \mathcal{N}^{-1}(P_k(p_q))\right\|_{L^p} \leq \left\|\mathcal{N}^{-1}\right\| \left\|P_k(p_q)\right\| \\ &\leq \left\|\mathcal{N}^{-1}\right\| [\|x_1\| + \|\overline{x}_1\| + M + K_T\left(\|x_0\| + 2\|\overline{x}_0\|\right) + 2K_T\|\varphi_n\|_{L^1}]. \end{aligned}$$

Therefore, by using the Hölder inequality, we get, for every $t \in [0, T]$

$$\|\Sigma_k(q)(t)\| \leq K_T\|x_0\| + K_T\|\overline{x}_0\| + K_T\|\varphi_n\|_{L^1} + K_T\,\|B\|\,T^{1-\frac{1}{p}}\|\tilde{N}^{-1}(P_k(p_q))\|_{L^p} \leq C_1 + C_2\|\varphi_n\|_{L^1}, \qquad (3.8)$$

where

$$C_1 = K_T\left(\|x_0\| + \|\overline{x}_0\|\right) + K_T\,\|B\|\,T^{1-\frac{1}{p}}\left\|\mathcal{N}^{-1}\right\|\left(\|x_1\| + \|\overline{x}_1\| + M + K_T\left(\|x_0\| + 2\|\overline{x}_0\|\right)\right)$$

and

$$C_2 = K_T\left(1 + 2K_T\,\|B\|\,T^{1-\frac{1}{p}}\left\|\mathcal{N}^{-1}\right\|\right). \qquad (3.9)$$

Moreover, for every $t \in [0, T]$,

$$\|\dot{\Sigma}_k(q)(t)\| \leq M + K_T\|\overline{x}_0\| + K_T\|\varphi_n\|_{L^1} + K_T\,\|B\|\,T^{1-\frac{1}{p}}\|\tilde{N}^{-1}(P_k(p_q))\|_{L^p} \leq D_1 + C_2\|\varphi_n\|_{L^1}, \qquad (3.10)$$

where

$$D_1 = M + K_T\|\overline{x}_0\| + K_T\,\|B\|\,T^{1-\frac{1}{p}}\left\|\mathcal{N}^{-1}\right\|\left(\|x_1\| + \|\overline{x}_1\| + M + K_T\left(\|x_0\| + 2\|\overline{x}_0\|\right)\right).$$

Consequently, assuming

$$L_1 = \max\{C_1,\ D_1\}, \qquad (3.11)$$

from (3.8), (3.10), and (3.11) we obtain

$$\|\Sigma_k(q)\|_{C^1} = \max\{\|\Sigma_k(q)\|_C, \|\dot{\Sigma}_k(q)\|_C\} \leq L_1 + C_2\|\varphi_n\|_{L^1} \qquad (3.12)$$

for every $q \in nB_k, n, k \in \mathbb{N}$.
Thus, according to $(f3)$, a subsequence exists (for the sake of simplicity it is denoted as the sequence) such that

$$\lim_{n\to\infty} \frac{L_1 + C_2\|\varphi_n\|_{L^1}}{n} = 0.$$

This ensures the existence of $N_0 > 0$ such that

$$\frac{L_1 + C_2\|\varphi_{N_0}\|_{L^1}}{N_0} < 1,$$

which, together with Eq (3.12), ensures that

$$\frac{1}{N_0}\|\Sigma_k(q)(t)\|_{C^1} < 1,$$

i.e., that $\Sigma_k(q)(t) \in N_0B_k$, for every $k \in \mathbb{N}$ and $q \in N_0B_k$, as required.
Subsequently, applying the Schauder fixed point theorem, we can see that, for every $k \in \mathbb{N}$, Σ_k has a fixed point q_k. Moreover, it follows from the proof that all fixed points belong to the set

$$N_0B = \left\{ q \in C^1([0,T],X) : \|q(t)\|, \|\dot{q}(t)\| \leq N_0, \text{ for all } t \in [0,T] \right\}.$$

Step 2. *Limiting procedure*

The sequence $\{q_k\}_k$ whose existence was proven in **Step 1** fulfills, for all $k \in \mathbb{N}$ and $t \in [0,T]$

$$\begin{aligned} q_k(t) = \; & P_kC(t,0)\,x_0 + P_kS(t,0)\overline{x}_0 + \int_0^t P_kS(t,s)P_kf(s,q_k(s),\dot{q}_k(s))\,ds \\ & + \int_0^t P_kS(t,s)B(\tilde{N}^{-1}(P_k(p_{q_k}))(s))\,ds, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} p_{q_k} = \; & \Big(x_1 - C(T,0)\,x_0 - S(T,0)\overline{x}_0 - \int_0^T S(T,s)f(s,q_k(s),\dot{q}_k(s))\,ds, \\ & \overline{x}_1 - \frac{\partial}{\partial t}C(t,0)\Big|_{t=T} x_0 - \frac{\partial}{\partial t}S(t,0)\Big|_{t=T}\overline{x}_0 - \int_0^T \frac{\partial}{\partial t}S(t,0)\Big|_{t=T} f(s,q_k(s),\dot{q}_k(s))\,ds \Big). \end{aligned} \tag{3.14}$$

Furthermore

$$\begin{aligned} \dot{q}_k(t) = \; & P_k\frac{\partial}{\partial t}C(t,0)x_0 + P_k\frac{\partial}{\partial t}S(t,0)\overline{x}_0 + \int_0^t P_k\frac{\partial}{\partial t}S(t,s)P_kf(s,q_k(s),\dot{q}_k(s))\,ds \\ & + \int_0^t P_k\frac{\partial}{\partial t}S(t,s)B(\tilde{N}^{-1}(P_k(p_{q_k}))(s))\,ds, \end{aligned}$$

where p_{q_k} is defined by Formula (3.14).
Let us now prove that the sequence $\{q_k\}_k$ has a subsequence that weakly converges pointwise to a solution $q \in C^1([0,T],X)$ to the problem (1.1) satisfying $q(T) = x_1$ and $\dot{q}(T) = \overline{x}_1$. Let us assume

$$f_k(\cdot) = f(\cdot, q_k(\cdot), \dot{q}_k(\cdot)) \text{ and } g_k(\cdot) = P_kf_k(\cdot),$$

for all $k \in \mathbb{N}$. Since $q_k \in N_0B$, for every $k \in \mathbb{N}$, we can obtain, from $(f3)$

$$\|f_k(s)\| \leq \varphi_{N_0}(s), \tag{3.15}$$

for a.a. $s \in [0,T]$.
Reasoning as in **Step 1** (a), we can deduce the existence of a subsequence, referred to as the sequence for simplicity, and of a function g satisfying $f_k \rightharpoonup g$ and $g_k \rightharpoonup g$ in $L^1([0,T],X)$ can be obtained, as well as

$$\int_0^t S(t,s)g_k(s)\,ds \to \int_0^t S(t,s)g(s)\,ds$$

and

$$\int_0^t \frac{\partial}{\partial t}S(t,s)g_k(s)\,ds \to \int_0^t \frac{\partial}{\partial t}S(t,s)g(s)\,ds$$

uniformly in $[0, T]$.
By Lemma 2.1 (c),

$$\int_0^t P_k S(t,s) g_k(s)\, ds \to \int_0^t S(t,s) g(s)\, ds \tag{3.16}$$

and

$$\int_0^t P_k \frac{\partial}{\partial t} S(t,s) g_k(s)\, ds \to \int_0^t \frac{\partial}{\partial t} S(t,s) g(s)\, ds \tag{3.17}$$

for every $t \in [0, T]$.
For all $k \in \mathbb{N}$, since $q_k \in N_0 B$, we can proceed analogously to **Step 1** (c) and prove that, for every $k \in \mathbb{N}$,

$$\|\tilde{N}^{-1}(P_k(p_{q_k}))\|_{L^p} \leq \|\mathcal{N}^{-1}\| L \tag{3.18}$$

where

$$L = \|x_1\| + \|\overline{x}_1\| + M + K_T\,(\|x_0\| + 2\|\overline{x}_0\|) + 2K_T \|\varphi_{N_0}\|_{L^1}.$$

Since U is a uniformly convex space, $L^p([0, T], U)$, $1 < p < \infty$, is uniformly convex as well, and thus is reflexive. Therefore, due to (3.18), there is a subsequence, still denoted as the sequence, such that $\tilde{N}^{-1}(P_k(p_{q_k}))$ converges weakly to u in $L^p([0, T], U)$.
Given $\phi \in X^*$ and $t \in [0, T]$, let us consider the operator $\Phi : L^1([0, t], U) \to \mathbb{R}$ defined by

$$\Phi(p) = \phi\left(\int_0^t S(t,s) B(\tilde{N}^{-1}(P_k(p))(s))\, ds\right).$$

Due to Lemma 2.2, $S(t, s)$ is bounded and linear for all $t, s \in [0, T]$. Thus, since B is continuous, Φ is also linear and bounded, and we obtain

$$\begin{aligned} \phi\left(\int_0^t S(t,s) B\left(\tilde{N}^{-1}(P_k(p_{q_k}))(s)\right) ds\right) &= \Phi(\tilde{N}^{-1}(P_k(p_{q_k}))) \to \Phi(u) \\ &= \phi\left(\int_0^t S(t,s) B\,(u(s))\, ds\right). \end{aligned}$$

Since ϕ is arbitrary, we then get

$$\int_0^t S(t,s) B\left(\tilde{N}^{-1}(P_k(p_{q_k}))(s)\right) ds \rightharpoonup \int_0^t S(t,s) B\,(u(s))\, ds$$

and hence

$$\int_0^t P_k S(t,s) B\left(\tilde{N}^{-1}(P_k(p_{q_k}))(s)\right) ds \rightharpoonup \int_0^t S(t,s) B\,(u(s))\, ds, \tag{3.19}$$

due to Lemma 2.1 (c), for every $t \in [0, T]$. Similarly, we can obtain

$$\int_0^t P_k \frac{\partial}{\partial t} S(t,s) B\left(\tilde{N}^{-1}(P_k(p_{q_k}))(s)\right) ds \rightharpoonup \int_0^t \frac{\partial}{\partial t} S(t,s) B\,(u(s))\, ds. \tag{3.20}$$

By (3.16), (3.17), (3.19), and (3.20) and the definition of P_k, we have, for every $t \in [0, T]$

$$q_k(t) \rightharpoonup q(t) = C(t,0)\, x_0 + S(t,0)\overline{x}_0 + \int_0^t S(t,s) g(s)\, ds + \int_0^t S(t,s) B(u(s))\, ds$$

and

$$\dot{q}_k(t) \rightharpoonup \dot{q}(t) = \frac{\partial}{\partial t}C(t,0)\,x_0 + \frac{\partial}{\partial t}S(t,0)\overline{x}_0 + \int_0^t \frac{\partial}{\partial t}S(t,s)g(s)\,ds + \int_0^t \frac{\partial}{\partial t}S(t,s)B(u(s))\,ds.$$

Hence, by $(f2)$, for a.a. $t \in [0,T]$

$$f(t,q_k(t),\dot{q}_k(t)) \rightharpoonup f(t,q(t),\dot{q}(t)),$$

and then

$$g_k(t) = P_k f(t,q_k(t),\dot{q}_k(t)) \rightharpoonup f(t,q(t),\dot{q}(t)),$$

by Lemma 2.1 (c). Thus, by [44, Theorem 2.1], according to (3.15),

$$g_k \rightharpoonup f(\cdot,q(\cdot),\dot{q}(\cdot))$$

in $L^1([0,T],X)$, and the uniqueness of the weak limit implies that $g(t) = f(t,q(t),\dot{q}(t))$ for a.a. $t \in [0,T]$.

In the remaining part of the proof, it is necessary to show that $q(T) = x_1$ and $\dot{q}(T) = \overline{x}_1$. For this purpose, let us denote $\pi_1 : Y_T \times X \to Y_T$ as the map defined as $\pi_1(u_1,u_2) = u_1$. Subsequently, according to the condition (WZ), (3.14) and the definition of $\tilde{N}^{-1}$, it follows that

$$\begin{aligned}
&\int_0^T S(T,s)B(\tilde{N}^{-1}(P_k(p_{q_k}))(s))\,ds \\
&= \pi_1(N(\tilde{N}^{-1}(P_k(p_{q_k})))) = \pi_1(P_k(p_{q_k})) \\
&= P_k\Big(x_1 - C(T,0)\,x_0 - S(T,0)\overline{x}_0 - \int_0^T S(T,s)f(s,q_k(s),\dot{q}_k(s))\,ds\Big).
\end{aligned}$$

Furthermore, we obtain, by (3.13), for all $k \in \mathbb{N}$

$$\begin{aligned}
q_k(T) = {} & P_kC(T,0)\,x_0 + P_kS(T,0)\overline{x}_0 + \int_0^T P_kS(T,s)P_kf(s,q_k(s),\dot{q}_k(s))\,ds \\
& + \int_0^T P_kS(T,s)B(\tilde{N}^{-1}(P_k(p_{q_k}))(s))\,ds \\
= {} & P_kC(T,0)\,x_0 + P_kS(T,0)\overline{x}_0 + \int_0^T P_kS(T,s)P_kf(s,q_k(s),\dot{q}_k(s))\,ds \\
& + P_k\Big(x_1 - C(T,0)\,x_0 - S(T,0)\overline{x}_0 - \int_0^T S(T,s)f(s,q_k(s),\dot{q}_k(s))ds\Big) \\
= {} & P_kx_1.
\end{aligned} \tag{3.21}$$

Similarly, by (3.14), we arrive, for all $k \in \mathbb{N}$, at

$$\begin{aligned}
\dot{q}_k(T) = {} & P_k\frac{\partial}{\partial t}C(t,0)\Big|_{t=T}x_0 + P_k\frac{\partial}{\partial t}S(t,0)\Big|_{t=T}\overline{x}_0 + \int_0^T P_k\frac{\partial}{\partial t}S(t,s)\Big|_{t=T}P_kf(s,q_k(s),\dot{q}_k(s))ds \\
& + \int_0^T P_k\frac{\partial}{\partial t}S(t,s)\Big|_{t=T}B(\tilde{N}^{-1}(P_k(p_{q_k}))(s))\,ds \\
= {} & P_k\frac{\partial}{\partial t}C(t,0)\Big|_{t=T}x_0 + P_k\frac{\partial}{\partial t}S(t,0)\Big|_{t=T}\overline{x}_0 + \int_0^T P_k\frac{\partial}{\partial t}S(t,s)\Big|_{t=T}P_kf(s,q_k(s),\dot{q}_k(s))ds \\
& + P_k\Big(\overline{x}_1 - \frac{\partial}{\partial t}C(t,0)\Big|_{t=T}x_0 - \frac{\partial}{\partial t}S(t,0)\Big|_{t=T}\overline{x}_0 - \int_0^T \frac{\partial}{\partial t}S(t,s)\Big|_{t=T}f(s,q_k(s),\dot{q}_k(s))ds\Big) \\
= {} & P_k\overline{x}_1.
\end{aligned} \tag{3.22}$$

Passing to the weak limit in (3.21) and (3.22), we get $q(T) = x_1$ and $\dot{q}(T) = \overline{x}_1$ as required.

In the second controllability result, the growth condition $(f3)$ is replaced by the growth condition $(f3')$; a detailed analysis of the differences between $(f3)$ and $(f3')$ is described in [36]. The technique used in the sketch of its proof is based on the method applied in [45] for the second-order problems with the r.h.s. not depending on the first derivative.

Theorem 3.2. *Let us consider the control problem* (1.1)*, where* $x_0 \in Y$, $x_1 \in Y_T, \overline{x}_0, \overline{x}_1 \in X$, *and let* $f : [0,T] \times X \times X \to X$ *fulfills the assumptions* $(f1)$ *and* $(f2)$*. Furthermore, let the conditions* (SB) *and* (WZ) *hold, and let the following assumption be satisfied:*

$(f3')$ $a, b \in L^1([0,T], \mathbb{R})$ *exist such that, for a.a.* $t \in [0,T]$ *and all* x, $y \in X$,

$$\|f(t,x,y)\| \leq a(t)\max\{\|x\|, \|y\|\} + b(t).$$

Then the problem (1.1) *is controllable on the interval* $[0,T]$.

Proof. When changing $\varphi_n(t)$ for $na(t) + b(t)$, the proof can proceed analogously to Theorem 3.1. The only significant difference occurs in **Step 1** (c). Therefore, we will focus on this step and prove that there is a set H_k that is bounded, closed, and convex, satisfying $\Sigma_k(H_k) \subset H_k$ for all $k \in \mathbb{N}$.
For this reason, define, for every $j \in \mathbb{N}$

$$q_j = \max_{t\in[0,T]} \int_0^T e^{-j(t-s)} \chi_{[0,t]}(s) a(s)\, ds.$$

Its existence is ensured by continuity. Furthermore, for every $j \in \mathbb{N}$, let t_j denote the point at which the maximum is reached. Since $\{t_j\}_j \subset [0,T]$, there is a $\bar{t}$ satisfying $t_j \to \bar{t}$ (optionally passing to a subsequence, if necessary). Therefore, the sequence $\{\phi_j\}_j \subset L^1([0,T], X)$ defined by $\phi_j(s) = e^{-j(t_j-s)}\chi_{[0,t_j]}(s)a(s)$ converges pointwise to 0. Since the convergence is dominated, $\phi_j \to 0$ in $L^1([0,T], X)$. In particular, there is a subsequence (for the sake of simplicity, it is denoted as the sequence) satisfying $q_j \to 0$. Let us take $R_0 \in \mathbb{R}$ and $\bar{j} \in \mathbb{N}$ fulfilling $1 - C_2 q_{\bar{j}} > 0$, and

$$R_0 > \frac{L_1 + C_2\|b\|_{L^1}}{1 - C_2 q_{\bar{j}}},$$

where C_2 and L_1 are the constants introduced in (3.9) and (3.11). Furthermore, let us define the bounded, closed, and convex set

$$H_k = \{x \in C^1([0,T], X_k) : \max_{t\in[0,T]} (e^{-\bar{j}t} \max\{\|x(t)\|, \|\dot{x}(t)\|\}) \leq R_0\}.$$

Now, using the previous notation and estimates from the proof of Theorem 3.1, we get, for every $q \in H_k, t \in [0,T]$

$$\begin{aligned}
e^{-\bar{j}t}\|\Sigma_k(q)(t)\| \leq\ & e^{-\bar{j}t}C_1 + e^{-\bar{j}t}C_2\|b\|_{L^1} + C_2 e^{-\bar{j}t}\int_0^t a(s)\max\{\|x(s)\|, \|\dot{x}(s)\|\}ds \\
\leq\ & e^{-\bar{j}t}[C_1 + C_2\|b\|_{L^1}] + C_2\int_0^t e^{-\bar{j}(t-s)}a(s)e^{-\bar{j}s}\max\{\|x(s)\|, \|\dot{x}(s)\|\}ds \\
\leq\ & e^{-\bar{j}t}[C_1 + C_2\|b\|_{L^1}] + C_2R_0\int_0^T e^{-\bar{j}(t-s)}\chi_{[0,t]}(s)a(s)ds \\
\leq\ & C_1 + C_2\|b\|_{L^1} + C_2R_0q_{\bar{j}} < R_0,
\end{aligned}$$

due to the definition of R_0 and since $L_1 = \max\{C_1, D_1\}$.
Similarly

$$\begin{aligned} e^{-\bar{j}t}\|\dot{\Sigma}_k(q)(t)\| \leq\ & e^{-\bar{j}t}D_1 + e^{-\bar{j}t}C_2\|b\|_{L^1} + C_2e^{-\bar{j}t}\int_0^t a(s)\max\{\|x(s)\|, \|\dot{x}(s)\|\}ds \\ \leq\ & D_1 + C_2\|b\|_{L^1} + C_2R_0q_{\bar{j}} < R_0. \end{aligned}$$

Therefore, $\Sigma_k(q) \in H_k$. Since H_k is a subset of the set

$$H = \{x \in C^1([0,T], X) : \max_{t\in[0,T]}(e^{-\bar{j}t}\max\{\|x(t)\|, \|\dot{x}(t)\|\}) \leq R_0\},$$

which is bounded, we can proceed similarly to the proof of Theorem 3.1 in order to get the required conclusion.

Remark 3.1. *We point out that the hypothesis* (SB) *can be dropped if* X *is a Hilbert space. In fact, in this case, Proposition 2.1 (iv) implies that* $\tilde{N}^{-1}$ *is also linear and it is therefore possible to define*

$$\begin{aligned} \Sigma_k(q)(t) =\ & P_kC(t,0)\,x_0 + P_kS(t,0)\,\overline{x}_0 + \int_0^t P_kS(t,s)g_k(s,q(s),\dot{q}(s))\,ds \\ & + \int_0^t P_kS(t,s)B(\tilde{N}^{-1}(p_q)(s))\,ds \end{aligned}$$

instead of as in (3.1)*. The only differences in the proof would be, in this case, related to* ***Step 1*** (a) *and* (b)*. More precisely, in the first case, we would get* $p_{q_m} \rightharpoonup p_0$*, and hence* $\tilde{N}^{-1}(p_{q_m}) \rightharpoonup \tilde{N}^{-1}(p_0)$*. Then, since* B *is a bounded and linear operator, with reasoning like that in Theorem 2.1, we obtain*

$$\int_0^t S(t,s)B(\tilde{N}^{-1}(p_{q_m})(s))ds \to \int_0^t S(t,s)B(\tilde{N}^{-1}(p_0)(s))ds$$

and

$$\int_0^t \frac{\partial}{\partial t}S(t,s)B(\tilde{N}^{-1}(p_{q_m})(s))ds \to \int_0^t \frac{\partial}{\partial t}S(t,s)B(\tilde{N}^{-1}(p_0)(s))ds$$

uniformly in $[0,T]$*, which allows us to conclude the proof of* ***Step 1*** (a) *as well. Similarly, it is possible to reason in the second case.*

However, we stress that the hypothesis (SB) *is not too restrictive. In fact, it is, for example, verified by the well known Klein–Gordon operator*

$$A : D(A) = \{y \in W^{2,2}([0,\pi]) : y(0) = y(\pi) = 0\} \subset L^2([0,\pi])) \to L^2([0,\pi]))$$

defined as

$$Ay = y'' - a^2y,$$

with $a \in \mathbb{R}$*, which corresponds to the Laplacian operator when* $a = 0$*. It is well known that the operator* A *generates a cosine family* $\{\tilde{C}(t)\}_t$*, and hence a fundamental system* $\{S(t,s)\}_{t,s}$ *defined as*

$$S(t,s)x = \int_0^{t-s}\tilde{C}(r)x\,dr.$$

In this case (see [20] and Remark 2.2)

$$Y_T = W^{1,2}([0,\pi]) \cap C_0([0,\pi])$$

and

$$e_k(x) = \sqrt{\frac{2}{\pi}} \sin(\sqrt{k^2 + a^2}x),$$

which clearly belongs to Y_T, for every $k \in \mathbb{N}$.

Remark 3.2. *Consider a Carathéodory function $q : [0,T] \times [0,a] \times \mathbb{R} \to \mathbb{R}$, i.e., such that*

- *for all $r \in \mathbb{R}$, $q(\cdot,\cdot,r) : [0,T] \times [0,a] \to \mathbb{R}$ is measurable,*
- *for a.a. $t \in [0,T]$ and $y \in [0,a]$, $q(t,y,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous,*

and the reflexive Banach space $X = L^p([0,a],\mathbb{R})$, with $p > 1$.
Assume furthermore that there are $\eta \in L^1([0,T],\mathbb{R})$ and $\lambda : [0,a] \to [0,\infty)$, where λ is increasing, such that, for a.a. $t \in [0,T]$ and every $y \in [0,a]$, $r \in \mathbb{R}$

$$|q(t,y,r)| \le \eta(t)\lambda(|r|)$$

with

$$\liminf_{r\to\infty} \frac{\lambda(r)}{r} = 0.$$

Then the nonlinear map

$$\begin{aligned} f : \quad [0,T] \times X \times X \to & \quad X \\ (t,x,y) \to & \quad q\left(t,\cdot,\int_0^a x(\xi)\,d\xi\right) \end{aligned}$$

simultaneously satisfy Hypotheses $(f1)$, $(f2)$, and $(f3)$ (see [46] for the proof).
On the other hand, if we further assume that $\varphi \in L^1([0,T],\mathbb{R})$ exists such that, for a.a. $t \in [0,T]$ and every $y \in [0,a]$ and $r \in \mathbb{R}$

$$|q(t,x,r)| \le \varphi(t),$$

the nonlinear map

$$\begin{aligned} f : \quad [0,T] \times X \times X \to & \quad X \\ (t,x,y) \to & \quad q\left(t,\cdot,\int_0^a x(\xi)\,d\xi\right)x, \end{aligned}$$

simultaneously satisfy Hypotheses $(f1)$, $(f2)$, and $(f3')$ (see [46] for the proof).
Such nonlinearities, as well as similar ones involving an integral kernel, arise in second-order one-dimensional Klein–Gordon equations and represent a viscous damping term of the nonlinear Balakrishnan–Taylor-type.

4. Discussion and conclusions

This paper has dealt with the problems related to the controllability of the second-order evolution Cauchy problem in a Banach space, provided that the nonlinear term depends on both the solution and its derivative and the linear term is time-dependent. These kinds of equations can be seen as an abstract formulation of a second-order partial differential equation, where the dependence of the linear term on the time represents properties like stiffness or tension that may vary with time, while the dependence of the nonlinear term on the derivative of the unknown describes effects such as damping. Thus, it is certainly interesting to approach such general models.

The first contribution of this paper has been the introduction of a definition of controllability that ensures the steering of both the solution and its derivative to the target values by means of a unique control function, in contrast with most papers in the literature, which only guarantee that the solution reaches a prescribed point, hence violating the controllability concept, because the derivative is a state variable for second-order problems. Our definition is the extension to the equation containing time-dependent linear terms of the definition given in [20], the only paper in literature where the derivative is considered in the controllability definition without requiring strong conditions which require the linear part A to be bounded on the whole Banach space and/or the Banach space to be finite-dimensional.

After the introduction of the new definition of controllability, sufficient conditions for achieving it have been studied. The results have been obtained by applying Schauder fixed point theorem together with the method of approximation solvability, and concept of weak topology. The procedure has allowed us to eliminate any conditions related to the compactness of the right-hand side and of the fundamental system generated by the linear operator.

Furthermore, let us note that in all previous papers concerning controllability for the second-order evolution problems, even if it is not explicitly stated in all cases, the control belongs to $L^2([0,T],U)$, and the control space U is a Hilbert space (see [36] for a discussion on this subject), while our results hold in $L^p([0,T],U)$, where it is enough to consider a uniformly convex control space U.

Some future research directions relevant to the topics under study are as follows:

- The abstract results have been derived in the paper by using a fixed point theorem, allowing us to assume only linear growth of the nonlinear term. It would therefore be interesting to relax the required growth of the nonlinear part, e.g., by replacing the application of a fixed point theorem by a suitable continuation principle.
- We have investigated the Cauchy problem associated with the equation; it would be interesting to consider some boundary conditions instead, e.g., Dirichlet, periodic, antiperiodic, or integral.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research was supported by the grant MIUR-PRIN 2020F3NCPX "Mathematics for industry 4.0 (Math4I4)".

The research was supported by the European Structural and Investment Funds (OP Research, Development and Education) and by the Ministry of Education, Youth, and Sports of the Czech Republic by Grant No. CZ.02.2.69/0.0/0.0/18 054/0014592 *The Advancement of Capacities for Research and Development at Moravian Business College Olomouc.*

V. Taddei is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Conflict of interest

The authors declare there are no conflicts of interest.

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