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Research article

Accuracy-preassigned fixed-time synchronization of inertial neural networks with time-varying leakage delays and proportional delays

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Abstract: This work established synchronization criteria for master-slave inertial neural networks with leakage time-varying delays and proportional delays, The solution employed a direct analysis method based on parameterized system solutions. The derived synchronization conditions consisted of only a few simple inequalities, which were easy to solve.Based on the adopted approach, a novel class of synchronization controllers was designed for proportional delays without constructing any complex functionals. Using the proposed method, leakage delays could be transformed into their maximum absolute values, enabling the derivation of delay-dependent conditions without any intricate treatment of leakage delays. Furthermore, it was noteworthy that this paper presented the first investigation into this problem using the proposed method, and the technique employed was novel. Finally, numerical simulations were provided to verify the effectiveness of the proposed method.

Keywords: inertial neural networks; leakage delay; proportional delays; a direct analysis method grounded in parameterized system solutions; fixed-time synchronization with preassigned accuracy

1. Introduction

For many years, research on neural networks (NNs) has been a popular topic [1–5]. Inertial neural networks (INNs), a specialized subclass of NNs, are characterized by the incorporation of inertial terms into their dynamics. Babcocka et al. [6] found that electronic neural networks with inertial couplings exhibit more complex dynamics than those with standard resistor-capacitor couplings. This integration leads to significantly more complex dynamic behaviors compared to conventional first-order neural models [7, 8]. Research on INNs has consistently remained an active and important area within computational neuroscience and nonlinear systems [9–12]. Recently, studies on INNs have demonstrated promising results in critical application domains such as secure communication and image encryption [13–16]. These advances highlight the substantial research value of investigating INNs, particularly in relation to their distinctive dynamic properties and practical implementation potential.

Proportional delay, as a special category of unbounded time delays, has attracted growing attention from an increasing number of scholars in the field of neural networks in recent years. When proportional delays are introduced into INNs, the combined influence of inertial terms and such delays substantially increases the complexity of the associated problems. Consequently, research on INNs with proportional delays has attracted substantial scholarly attention. Several well-established methodologies have been developed to address these challenges, including: Lyapunov functional analysis [17]; the delay-free functional approach by Zhang et al. [18]; proportional delay differential inequalities proposed by Zhou et al. [19]; the Halanay inequality technique documented in [20]; etc. Given these developments, investigating INNs with proportional delays demonstrates significant theoretical value and practical relevance. The pursuit of novel methodologies to resolve these complex issues represents a compelling and substantial scientific challenge.

Leakage delays, recognized as a particularly challenging category of time delays, have attracted considerable research attention due to their significant theoretical implications. Consequently, their integration into the study of INNs has become an active research frontier. Wang et al. [21] established convergence criteria for such systems using Lyapunov stability theory; Yu et al. [22] derived delay-dependent stability conditions by combining Lyapunov functionals with the Jensen inequality; reference [23] developed synchronization criteria through an effective sliding mode control law without requiring delay differentiability; etc. Nevertheless, a fundamental question remains: Can novel methodologies be formulated to analyze INNs with leakage delays while circumventing the need for intricate delay-specific treatments? Therefore, the study of INNs with leakage delays continues to hold significant research value.

Fixed-time synchronization (FTS), as an advanced synchronization technique, has demonstrated substantial success across diverse domains in recent years [24,25], including image encryption [26–28], secure communication [29], chaotic cryptosystems, 3D point cloud data protection, Chua's circuit networks [30], robotic trajectory tracking, Chua oscillator metrics, and multi-agent systems [31–34]. Consequently, research on FTS for NNs possesses significant theoretical and practical value, attracting considerable scholarly attention. Representative studies include: Yao et al. [35] investigated fixed-time projective synchronization for INNs; reference [36] addressed FTS control for fuzzy inertial cellular NNs; Ran et al. [37] studied FTS in INNs with parametric and structural mismatches; and Long et al. [38] resolved FTS for delayed inertial complex-valued NNs.

However, conventional FTS often proves overly idealized for real-world applications due to practical constraints. Motivated by this limitation, Su et al. [39] introduced the concept of practical FTS, which was subsequently refined by Yuan et al. [40] into the framework of FTS with pre-specified accuracy. Both approaches explicitly account for initial condition dependencies, with the latter establishing rigorous quantitative precision guarantees.

Furthermore, to the authors' knowledge, no existing studies have addressed the synchronization problem for INNs that include both leakage and proportional delays. Despite increasing attention to complex delay structures, this gap remains unaddressed. Motivated by the importance of these delays, the value of FTS, and the limitations of conventional methods, this paper proposes a direct analysis approach based on parameterized system solutions to study accuracy-preassigned FTS (AP-FTS) for INNs with time-varying leakage and proportional delays. This work fills a critical research gap and advances the methodology for FTS in delay-sensitive INNs.

The main contributions of this paper are as follows:

1) This paper presents a novel method for studying synchronization problems and applies it to INNs with proportional delays.

2) The derived synchronization conditions consist of only a few simple inequalities, ensuring computational tractability.

3) A novel controller designed specifically for proportional delays has been developed, which avoids complex delay-handling procedures.

4) The proposed methodology effectively mitigates leakage delay impacts while streamlining the proof process.

5) The coupled dynamics of leakage and proportional delays are concurrently addressed, significantly broadening applicability.

6) Inertial term processing is achieved through tailored tuning parameters without resorting to variable substitution.

Notations: Denote by $\mathbb{R}$ the real number field. The $\mathbb{R}$-linear space of continuous functions mapping $\mathbb{S}_1$ to $\mathbb{S}_2$ is denoted $C(\mathbb{S}_1, \mathbb{S}_2)$. Additionally, define $\langle n\rangle = 1, 2, \ldots, n$.

2. Model preparation

Fist of all, we introduce an INN model characterized by time-varying leakage delays and proportional delays:

$$\begin{aligned}\ddot{H}_i(t) = &-\xi_i\dot{H}_i(t) - \bar{a}_i H_i(t - o_i(t)) + \sum_{j=1}^{n} \bar{b}_{ij}\bar{f}_j^{(1)}(H_j(t)) \\ &+ \sum_{j=1}^{n} \bar{c}_{ij}\bar{f}_j^{(2)}(H_j(b_{ij}t)),\ i \in \langle n\rangle, t \geq t_0,\end{aligned} \tag{2.1a}$$

$$\begin{aligned}&H_i(s) = \varphi_i^{(1)}(s),\ \dot{H}_i(s) = \psi_i^{(1)}(s), \\ &\qquad i \in \langle n\rangle, s \in [\min\{bt_0, t_0 - \hat{o}\}, t_0],\end{aligned} \tag{2.1b}$$

in which n indicates the number of neurons and i denotes the i-th neuron, H_i is the neuronal state, $\ddot{H}_i$ is the inertial term, $o_i(t) \in C([t_0, +\infty), [\check{o}_i, \hat{o}_i])$ refers to the leakage delay, $t_0 \geq 0$ represents the initial time, $0 \leq \check{o}_i \leq \hat{o}_i$, $0 < b_{ij} < 1$ stands for the proportional coefficient, $\xi_i > 0$, $\bar{a}_i > 0$, connection weights are given by $\bar{b}_{ij}$ and $\bar{c}_{ij}$, while the mappings $\bar{f}_j^{(q)} : \mathbb{R} \to \mathbb{R}$ ($q \in \langle 2\rangle$) define the activation functions,

$\varphi_i^{(1)}$ and $\psi_i^{(1)} \in C([\min\{bt_0, t_0 - \hat{o}\}, t_0], \mathbb{R})$ appertain to the initial functions, $\hat{o} = \max\limits_{1\le i\le n} \hat{o}_i$ represents the upper bound of the absolute values of leakage delays, and $b = \min\limits_{1\le i,j\le n} b_{ij}$ is the minimum value of the proportional coefficients.

Let the model (2.1) be the master INN, and assume its slave INN as follows:

$$\begin{aligned}\ddot{I}_i(t) = &- \xi_i \dot{I}_i(t) - \bar{a}_i I_i(t - o_i(t))\\ &+ \sum_{j=1}^{n} \bar{b}_{ij} \bar{f}_j^{(1)}(I_j(t)) + \sum_{j=1}^{n} \bar{c}_{ij} \bar{f}_j^{(2)}(I_j(b_{ij}t))\\ &+ u_i(t),\ i \in \langle n\rangle, t \ge t_0,\end{aligned} \tag{2.2a}$$

$$\begin{aligned}&I_i(s) = \varphi_i^{(2)}(s),\ \dot{I}_i(s) = \psi_i^{(2)}(s),\\ &\quad i \in \langle n\rangle, s \in [\min\{bt_0, t_0 - \hat{o}\}, t_0],\end{aligned} \tag{2.2b}$$

where I_i represents the neuronal state, $\ddot{I}_i$ indicates the inertial term, $\varphi_i^{(2)}$ and $\psi_i^{(2)} \in C([\min\{bt_0, t_0 - \hat{o}\}, t_0], \mathbb{R})$ stand for the initial functions, and the i-th controller component is denoted by $u_i(t)$.

Consequently, the error system below is derived from (2.1) and (2.2).

$$\begin{aligned}\ddot{E}_i(t) = &- \xi_i \dot{E}_i(t) - \bar{a}_i E_i(t - o_i(t))\\ &+ \sum_{j=1}^{n} \bar{b}_{ij} F_j^{(1)}(E_j(t)) + \sum_{j=1}^{n} \bar{c}_{ij} F_j^{(2)}(E_j(b_{ij}t))\\ &+ u_i(t),\ i \in \langle n\rangle, t \ge t_0,\end{aligned} \tag{2.3a}$$

$$\begin{aligned}&E_i(s) = \varphi_i(s),\ \dot{E}_i(s) = \psi_i(s),\\ &\quad i \in \langle n\rangle, s \in [\min\{bt_0, t_0 - \hat{o}\}, t_0],\end{aligned} \tag{2.3b}$$

where $E_i(t) = I_i(t) - H_i(t)$, $\varphi_i(s) = \varphi_i^{(2)}(s) - \varphi_i^{(1)}(s)$, $\psi_i(s) = \psi_i^{(2)}(s) - \psi_i^{(1)}(s)$, and

$$F_j^{(q)}(E_i(t)) = \bar{f}_j^{(q)}(I_i(t)) - \bar{f}_j^{(q)}(H_i(t)),\ q \in \langle 2\rangle.$$

Furthermore, the error system is assumed to satisfy the following assumptions.

Assumption 1. *A nonnegative constant $\tilde{F}$ exists satisfying $|F_i^{(q)}(u)| \le \tilde{F}$ for all $i \in \langle n\rangle$, $u \in \mathbb{R}$, and $q \in \langle 2\rangle$.*

The controller $u_i(t)$ is designed as follows to address the posed problem:

$$\begin{aligned}u_i(t) = &(\xi_i + P_i)\, \dot{E}_i(t) + Q_i E_i(t)\\ &- \operatorname{sgn}(\pi_i(t))\, \blacktriangle_i, t \ge t_0, i \in \langle n\rangle,\end{aligned} \tag{2.4}$$

where the controller gains requiring determination are $P_i < 0$ and $Q_i \in \left[-\frac{(P_i)^2}{4}, 0\right]$, and

$$\blacktriangle_i = \sum_{j=1}^{n} \tilde{F}\left(|\bar{b}_{ij}| + |\bar{c}_{ij}|\right),$$

$$\pi_i(t) = \ddot{E}_i(t) - P_i \dot{E}_i(t) - Q_i E_i(t).$$

Remark 1. *Distinct from conventional approaches in references [17–20] that handle proportional delays through Lyapunov functional construction or inequality techniques, this study designs controller* (2.4) *based on the proposed methodology to address proportional delays, resulting in significantly streamlined proof procedures.*

Let $\vec{A}_i$ and $\vec{B}_i$ satisfy the following relationship:

$$-\vec{A}_i\vec{B}_i = Q_i,\ \vec{A}_i + \vec{B}_i = P_i \text{ and } \vec{B}_i < \vec{A}_i < 0.$$

Based on this, the error system (2.3) can be transformed into the following form:

$$\begin{aligned}\ddot{E}_i(t) =&\vec{A}_i\dot{E}_i(t) + \vec{B}_i\dot{E}_i(t) - \vec{A}_i\vec{B}_iE_i(t) - \bar{a}_iE_i(t - o_i(t)) \\ &+ \sum_{j=1}^{n}\bar{b}_{ij}F_j^{(1)}(E_j(t)) + \sum_{j=1}^{n}\bar{c}_{ij}F_j^{(2)}(E_j(b_{ij}t)) - \operatorname{sgn}(\pi_i(t))\,\blacktriangle_i, \\ &i \in \langle n\rangle, t \geq t_0,\end{aligned} \tag{2.5a}$$

$$E_i(s) = \varphi_i(s),\ \dot{E}_i(s) = \psi_i(s), i \in \langle n\rangle, s \in [\min\{bt_0, t_0 - \hat{o}\}, t_0]. \tag{2.5b}$$

Remark 2. *By constraining the magnitude relationships among controller gains, we design regulation parameters $\vec{A}_i$ and $\vec{B}_i$ to process inertial terms directly. This approach eliminates the need for variable substitutions while substantially reducing theoretical derivation complexity.*

Assumption 2. *The mentioned $\vec{A}_i$, $\vec{B}_i$ $(i \in \langle n\rangle)$ and $\lambda > 0$ satisfy the following inequalities:*

$$2\lambda + 1 < -\vec{A}_i < -\vec{B}_i \text{ and } \vec{A}_i - \vec{B}_i - 2\bar{a}_i e^{\lambda\hat{o}_i} > 0.$$

Remark 3. *Assumption 2 yields the inequalities $\bar{F}_i(\lambda) := \frac{1}{\vec{A}_i+\lambda} + \frac{1}{\vec{B}_i+\lambda} > -\frac{2}{\lambda+1} > -2$ and $\frac{2\bar{a}_i e^{\lambda\hat{o}_i}}{\vec{A}_i-\vec{B}_i} < 1$ $(i \in \langle n\rangle)$. Moreover, there exists $L > 1$ such that*

$$R_{1i}(\lambda) := \frac{-\vec{B}_i}{(\vec{A}_i - \vec{B}_i)L} - \frac{2\bar{a}_i e^{\lambda\hat{o}_i}\bar{F}_i(\lambda)}{\vec{A}_i - \vec{B}_i} < 1, i \in \langle n\rangle.$$

Remark 4. *The inequalities presented in Assumption 2 and Remark 3 constitute the synchronization conditions in this study. It should be noted that these conditions consist solely of simple scalar inequalities, making them highly tractable for computational implementation.*

Definition 1. *The inertial neural networks* (2.1) *and* (2.2) *are said to achieve FTS with accuracy ϵ' under controller $u_i(t)$ if, for arbitrary initial states φ_i and ψ_i $(i \in \langle n\rangle)$, there exist $\epsilon' > 0$ and $t_* > 0$ satisfying*

$$\sum_{i=1}^{n}(|E_i(t)| + |\dot{E}_i(t)|) < \epsilon', \quad t > t_*.$$

Assumption 3. *Given fixed L and λ, there exists $\epsilon > 0$ satisfying*

$$\ln\tilde{L}(\lambda) + \ln\|\vec{E}\| - \ln\frac{\epsilon}{n} > 0,$$

where

$\tilde{L}(\lambda) = \max_{1\le i\le n} \vec{L}_i(\lambda),$

$\vec{L}_i(\lambda) = L\left(1 + 4\bar{a}_i e^{\lambda\hat{o}_i}\right) + \vec{A}_i\vec{B}_i,$

$\|\vec{E}\| = \max_{1\le i\le n}\left(\|\tilde{E}_i\| + \|\tilde{\dot{E}}_i\|\right),$

$\|\tilde{E}_i\| = \sup_{\min\{bt_0, t_0-\hat{o}\}\le s\le t_0} |\tilde{E}_i(s)|,\ \tilde{E}_i(s) = E_i(t_0 + s),$

$\|\tilde{\dot{E}}_i\| = \sup_{\min\{bt_0, t_0-\hat{o}\}\le s\le t_0} |\tilde{\dot{E}}_i(s)|,\ \tilde{\dot{E}}_i(s) = \dot{E}_i(t_0 + s).$

This paper will design controller (2.4) for guaranteeing AP-FTS of INNs (2.1) and (2.2).

3. Main results

Next, a controller (2.4) is designed to ensure AP-FTS for INNs (2.1) and (2.2).

Theorem 1. *Given Assumption 1, any solution* $E(t) := (E_1(t), E_2(t), \ldots E_n(t))^T$ *to the error system* (2.5a) *satisfies:*

$$|E_i(t)| \le \frac{J_{i1}(t) + J_{i2}(t)}{\vec{A}_i - \vec{B}_i},\ t \ge t_0, i \in \langle n\rangle, \tag{3.1}$$

where

$$J_{i1}(t) = \left(e^{\vec{A}_i t} - e^{\vec{B}_i t}\right)|\dot{E}_i(0)| + \left(\vec{A}_i e^{\vec{B}_i t} - \vec{B}_i e^{\vec{A}_i t}\right)|E_i(0)|,$$

$$J_{i2}(t) = \bar{a}_i \int_0^t \left(e^{\vec{A}_i(t-s)} + e^{\vec{B}_i(t-s)}\right)|E_i(s - o_i(s))| \mathrm{d}s.$$

Proof. For any $t \ge t_0$, according to (2.5a), there is

$$\begin{aligned}
&\ddot{E}_i(t) - \vec{A}_i\dot{E}_i(t) - \vec{B}_i\left[\dot{E}_i(t) - \vec{A}_i E_i(t)\right] \\
=& -\bar{a}_i E_i(t - o_i(t)) + \sum_{j=1}^n \bar{b}_{ij} F_j^{(1)}(E_j(t)) \\
&+ \sum_{j=1}^n \bar{c}_{ij} F_j^{(2)}(E_j(b_{ij}t)) - \mathrm{sgn}(\pi_i(t))\blacktriangle_i,\ i \in \langle n\rangle.
\end{aligned}$$

Next, it can be obtained that

$$\begin{aligned}
e^{-\vec{B}_i t}\pi_i(t) =& \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\vec{B}_i t}\left[\dot{E}_i(t) - \vec{A}_i E_i(t)\right]\right) \\
=& -e^{-\vec{B}_i t}\bar{a}_i E_i(t - o_i(t)) + e^{-\vec{B}_i t}\sum_{j=1}^n \bar{b}_{ij} F_j^{(1)}(E_j(t)) \\
&+ e^{-\vec{B}_i t}\sum_{j=1}^n \bar{c}_{ij} F_j^{(2)}(E_j(b_{ij}t) - e^{-\vec{B}_i t}\,\mathrm{sgn}(\pi_i(t))\blacktriangle_i,\ i \in \langle n\rangle.
\end{aligned}$$

Then, according to Assumption 1, there will be

$$\begin{cases} 0 \leq \mathrm{e}^{-\vec{B}_i t}\pi_i(t) \leq \mathrm{e}^{-\vec{B}_i t}\bar{a}_i|E_i(t-o_i(t))|, \text{when } \pi_i(t) \geq 0,\ i \in \langle n \rangle, \\ -\mathrm{e}^{-\vec{B}_i t}\bar{a}_i|E_i(t-o_i(t))| \leq \mathrm{e}^{-\vec{B}_i t}\pi_i(t) < 0, \text{when } \pi_i(t) < 0,\ i \in \langle n \rangle. \end{cases} \tag{3.2}$$

Combining the two cases in (3.2), we can get

$$\begin{aligned} -\mathrm{e}^{-\vec{B}_i t}\bar{a}_i|E_i(t-o_i(t))| &\leq \mathrm{e}^{-\vec{B}_i t}\pi_i(t) \\ &\leq \mathrm{e}^{-\vec{B}_i t}\bar{a}_i|E_i(t-o_i(t))|, \\ &t \geq t_0, i \in \langle n \rangle. \end{aligned} \tag{3.3}$$

Then, exchanging $\vec{A}_i$ and $\vec{B}_i$ in the process of deriving (3.3), we can get

$$\begin{aligned} -\mathrm{e}^{-\vec{A}_i t}\bar{a}_i|E_i(t-o_i(t))| &\leq \mathrm{e}^{-\vec{A}_i t}\pi_i(t) \\ &\leq \mathrm{e}^{-\vec{A}_i t}\bar{a}_i|E_i(t-o_i(t))|, \\ &t \geq t_0, i \in \langle n \rangle. \end{aligned} \tag{3.4}$$

Integrating both sides of (3.2) and (3.3) from 0 to t yields

$$\begin{aligned} &-\bar{a}_i \int_0^t \mathrm{e}^{\vec{B}_i(t-s)}|E_i(s-o_i(s))|\mathrm{d}s \\ \leq &\dot{E}_i(t) - \vec{A}_i E_i(t) - \mathrm{e}^{\vec{B}_i t}\left[\dot{E}_i(0) - \vec{A}_i E_i(0)\right] \\ \leq &\bar{a}_i \int_0^t \mathrm{e}^{\vec{B}_i(t-s)}|E_i(s-o_i(s))|\mathrm{d}s,\ t \geq t_0, i \in \langle n \rangle \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &-\bar{a}_i \int_0^t \mathrm{e}^{\vec{A}_i(t-s)}|E_i(s-o_i(s))|\mathrm{d}s \\ \leq &\dot{E}_i(t) - \vec{B}_i E_i(t) - \mathrm{e}^{\vec{A}_i t}\left[\dot{E}_i(0) - \vec{B}_i E_i(0)\right] \\ \leq &\bar{a}_i \int_0^t \mathrm{e}^{\vec{A}_i(t-s)}|E_i(s-o_i(s))|\mathrm{d}s,\ t \geq t_0, i \in \langle n \rangle. \end{aligned} \tag{3.6}$$

Therefore,

$$-J_{i1}(t) - J_{i2}(t) \leq (A_i^{\alpha_k} - B_i^{\alpha_k})\omega_i(t) \leq J_{i1}(t) + J_{i2}(t),$$

and, hence, (3.1) holds.

Remark 5. *Theorem 1 establishes an inequality governing state-related scalar function within the error system. This result fundamentally characterizes solution properties of the error dynamics and serves as the foundational basis for all subsequent derivation procedures throughout this paper.*

The following theorem provides sufficient conditions for AP-FTS of INNs (2.1) and (2.2).

Theorem 2. *With Assumptions 1–3 holding, the following hold:*

(i) $|E_i(t)| \leq L\|\vec{E}\|\mathrm{e}^{-\lambda t}$ *for any* $t \geq t_0, i \in \langle n \rangle$, *where* $\|\vec{E}\|$ *is the same as in Assumption 3;*

(ii) *The controller* (2.4) *with* $P_i = \vec{A}_i + \vec{B}_i$ *and* $Q_i = -\vec{A}_i\vec{B}_i$ *can guarantee that the INNs* (2.1) *and* (2.2) *realize FTS with a preassigned accuracy* ϵ.

Proof. (i) According to (3.1) and Assumption 2, it is obtained that

$$J_{i0}(t) \leq -\vec{B}_i\|\vec{E}\|\mathrm{e}^{\vec{A}_i t} \leq -\vec{B}_i\|\vec{E}\|\mathrm{e}^{-\lambda t},$$

Then, there will be

$$|E_i(t)| \leq \frac{1}{\vec{A}_i - \vec{B}_i}\left[-\vec{B}_i\|\vec{E}\|\mathrm{e}^{-\lambda t} + \bar{a}_i \int_0^t \left(\mathrm{e}^{\vec{A}_i(t-s)} + \mathrm{e}^{\vec{B}_i(t-s)}\right)|E_i(s - o_i(s))|\mathrm{d}s\right]. \tag{3.7}$$

When $t = t_0$, (i) is obvious. If (i) is incorrect for some $t \geq t_0,\ v' \in \langle n \rangle$, set $\tilde{T} = \inf\{t \geq t_0 : |E_{v'}(t)| > L\|\vec{E}\|\mathrm{e}^{-\lambda t}$ for some $v' \in \langle n \rangle\}$. By continuity of $E_i(t)$, there exists $v \in \langle n \rangle$ satisfying

$$|E_i(t)| \leq L\|\vec{E}\|\mathrm{e}^{-\lambda t},\ t \in [t_0, \tilde{T}), i \in \langle n \rangle, \tag{3.8}$$

and

$$|E_v(\tilde{T})| = L\|\vec{E}\|\mathrm{e}^{-\lambda \tilde{T}}. \tag{3.9}$$

By (3.7)–(3.9), we have

$$|E_v(\tilde{T})| \leq \frac{\left(-\vec{B}_v\|\vec{E}\|\mathrm{e}^{-\lambda \tilde{T}} + \bar{a}_v L\|\vec{E}\|\mathrm{e}^{\lambda \hat{o}_v} \int_0^{\tilde{T}} \left(\mathrm{e}^{\vec{A}_v(\tilde{T}-s)} + \mathrm{e}^{\vec{B}_v(\tilde{T}-s)}\right)\mathrm{e}^{-\lambda s}\mathrm{d}s\right)}{\vec{A}_v - \vec{B}_v}. \tag{3.10}$$

Using the integral inequality technique, we can obtain that

$$\begin{aligned}
&\int_0^{\tilde{T}} \left(\mathrm{e}^{\vec{A}_v(\tilde{T}-s)} + \mathrm{e}^{\vec{B}_v(\tilde{T}-s)}\right)\mathrm{e}^{-\lambda s}\mathrm{d}s \\
&= \frac{1}{-\vec{A}_v - \lambda}\left(\mathrm{e}^{-\lambda \tilde{T}} - \mathrm{e}^{\vec{A}_v \tilde{T}}\right) + \frac{1}{-\vec{B}_v - \lambda}\left(\mathrm{e}^{-\lambda \tilde{T}} - \mathrm{e}^{\vec{B}_v \tilde{T}}\right) \\
&\leq -\bar{F}_v(\lambda)\mathrm{e}^{-\lambda \tilde{T}},
\end{aligned}$$

and, hence, (3.10) implies that $|E_v(\tilde{T})| \leq R_{1v}(\lambda)L\|\vec{E}\|\mathrm{e}^{-\lambda \tilde{T}}$. This, together with Remark 3, gives

$$|E_v(\tilde{T})| < L\|\vec{E}\|\mathrm{e}^{-\lambda \tilde{T}},$$

which contradicts with (3.9). Therefore, (i) holds.

(ii) It follows from (i) that

$$|E_i(t)| \leq L\|\vec{E}\|\mathrm{e}^{-\lambda t},\ t \geq t_0, i \in \langle n \rangle. \tag{3.11}$$

Derivation from (3.5) and (3.6) yields

$$
\begin{aligned}
&-\bar{a}_i\int_0^t(-\vec{B}_i)\mathrm{e}^{\vec{B}_i(t-s)}|E_i(s-o_i(s))|\mathrm{d}s\\
\leq&(-\vec{B}_i)[\dot{E}_i(t)-\vec{A}_iE_i(t)]-(-\vec{B}_i)\mathrm{e}^{\vec{B}_it}\left[\dot{E}_i(0)-\vec{A}_iE_i(0)\right]\\
\leq&\bar{a}_i\int_0^t(-\vec{B}_i)\mathrm{e}^{\vec{B}_i(t-s)}|E_i(s-o_i(s))|\mathrm{d}s,\ i\in\langle n\rangle,t\geq t_0,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\bar{a}_i\int_0^t(-\vec{A}_i)\mathrm{e}^{\vec{A}_i(t-s)}|E_i(s-o_i(s))|\mathrm{d}s\\
\leq&\vec{A}_i[\dot{E}_i(t)-\vec{B}_iE_i(t)]-\vec{A}_i\mathrm{e}^{\vec{A}_it}\left[\dot{E}_i(0)-\vec{B}_iE_i(0)\right]\\
\leq&\bar{a}_i\int_0^t(-\vec{A}_i)\mathrm{e}^{\vec{A}_i(t-s)}|E_i(s-o_i(s))|\mathrm{d}s,\ i\in\langle n\rangle,t\geq t_0,
\end{aligned}
$$

hence,

$$
\begin{aligned}
&-\bar{a}_i\int_0^t\left((-\vec{A}_i)\mathrm{e}^{\vec{A}_i(t-s)}+(-\vec{B}_i)\mathrm{e}^{\vec{B}_i(t-s)}\right)|E_i(s-o_i(s))|\mathrm{d}s\\
\leq&(\vec{A}_i-\vec{B}_i)\dot{E}_i(t)-\left(\vec{A}_i\mathrm{e}^{\vec{A}_it}-\vec{B}_i\mathrm{e}^{\vec{B}_it}\right)\dot{E}_i(0)\\
&+\vec{A}_i\vec{B}_i\left(\mathrm{e}^{\vec{A}_it}-\mathrm{e}^{\vec{B}_it}\right)E_i(0)\\
\leq&\bar{a}_i\int_0^t\left((-\vec{A}_i)\mathrm{e}^{\vec{A}_i(t-s)}+(-\vec{B}_i)\mathrm{e}^{\vec{B}_i(t-s)}\right)|E_i(s-o_i(s))|\mathrm{d}s,\\
&i\in\langle n\rangle,t\geq t_0.
\end{aligned}
$$

From this, we can obtain that

$$
-\sum_{l=1}^{2}\tilde{J}_{il}(t)\leq(\vec{A}_i-\vec{B}_i)\dot{E}_i(t)\leq\sum_{l=1}^{2}\tilde{J}_{il}(t),\ t\geq t_0,i\in\langle n\rangle,
$$

where

$$
\tilde{J}_{i1}(t)=\left|\vec{A}_i\mathrm{e}^{\vec{A}_it}-\vec{B}_i\mathrm{e}^{\vec{B}_it}\right||\dot{E}_i(0)|+\vec{A}_i\vec{B}_i\left(\mathrm{e}^{\vec{A}_it}-\mathrm{e}^{\vec{B}_it}\right)|E_i(0)|,
$$

$$
\tilde{J}_{i2}(t)=\bar{a}_i\int_0^t\left((-\vec{A}_i)\mathrm{e}^{\vec{A}_i(t-s)}+(-\vec{B}_i)\mathrm{e}^{\vec{B}_i(t-s)}\right)|E_i(s-o_i(s))|\mathrm{d}s.
$$

It is obvious that

$$
\tilde{J}_{i1}(t)\leq\vec{A}_i\vec{B}_i\|\vec{E}\|\mathrm{e}^{\vec{A}_it}\leq\vec{A}_i\vec{B}_i\|\vec{E}\|\mathrm{e}^{-\lambda t},\tag{3.12}
$$

$$
\begin{aligned}
\tilde{J}_{i2}(t)&\leq\bar{a}_iL\|\vec{E}\|\mathrm{e}^{\lambda\hat{o}_i}\int_0^t\left((-\vec{A}_i)\mathrm{e}^{\vec{A}_i(t-s)}+(-\vec{B}_i)\mathrm{e}^{\vec{B}_i(t-s)}\right)\mathrm{e}^{-\lambda s}\mathrm{d}s\\
&\leq\bar{a}_iL\|\vec{E}\|\mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda\hat{o}_i}\int_0^t\left((-\vec{A}_i)\mathrm{e}^{(\vec{A}_i+\lambda)(t-s)}+(-\vec{B}_i)\mathrm{e}^{(\vec{B}_i+\lambda)(t-s)}\right)\mathrm{d}s.
\end{aligned}\tag{3.13}
$$

In addition,

$$\begin{aligned}
&\int_0^t \left((-\vec{A}_i)e^{(\vec{A}_i+\lambda)(t-s)} + (-\vec{B}_i)e^{(\vec{B}_i+\lambda)(t-s)}\right) ds \\
=&\frac{-\vec{A}_i}{-\vec{A}_i-\lambda}\left(1-e^{(\vec{A}_i+\lambda)t}\right) + \frac{-\vec{B}_i}{-\vec{B}_i-\lambda}\left(1-e^{(\vec{B}_i+\lambda)t}\right) \\
\leq&\frac{\vec{A}_i}{\vec{A}_i+\lambda} + \frac{\vec{B}_i}{\vec{B}_i+\lambda} \\
=&2-\lambda\bar{F}_i(\lambda) < 2+\frac{2\lambda}{\lambda+1} < 4.
\end{aligned} \tag{3.14}$$

Therefore, according to (3.11)–(3.14), we have

$$|\dot{E}_i(t)| \leq \left(\vec{A}_i\vec{B}_i + 4\bar{a}_i e^{\lambda\hat{\sigma}_i}L\right)||\vec{E}||e^{-\lambda t}.$$

This, together with (3.11), gives

$$|E_i(t)| + |\dot{E}_i(t)| \leq \vec{L}_i(\lambda)||\vec{E}||e^{-\lambda t} \leq \tilde{L}(\lambda)||\vec{E}||e^{-\lambda t},\ t \geq t_0, i \in \langle n\rangle. \tag{3.15}$$

Then, we can get that

$$\sum_{i=1}^{n}(|E_i(t)| + |\dot{E}_i(t)|) \leq n\tilde{L}(\lambda)||\vec{E}||e^{-\lambda t},\ t \geq t_0, i \in \langle n\rangle.$$

Note that when $t > T^* := \frac{\ln\tilde{L}(\lambda)+\ln||\vec{E}||-\ln\frac{\epsilon}{n}}{\lambda}$, it is obvious that

$$\tilde{L}(\lambda)||\vec{E}||e^{-\lambda t} < \frac{\epsilon}{n}.$$

Thus, we can obtain

$$|E_i(t)| + |\dot{E}_i(t)| < \frac{\epsilon}{n},\ t > \frac{\ln\tilde{L}(\lambda)+\ln||\vec{E}||-\ln\frac{\epsilon}{n}}{\lambda}, i \in \langle n\rangle.$$

Hence

$$\sum_{i=1}^{n}(|E_i(t)| + |\dot{E}_i(t)|) < \epsilon,\ t > T^*, i \in \langle n\rangle,$$

which implies that (ii) holds. The proof was completed.

Remark 6. *In contrast to the leakage delay treatment in references [21–23], the methodology adopted in this work enables direct transformation of leakage delays into the maximum of their absolute values. This approach thereby yields delay-dependent synchronization conditions without requiring intricate processing procedures.*

4. Numerical simulations

The example is given below to illustrate the applicability of the research.

Example 1. *Consider INNs* (2.1) *and* (2.2) *under parameters:*
$n = 2$, $\bar{f}_i^{(q)}(u) = \tanh|u + 1|$ $(u \in \mathbb{R}, q \in \langle 2\rangle)$,
$o_i(t) = 0.1(\sin(t) + 2)$, $b_{ij} = 0.1$ $(i, j \in \langle 2\rangle, t \geq 0)$,
$\xi_1 = 3$, $\xi_2 = 3$, $\bar{a}_1 = 1.5$, $\bar{a}_2 = 1.5$,
$\bar{b}_{11} = 2$, $\bar{b}_{12} = 5$, $\bar{b}_{21} = 7$, $\bar{b}_{22} = 2$,
$\bar{c}_{11} = 2$, $\bar{c}_{12} = 3$, $\bar{c}_{21} = 1$, $\bar{c}_{22} = 2$.

Set $\tilde{F} = 1$, $\hat{o}_i = 0.3$ $(i \in \langle 2\rangle)$, *satisfying Assumption 1.*

For $\lambda = 1$, *Assumption 2 yields* $\vec{A}_1 = -3.5$, $\vec{A}_2 = -3.5$, $\vec{B}_1 = -87$, $\vec{B}_2 = -87$, *leading to* $P_1 = -90.5$, $P_2 = -90.5$, $Q_1 = -304.5$, $Q_2 = -304.5$. *With* $L = 7$, *Remark 3's inequalities hold. Setting* $\epsilon = 0.1$ *satisfies Assumption 3.*

Under initial conditions:
$\varphi_1^{(1)}(t) \equiv 2$, $\varphi_2^{(1)}(t) \equiv -2$, $\psi_1^{(1)}(t) \equiv 4$, $\psi_2^{(1)}(t) \equiv -4$,
$\varphi_1^{(2)}(t) \equiv 8$, $\varphi_2^{(2)}(t) \equiv -8$, $\psi_1^{(2)}(t) \equiv 4$, $\psi_2^{(2)}(t) \equiv -4$,
Figures 1–4 depict state trajectories of INNs (2.1), (2.2), *and* (2.3).

Theorem 2 yields $T^* = 10.7$, *with actual synchronization occurring at* $t \approx 2.5 < T^*$. *Settling-times for varying preassigned accuracies are documented in Table 1. Collectively, the table and figures demonstrate that the proposed methodology successfully achieves AP-FTS in master-slave INNs* (2.1) *and* (2.2), *thus conclusively establishing its efficacy in solving the AP-FTS problem.*

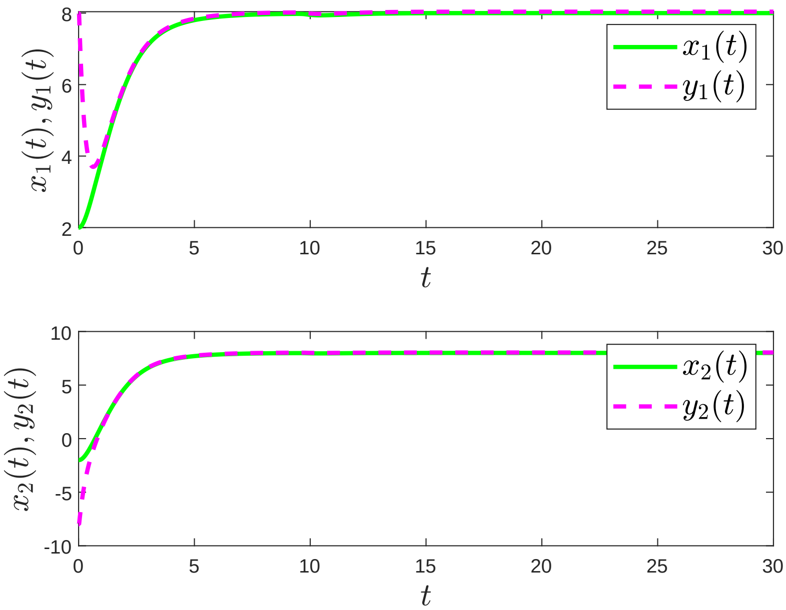

Figure 1. Response curves of master and slave INNs in Example 1.

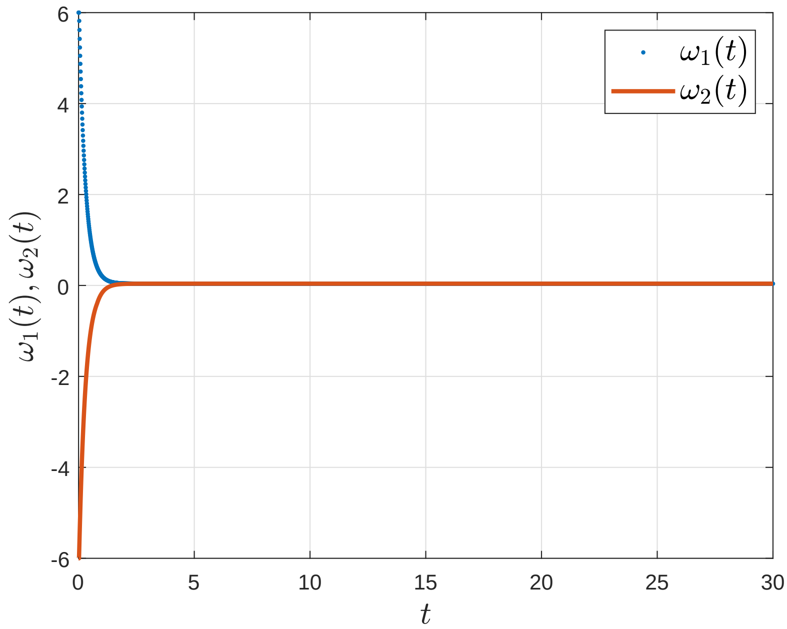

Figure 2. Response curves of error INN in Example 1.

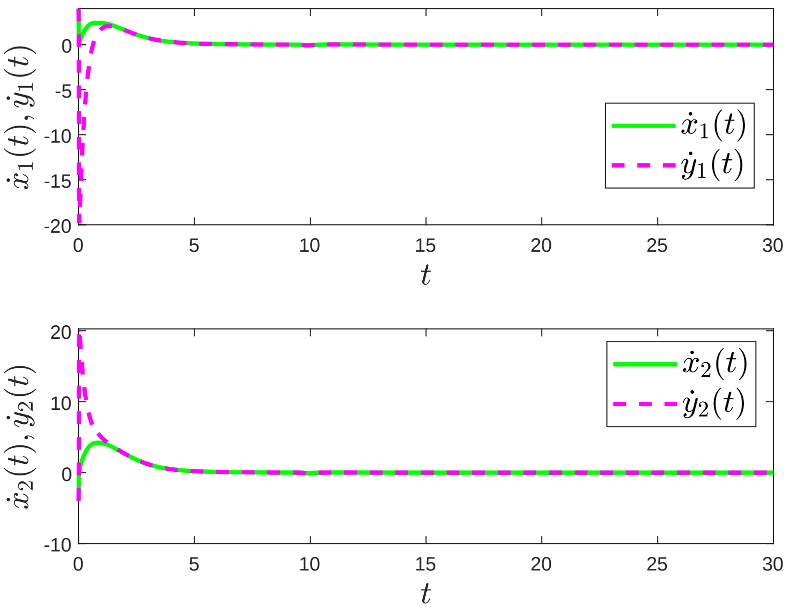

Figure 3. Response curves of master and slave INNs in Example 1.

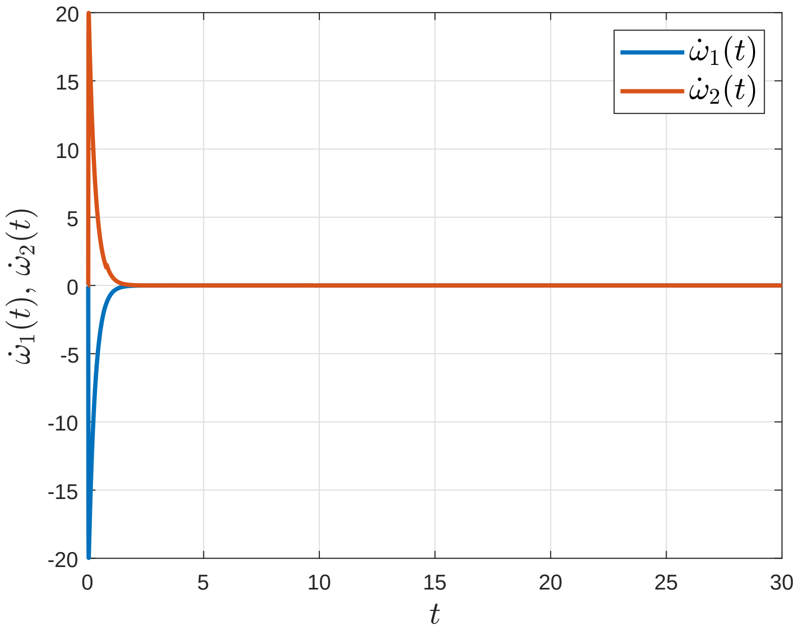

Figure 4. Response curves of error INN in Example 1.

Table 1. Preassigned-accuracy-dependent settling-times in Example 1.

Accuracy	Time
$\epsilon = 1$	8.4
$\epsilon = 0.1$	10.7
$\epsilon = 0.01$	13.0
$\epsilon = 0.001$	15.3
$\epsilon = 0.0001$	17.6
$\epsilon = 0.00001$	19.9

Example 2. *Consider INNs* (2.1) *and* (2.2) *under parameters:*
$n = 3$, $\bar{f}_i^{(q)}(u) = \tanh|u + 1|$ $(u \in \mathbb{R}, q \in \langle 3\rangle)$,
$o_i(t) = 0.1(\sin(t) + 2)$, $b_{ij} = 0.3$ $(i, j \in \langle 3\rangle, t \geq 0)$,
$\xi_1 = 3$, $\xi_2 = 3.5$, $\xi_3 = 4$, $\bar{a}_1 = 1.5$, $\bar{a}_2 = 1.8$, $\bar{a}_3 = 2$,
$\bar{b}_{11} = 2$, $\bar{b}_{12} = 1.5$, $\bar{b}_{13} = -0.8$, $\bar{b}_{21} = -1.2$, $\bar{b}_{22} = 2.5$, $\bar{b}_{23} = 1$, $\bar{b}_{31} = 0.7$, $\bar{b}_{32} = -1$, $\bar{b}_{33} = 3$,
$\bar{c}_{11} = 1.5$, $\bar{c}_{12} = -0.6$, $\bar{c}_{13} = 0.9$, $\bar{c}_{21} = -0.8$, $\bar{c}_{22} = 2$, $\bar{c}_{23} = -0.5$, $\bar{c}_{31} = 1.2$, $\bar{c}_{32} = 0.4$, $\bar{c}_{33} = 1.8$.

Set $\tilde{F} = 1$, $\hat{o}_i = 0.3$ $(i \in \langle 3\rangle)$, *satisfying Assumption 1.*

For $\lambda = 1$, *Assumption 2 yields* $\vec{A}_1 = -73.04$, $\vec{A}_2 = -72.84$, $\vec{A}_3 = -72.70$, $\vec{B}_1 = -164.1$, $\vec{B}_2 = -164.3$, $\vec{B}_3 = -164.4$. *With* $L = 7$, *Remark 3's inequalities hold. Setting* $\epsilon = 0.1$ *satisfies Assumption 3.*

We choose the initial values of the state variables as $\varphi_1^{(1)}(t) \equiv 2$, $\varphi_2^{(1)}(t) \equiv -2$, $\varphi_3^{(1)}(t) \equiv 1$, $\psi_1^{(1)}(t) \equiv 4$, $\psi_2^{(1)}(t) \equiv -4$, $\psi_3^{(1)}(t) \equiv 2$, $\varphi_1^{(2)}(t) \equiv 8$, $\varphi_2^{(2)}(t) \equiv -8$, $\varphi_3^{(2)}(t) \equiv 4$, $\psi_1^{(2)}(t) \equiv 4$, $\psi_2^{(2)}(t) \equiv -4$, $\psi_3^{(2)}(t) \equiv 2$. *Figures 5–8 depict state trajectories of INNs* (2.1), (2.2), *and* (2.3). *Collectively, the figures* *demonstrate that the proposed methodology successfully achieves AP-FTS in master-slave INNs* (2.1) *and* (2.2)*, thus conclusively establishing its efficacy in solving the AP-FTS problem.*

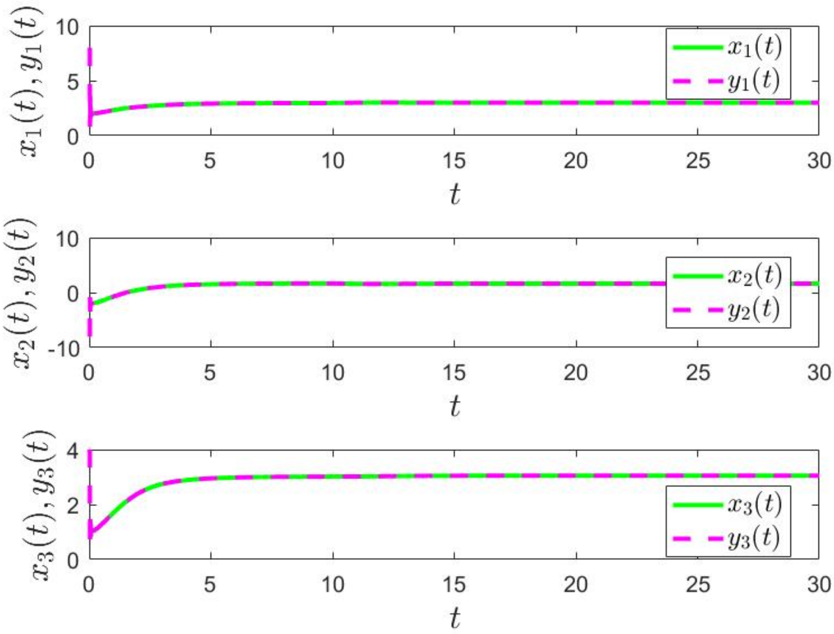

Figure 5. Response curves of master and slave INNs in Example 2.

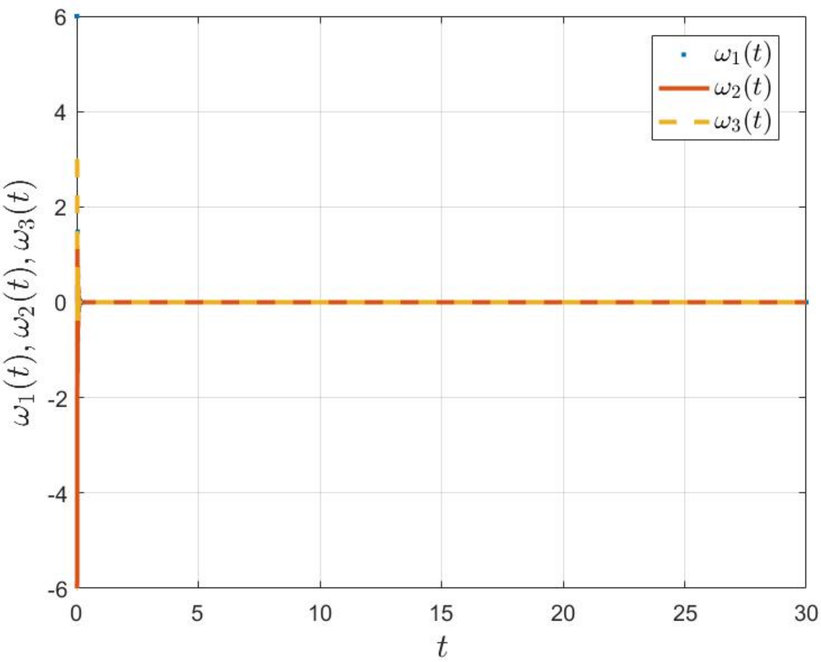

Figure 6. Response curves of error INN in Example 2.

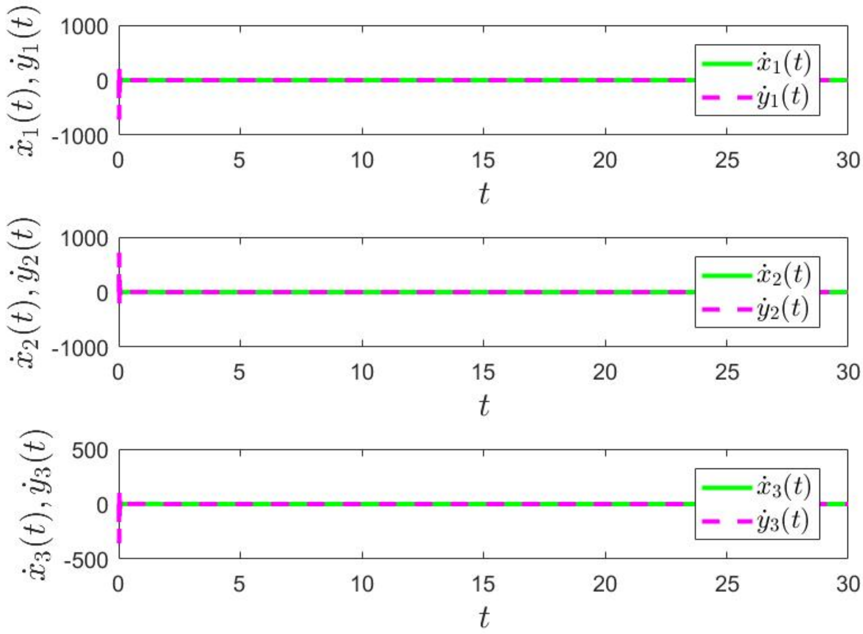

Figure 7. Response curves of master and slave INNs in Example 2.

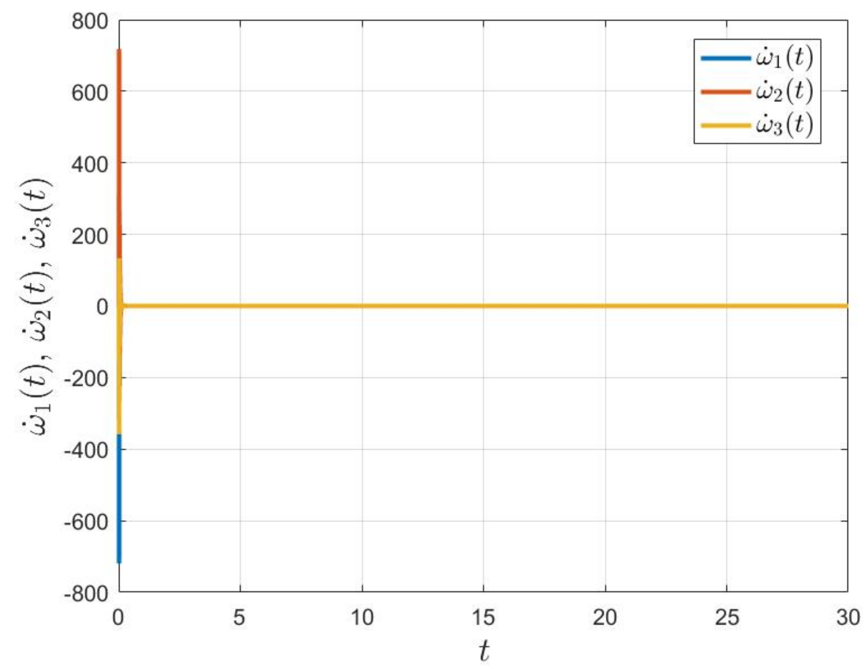

Figure 8. Response curves of error INN in Example 2.

5. Conclusions

This study presents the first investigation into AP-FTS for INNs incorporating both time-varying leakage delays and proportional delays. By introducing a direct analysis method grounded in parameterized system solutions, we establish synchronization criteria for master-slave systems. The proposed methodology offers these distinctive advantages:

i) Simplified delay treatment: Our approach enables straightforward handling of delays, significantly streamlining the proof process.

ii) Computationally efficient criteria: The derived synchronization conditions are readily solvable, reducing computational complexity.

iii) Inertial term processing without substitution: The methodology eliminates the need for variable substitutions when addressing inertial terms.

iv) Comprehensive delay coverage: Concurrent consideration of leakage and proportional delays extends the applicability of INN synchronization frameworks.

v) Explicit settling-time expression: The settling-time bound is derived explicitly without constructing complex auxiliary functions.

Finally, regarding future research directions:

(a) The present framework can be extended to include unbounded distributed delays.

(b) L_p AP-FTS for INNs represents a promising avenue for further investigation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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