Nonradial singular solutions for elliptic equations with exponential nonlinearity

Jingyue Cao, Yunkang Shao, Fangshu Wan*, Jiaqi Wang and Yifei Zhu

School of Mathematical Sciences, Anhui University, Hefei 230601, China

* Correspondence: Email: wfangshu@ustc.edu.cn.

Abstract: For any $R > 0$, infinitely many nonradial singular solutions can be constructed for the following equation:

$$-\Delta u = e^u \text{ in } B_R \backslash \{0\},$$

where $B_R = \{ x \in \mathbb{R}^N (N \geq 3) : |x| < R \}$. To construct nonradial singular solutions, we need to consider asymptotic expansion at the isolated singular point $x = 0$ of a prescribed solution of (0.1). Then, nonradial singular solutions of (0.1) can be constructed by using the asymptotic expansion and introducing suitable weighted Hölder spaces.

Keywords: nonradial singular solutions; asymptotic expansions; exponential nonlinearity; weighted spaces

1. Introduction

We are interested in singular solutions of the following equation with exponential nonlinearity:

$$\Delta u + e^u = 0 \text{ in } B_R \backslash \{0\},$$

where $R > 0$ and $B_R = \{ x \in \mathbb{R}^N (N \geq 3) : |x| < R \}$ is a ball.

By a singular solution of (1.1) we mean that $u \in C^2(B_R \backslash \{0\})$ and 0 is a nonremovable singular point of $u$.

It is easily known that (1.1) admits a (trivial) radial singular solution:

$$U_s(x) = U_s(|x|) := -2 \ln |x| + \ln[2(N - 2)].$$

We are mainly concerned with nonradial singular solutions of (1.1) in this paper.
When $N = 2$, by using the moving plane method, the authors of [1] proved that every solution of

$$\begin{cases}
\Delta u + e^u = 0 \quad \text{in} \quad \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^u \, dx < \infty
\end{cases}$$

(1.3)

has the form

$$u(x) = \ln \frac{32\lambda^2}{(4 + \lambda^2|x - x_0|^2)^2}, \quad \lambda > 0, \quad x_0 \in \mathbb{R}^2.$$  

For $n > 0$, symmetry and uniqueness results were obtained in [2] for the solutions of the following problem:

$$\begin{cases}
\Delta u + |x|^{2(n-1)}e^u = 0 \quad \text{in} \quad \mathbb{R}^2, \\
\int_{\mathbb{R}^2} |x|^{2(n-1)}e^u \, dx < \infty
\end{cases}$$

(1.4)

If $n = 1$, problem (1.4) reduces to (1.3); also, classification of solutions of (1.4) can be found in [1]. Under the condition that $n \geq 2$ is an integer, the authors of [2] showed that problem (1.4) admits radial and nonradial solutions, but, when $n > 0$ is not an integer, problem (1.4) only has radial solutions. Note that, for each $n > 0$, if $u(x)$ is a solution of (1.4), we can perform the following transformation:

$$v(x) = 2(n - 1) \ln |x| + u(x),$$

we see that $v(x)$ satisfies the following equation

$$\begin{cases}
\Delta v + e^v = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}, \\
\int_{\mathbb{R}^2 \setminus \{0\}} e^v \, dx < \infty
\end{cases}$$

(1.5)

The results in [2] imply that (1.5) admits a family of radial and nonradial singular solutions.

The asymptotic behavior of singular solutions of the problem given by

$$\begin{cases}
\Delta u + |x|^{2(n-1)}e^u = 0 \quad \text{in} \quad D_1 \setminus \{0\}, \\
\int_{D_1 \setminus \{0\}} e^u \, dx < \infty
\end{cases}$$

(1.6)

where $D_1 \subset \mathbb{R}^2$ is the unit disc, was studied in [3]. The authors of [3] obtained that if $u \in C^2(D_1 \setminus \{0\})$ is a singular solution of (1.6), then there is $\alpha > -2$ such that

$$u(x) = \alpha \ln |x| + O(1) \quad \text{as} \quad |x| \to 0.$$  

In a recent paper [4], the authors continued the study in [3] and obtained asymptotic expansions up to arbitrary orders for $u(x)$ as $|x| \to 0$.

Under the condition that $N \geq 2$, the structure of finite Morse index solutions of the equation

$$\Delta u + e^u = 0 \quad \text{in} \quad \mathbb{R}^N$$

(1.7)

was studied in [5–7]. In particular, under the condition that $N = 3$, the asymptotic behavior at $x = 0$ of solutions $u$ with $|x|^2 e^u \in L^\infty(\mathbb{R}^3)$ of the equation

$$\Delta u + e^u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \{0\},$$

(1.8)
was classified in [8]. For the case that $N = 3$, if we write

$$u(x) = -2 \ln |x| + \Theta(\theta)$$

where $([x], \theta) \in (0, \infty) \times S^2$ denotes the spherical coordinates in $\mathbb{R}^3 \setminus \{0\}$, we find that $\Theta(\theta)$ must satisfy

$$\Delta_{S^2} \Theta - 2 + e^\Theta = 0$$

(1.9)
on $S^2$ where $\Delta_{S^2}$ is the Laplace-Beltrami operator on $(S^2, g_0)$ and $g_0$ is the standard round metric. It means that the Gaussian curvature of the metric $g = e^\Theta g_0$ on $S^2$ is $\frac{1}{2}$. This and related equations have been studied for more than three decades. Chang and Yang [9] and Onofri [10] described all regular solutions of (1.9). Specifically, axially symmetric solutions of (1.9) can be written explicitly as

$$\Theta(\theta) = \log 2 - 2 \log(\sqrt{c^2 + 1 - c \cos \theta})$$

where $c \in \mathbb{R}$ is constant and $\theta \in [0, \pi]$ is the geodesic distance from the north pole of $S^2$. Hence,

$$u(x) = -2 \ln |x| + \log 2 - 2 \log(\sqrt{c^2 + 1 - c \cos \theta})$$

is a one-parameter family of non-radial singular solutions of (1.8).

Recently, singular solutions in different settings have also been studied in [11] and [12]. The authors of [12] obtained the existence and asymptotic behavior of singular solutions to quasilinear elliptic inequalities with nonlocal terms. Moreover, by using mini-max and asymptotic approximation methods, the existence of positive singular solutions to the planar logarithmic Choquard equation with exponential nonlinearity was established in [11].

In this paper, we study singular solutions of (1.1) in $B_R \subset \mathbb{R}^N$ $(N \geq 3)$. We are interested in not only the asymptotic behavior of singular solutions of (1.1) at $x = 0$, but also the existence of nonradial singular solutions of (1.1). The structure of nonradial singular solutions of the equation

$$\Delta u + e^u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\} \quad \text{with} \quad N \geq 3$$

(1.10)

remains largely open. Motivated by the main ideas in [13], the authors of [14, 15] obtained infinitely many nonradial singular solutions of (1.10) of the following form given $4 \leq N \leq 10$:

$$u(x) = -2 \ln |x| + \Theta(\theta),$$

(1.11)

where $\Theta(\theta)$ is a non-constant solution of the equation

$$\Delta_{S^{N-1}} \Theta - 2(N - 2) + e^\Theta = 0$$

(1.12)
on $S^{N-1}$, where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on $(S^{N-1}, g_0)$. They constructed infinitely many axially symmetric non-constant classical solutions of (1.12). The only singular solutions to (1.10) known so far are the (trivial) radial singular solution $U_s(x)$ and the solutions given in (1.11). It is clear that they are also the singular solutions to (1.1).

We will construct a new type of singular solutions of (1.1) in the following form:

$$u(x) - U_s(x) = O(|x|^\epsilon) \quad \text{as} \quad |x| \to 0,$$

(1.13)

for some $\epsilon > 0$.

Our main result is as follows.
Theorem 1.1. For any $R > 0$, Eq (1.1) admits infinitely many nonradial singular solutions $u(x)$ of the following form:

$$u(x) = U_s(x) + O(|x|^{-\sigma_+^{(2)}}) \quad \text{as} \ |x| \to 0,$$

where

$$\sigma_+^{(2)} = -\frac{1}{2}(N-2) + \frac{1}{2}\sqrt{N^2 - 4N + 20} > 0.$$ 

It is known from Theorem 1.1 that the parameter $\epsilon$ in (1.13) is $\sigma_+^{(2)}$.

To prove Theorem 1.1, we firstly study the detailed asymptotic behavior at $x = 0$ of a prescribed singular solution $u$ of

$$\Delta u + e^u = 0 \quad \text{in} \ B \setminus \{0\}$$

with the form (1.13), where $B := B_1 = \{x \in \mathbb{R}^N \ (N \geq 3) : \ |x| < 1\}$ is the unit ball. Then, the infinitely many nonradial singular solutions of the form (1.14) can be constructed by using the asymptotic expansion and introducing suitable Hölder spaces.

This paper is organized as follows. In Section 2, we obtain asymptotic expansions near the isolated singular point $x = 0$ of solutions of (1.15). In Section 3, we establish weighted Hölder spaces and invertible operators related to our Equation (1.15). In Section 4, we construct infinitely many singular solutions of (1.1) and show that the singular solutions that we have constructed are non-radial singular solutions.

2. Asymptotic expansion of a prescribed singular solution $u \in C^2(B \setminus \{0\})$ of (1.15) that satisfies (1.13)

We will establish asymptotic expansions of the singular solution $u \in C^2(B \setminus \{0\})$ of (1.15) such that (1.13) is satisfied.

Let $v(x) = u(x) - U_s(x)$. Then $v(x)$ satisfies

$$-\Delta v = 2(N-2)|x|^{-2}(e^v - 1) \quad \text{in} \ B \setminus \{0\}.$$ 

(2.1)

Making the following transformations:

$$t = \ln r, \quad w(t, \theta) = v(r, \theta),$$

we see from (2.1) that $w(t, \theta)$ satisfies

$$w_{tt} + (N-2)w_t + \Delta g^{-1}w + 2(N-2)(e^w - 1) = 0 \quad \text{in} \ (-\infty, 0) \times S^{N-1}.$$ 

(2.2)

We write (2.2) in the following forms

$$w_{tt} + (N-2)w_t + \Delta g^{N-1}w + 2(N-2)w + 2(N-2)(e^w - 1 - w) = 0 \quad \text{in} \ (-\infty, 0) \times S^{N-1}$$

(2.3)

and

$$\mathcal{L}w + \mathcal{F}(w) = 0 \quad \text{in} \ (-\infty, 0) \times S^{N-1},$$

(2.4)

where

$$\mathcal{L}w = w_{tt} + (N-2)w_t + \Delta g^{N-1}w + 2(N-2)w, \quad \mathcal{F}(w) = 2(N-2)(e^w - 1 - w).$$
Moreover, (1.13) implies that
\[ w(t, \theta) = O(e^{\epsilon t}) \] uniformly for \( \theta \in S^{N-1} \) as \( t \to -\infty \).

(2.5)

Define a linearized operator
\[ \mathcal{L} = \frac{\partial^2}{\partial t^2} + (N - 2) \frac{\partial}{\partial t} + \Delta_{S^{N-1}} + 2(N - 2). \]

(2.6)

Obviously, \( \mathcal{L} \) can decouple into infinitely many ordinary differential operators, i.e.,
\[ \mathcal{L}_k = \frac{d^2}{dt^2} + (N - 2) \frac{d}{dt} - \lambda_k + 2(N - 2) \]
for \( k = 0, 1, 2, \ldots \), where \( \lambda_k \) is the \( k \)-th eigenvalue of
\[ -\Delta_{S^{N-1}} Q = \lambda Q \]
and \( \lambda_k = k(N - 2 + k) \) with the following multiplicity:
\[ m_k = \frac{(N - 2 + 2k)(N - 3 + k)!}{k!(N - 2)!}. \]

The \( \{Q_{1}^{k}(\theta), \ldots, Q_{m_k}^{k}(\theta)\} \) with \( \|Q_{i}^{k}\|_{L^2(S^{N-1})} = 1 \) denotes the basis of the eigenspace \( H_k(S^{N-1}) \subset L^2(S^{N-1}) \) corresponding to \( \lambda_k \). Then two roots of characteristic polynomial of (2.7) are as follows:
\[ \sigma_{\pm}^{(k)} = -\frac{1}{2}(N - 2) \pm \frac{1}{2} \sqrt{(N - 2)(N - 10) + 4k(N - 2 + k)}. \]

(2.9)

For \( k = 0 \), we have from (2.9) that
\[ \sigma_{\pm}^{(0)} = \begin{cases} 
-\frac{1}{2}(N - 2) \pm \frac{i}{2} \sqrt{(N - 2)(10 - N)}, & \text{for } 3 \leq N \leq 9, \\
-\frac{1}{2}(N - 2) < 0, & \text{for } N = 10, \\
-\frac{1}{2}(N - 2) \pm \frac{i}{2} \sqrt{(N - 2)(N - 10)} < 0, & \text{for } N \geq 11.
\end{cases} \]

(2.10)

For \( k = 1 \),
\[ \sigma_{\pm}^{(1)} = -\frac{(N - 2)}{2} \pm \frac{|N - 4|}{2}. \]

(2.11)

Then,
\[ \sigma_{+}^{(1)} = \begin{cases} 
0, & \text{for } N = 3, \\
-\frac{(N - 2)}{2} + \frac{|N - 4|}{2} < 0, & \text{for } N \geq 4
\end{cases} \]
and
\[ \sigma_{-}^{(1)} = -\frac{(N - 2)}{2} - \frac{|N - 4|}{2} < 0, & \text{for } N \geq 3. \]

(2.12)

(2.13)

For \( k \geq 2 \), the fact that \( k(N - 2 + k) > 2(N - 2) \) implies that
\[ (N - 2)(N - 10) + 4k(N - 2 + k) > (N - 2)^2, \]
we see from (2.9) that
\[ \sigma_{+}^{(k)} > 0, \quad \sigma_{-}^{(k)} < 0. \]

(2.14)

It is clear that
\[ \sigma_{+}^{(k+1)} > \sigma_{+}^{(k)} > 0, \quad \sigma_{-}^{(k+1)} < \sigma_{-}^{(k)} < 0 \] for any \( k \geq 2 \).
Proposition 2.1. Assume that $N \geq 3$ and $u = u(r)$ is a radial solution of (1.15) that satisfies

$$u(r) - U_\epsilon(r) = O(r^\epsilon) \text{ for } r \text{ near } 0 \text{ and some } \epsilon > 0.$$  

Then

$$u(r) \equiv U_\epsilon(r) \text{ for } r \in (0, 1].$$

Proof. By applying the following transformations:

$$v(r) = u(r) - U_\epsilon(r), \quad w(t) = v(r), \quad t = \log r,$$

by (2.3), $w(t)$ satisfies the following ordinary differential equation (ODE):

$$w_{tt} + (N - 2)w_t + 2(N - 2)w + f(w) = 0 \quad \text{in } (-\infty, 0),$$

(2.15)

where

$$f(w) = 2(N - 2)(e^w - 1 - w) = O(w^2) = O(e^{2t}).$$

Note that $w(t) = O(e^{2t})$ for $t$ near $-\infty$. Therefore, for $3 \leq N \leq 9$,

$$w(t) = A_1 e^{\gamma t} \cos \gamma t + A_2 e^{\gamma t} \sin \gamma t - B_1 e^{\gamma t} \cos \gamma t \int_{-\infty}^t e^{-\gamma s} [-f(w(s))] \sin \gamma sds$$

$$-B_2 e^{\gamma t} \sin \gamma t \int_{-\infty}^t e^{-\gamma s} [-f(w(s))] \cos \gamma sds,$$

(2.16)

where $|B_1| = |B_2| = 1/\gamma$, $|f(w(t))| = O(e^{2t})$,

$$\tau = -\frac{1}{2}(N - 2), \quad \gamma = \frac{1}{2} \sqrt{(N - 2)(10 - N)}.$$

Since $w(t) \to 0$ as $t \to -\infty$, we obtain from (2.16) that $A_1 = A_2 = 0$ and

$$w(t) = B_1 e^{\gamma t} \cos \gamma t \int_{-\infty}^t e^{-\gamma s} O(w^2(s)) \sin \gamma sds + B_2 e^{\gamma t} \sin \gamma t \int_{-\infty}^t e^{-\gamma s} O(w^2(s)) \cos \gamma sds,$$

(2.17)

and

$$|w(t)| \leq Ce^{2\beta t} := e^{\beta t + e^{2t}}$$

for some fixed $\beta > 0$ with $C = e^\beta$ and $t$ near $-\infty$.  

(2.18)

Substituting (2.18) into (2.17), we see that

$$|w(t)| \leq e^{3\beta t + 4te^{2t}} \text{ for } t \text{ near } -\infty.$$  

(2.19)

We can do the same process to obtain that $w(t) \equiv 0$ for $t \leq -\frac{4\beta}{e}$. Since $w(t)$ satisfies the ODE in (2.15), then $w(t) \equiv 0$ for $t \in (-\infty, 0)$.

For $N = 10$, we see that

$$w(t) = A_1 e^{\gamma t} + A_2 e^{\gamma t}$$

$$-B_1 e^{\gamma t} \int_{-\infty}^t e^{-\gamma s} [-f(w(s))] ds - B_2 e^{\gamma t} \int_{-\infty}^t e^{-\gamma s} [-f(w(s))] ds,$$

(2.20)
where $|B_1| = |B_2| = 1$. Note that $\tau = \frac{(N-2)}{2} < 0$. Since $w(t) \to 0$ as $t \to -\infty$, we see that $A_1 = A_2 = 0$. Arguments similar to those in the proof for the case of $3 \leq N \leq 9$ imply that $w(t) \equiv 0$ for $t \in (-\infty, 0)$.

For $N \geq 11$, we see that

$$w(t) = A_1 e^{\sigma_+(0)t} + A_2 e^{\sigma_-(0)t} - B_1 e^{\sigma_+(0)} \int_{-\infty}^{t} e^{-\sigma_-(0)s} [-f(w(s))] ds - B_2 e^{\sigma_-(0)} \int_{-\infty}^{t} e^{-\sigma_-(0)s} [-f(w(s))] ds,$$

(2.21)

where $|B_1| = |B_2| = \left| \frac{1}{\sigma_+ - \sigma_-} \right|$. Note that $\sigma_+ < 0$ and $\sigma_- < 0$. Since $w(t) \to 0$ as $t \to -\infty$, we see that $A_1 = A_2 = 0$. Arguments similar to those in the proof for the case of $3 \leq N \leq 9$ imply that $w(t) \equiv 0$ for $t \in (-\infty, 0)$. This completes the proof of this proposition.

**Lemma 2.2.** Assume that $N \geq 3$ and $u \in C^2(B \setminus \{0\})$ is a singular solution of (1.15) that satisfies (1.13). Defining $w(t, \theta) = u(x) - U_s(x)$ and $t = \ln r$, it follows that $w(t, \theta) = O(e^{\sigma t})$ for $t \in (-\infty, -1)$, and

$$\max_{S^{N-1}} |w(t, \theta)| \leq C e^{\sigma t} \quad \text{for } t \in (-\infty, -1].$$

(2.22)

**Proof.** Let

$$w(t, \theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} w_j^k(t) Q_j^k(\theta).$$

Then $w_j^k(t)$ satisfies the following equation:

$$(w_j^k)'(t) + (N-2)(w_j^k)'(t) + [2(N-2) - A_k]w_j^k(t) = -g_j^k(t),$$

(2.23)

where

$$g_j^k(t) = \int_{S^{N-1}} F(w(t, \theta)) Q_j^k(\theta) d\theta.$$

Note that

$$||w||_{L^2(S^{N-1})}^2 = \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} ||w_j^k(t)||^2, \quad ||F(w)||_{L^2(S^{N-1})}^2 = \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} ||g_j^k(t)||^2.$$

Since $F(w) = O(w^2)$ and $w(t, \theta) = O(e^{\sigma t})$, we see that

$$||F(w)||_{L^2(S^{N-1})} = O(e^{\sigma t} ||w||_{L^2(S^{N-1})}).$$

(2.24)

On the other hand, it follows from (2.23) that for $k \geq 2$, $T \ll -1$ and $t < T$,

$$w_j^k(t) = A_{k,1} e^{\sigma_+(0)t} + A_{k,2} e^{\sigma_-(0)t} + B_{j,1}^k \int_{-\infty}^{t} e^{\sigma_+(s)} \int_{-\infty}^{\sigma_+(s)} [-g_j^k(s)] ds - B_{j,2}^k \int_{-\infty}^{t} e^{\sigma_+(s)} \int_{-\infty}^{\sigma_+(s)} [-g_j^k(s)] ds,$$

where

$$|B_{j,1}^k| = |B_{j,2}^k| = \left| \frac{1}{\sigma_+} - \frac{1}{\sigma_-} \right|.$$
Since $w_j^k(t) \to 0$ as $t \to -\infty$, we have that $A_{j,2}^k = 0$. Moreover,

\[
w_j^k(T) = A_{j,1}^k e^{\int_0^T \gamma_0(s) ds} - B_{j,2}^k \int_{-\infty}^T e^{\int_0^{T-s} \gamma_0(s) ds} [-g_j^k(s)] ds
\]

and

\[
A_{j,1}^k = O(e^{-\alpha_{\gamma_0}^k T}).
\]

Then,

\[
w_j^k(t) = O(e^{\alpha_{\gamma_0}^k (T-t)}) + B_{j,1}^k \int_t^T e^{\int_0^{T-s} \gamma_0(s) ds} [-g_j^k(s)] ds - B_{j,2}^k \int_{-\infty}^T e^{\int_0^{T-s} \gamma_0(s) ds} [-g_j^k(s)] ds
\]

and for small enough $\delta > 0$,

\[
[w_j^k(t)]^2 \leq O(e^{2\alpha_{\gamma_0}^k (T-t)}) + 4(B_{j,1}^k)^2 \left( \int_0^T e^{\int_0^{T-s} \gamma_0(s) ds} \left( \int_0^T e^{\int_0^{T-s} \gamma_0(s) ds} g_j^k(s)^2 ds \right) ds \right) + 4(B_{j,1}^k)^2 \left( \int_{-\infty}^0 e^{\int_0^{T-s} \gamma_0(s) ds} \left( \int_{-\infty}^T e^{\int_0^{T-s} \gamma_0(s) ds} g_j^k(s)^2 ds \right) ds \right) \leq C e^{2\alpha_{\gamma_0}^k (T-t)} + C \delta \int_t^T e^{2\alpha_{\gamma_0}^k (T-s)} g_j^k(s)^2 ds + C \delta \int_{-\infty}^T e^{2\alpha_{\gamma_0}^k (t-s)} g_j^k(s)^2 ds,
\]

with constants $C > 0$ and $C_\delta > 0$ being dependent on $\delta$ and independent of $(j,k)$.

For $k = 1$, $T \ll -1$ and $t < T$,

\[
w_j^1(t) = A_{j,1}^1 e^{\alpha_{\gamma_0}^1 t} + A_{j,2}^1 e^{\alpha_{\gamma_0}^1 t} - B_{j,1}^1 \int_{-\infty}^t e^{\int_0^{t-s} \gamma_0(s) ds} [-g_j^1(s)] ds - B_{j,2}^1 \int_{-\infty}^t e^{\int_0^{t-s} \gamma_0(s) ds} [-g_j^1(s)] ds,
\]

where

\[
|B_{j,1}^1| = |B_{j,2}^1| = \left| \frac{1}{\sigma_{\gamma_0} - \alpha_{\gamma_0}^1} \right|.
\]

Note that $\sigma_{\gamma_0}^1 \leq 0$ and $\sigma_{\gamma_0}^1 < 0$. Since $w_j^1(t) \to 0$ as $t \to -\infty$, we have that $A_{j,1}^1 = A_{j,2}^1 = 0$,

\[
w_j^1(t) = -B_{j,1}^1 \int_{-\infty}^t e^{\int_0^{t-s} \gamma_0(s) ds} [-g_j^1(s)] ds - B_{j,2}^1 \int_{-\infty}^t e^{\int_0^{t-s} \gamma_0(s) ds} [-g_j^1(s)] ds
\]

and

\[
(w_j^1(t))^2 = O(e^{4\alpha_{\gamma_0}}).
\]

Note that

\[
(g_j^1(t))^2 \leq C\|F(w)\|_{L^2(\mathbb{R}^{N-1})}^2 \leq Ce^{4\alpha_{\gamma_0}}.
\]

Similarly, we have

\[
(g_j^0(t))^2 = O(e^{4\alpha_{\gamma_0}}), \quad (w_j^0(t))^2 = O(e^{4\alpha_{\gamma_0}}).
\]
Then,
\[
\sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (w^j_k(t))^2 \leq C \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} e^{2\sigma^*_+(t-T)} + C_{\delta} \int_t^T e^{2(\sigma^*_+ - \delta)(t-s)} \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (g^j_k(s))^2 ds \\
+ C_{\delta} \int_{-\infty}^{t} e^{2(\sigma^*_+ + \delta)(t-s)} \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (g^j_k(s))^2 ds \\
\leq C \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} e^{2\sigma^*_+(t-T)} \\
+ C \int_t^T e^{2(\sigma^*_+ - \delta)(t-s)} e^{4\epsilon s} ds + C \int_t^T e^{2(\sigma^*_+ - \delta)(t-s)} e^{2\epsilon s} \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (w^j_k(s))^2 ds \\
+ C \int_{-\infty}^{t} e^{2(\sigma^*_+ + \delta)(t-s)} e^{4\epsilon s} ds + C \int_{-\infty}^{t} e^{2(\sigma^*_+ + \delta)(t-s)} e^{2\epsilon s} \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (w^j_k(s))^2 ds.
\]

Notice that
\[
\sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (g^j_k(t))^2 = \|F(w)\|_{L^2(S^{N-1})}^2 - \left[ (g^1_0(t))^2 + \sum_{j=1}^{N} (g^j_1(t))^2 \right]
\]
and
\[
\|F(w)\|_{L^2(S^{N-1})}^2 = O(e^{2\epsilon t})\|w\|_{L^2(S^{N-1})} = O(e^{2\epsilon t}) \left[ \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (w^j_k(t))^2 + (w^0_1(t))^2 + \sum_{j=1}^{N} (w^j_1(t))^2 \right].
\]

since $F(w) = O(w^2)$. We also know that
\[
\sum_{k=2}^{\infty} \sum_{j=1}^{m_k} e^{2\sigma^*_+(t-T)} = \sum_{k=2}^{\infty} m_k e^{2\sigma^*_+(t-T)} = O(e^{2\sigma^*_+(t-T)}),
\]

since
\[
\lim_{k \to \infty} \frac{m_{k+1} e^{2(\sigma^*_+ + \epsilon)(t-T)}}{m_k e^{2(\sigma^*_+ - \epsilon)(t-T)}} = \lim_{k \to \infty} \left[ \frac{m_{k+1}}{m_k} e^{2(\sigma^*_+ + \epsilon)(t-T)} \right] = e^{2(t-T)} < \frac{1}{2}.
\]

Let $[W(t)]^2 = \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (w^j_k(t))^2$. We have that, if $4\epsilon \neq 2\sigma^*_+ - \delta$,
\[
[W(t)]^2 \leq C e^{2\sigma^*_+(t-T)} + C e^{4\epsilon t} + C \int_t^T e^{2(\sigma^*_+ - \delta)(t-s)} e^{4\epsilon s} ds \\
+ C \int_t^T e^{2(\sigma^*_+ - \delta)(t-s)} e^{2\epsilon s} [W(s)]^2 ds + C \int_{-\infty}^{t} e^{2(\sigma^*_+ + \delta)(t-s)} e^{2\epsilon s} [W(s)]^2 ds.
\]

Now we show that, for $t < T \ll -1$,
\[
\|w\|_{L^2(S^{N-1})} = \left[ \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} (w^j_k(t))^2 \right]^{\frac{1}{2}} = O(e^{t^{0.5}}). \tag{2.29}
\]
In fact, two cases can occur: (i) $4\epsilon \geq [2\sigma_+^{(2)} - \delta]$ and (ii) $4\epsilon < [2\sigma_+^{(2)} - \delta]$. It suffices to consider that $4\epsilon > [2\sigma_+^{(2)} - \delta]$ in the case (i). So

$$[W(t)]^2 \leq Ce^{(2\sigma_+^{(2)} - \delta)(t-T)} + C \int_t^T e^{(2\sigma_+^{(2)} - \delta)(t-s)} e^{2\epsilon s} [W(s)]^2 ds$$

$$+ C \int_{-\infty}^t e^{(2\sigma_+^{(2)} - \delta)(t-s)} e^{2\epsilon s} [W(s)]^2 ds.$$

Set

$$K_1(t) = \int_t^T e^{(2\sigma_+^{(2)} - \delta)(t-s)} [W(s)]^2 ds, \quad K_2(t) = \int_{-\infty}^t e^{(2\sigma_+^{(2)} - \delta)(t-s)} [W(s)]^2 ds.$$

Then, if $|T|$ is large,

$$(K_2 - K_1)'(t) = (2\sigma_-^{(2)} + \delta)K_2(t) - (2\sigma_+^{(2)} - \delta)K_1(t) + 2[W(t)]^2$$

$$\leq (2\sigma_-^{(2)} + \delta)K_2(t) - (2\sigma_+^{(2)} - \delta)K_1(t) + Ce^{2\epsilon T} (K_1(t) + K_2(t)) + Ce^{(2\sigma_+^{(2)} - \delta)(t-T)}$$

where $\sigma_-^{(2)} < 0$, $\sigma_+^{(2)} > 0$. Since $K_1(t) \to 0$ and $K_2(t) \to 0$ as $t \to -\infty$, then for $t < T$,

$$K_2(t) \leq K_1(t) + Ce^{(2\sigma_+^{(2)} - \delta)t}.$$

Substituting (2.31) into (2.30), we have

$$[W(t)]^2 \leq Ce^{(2\sigma_+^{(2)} - \delta)t} + Ce^{2\epsilon T} \int_t^T e^{(2\sigma_+^{(2)} - \delta)(t-s)} [W(s)]^2 ds.$$

It follows from arguments similar to those in [16, 17] that, for $t < T$ (enlarge $|T|$ if necessary),

$$[W(t)]^2 \leq Ce^t e^{(2\sigma_+^{(2)} - \delta - \epsilon \epsilon[T])},$$

where $\epsilon_T = Ce^{2\epsilon T}$ (i.e., $C$ is independent of $\epsilon$). On the other hand, (2.27) and (2.28) imply that

$$(w_j^0(t))^2 = O(e^{4\epsilon t}) = O(e^{(2\sigma_+^{(2)} - \delta)t}), \quad j = 1, 2, \ldots, m_1,$$

$$(w_1^0(t))^2 = O(e^{4\epsilon t}) = O(e^{(2\sigma_+^{(2)} - \delta)t})$$

where $4\epsilon > 2\sigma_+^{(2)} - \delta$. Therefore, for $t < T$ (i.e., $T$ is sufficiently negative),

$$\sum_{k=0}^\infty \sum_{j=1}^{m_k} (w_j^k(t))^2 = O(e^{(2\sigma_+^{(2)} - \delta - \epsilon \epsilon[T])})$$

and

$$\|w\|_{L^2(S^{N-1})} = O(e^{(\sigma_+^{(2)} - 4\epsilon\epsilon[T])}).$$

Using (2.35), we obtain

$$\|\mathcal{F}(w)\|_{L^2(S^{N-1})}^2 = O(e^{2\epsilon T}) \|w\|_{L^2(S^{N-1})}^2 = O(e^{(2\sigma_+^{(2)} - \delta - \epsilon \epsilon + 2\epsilon T)}).$$


Choosing $\delta$ sufficiently small such that $\delta < 2\epsilon - \epsilon_T$, then

\[
\sum_{j=1}^{m_2} |w_j^2(t)| \leq Ce^{2\sigma_+^{(2)}(t-T)} + C \int_t^T e^{2\sigma_+^{(2)}(t-s)} \sum_{j=1}^{m_2} |g_j^2(s)| ds \\
+ C \int_{-\infty}^t e^{2\sigma_+^{(2)}(t-s)} \sum_{j=1}^{m_2} |g_j^2(s)| ds \\
\leq Ce^{2\sigma_+^{(2)t}} + C \int_t^T e^{2\sigma_+^{(2)}(t-s)} e^{(2\sigma_+^{(2)} - \frac{4}{3} - \frac{\epsilon_T}{2})s} ds \\
+ C \int_{-\infty}^t e^{2\sigma_+^{(2)}(t-s)} e^{(2\sigma_+^{(2)} - \frac{4}{3} - \frac{\epsilon_T}{2})s} ds \\
\leq Ce^{2\sigma_+^{(2)t}}.
\]

So for $t < T$,

\[
\sum_{j=1}^{m_2} |w_j^2(t)|^2 \leq \left[ \sum_{j=1}^{m_2} |w_j^2(t)| \right]^2 \leq Ce^{2\sigma_+^{(2)t}}. \tag{2.37}
\]

Moreover, we choose $0 < \delta < \min\{2\epsilon - \epsilon_T, 2\sigma_+^{(3)} - 2\sigma_+^{(2)}\}$; then,

\[
\sum_{k=3}^{\infty} \sum_{j=1}^{m_k} (w_j^k(t))^2 \leq C \sum_{k=3}^{\infty} \sum_{j=1}^{m_k} e^{2\sigma_+^{(3)}(t-T)} + C_\delta \int_t^T e^{(2\sigma_+^{(3)} - \delta)(t-s)} \sum_{k=3}^{\infty} \sum_{j=1}^{m_k} (g_j^k(s))^2 ds \\
+ C_\delta \int_{-\infty}^t e^{(2\sigma_+^{(3)} + \delta)(t-s)} \sum_{k=3}^{\infty} \sum_{j=1}^{m_k} (g_j^k(s))^2 ds \\
\leq Ce^{2\sigma_+^{(3)t}} + C \int_t^T e^{(2\sigma_+^{(3)} - \delta)(t-s)} e^{(2\sigma_+^{(2)} - \delta - \epsilon_T + 2\epsilon)s} ds \\
+ C \int_{-\infty}^t e^{(2\sigma_+^{(3)} + \delta)(t-s)} e^{(2\sigma_+^{(2)} - \delta - \epsilon_T + 2\epsilon)s} ds \\
\leq Ce^{2\sigma_+^{(3)t}} + C \max\{e^{(2\sigma_+^{(3)} - \delta)T}, e^{(2\sigma_+^{(2)} - \delta - \epsilon_T + 2\epsilon)T}\} + Ce^{(2\sigma_+^{(2)} - \delta - \epsilon_T + 2\epsilon)T} \\
\leq Ce^{2\sigma_+^{(2)t}}. \tag{2.38}
\]

It is known from (2.36) and $\delta < 2\epsilon - \epsilon_T$ that

\[
\|F(w)\|_{(G^{N-1})}^2 = O(e^{2\sigma_+^{(2)t}}), \quad \sum_{j=1}^{m_1} (g_j^1)^2 = O(e^{2\sigma_+^{(2)t}}), \quad (g_1^0)^2 = O(e^{2\sigma_+^{(2)t}}).
\]

By (2.26) and $g_j^1(t) = O(e^{\sigma_+^{(2)t}})$, we obtain

\[
\sum_{j=1}^{m_1} (w_j^1(t))^2 = O(e^{2\sigma_+^{(2)t}}). \tag{2.39}
\]

Similarly,

\[
(w_0^1(t))^2 = O(e^{2\sigma_+^{(2)t}}). \tag{2.40}
\]
We obtain from (2.37)–(2.40) that
\[ \|w\|_{L^2(\mathbb{S}^{N-1})} = O(e^{\rho^{(2)}_+ t}). \tag{2.41} \]

For any fixed \((t, \theta) \in (-\infty, T - 1) \times \mathbb{S}^{N-1}\), by applying the interior \(L^\infty\)-estimate to (2.4) with \((t - 1, t + 1) \times \mathbb{S}^{N-1}\), we obtain from (2.41) and (2.24) that
\[ |w(t, \theta)| \leq C(\|w\|_{L^2((t-1,t+1)\times\mathbb{S}^{N-1})} + \|F(w)\|_{L^2((t-1,t+1)\times\mathbb{S}^{N-1})}) \leq C e^{\rho^{(2)}_+ t}, \tag{2.42} \]

where \(C > 0\) is independent of \(t\). Note that we can also use arguments similar to those in the proof of [18] to obtain
\[ \max_{\theta \in \mathbb{S}^{N-1}} |w(t, \theta)| \leq Me^{\rho^{(2)}_+ t} \quad \text{for } t \in (-\infty, T - 1). \tag{2.43} \]

Defining
\[ v(r, \theta) = w(t, \theta), \quad r = e^t, \]

it follows that \(v(r, \theta)\) satisfies
\[ \Delta v + \frac{2(N-2)v}{r^2} + \frac{F(v)}{r^2} = 0 \quad \text{in } B_R \setminus \{0\}, \tag{2.44} \]

where \(R = e^{T-1}\). For any \(x_0 \in B_R \setminus \{0\}\), denote \(r_0 = |x_0| > 0\) and \(\Omega = B_{r_0/2}(x_0)\). Consider (2.44) to be a linear equation in \(\Omega\) as in Lemma 5.1 and Theorem 5.1 of [18] with
\[ k = k_1 = 1, \quad h(x) \equiv 0, \quad |c| = \frac{Q}{r_0^2}, \quad k_2 = \frac{Q}{r_0^2} \]

where \(Q = Q(v) > 0\). Then, (2.41) implies that there is a positive constant (independent of \(r_0\))
\[ M = M(k_1/k, k_2 r_0^2) = M(Q) = M(v) \]

such that
\[ \sup_{x \in B_{r_0/2}(x_0)} |v(x)| \leq M r_0^{\rho^{(2)}_+}. \]

In particular, we have
\[ |v(x_0)| \leq M r_0^{\rho^{(2)}_+}, \]
\[ \max_{|x|=r} |v(x)| \leq M r^{\rho^{(2)}_+}. \]

Hence, (2.43) follows for \(t \in (-\infty, T - 1)\).

For the case of \(4\epsilon = 2\rho^{(2)}_+ - \delta\), we may choose \(\delta'\) a little larger than \(\delta\) such that \(0 < \delta < \delta'\) and \(4\epsilon > 2\rho^{(2)}_+ - \delta'\). By similar arguments, we can prove (2.41) and (2.43).

For the case (ii), by \(F(w) = O(e^{2\epsilon t})\), \(\sum_{k=2}^{\infty} \sum_{j=1}^{m_k} \rho^{(2)}_+(g^k_j(s))^2 = O(e^{4\epsilon t})\) and \(4\epsilon < 2\rho^{(2)}_+ - \delta < 2\rho^{(2)}_+\), we can obtain
\[ \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (w^k_j(t))^2 \leq C \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} e^{2\rho^{(2)}_+(t-T)} + C \int_{t}^{T} e^{(2\rho^{(2)}_+ - \delta)(t-s)} \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (g^k_j(s))^2 ds \]
\[ + C \int_{-\infty}^{t} e^{(2\rho^{(2)}_+ + \delta)(t-s)} \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (g^k_j(s))^2 ds \]
\[ W(t) \leq Ce^{4t} \text{ for } t < T. \]

Together with (2.27) and (2.28), we know that
\[ \|w\|_{L^2(S^{N-1})} = O(e^{2et}). \]

Arguments similar to those in the proof of (2.43) imply that
\[ \max_{\theta \in S^{N-1}} |w(t, \theta)| \leq Me^{2t} \text{ for } t \in (-\infty, -1], \]

where \( M := M(w) > 0 \). As a consequence,
\[ \max_{S^{N-1}} |\mathcal{F}(w)| \leq Ce^{4t} \text{ for } t \in (-\infty, -1]. \]

Then (2.26) implies that
\[ w_j^1(t) = O(e^{4et}), \quad j = 1, 2, \ldots, m_1. \]

Similarly,
\[ w_0^1(t) = O(e^{4et}). \]

Therefore,
\[ [W(t)]^2 \leq Ce^{2\sigma_2^2 (t-T)} + C \int_t^T e^{(2\sigma_2^2 - \delta)(t-s)} e^{8es} ds + C \int_t^T e^{(2\sigma_2^2 - \delta)(t-s)} e^{2es} [W(s)]^2 ds \]
\[ + C \int_{-\infty}^t e^{(2\sigma_2^2 + \delta)(t-s)} e^{8es} ds + C \int_{-\infty}^T e^{(2\sigma_2^2 + \delta)(t-s)} e^{2es} [W(s)]^2 ds. \]

(2.49)

Note that
\[ \sum_{k=2}^{m_2} \sum_{j=1}^{m_1} (g_j^k(t))^2 \leq Ce^{5t} ([W(t)]^2 + e^{8et}). \]

In what follows, there are also two cases: (a) \( 8e \geq [2\sigma_2^2 - \delta] \) and (b) \( 8e < [2\sigma_2^2 - \delta] \).

For the case (a), using (2.49) and arguments similar to those in the proof of (i), we can obtain (2.43). The case (b) implies that \( \mathcal{F}(w) = O(e^{4et}). \) Then
\[ [W(t)]^2 \leq Ce^{8et} \text{ for } t < T. \]

This, (2.47) and (2.48) imply that
\[ \|w\|_{L^2(S^{N-1})} = O(e^{4et}) \]

and
\[ \|\mathcal{F}(w)\|_{L^2(S^{N-1})} = O(e^{5et}). \]

Therefore,
\[ g_j^1(t) = O(e^{5et}), \quad w_j^1(t) = O(e^{5et}), \quad w_0^1(t) = O(e^{5et}). \]
Then we have
\[
\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} (g_j^k(t))^2 \leq Ce^{2\epsilon t}([W(t)]^2 + e^{10\epsilon t})
\]
and
\[
[W(t)]^2 \leq Ce^{2\sigma(t-T)} + C \int_T^t e^{2\sigma(t-s)}e^{10\epsilon s} ds + C \int_T^t e^{2\sigma(t-s)}e^{2\epsilon s}[W(s)]^2 ds + C \int_{-\infty}^t e^{(2\sigma+\delta)t-s}e^{10\epsilon s} ds + C \int_{-\infty}^t e^{(2\sigma+\delta)t-s}e^{2\epsilon s}[W(s)]^2 ds. \tag{2.53}
\]
Similarly, we still consider two cases: \(10\epsilon \geq [2\sigma_+ - \delta]\) and \(10\epsilon < [2\sigma_+ - \delta]\); then, we obtain (2.43). The proof of this lemma is complete.

**Theorem 2.3.** Assume that \(N \geq 3\) and \(u \in C^2(B \setminus \{0\})\) is a singular solution of (1.15) that satisfies (1.13). Defining \(w(t, \theta) = u(x) - U_\jmath(x)\) and \(t = \ln r\), there is a positive number sequence \(\{\mu_k\}_{k \geq 1}\), strictly increasing and converging to \(\infty\) with
\[
\mu_1 = \sigma_+^{(2)} \tag{2.54}
\]
such that for any positive integer \(n \gg 1\) and any \((t, \theta) \in (-\infty, -1) \times S^{N-1}\),
\[
w(t, \theta) = \sum_{k=1}^{n} \sum_{\ell=0}^{k-1} c_{k\ell}(\theta) t^{\ell} e^{\mu_{k+}} + O(|t|^{n} e^{\mu_{n+1}}), \tag{2.55}
\]
where
\[
c_{k\ell}(\theta) = \sum_{i=0}^{M_{k\ell}} a_{k\ell i} Q_i(\theta) \tag{2.56}
\]
and \(M_{k\ell}\) is a nonnegative integer depending on \(N, k, \ell\); \(a_{k\ell i}\) is constant and \(Q_i(\theta)\) is a linear combination of \([Q_1(\theta), Q_2(\theta), \ldots, Q_{m_i}(\theta)]\). Especially, for \(k = 1\),
\[
c_{10}(\theta) = a_{102} Q_2(\theta),
\]
where \(a_{102}\) is a constant.

**Proof.** By using the starting estimate (2.22), constructing the index set \(I\), and examining the equation of \(w\), the expansion of \(w(t, \theta)\) can be established via similar arguments to those in Theorem 1.1 of [19].

Let \(\{\rho_k\}_{k \geq 1}\) be positive strictly increasing and converging to \(\infty\):
\[
\rho_1 = \sigma_+^{(2)}, \rho_2 = \sigma_+^{(3)}, \ldots, \rho_k = \sigma_+^{(k+1)}, \ldots
\]
Also, let \(\mathbb{Z}_+\) be the collection of nonnegative integers. Define the index set \(I\) by
\[
I = \{ \sum_{k \geq 1} n_k \rho_k : n_k \in \mathbb{Z}_+ \text{ with finitely many } n_k > 0 \}. \tag{2.57}
\]
Set
\[
I_p = \{ \rho_k : k \geq 1 \} \tag{2.58}
\]
and
\[
\mathcal{I}_\beta = \left\{ \sum_{k=1}^{n} n_k \rho_k : \ n_k \in \mathbb{Z}_+, \ \sum_{k=1}^{n} n_k \geq 2 \right\}. \tag{2.59}
\]

Assume that \(\mathcal{I}_\rho\) is given by a strictly increasing sequence \(\{\rho_k\}_{k \geq 1}\) with \(\rho_1 = 2\rho_1\). There may be identical elements in \(\mathcal{I}_\rho\) and \(\mathcal{I}_\beta\).

For \(\tilde{\rho}_k \in \mathcal{I}_\beta\), there are nonnegative integers \(n_1, \ldots, n_{i_k}\) such that
\[
n_1 + \cdots + n_{i_k} \geq 2, \quad n_1 \rho_1 + \cdots + n_{i_k} \rho_{i_k} = \tilde{\rho}_k. \tag{2.60}\]

The collections of nonnegative integers \(n_1, \ldots, n_{i_k}\) that satisfy (2.60) are finite. Set
\[
\tilde{M} = \max\{2n_1 + 3n_2 + \cdots + (i_1 + 1)n_{i_1} : \ n_1, \ldots, n_{i_1} \text{ are nonnegative integers satisfying (2.60)}\}. \tag{2.61}
\]

Arrange \(\mathcal{I}\) as follows:
\[
\rho_1 < \cdots < \rho_{i_1} \leq \tilde{\rho}_1 < \cdots < \tilde{\rho}_{i_1} < \cdots < \rho_{i_2} \leq \tilde{\rho}_{i_1} < \cdots < \tilde{\rho}_{i_2} \leq \rho_{i_2 + 1} < \cdots. \tag{2.62}\]

Note that if \(\rho_1 < \tilde{\rho}_1 < \rho_2\), we choose \(i_1 = 1\) and \(l_1 = 1\) and the arrangement of (2.62) becomes \(\rho_1 < \tilde{\rho}_1 < \rho_2 < \cdots\). Similarly, if \(\rho_{i_k + 1} \leq \tilde{\rho}_{i_k + 1} < \rho_{i_{k+2}}\) for some \(k \geq 1\), define \(i_k + 1 = i_{k+1}\) and \(l_k + 1 = l_{k+1}\). We do not consider the multiplicity of \(\rho_k\) here, since all terms containing \(e^{\alpha t}\) in the expansions of \(w(t, \theta)\) can be combined as one term. We know
\[
L(w) = -\mathcal{F}(w),
\]
where \(\mathcal{F}(w) = \sum_{k=2}^{\infty} b_k w^k\) for \(|w| < \hat{\epsilon}\) with some sufficiently small \(\hat{\epsilon} > 0\); also, the expansion of \(\mathcal{F}(w)\) consists of terms including \(\sum_{l=0}^{l(\tilde{k})} \left( \sum_{i=0}^{M_l} c_{k\ell} Q_i(\theta) \right) \ell^l e^{\beta \ell t}\) for \(\tilde{\rho}_k \in \mathcal{I}_\beta\) in (2.59).

Define
\[
\mu_1 = \rho_1, \mu_2 = \rho_2, \mu_3 = \rho_3, \ldots, \mu_{i_1} = \rho_{i_1}, \mu_{i_1 + 1} = \tilde{\rho}_1, \ldots \tag{2.63}
\]
according to the arrangement in (2.62). To ensure that \(\{\mu_k\}_{k \geq 1}\) is a strictly increasing sequence of positive constants, when \(\rho_{i_1} = \tilde{\rho}_1\), define \(\mu_{i_1} = \rho_{i_1}\) and \(\mu_{i_1 + 1} = \tilde{\rho}_2\). In this case, an extra power of \(t\) term corresponding to \(\mu_{i_1}\) may appear in the expansion of \(w(t, \theta)\). Similarly, make the same choices of \(\mu_k\) for the cases \(\tilde{\rho}_{i_1} = \rho_{i_1 + 1}, \tilde{\rho}_{i_2} = \rho_{i_1 + 1}, \tilde{\rho}_{i_2} = \rho_{i_2 + 1}, \) etc. As a consequence, for any positive integer \(n \gg 1\) and any \((t, \theta) \in (-\infty, -1) \times S^{N-1}\),
\[
w(t, \theta) = \sum_{k=1}^{n} \sum_{l=0}^{k-1} c_{k\ell}(\theta) t^l e^{\alpha \ell t} + \mathcal{O}(|t|^n e^{\alpha n t}), \tag{2.64}
\]
where
\[
c_{k\ell}(\theta) = \sum_{i=0}^{M_{k\ell}} a_{k\ell i} Q_i(\theta) \tag{2.65}
\]
and \(M_{k\ell}\) is a nonnegative integer that is dependent on \(N, k, \ell, a_{k\ell i}\) is constant and \(Q_i(\theta)\) is in the span of \(Q_1(\theta), Q_2(\theta), \ldots, Q_m(\theta)\). Especially, for \(k = 1,\)
\[
c_{10}(\theta) = a_{102} Q_2(\theta).
\]

The proof of this theorem is complete.

3. Linearized equations and inverse of the operator

We will introduce the appropriate weighted Hölder spaces and obtain the inverse of the operator $\mathcal{L}$ on those spaces, where $\mathcal{L}$ is given in (2.4). We use some ideas from [20], where the authors derived singular solutions to the following equation:

$$\Delta u + \frac{N(N-2)}{4} u^{\frac{N+2}{N-2}} = 0 \quad \text{in } B\setminus\{0\}.$$ 

Fix a $t_0 < 0$. For a nonnegative integer $i$, $\alpha \in (0, 1)$, and $\mu \in \mathbb{R}$, define

$$\|v\|_{C^i_\mu((t_0, t_0] \times S^{N-1})} = \sum_{j=0}^i \sup_{(t, \theta) \in (t_0, t_0] \times S^{N-1}} e^{-\mu t} |\nabla^j v(t, \theta)|,$$

and

$$\|v\|_{C^0_\mu((t_0, t_0] \times S^{N-1})} = \|v\|_{C^i_\mu((t_0, t_0] \times S^{N-1})} + \sup_{t \leq t_0-1} e^{-\mu t} \|\nabla v\|_{C^0([t-1, t+1] \times S^{N-1})},$$

where $[\cdot]_{C^\alpha}$ is the usual Hölder semi-norm.

**Definition 3.1.** The collection of functions $v$ in $C^i((t_0, t_0] \times S^{N-1})$ with a finite norm $\|v\|_{C^i_\mu((t_0, t_0] \times S^{N-1})}$ is the weighted Hölder space $C^i_\mu((t_0, t_0] \times S^{N-1})$.

For $\mu > 0$ and some $g \in C^0_\mu((t_0, t_0] \times S^{N-1})$, to consider the linear equation given by

$$\mathcal{L}v = g \quad \text{in } (t_0, t_0] \times S^{N-1}, \quad (3.1)$$

we introduce a boundary condition on $t = t_0$ such that

$$\mathcal{L} : C^2_\mu((t_0, t_0] \times S^{N-1}) \to C^0_\mu((t_0, t_0] \times S^{N-1})$$

has a bounded inverse. However, since signs of coefficients of zero order terms are inappropriate, we cannot directly apply the maximum principle to the following Dirichlet boundary-value problem

$$\begin{cases} 
\mathcal{L}v = g & \text{in } (t_0, t_0] \times S^{N-1}, \\
v = \varphi & \text{on } \{0\} \times S^{N-1}. 
\end{cases} \quad (3.2)$$

**Lemma 3.1.** Let $\mu > 0$, $g \in C^0_\mu((t_0, t_0] \times S^{N-1})$, and $\varphi \in C^0(S^{N-1})$. Then, there is at most one solution $v \in C^2_\mu((t_0, t_0] \times S^{N-1})$ of (3.2).

**Proof.** Assume that $g = 0$, $\varphi = 0$ and $v \in C^2_\mu((t_0, t_0] \times S^{N-1})$ is a solution of (3.2). For each $k \geq 0$, define

$$v_k(t) = \int_{S^{N-1}} v(t, \theta) Q_k(\theta) d\theta.$$ 

So $\mathcal{L}_k(v_k) = 0$ on $(t_0, t_0)$ and $v_k(t_0) = 0$. This implies that $v_k$ is a linear combinations of the basis of $\text{Ker}(\mathcal{L}_k)$. In particular, for $k = 0$,

$$v_0(t) = \begin{cases} 
 c_0^1 e^{\cos(\sigma^0 t)} \cos \gamma t + c_0^2 e^{\cos(\sigma^0 t)} \sin \gamma t, & \text{for } 3 \leq N \leq 9, \\
 c_0^1 e^{\cos(\sigma^0 t)} + c_0^2 e^{\cos(\sigma^0 t)}, & \text{for } N = 10, \\
 c_0^1 e^{\cos(\sigma^0 t)} + c_0^2 e^{\cos(\sigma^0 t)}, & \text{for } N \geq 11,
\end{cases}$$

and for $k \geq 1$,
\[ v_k(t) = c^1_k e^{\sigma^{(k)}_0 t} + c^2_k e^{\sigma^{(k)}_0 t}, \]
where $c^1_k$ and $c^2_k$ are constants for $k = 0, 1, 2, \ldots$. By the assumption, we have the following for any $t \in (−\infty, t_0)$:
\[ |e^{-\mu t} v_k(t)| \leq C. \]  
(3.3)

Hence, $v_k = 0$ for $k = 0$ and $k = 1$. Note that
\[ \{ \begin{array}{ll}
\Re(\sigma^{(0)}_+) < 0, & \text{for } 3 \leq N \leq 9, \\
\Re(\sigma^{(0)}_+) < 0, & \text{for } N \geq 10,
\end{array} \]
\[ \sigma^{(1)}_+ \leq 0 \text{ and } \sigma^{(1)}_- < 0 \text{ for } N \geq 3. \]  
Moreover, $v_k(t) = c^1_k e^{\sigma^{(k)}_0 t}$ for $k \geq 2$, which decays exponentially as $t \to -\infty$ (note that $c^2_k = 0$ since $\sigma^{(k)}_+ < 0$ for $k \geq 2$). Since $v_k(t_0) = 0$, we can directly obtain $c^1_k = 0$ and $v_k(t) \equiv 0$ for $k \geq 2$. In conclusion, $v_k = 0$ for all $k \geq 0$, i.e., $v \equiv 0$.

**Lemma 3.2.** Let $\alpha \in (0, 1)$, $\mu > 0$, $g \in C^0_{\mu,\alpha}((−\infty, t_0] \times S^{N-1})$, and $\varphi \in C^2_{\mu}(S^{N-1})$. Suppose that $v \in C^2_{\mu}((−\infty, t_0] \times S^{N-1})$ is a solution of (3.2). Then
\[ \|v\|_{C^2_{\mu}((−\infty, t_0] \times S^{N-1})} \leq C \left[ \|v\|_{C^0_{\mu}((−\infty, t_0] \times S^{N-1})} + \|g\|_{C^0_{\mu}((−\infty, t_0] \times S^{N-1})} + e^{-\mu t_0} \|\varphi\|_{C^2_{\mu}(S^{N-1})} \right], \]  
(3.4)

where $C$ is a positive constant that is only dependent on $N, \alpha, \mu$ and is independent of $t_0$.

**Proof.** Using similar arguments to that of Lemma 2.5 of [20], consider two cases:
(i) $t < t_0 - 2$. We have
\[ \sum_{j=0}^{2} \sup_{S^{N-1}} |\nabla^j v(t, \cdot)| + |\nabla^2 v|_{C^0([t-1, t+1] \times S^{N-1})} \leq C \left[ \|v\|_{L^\infty([t-2, t+2] \times S^{N-1})} + \|g\|_{L^\infty([t-2, t+2] \times S^{N-1})} \right], \]
where $C$ is a positive constant that is independent of $t$. We estimate $\|g\|_{L^\infty([t-2, t+2] \times S^{N-1})}$, by setting $(t_1, \theta_1), (t_2, \theta_2) \in [t-2, t+2] \times S^{N-1}$ with $(t_1, \theta_1) \neq (t_2, \theta_2)$. There are two cases: $|t_1 - t_2| \leq 2$ and $|t_1 - t_2| > 2$.

When $|t_1 - t_2| \leq 2$, choose $t' \in [t-1, t+1]$ such that $t_1, t_2 \in [t'-1, t'+1]$ is satisfied. Then,
\[ [g]_{C^0([t-2, t+2] \times S^{N-1})} \leq \max \left\{ \sup_{r \in [t-1, t+1]} [g]_{C^0([t'-1, t'+1] \times S^{N-1})}, \|g\|_{L^\infty([t-2, t+2] \times S^{N-1})} \right\}. \]

So,
\[ \sum_{j=0}^{2} \sup_{S^{N-1}} |\nabla^j v(t, \cdot)| + |\nabla^2 v|_{C^0([t-1, t+1] \times S^{N-1})} \leq C \left[ \|v\|_{L^\infty([t-2, t+2] \times S^{N-1})} + \|g\|_{L^\infty([t-2, t+2] \times S^{N-1})} + \sup_{r \in [t-1, t+1]} [g]_{C^0([t'-1, t'+1] \times S^{N-1})} \right]. \]

We multiply both sides by $e^{-\mu t}$ and take the supremum over $t \in (−\infty, t_0 - 2)$. The following holds
\[ \sum_{j=0}^{2} \sup_{t \in (−\infty, t_0 - 2)} e^{-\mu t} |\nabla^j v(t, \cdot)| + \sup_{t \in (−\infty, t_0 - 2)} e^{-\mu t} |\nabla^2 v|_{C^0([t-1, t+1] \times S^{N-1})} \]
Similarly, there exists a positive constant dependent only on $N$

Then, by arguments similar to those in [20–22], we obtain the $L^\infty$ estimates of solutions on finite cylinders to (3.2) with a 0 boundary value.

**Lemma 3.3.** Let $\mu > \rho_1$ and $\mu \neq \rho_k$ for $k \geq 1$, $T$ and $t_0$ be constants with $t_0 \leq 0$ and $T - t_0 \leq -4$, and $g \in C^0([T, t_0] \times \mathbb{S}^{N-1})$. Suppose that $v \in C^2([T, t_0] \times \mathbb{S}^{N-1})$ satisfies the following:

$$
\begin{cases}
    \mathcal{L}v = g & \text{in } (T, t_0) \times \mathbb{S}^{N-1}, \\
    v = 0 & \text{on } ((T) \cup \{t_0\}) \times \mathbb{S}^{N-1},
\end{cases}
$$

and $\int_{\mathbb{S}^{N-1}} v(t, \theta) Q_k(\theta) d\theta = 0$ for $k = 0, 1, \cdots, K$, where $K$ is the largest integer satisfying that $\rho_{K-1} < \mu$. Then,

$$
\sup_{(t, \theta) \in [T, t_0] \times \mathbb{S}^{N-1}} e^{-\mu t} |v(t, \theta)| \leq C \sup_{(t, \theta) \in [T, t_0] \times \mathbb{S}^{N-1}} e^{-\mu t} |g(t, \theta)|,
$$

where $C$ is a positive constant dependent only on $N, \mu$ and independent of $T$ and $t_0$.

**Proof.** Note that $\rho_1 = \sigma_+^{(2)}$.

Let the sequences $\{T_i, t_i, v_i\}$ and $\{g_i\}$ with $t_i \leq 0$ and $T_i - t_i \leq -4$, satisfy the following:

$$
\begin{cases}
    \mathcal{L}v_i = g_i & \text{in } (T_i, t_i) \times \mathbb{S}^{N-1}, \\
    v_i = 0 & \text{on } ((T_i) \cup \{t_i\}) \times \mathbb{S}^{N-1},
\end{cases}
$$

and

$$
\sup_{(t, \theta) \in [T_i, t_i] \times \mathbb{S}^{N-1}} \sup_{(t, \theta) \in [T_i, t_i] \times \mathbb{S}^{N-1}} e^{-\mu t} |g_i(t, \theta)| = 1,
$$

$$
\sup_{(t, \theta) \in [T_i, t_i] \times \mathbb{S}^{N-1}} e^{-\mu t} |v_i(t, \theta)| \to \infty \text{ as } i \to \infty.
$$

There exists $t_i^* \in (T_i, t_i)$ that satisfies

$$
M_i = \sup_{\mathbb{S}^{N-1}} e^{-\mu t} |v_i(t_i^*, \cdot)| = \sup_{\mathbb{S}^{N-1}} e^{-\mu t} |v_i(t, \theta)| \to \infty \text{ as } i \to \infty.
$$
From (3.10) we have
\[ \hat{\nu}(t, \theta) = M_i^{-1} e^{-\mu t} \nu_i(t + \tau, \theta), \] (3.8)
\[ \bar{g}_i(t, \theta) = M_i^{-1} e^{-\mu t} g_i(t + \tau, \theta). \] (3.9)

The following holds:
\[ \sup_{S^{N-1}} |\hat{\nu}(0, \cdot)| = 1, \] for any \((t, \theta) \in [T_i - t'_i, t_i - t'_i] \times S^{N-1},\]
\[ |e^{-\mu t} \hat{\nu}(t, \theta)| \leq 1 \] (3.10)
and
\[ \mathcal{L} \hat{\nu}_i = \bar{g}_i \text{ for } t \in (T_i - t'_i, t_i - t'_i) \times S^{N-1}. \]

Assume the following for some \( \tau_- \in \mathbb{R} - \{ -\infty \} \) and \( \tau^+ \in \mathbb{R}^+ \cup \{ \infty \} \):
\[ T_i - t'_i \rightarrow \tau_-, \quad t_i - t'_i \rightarrow \tau^+. \] (3.11)

From (3.10) we have
\[ |\hat{\nu}_i| \leq Ce^{\mu(t-r-T_i)} \text{ on } (T_i - t'_i, T_i - t'_i + 2) \times S^{N-1}, \]
and hence
\[ \left| \frac{d^2 \hat{\nu}_i}{dt^2} + (N - 2) \frac{d \hat{\nu}_i}{dt} + \Delta_\nu \hat{\nu}_i \right| \leq Ce^{\mu(t-r-T_i)} \text{ on } (T_i - t'_i, T_i - t'_i + 2) \times S^{N-1}. \]

Since \( \bar{\nu}_i = 0 \) on \((T_i - t'_i) \times S^{N-1}\), we have
\[ |\nabla \bar{\nu}_i| \leq Ce^{\mu(t-r-T_i)} \text{ on } (T_i - t'_i, T_i - t'_i + 1) \times S^{N-1}. \]

This implies that \( T_i - t'_i \) remains bounded away from zero. Similar arguments imply that \( t_i - t'_i \) is bounded away from zero. As a consequence, \( 0 \in (\tau_-, \tau^+) \). Let
\[ \hat{\nu}_i \rightarrow \hat{\nu} \text{ in the compact set of } (\tau_-, \tau^+). \] (3.12)

Moreover, \( \bar{g}_i \rightarrow 0 \) in every compact set of \((\tau_-, \tau^+)\). So the following holds:
\[ \hat{\nu} \neq 0, \]
\[ |e^{-\mu t} \hat{\nu}(t, \theta)| \leq 1 \text{ for any } (t, \theta) \in (\tau_-, \tau^+) \times S^{N-1}, \] (3.13)
and
\[ \mathcal{L} \hat{\nu} = 0 \text{ on } (\tau_-, \tau^+) \times S^{N-1}, \] (3.14)

where \( \tau_* = \tau_- \text{ or } \tau^+ \) if it is finite.

Let
\[ \hat{\nu}_k(t) = \int_{S^{N-1}} \hat{\nu}(t, \theta) Q_k(\theta) d\theta. \] (3.15)

Then \( \mathcal{L}_k(\hat{\nu}_k) = 0 \); hence, \( \hat{\nu}_k \) is the linear combination of the basis of \( \text{Ker}(\mathcal{L}_k) \). We now take \( k \geq 2 \) with \( \rho_k > \mu \). Then
\[ \hat{\nu}_{k+1}(t) = c^1_{k+1} e^{\rho_k t} + c^2_{k+1} e^{\rho_k(1-t)}, \]
where $c^1_{k+1}$ and $c^2_{k+1}$ are constants. From (3.13), we obtain the following for any $t \in (\tau_-, \tau_+)$:

$$|e^{-\mu t} \hat{v}_{k+1}(t)| \leq C.$$ 

When $\tau_+ = \infty$, $c^1_{k+1} = 0$ and hence $\hat{v}_{k+1}(t) = \frac{2}{\mu}e^{(\sigma_{k+1}^2/2)t}$. When $\tau_+$ is finite, $\lim_{t \to \tau_+} \hat{v}_{k+1}(t) = 0$ by (3.14). Similarly, when $\tau_- = -\infty$, $\hat{v}_{k+1}(t) = c^1_{k+1}e^{\alpha t} = c^1_{k+1}e^{\alpha_0 t}$. When $\tau_-$ is finite, $\lim_{t \to \tau_-} \hat{v}_{k+1}(t) = 0$ by (3.14). Thus,

$$\int_{\tau_-}^{\tau_+} \left( (\partial_t \hat{v}_{k+1})^2 + [\lambda_{k+1} - 2(N-2)]\hat{v}_{k+1}^2 \right) dt = 0.$$ 

Since $\rho_k > \mu > 0$, it follows that $\lambda_{k+1} > 2N$ for each $k$. This implies that $\lambda_{k+1} - 2(N-2) > 0$ for each $k$. Therefore, $\hat{v}_{k+1} = 0$ for each $k$. By the assumption, we have

$$\hat{v}_0 = \hat{v}_1 = \ldots = \hat{v}_k = 0$$

provided that $\rho_{k-1} < \mu$. In conclusion, $\hat{v}_k = 0$ for any $k \geq 0$; hence, $\hat{v} \equiv 0$. This is a contradiction.

**Lemma 3.4.** Let $\alpha \in (0, 1)$, $\mu > \rho_k$ for some $K \geq 1$, and $g \in C^{0,\alpha}_\mu((-\infty, t_0] \times S^{N-1})$ with $g(t, \cdot, \cdot) \in \text{span}\{Q_0, Q_1, \ldots, Q_{K+1}\}$ for $t \leq t_0$. Then, there is a unique solution $v \in C^{2,\alpha}_\mu((-\infty, t_0] \times S^{N-1})$ of (3.1) with $v(t, \cdot, \cdot) \in \text{span}\{Q_0, Q_1, \ldots, Q_{K+1}\}$ for $t \leq t_0$. Furthermore, $g \mapsto v$ is linear, and

$$\|v\|_{C^{2,\alpha}_\mu((-\infty, t_0] \times S^{N-1})} \leq C\|g\|_{C^{0,\alpha}_\mu((-\infty, t_0] \times S^{N-1})},$$

where $C$ is a positive constant that is only dependent on $N, \alpha, \mu$ and independent of $t_0$.

**Proof.** For $k = 0, 1, \ldots, K + 1$, define

$$g_k(t) = \int_{S^{N-1}} g(t, \theta) Q_k(\theta) d\theta.$$

Then,

$$\|g_k\|_{C^{0,\alpha}_\mu((-\infty, t_0])} \leq \|g\|_{C^{0,\alpha}_\mu((-\infty, t_0] \times S^{N-1})},$$

and

$$g(t, \theta) = \sum_{k=0}^{K+1} g_k(t) Q_k(\theta). \quad (3.16)$$

Consider the following ODE:

$$\mathcal{L}_k v_k = g_k. \quad (3.17)$$

Suppose that there is a solution $v_k \in C^{2,\alpha}_\mu((-\infty, t_0])$ of (3.17) and

$$\|v_k\|_{C^{2,\alpha}_\mu((-\infty, t_0])} \leq C\|g_k\|_{C^{0,\alpha}_\mu((-\infty, t_0])}, \quad (3.18)$$

where $C$ is a constant that depends only on $N, \alpha, \mu$ and independent of $t_0$.

If $k = 0$, it is known from Section 2 that $\text{Ker}(\mathcal{L}_0)$ encompasses $e^{\tau t} \cos \gamma t$ and $e^{\tau t} \sin \gamma t$ with

$$\tau = -\frac{1}{2}(N-2) < 0, \quad \gamma = \frac{1}{2}\sqrt{(N-2)(10-N)}.$$
provided that $3 \leq N \leq 9$. $\text{Ker}(L_0)$ encompasses $e^{\sigma_0(t)}$ and $te^{\sigma_0(t)}$ with $\sigma^{(0)}_+ = \sigma^{(0)}_- < 0$ provided that $N = 10$. $\text{Ker}(L_0)$ encompasses $e^{\sigma_1(t)}$ and $e^{\sigma_1(t)}t$ with $\sigma^{(0)}_+ < 0, \sigma^{(0)}_- < 0$ provided that $N \geq 11$. If $k = 1$, $\text{Ker}(L_1)$ encompasses $e^{\sigma_1(t)}$ and $e^{\sigma_1(t)}t$ with $\sigma^{(1)}_+ \leq 0, \sigma^{(1)}_- < 0$ provided that $N \geq 3$.

We now consider $v_0(t)$ for $k = 0$ and $k = 1$. For $k = 0$, we see that

$$v_0(t) = \begin{cases} B_0^1 \int_{-\infty}^{t} e^{\sigma_1(t-s)} \sin \gamma(t-s) g_0(s) ds, & \text{for } 3 \leq N \leq 9, \\ \int_{-\infty}^{t} e^{\sigma_1(t-s)} g_0(s) ds - t \int_{-\infty}^{t} e^{\sigma_1(t-s)} g_0(s) ds, & \text{for } N = 10, \\ B_0^1 \int_{-\infty}^{t} e^{\sigma_1(t-s)} g_0(s) ds - B_0^1 \int_{-\infty}^{t} e^{\sigma_1(t-s)} g_0(s) ds, & \text{for } N \geq 11, \end{cases}$$

(3.19)

where

$$|B_0^1| = \begin{cases} \frac{1}{\sigma_-^{(0)} - \sigma_+^{(0)}}, & \text{for } 3 \leq N \leq 9, \\ \frac{1}{\sigma_-^{(1)} - \sigma_+^{(1)}}, & \text{for } N \geq 11. \end{cases}$$

We only consider $v_0(t)$ for $N \geq 11$ and $v_1(t)$. The other cases for $v_0(t)$ can be demonstrated similarly since $\tau < 0$ and $\sigma^{(0)}_+ < 0$ in these cases. Let

$$v_k(t) = B_k^1 \int_{-\infty}^{t} e^{\sigma_1(t-s)} g_k(s) ds - B_k^1 \int_{-\infty}^{t} e^{\sigma_1(t-s)} g_k(s) ds,$$

(3.20)

where

$$|B_k^1| = \left| \frac{1}{\sigma_-^{(0)} - \sigma_+^{(0)}} \right|,$$

Direct calculation implies the following for $t \leq t_0$:

$$e^{-\mu t} |v_k(t)| \leq C \sup_{t \leq t_0} e^{-\mu t} |g_k(t)| = C \|g_k\|_{C^0_\mu((0, t_0))},$$

(3.21)

$$e^{-\mu t} (|v'_k(t)| + |v''_k(t)|) \leq C \|g_k\|_{C^0_\mu((0, t_0))},$$

(3.22)

Set

$$v_k''(t) = R_1(t) + R_2(t),$$

where

$$R_1(t) = B_k^1 (e^{\sigma_1(t)})'' \int_{-\infty}^{t} e^{-\sigma_1(s)} g_k(s) ds - B_k^1 (e^{\sigma_1(t)})'' \int_{-\infty}^{t} e^{-\sigma_1(s)} g_k(s) ds,$$

and

$$R_2(t) = B_k^1 (e^{\sigma_1(t)})' e^{-\sigma_1(t)} g_k(t) - B_k^1 (e^{\sigma_1(t)})' e^{-\sigma_1(t)} g_k(t).$$

Then,

$$R_1'(t) = B_k^1 (e^{\sigma_1(t)})'' \int_{-\infty}^{t} e^{-\sigma_1(s)} g_k(s) ds - B_k^1 (e^{\sigma_1(t)})'' \int_{-\infty}^{t} e^{-\sigma_1(s)} g_k(s) ds$$

$$+ B_k^1 (e^{\sigma_1(t)})' e^{-\sigma_1(t)} g_k(t) - B_k^1 (e^{\sigma_1(t)})' e^{-\sigma_1(t)} g_k(t).$$

Then we have

$$e^{-\mu t} |R_1'(t)| \leq C \|g_k\|_{C^0_\mu((0, t_0))},$$

and hence, for $t \leq t_0 - 1$,

$$e^{-\mu t} |R_1(t)| \leq C \|g_k\|_{C^0_\mu((0, t_0))}.$$
\[ e^{-\mu t} |R_2| C^{\alpha(1)}_{(1, t+1)}(1, t+1) | \leq C \| g_k \|_{C^{2,\alpha}_{\mu}((\infty, t_0])}, \]

Therefore, for \( t \leq t_0 - 1 \),

\[ e^{-\mu t} |v'_k| C^{\alpha(1)}_{(1, t+1)}(1, t+1) | \leq C \| g_k \|_{C^{2,\alpha}_{\mu}((\infty, t_0])}, \]

(3.23)

Combining (3.21), (3.22) and (3.23), (3.18) holds for \( k = 0 \) with \( N \geq 11 \) and \( k = 1 \).

For \( 2 \leq k \leq K + 1 \), since \( e^{-\mu t} |g_k(t)| \leq C \) and \( \mu > \rho_K \), we can also define

\[ v_k(t) = B_k^1 \int_{-\infty}^t e^{\sigma^+(t-s)} g_k(s) ds - B_k^1 \int_{-\infty}^t e^{\sigma^-(t-s)} g_k(s) ds. \]

(3.24)

Note that, at this time, \( \sigma^+_k > 0 \) and \( \sigma^-_k < 0 \) for \( k = 2, \ldots, K + 1 \) and \( \rho_k = \sigma^+_k \). By arguments similar to those in the proof of (3.18) for \( k = 0 \) with \( N \geq 11 \) and \( k = 1 \), we obtain (3.18) for \( 2 \leq k \leq K + 1 \).

With the solution \( v_k \) of (3.17) for \( k = 0, 1, \ldots, K + 1 \), we set

\[ v(t, \theta) = \sum_{k=0}^{K+1} v_k(t) Q_k(\theta). \]

Then, \( \mathcal{L}v = g \) and, by (3.18), we have

\[ \|v\|_{C^{2,\alpha}_{\mu}((\infty, t_0]) \times \mathbb{S}^{N-1}} \leq C \sum_{k=0}^{K+1} \|v_k\|_{C^{2,\alpha}_{\mu}((\infty, t_0])}, \]

Then the extra requirement \( v(t, \cdot) \in \text{span}\{Q_0, Q_1, \ldots, Q_{K+1}\} \) implies the uniqueness of \( v \).

**Lemma 3.5.** Let \( \alpha \in (0, 1) \), \( \mu > \rho_1 \) and \( \mu \neq \rho_k \) for any \( k \geq 1 \); also, \( g \in C^{2,\alpha}_{\mu}((\infty, t_0] \times \mathbb{S}^{N-1}) \), with \( \int_{\mathbb{S}^{N-1}} g(t, \cdot) Q_k(\theta) d\theta = 0 \) for any \( k = 0, 1, \ldots, K, K + 1 \), where \( K \) is the largest integer such that \( \rho_K < \mu \), and \( t \leq t_0 \). Then, there exists a unique solution \( v \in C^{2,\alpha}_{\mu}((\infty, t_0] \times \mathbb{S}^{N-1}) \) of (3.1) with \( v = 0 \) on \( [t_0] \times \mathbb{S}^{N-1} \). Moreover,

\[ \|v\|_{C^{2,\alpha}_{\mu}((\infty, t_0]) \times \mathbb{S}^{N-1})} \leq \|g\|_{C^{2,\alpha}_{\mu}((\infty, t_0]) \times \mathbb{S}^{N-1})}, \]

(3.25)

where \( C \) is a positive constant that is dependent on \( N, \alpha, \mu \) and \( \rho \) independent of \( t_0 \).

**Proof.** Assume that \( T \leq t_0 - 4 \). We claim that there is a solution \( v_T \in C^{2,\alpha}(T, t_0] \times \mathbb{S}^{N-1}) \) to the following problem:

\[ \begin{aligned}
\mathcal{L}v_T &= g \quad \text{in } (T, t_0] \times \mathbb{S}^{N-1}, \\
v_T &= 0 \quad \text{on } ([T] \cup \{t_0\}) \times \mathbb{S}^{N-1}.
\end{aligned} \]

(3.26)

The problem described by (3.26) can be written as follows:

\[ \begin{aligned}
\frac{\partial}{\partial \tau} (e^{\tau T} v_T) + e^{\tau T} \Delta_0 v_T + 2(2N - 2)e^{\tau T} v_T = e^{\tau T} g \quad &\text{in } (T, t_0] \times \mathbb{S}^{N-1}, \\
v_T &= 0 \quad &\text{on } ([T] \cup \{t_0\}) \times \mathbb{S}^{N-1}.
\end{aligned} \]

(3.27)

where \( \tau = N - 2 \). Consider the energy function

\[ \mathcal{G}_T(v) = \int_{T}^{t_0} \int_{\mathbb{S}^{N-1}} \left[ e^{\tau(r)} (\partial^2 v)^2 + e^{\tau(r)} |\nabla \partial v|^2 - 2(2N - 2) e^{\tau(r)} v^2 + 2e^{\tau(r)} g v \right] dt d\theta. \]
where \( t \in \mathbb{R}^N \) there is a minimizer \( v_{\lambda} \) since \( \phi \). Set

\[
\phi(\theta)Q_k(\theta) = 0 \quad \text{for} \quad k = 0, 1, \ldots, K, K + 1 \quad \text{with} \quad \rho_k < \mu.
\]

Then, for any \( \phi \in \Gamma \),

\[
\int_{\mathbb{R}^N} |\nabla \phi|^2 d\theta \geq 2N \int_{\mathbb{R}^N} \phi^2 d\theta,
\]

since \( \lambda_{k+1} \geq \lambda_2 = 2N \) with \( \mu > \rho_k \geq \rho_1 \). Hence, for any \( v \in H_0^1((T, t_0) \times \mathbb{S}^{N-1}) \) with \( v(t, \cdot) \in \Gamma \) for any \( t \in (T, t_0) \), we have

\[
\mathcal{G}_T(v) \geq \int_T^{t_0} \int_{\mathbb{R}^N} \left(e^{\nu}(\partial_t v)^2 + [2N - 2(N - 2)]e^{\nu}v^2 + 2e^{\nu}gv\right) d\theta dt.
\]

The inequality \( 2N - 2(N - 2) > 0 \) implies that \( \mathcal{G}_T \) is coercive and weakly lower semi-continuous. Then there is a minimizer \( v_T \) of \( \mathcal{G}_T \) in the following statement:

\[
\{ v \in H_0^1((T, t_0) \times \mathbb{S}^{N-1}) : v(t, \cdot) \in \Gamma \quad \text{for any} \quad t \in (T, t_0) \}.
\]

So \( v_T \) is a solution of (3.26) that satisfies \( v_T(t, \cdot) \in \Gamma \) for any \( t \in (T, t_0) \).

We obtain that by Lemma 3.3,

\[
\sup_{(t, \theta) \in [T, t_0] \times \mathbb{S}^{N-1}} e^{-\mu t} |v_T(t, \theta)| \leq C \sup_{(t, \theta) \in [T, t_0] \times \mathbb{S}^{N-1}} e^{-\mu t} |g(t, \theta)|,
\]

where \( C \) is a positive constant that is dependent on \( N, \mu \) and independent of \( T \) and \( t_0 \). Fix \( T_0 < t_0 \). By the interior and boundary Schauder estimates in \( [t_0 + T_0, t_0] \times \mathbb{S}^{N-1} \subset [t_0 + T_0 - 1, t_0] \times \mathbb{S}^{N-1} \) and \( v_T(t_0, \theta) = 0 \), there exists a subsequence \( v_T \) that converges to a \( C^{2, \alpha} \)-solution \( v \) of (3.1) in \( [t_0 + T_0, t_0] \times \mathbb{S}^{N-1} \) satisfying \( v = 0 \) on \( [t_0] \times \mathbb{S}^{N-1} \), as \( T \to -\infty \). Then, we can obtain that \( v_T \) converges to a \( C^{2, \alpha} \)-solution \( v \) of (3.1) in \( (-\infty, t_0] \times \mathbb{S}^{N-1} \) such that the following is satisfied: \( v = 0 \) on \( [t_0] \times \mathbb{S}^{N-1} \).

\[
\sup_{(t, \theta) \in [T, t_0] \times \mathbb{S}^{N-1}} e^{-\mu t} |v(t, \theta)| \leq C \sup_{(t, \theta) \in [T, t_0] \times \mathbb{S}^{N-1}} e^{-\mu t} |g(t, \theta)|,
\]

or

\[
\|v\|_{L^2((-\infty, t_0] \times \mathbb{S}^{N-1})} \leq C \|g\|_{L^2((-\infty, t_0] \times \mathbb{S}^{N-1})}, \quad (3.28)
\]

Combining (3.28) and (3.4) with \( \varphi = 0 \), (3.25) holds.

**Theorem 3.6.** Let \( \alpha \in (0, 1) \), \( \mu > \rho_1 \), \( \mu \neq \rho_k \) for any \( k \geq 1 \) and \( g \in C^{0, \alpha}_\mu ((-\infty, t_0] \times \mathbb{S}^{N-1}) \). Then (3.1) admits a solution \( v \in C^{2, \alpha}_\mu ((-\infty, t_0] \times \mathbb{S}^{N-1}) \) and

\[
\|v\|_{C^{2, \alpha}_\mu ((-\infty, t_0] \times \mathbb{S}^{N-1})} \leq C \|g\|_{C^{0, \alpha}_\mu ((-\infty, t_0] \times \mathbb{S}^{N-1})}, \quad (3.29)
\]

where \( C \) is a positive constant that is dependent on \( N, \alpha, \mu \), and independent of \( t_0 \). Also, \( g \mapsto v \) is linear.

**Proof.** Assume that \( K \geq 1 \) is the largest integer with \( \rho_k < \mu \). Define

\[
g_k(t) = \int_{\mathbb{S}^{N-1}} g(t, \theta)Q_k(\theta) d\theta \quad \text{for} \quad k = 0, 1, \ldots, K + 1.
\]
Then \( v_1 \in C^{2,\alpha}_{\mu}((-\infty, t_0] \times \mathbb{S}^{N-1}) \) is a solution of

\[
\mathcal{L}(v_1) = \sum_{k=0}^{K+1} g_k(t)Q_k(\theta) \quad \text{in } (-\infty, t_0] \times \mathbb{S}^{N-1}
\]

as in Lemma 3.4. By Lemma 3.5, let \( v_2 \in C^{2,\alpha}_{\mu}((-\infty, t_0] \times \mathbb{S}^{N-1}) \) be the unique solution of the following problem:

\[
\begin{cases}
\mathcal{L}v = g - \sum_{k=0}^{K+1} g_kQ_k & \text{in } (-\infty, t_0] \times \mathbb{S}^{N-1}, \\
v = 0 & \text{on } \{t_0\} \times \mathbb{S}^{N-1}.
\end{cases}
\]

Then \( v = v_1 + v_2 \) is a solution of (3.1) satisfying that 
\( v(t_0, \theta) = v_1(t_0, \theta) = \sum_{k=0}^{K+1} v_k(t_0)Q_k(\theta) \), where \( v_k(t) \) for \( k = 0, 1, \ldots, K, K + 1 \) is given in Lemma 3.4.

**Remark 3.7.** Theorem 3.6 implies that the bound of

\[
\mathcal{L}^{-1} : C^{2,\alpha}_{\mu}((-\infty, t_0] \times \mathbb{S}^{N-1}) \to C^{2,\alpha}_{\mu}((-\infty, t_0] \times \mathbb{S}^{N-1})
\]

is independent of \( t_0 \).

**4. Nonradial singular solutions of (1.1)**

In what follows, singular solutions of (1.1) will be constructed.

We set

\[
N(w) = w_{\mu} + (N - 2)w_{\gamma} + \Delta_{\mathbb{S}^{N-1}}w + 2(N - 2)(e^w - 1). \tag{4.1}
\]

Then \( w \) satisfies

\[
w_{\mu} + (N - 2)w_{\gamma} + \Delta_{\mathbb{S}^{N-1}}w + 2(N - 2)(e^w - 1) = 0 \quad \text{in } (-\infty, 0) \times \mathbb{S}^{N-1} \tag{4.2}
\]

if \( N(w) = 0 \) in \((-\infty, 0) \times \mathbb{S}^{N-1}\). This also implies that \( u(x) = U_{\gamma}(x) + w(\ln |x|, \theta) \) is a solution of (1.15) in \( B \setminus \{0\} \).

**Theorem 4.1.** Let \( U_{\gamma}(x) \) be given as in (1.2), the index set \( I_\rho \) be given as in (2.58), and \( \mu > \rho_1 \) with \( \mu \notin I_\rho \). Suppose that \( \hat{w} \in C^{2,\alpha}((-\infty, 0] \times \mathbb{S}^{N-1}) \) satisfies

\[
|\hat{w}(t, \theta)| + |\nabla \hat{w}(t, \theta)| \to 0 \quad \text{as } t \to -\infty \text{ uniformly in } \theta \in \mathbb{S}^{N-1},
\]

and for \((t, \theta) \in (-\infty, 0] \times \mathbb{S}^{N-1}\),

\[
|N(\hat{w})(t, \theta)| + |\nabla(N(\hat{w}))(t, \theta)| \leq Ce^{\mu t}, \tag{4.4}
\]

where \( C \) is a positive constant. Then, there exist \( t_0 < 0 \) and a solution \( w \in C^{2,\alpha}((-\infty, t_0] \times \mathbb{S}^{N-1}) \) of the equation in (4.2) such that the following is satisfied for \((t, \theta) \in (-\infty, t_0] \times \mathbb{S}^{N-1}\):

\[
|w(t, \theta) - \hat{w}(t, \theta)| \leq Ce^{\mu t}, \tag{4.5}
\]

where \( C \) is a positive constant.
Proof. The proof consists of 4 steps.

Step 1. We claim that there is \( z \in C^2_{\mu}(\infty, t_0) \times S^{N-1} \) such that

\[
N(\hat{w} + z) = 0. \tag{4.6}
\]

Rewrite this equation as

\[
\mathcal{L}z + N(\hat{w}) + P(z) = 0, \tag{4.7}
\]

and

\[
z = \mathcal{L}^{-1}\left[ -N(\hat{w}) - P(z) \right]. \tag{4.8}
\]

where

\[
P(z) = 2(N-2)[e^{\hat{w}z} - e^{\hat{w}} - z]. \tag{4.9}
\]

Defining \( \mathcal{T} \) by

\[
\mathcal{T}(z) = \mathcal{L}^{-1}\left[ -N(\hat{w}) - P(z) \right], \tag{4.10}
\]

we prove that \( \mathcal{T} \) is a contraction on some ball in \( C^2_{\mu}(\infty, t_0) \times S^{N-1} \), for some \( t_0 < 0 \) with \( |t_0| \) large. We set

\[
\Gamma_{B,t_0} = \{ z \in C^2_{\mu}(\infty, t_0) \times S^{N-1} : \|z\|_{C^2_{\mu}(\infty, t_0) \times S^{N-1}} \leq B \}.
\]

Step 2. We claim that \( \mathcal{T} \) maps \( \Gamma_{B,t_0} \) to itself, for some fixed \( B \) and any \( t_0 \) with \( |t_0| \) sufficiently large, namely, for any \( z \in C^2_{\mu}(\infty, t_0) \times S^{N-1} \) with \( \|z\|_{C^2_{\mu}(\infty, t_0) \times S^{N-1}} \leq B \), we have that \( \mathcal{T}(z) \in C^2_{\mu}(\infty, t_0) \times S^{N-1} \) and \( \|\mathcal{T}(z)\|_{C^2_{\mu}(\infty, t_0) \times S^{N-1}} \leq B \).

First, it follows from (4.4) that

\[
\|N(\hat{w})\|_{C^1_{\mu}(\infty, t_0) \times S^{N-1}} \leq C_1.
\]

Next, set

\[
E(z) = 2(N-2) \int_0^1 (e^{\hat{w}+zs} - 1) ds. \tag{4.11}
\]

Then, \( P(z) = zE(z) \). Take any \( z \in C^2_{\mu}(\infty, t_0) \times S^{N-1} \) with \( \|z\|_{C^2_{\mu}(\infty, t_0) \times S^{N-1}} \leq B \) where \( B \) will be determined later. Because

\[
|\hat{w}| + |\nabla \hat{w}| \leq \epsilon(t),
\]

where \( \epsilon \) is increasing such that \( \epsilon(t) \to 0 \) as \( t \to -\infty \) and

\[
|z| + |\nabla z| \leq Be^{\mu t}.
\]

Then, for \( t \leq t_0 \),

\[
|E(z)| + |\nabla E(z)| \leq C_2(\epsilon(t) + Be^{\mu t}), \tag{4.12}
\]

and hence,

\[
\|P(z)\|_{C^1_{\mu}(\infty, t_0) \times S^{N-1}} \leq C_2(\epsilon(t_0) + Be^{\mu t_0}) \|z\|_{C^1_{\mu}(\infty, t_0) \times S^{N-1}} \leq C_2(\epsilon(t_0) + Be^{\mu t_0})B.
\]

By Theorem 3.6, we have

\[
\|\mathcal{T}(z)\|_{C^2_{\mu}(\infty, t_0) \times S^{N-1}} \leq C\|N(\hat{w}) - P(z)\|_{C^0_{\mu}(\infty, t_0) \times S^{N-1}}
\]
where \( C, C_1 \) and \( C_2 \) are constants that are independent of \( t_0 \). Choose \( B \geq 2C_1 \) and \( t_0 \) with \(|t_0|\) sufficiently large such that \( C_2(e(t_0) + Be^{\mu t}) \leq 1/2 \). Then,

\[
\|T(z)\|_{C^2_{\mu,\alpha}}(\mathbb{R} \times \mathbb{S}^{N-1}) \leq B.
\]

**Step 3.** We claim that \( T : \Gamma_{B,t_0} \to \Gamma_{B,t_0} \) is a contraction, i.e., for any \( z_1, z_2 \in \Gamma_{B,t_0} \),

\[
\|T(z_1) - T(z_2)\|_{C^2_{\mu,\alpha}}(\mathbb{R} \times \mathbb{S}^{N-1}) \leq \kappa \|z_1 - z_2\|_{C^2_{\mu,\alpha}}(\mathbb{R} \times \mathbb{S}^{N-1}),
\]

for some constant \( \kappa \in (0, 1) \).

Note that

\[
|T(z_1) - T(z_2)| = |z_1E(z_1) - z_2E(z_2)| = (z_1 - z_2)E(z_1) + z_2E(z_1) - E(z_2).
\]

By (4.11), we have

\[
E(z_1) - E(z_2) = 2(N - 2) \int_0^1 [e^{\hat{w} + sz_1} - e^{\hat{w} + sz_2}] ds.
\]

Then,

\[
|E(z_1) - E(z_2)| + |\nabla(E(z_1) - E(z_2))| \leq C(|z_1 - z_2| + |\nabla(z_1 - z_2)|).
\]

By (4.12), we have the following for any \( t \leq t_0 \),

\[
|P(z_1) - P(z_2)| + |\nabla(P(z_1) - P(z_2))| \leq C(e(t) + Be^{\mu t})(|z_1 - z_2| + |\nabla(z_1 - z_2)|),
\]

and hence

\[
|P(z_1) - P(z_2)|_{C^2_{\mu,\alpha,\alpha}}(\mathbb{R} \times \mathbb{S}^{N-1}) \leq C(e(t_0) + Be^{\mu t_0})|z_1 - z_2|_{C^2_{\mu,\alpha,\alpha}}(\mathbb{R} \times \mathbb{S}^{N-1}).
\]

By Theorem 3.6, we obtain

\[
\|T(z_1) - T(z_2)\|_{C^2_{\mu,\alpha}}(\mathbb{R} \times \mathbb{S}^{N-1}) \leq C\|P(z_1) - P(z_2)\|_{C^2_{\mu,\alpha}}(\mathbb{R} \times \mathbb{S}^{N-1}) \leq C(e(t_0) + Be^{\mu t_0})|z_1 - z_2|_{C^2_{\mu,\alpha,\alpha}}(\mathbb{R} \times \mathbb{S}^{N-1}).
\]

We derive (4.13) by choosing \( t_0 \) with \( t_0 \) sufficiently negative.

**Step 4.** The contraction mapping principle implies that there exists \( z \in C^2_{\mu,\alpha}((\mathbb{R}, t_0) \times \mathbb{S}^{N-1}) \) such that \( T(z) = z \). Then \( z \in C^2_{\mu,\alpha}((\mathbb{R}, t_0) \times \mathbb{S}^{N-1}) \) is a solution of (4.6) and hence \( w = \hat{w} + z \) is a solution of (4.2).

We call a function \( \hat{w} \) that satisfies (4.3) and (4.4) an approximate solution to (4.2) with order \( \mu \).

**Lemma 4.2.** Assume that \( Q_k(\theta) \) is the combination of \( Q_{k1}(\theta), \ldots, Q_{mk}(\theta) \). Then,

\[
Q_{k+1} = \sum_{i=0}^{k+1} Q_i \tag{4.14}
\]
Proof. The proof is similar to that of Lemma 2.4 in [19]. We omit the details here.

Proposition 4.3. Let the index sets $I_{\rho}$ and $I_{\tilde{\rho}}$ be respectively given as in (2.58) and (2.59) and $\mu > \rho_1$ with $\mu \notin I_{\rho} \cup I_{\tilde{\rho}}$. Let $\eta$ be a solution of $\mathcal{L}(\eta) = 0$ on $\mathbb{R} \times \mathbb{S}^{N-1}$ such that $\eta(t, \cdot) \to 0$ as $t \to -\infty$ uniformly on $\mathbb{S}^{N-1}$. Therefore for some $t_0 < 0$, there exists a smooth function $\tilde{\eta}$ on $(-\infty, t_0] \times \mathbb{S}^{N-1}$ such that $\tilde{\eta} = \eta + \tilde{\eta}$ satisfies (4.3) and (4.4).

Proof. Choose $\phi$ with $|\phi| < \varepsilon$ on $(-\infty, 0) \times \mathbb{S}^{N-1}$, where $\varepsilon > 0$ is a sufficiently small number. Then, a simple computation yields
\[
\mathcal{N}(\phi) = \mathcal{L}\phi + 2(N-2)(e^\phi - 1 - \phi).
\]

Therefore,
\[
\mathcal{N}(\phi) = \mathcal{L}\phi + \sum_{i=2}^{\infty} a_i \phi^i. \tag{4.15}
\]

Assume $K \geq 1$ and $\tilde{K}$ represent the largest corresponding integer with $\rho_K < \mu$ and $\tilde{\rho}_{\tilde{K}} < \mu$, respectively. There is no function that converges to 0 as $t \to -\infty$ in $\text{Ker} \mathcal{L}_0$ and $\text{Ker} \mathcal{L}_1$; for $k \geq 2$, $\psi_k(t) = e^{\rho_k t}$ and $\tilde{\psi}_{\tilde{k}}(t) = e^{\tilde{\rho}_{\tilde{k}} t}$ in $\text{Ker} \mathcal{L}_k$. Any term $e^{\rho_i t}$ with $k > K$ in $\eta$ will produce the term $e^{\tilde{\rho}_{\tilde{k}} t}$ with $\tilde{\rho}_{\tilde{k}} > \mu$ in $\mathcal{N}(\eta)$; set
\[
\eta(t, \theta) = \sum_{k=1}^{K} c_k Q_{k+1}(\theta)e^{\rho_k t}, \tag{4.16}
\]

where $c_k$ denotes constants.

For the following case:
\[
I_{\rho} \cap I_{\tilde{\rho}} = \emptyset, \tag{4.17}
\]

we will show that there $\tilde{\eta}_0, \tilde{\eta}_1, \ldots, \tilde{\eta}_{\tilde{K}}$ exists in succession such that, for any $i = 0, 1, \ldots, \tilde{K}$,
\[
\mathcal{N}(\eta + \tilde{\eta}_0 + \ldots + \tilde{\eta}_i) = O(e^{\tilde{\rho}_{\tilde{K}+1} t}). \tag{4.18}
\]

Set $\phi = \eta$. From (4.15) and $\mathcal{L}(\eta) = 0$, we obtain
\[
\mathcal{N}(\eta) = \sum_{n_1 + \ldots + n_i \geq 2} a_{n_1 \ldots n_i} e^{(n_1 \rho_1 + \ldots + n_i \rho_i) t} Q_2^{n_2} \ldots Q_{n_i+1}^{n_i},
\]

where $n_1, \ldots, n_i$ are nonnegative integers and $a_{n_1 \ldots n_i}$ is a constant. By the definition of $I_{\tilde{\rho}}$, $n_1 \rho_1 + \ldots + n_i \rho_i$ is some $\tilde{\rho}_{\tilde{k}}$. Hence, by Lemma 4.2,
\[
\mathcal{N}(\eta) = \sum_{k=1}^{\tilde{K}} \left( \sum_{m=0}^{\tilde{M}_k} a_{km} Q_m(\theta) \right) e^{\tilde{\rho}_{\tilde{k}} t} + O(e^{\tilde{\rho}_{\tilde{K}+1} t}), \tag{4.19}
\]

where $\tilde{M}_k$ is defined as in (2.61) and $a_{km}$ is constant. We see that
\[
\mathcal{N}(\eta) = O(e^{\tilde{\rho}_{\tilde{k}} t}),
\]

where $\tilde{\rho}_1 = 2 \rho_1$. Then $\tilde{\eta}_0 = 0$. 

Assume that (4.18) holds for 0, 1, \ldots, i - 1. Set
\[
\tilde{\eta}_i(t, \theta) = \left( \sum_{m=0}^{\hat{M}_i} c_{im} Q_m(\theta) \right) e^\tilde{\rho}_i t,
\]  
where $c_{im}$ is constant. This implies that
\[
N(\eta + \tilde{\eta}_0 + \ldots + \tilde{\eta}_i) = \mathcal{L}(\tilde{\eta}_i) + \ldots + \mathcal{L}(\tilde{\eta}_1) + \sum_{k=1}^{\hat{K}} \left( \sum_{m=0}^{\hat{M}_i} a_{km} Q_m(\theta) \right) e^\tilde{\rho}_i t + O(e^{\tilde{\rho}_i t + 1}),
\]
where $a_{km}$ is different from that in (4.19). The induction hypothesis implies that
\[
N(\eta + \tilde{\eta}_0 + \ldots + \tilde{\eta}_i) = \mathcal{L}(\tilde{\eta}_i) + \ldots + \mathcal{L}(\tilde{\eta}_I) + \sum_{k=1}^{\hat{K}} \left( \sum_{m=0}^{\hat{M}_i} a_{km} Q_m(\theta) \right) e^\tilde{\rho}_i t + O(e^{\tilde{\rho}_i t + 1}).
\]
Choose $\tilde{\eta}_i$ such that
\[
\mathcal{L}(\tilde{\eta}_i) = -\left( \sum_{m=0}^{\hat{M}_i} a_{im} Q_m(\theta) \right) e^\tilde{\rho}_i t.
\]
So (4.18) holds for $i$. For $m = 0, 1, \ldots, \hat{M}_i$, solve
\[
\mathcal{L}_m(c_{im} e^{\tilde{\rho}_i t}) = -a_{im} e^{\tilde{\rho}_i t}.
\]
Similar to that in Lemma 3.4, there is a formula for $c_{im} e^{\tilde{\rho}_i t}$ in terms of $a_{im} e^{\tilde{\rho}_i t}$. For $0 < \rho_m < \tilde{\rho}_i$, the expression is in the form of (3.24). For $\rho_m > \tilde{\rho}_i$, the expression is similar. If $m = 0$ or 1, the expression is (3.19) or (3.20). Therefore, define
\[
\tilde{\eta}(t, \theta) = \sum_{k=1}^{\hat{K}} \left( \sum_{m=0}^{\hat{M}_i} c_{km} Q_m(\theta) \right) e^{\tilde{\rho}_i t},
\]  
where $c_{km}$ is a constant. We obtain
\[
N(\eta + \tilde{\eta}) = O(e^{\tilde{\rho}_i t + 1}) = O(e^{\mu t}).
\]
The estimate of $\nabla N(\eta + \tilde{\eta})$ is similar.

Then for the general case, $\rho_k$ can be some $\tilde{\rho}_i$. We only need to modify (4.21). When $\rho_m = \tilde{\rho}_i$, there exist constants $c_{i0m}$ and $c_{i1m}$ such that
\[
\mathcal{L}_m(c_{i0m} + tc_{i1m}) e^{\tilde{\rho}_i t} = -a_{im} e^{\tilde{\rho}_i t}.
\]
By iteration, there are more powers of $t$. Therefore, there exist constants denoted by $c_{ijm}$ with $j = 0, 1, \ldots, J + 1$ such that
\[
\mathcal{L}_m \left( \sum_{j=0}^{J+1} c_{ijm} t^j e^{\tilde{\rho}_i t} \right) = \sum_{j=0}^{J} a_{ijm} t^j e^{\tilde{\rho}_i t}.
\]
In conclusion, instead of (4.22), apply
\[
\eta(t, \theta) = \sum_{k=1}^{K} \sum_{j=0}^{k} \{ \sum_{m=0}^{\infty} c_{kjm} Q_m(\theta) \} t^j \delta^k, \tag{4.23}
\]
where \( c_{kjm} \) is a constant.

**Proof of Theorem 1.1**

Theorem 4.1 and Proposition 4.3 imply that we can obtain \( t_0 \) and a solution \( w \in C^{2,\alpha}((-\infty, t_0] \times \mathbb{S}^{N-1}) \) of the equation in (4.2) such that, for \( \mu > \rho \) with \( \mu \notin \mathcal{I}_\rho \cup \mathcal{I}_\beta \) and any \( (t, \theta) \in (-\infty, t_0) \times \mathbb{S}^{N-1} \),
\[
w(t, \theta) = \tilde{\eta}(t, \theta) + O(e^{\alpha t}), \tag{4.24}
\]
\[
\hat{\eta}(t, \theta) = \tilde{\eta}(t, \theta) + \tilde{\eta}(t, \theta).
\]

Let \( r_0 = e^{\alpha t_0} \). Then \( 0 < r_0 < 1 \). Since \( u(x) = U_s(x) + w(\ln |x|, \theta) \), we add easily see that
\[
u(x) = U_s(x) + O(|x|^{2\alpha}) \quad \text{for} \quad x \in B_{r_0} \setminus \{0\}. \tag{4.25}
\]
On the other hand, for any \( R > 0 \), we see that \( \tilde{u}(y) := u(x) + 2 \ln(R^{-1} r_0), \quad y = R r_0^{1-\alpha} x \) satisfies the following equations:
\[
- \Delta \tilde{u} = e^{\alpha} \quad \text{in} \quad B_R \setminus \{0\} \tag{4.26}
\]
and
\[
\tilde{u}(y) = U_s(y) + O(|y|^{2\alpha}) \quad \text{as} \quad |y| \to 0^+. \tag{4.27}
\]
This implies that \( \tilde{u} \) is a singular solution of (1.1).

Now, suppose that \( \tilde{u} \) is non-radial. It suffices to prove that \( u \) is non-radial in \( B_{r_0} \setminus \{0\} \). Suppose that \( u(x) = u(|x|) \). Then from Proposition 2.1 we have
\[
u(x) \equiv U_s(x) \quad \text{for} \quad x \in B_{r_0} \setminus \{0\}. \tag{4.28}
\]
Moreover, we can easily see from (4.24) and (4.25) that
\[
w(t, \theta) \neq 0.
\]
This implies that \( u \) and \( \tilde{u} \) are non-radial singular solutions of (1.15) in \( B_{r_0} \setminus \{0\} \) and (1.1) in \( B_R \setminus \{0\} \), respectively.

Since \( w(t, \theta) \) depends on the parameter \( \mu \), for each \( \mu > \rho_1 \), different coefficients \( c_k \) of \( \eta(t, \theta) \) in (4.16) can determine infinitely many \( \tilde{u}_\mu \) and \( t_0 \) may change. When restricting all coefficients of \( \eta(t, \theta) \) in (4.16) in a bounded interval, there is a minimal \( t_0 < 0 \). Since \( K \) of \( \eta(t, \theta) \) in (4.16) depends on \( \mu \), we can obtain infinitely many \( \tilde{u}(y) \) by choosing a sequence of parameters \( \mu > \rho_1 \) with \( \mu \to \infty \). Hence, a family of nonradial singular solutions of (1.1) can be constructed.

5. Conclusions

In this manuscript, infinitely many nonradial singular solutions have been constructed for the equation
\[
- \Delta u = e^{\alpha} \quad \text{in} \quad B_R \setminus \{0\},
\]
where \( B_R = \{ x \in \mathbb{R}^N \ (N \geq 3) : \ |x| < R \} \).
Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflicts of interest.

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