Research article

Multiple positive solutions for a singular tempered fractional equation with lower order tempered fractional derivative

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Abstract: Let $\alpha \in (1, 2], \beta \in (0, 1)$ with $\alpha - \beta > 1$. This paper focused on the multiplicity of positive solutions for a singular tempered fractional boundary value problem

\[
\begin{aligned}
&-R_0D_t^{\alpha-\lambda} u(t) = p(t)h \left( e^{\lambda t} u(t) , R_0D_t^{\beta-\lambda} u(t) \right), \\ &R_0D_t^{\beta-\lambda} u(0) = 0, \\ &R_0D_t^{\beta-\lambda} u(1) = 0,
\end{aligned}
\]

where $h \in C([0, +\infty) \times [0, +\infty), [0, +\infty))$ and $p \in L^1([0, 1], (0, +\infty))$. By applying reducing order technique and fixed point theorem, some new results of existence of the multiple positive solutions for the above equation were established. The interesting points were that the nonlinearity contained the lower order tempered fractional derivative and that the weight function can have infinite many singular points in $[0, 1]$.

Keywords: singularity; multiplicity; tempered fractional equation; reducing order technique

1. Introduction

In this paper, we consider the multiplicity of positive solutions for the following singular tempered fractional equation with lower order tempered fractional derivative

\[
\begin{aligned}
&-R_0D_t^{\alpha-\lambda} u(t) = p(t)h \left( e^{\lambda t} u(t) , R_0D_t^{\beta-\lambda} u(t) \right), \\ &R_0D_t^{\beta-\lambda} u(0) = 0, \\ &R_0D_t^{\beta-\lambda} u(1) = 0,
\end{aligned}
\]
where $\alpha \in (1, 2], \beta \in (0, 1)$ with $\alpha - \beta > 1$, $h \in C([0, +\infty) \times [0, +\infty), [0, +\infty]), p \in L^1((0, 1), (0, +\infty))$, which implies that the weight function can have infinite many singular points in $[0, 1]$.  

The equation (1.1) contains a tempered fractional derivative $^\alpha_0 D_\alpha$, which is actually obtained by multiplying an exponential factor in the Riemann-Liouville fractional derivative $^\alpha_0 \mathcal{D}_\alpha$, i.e., the following relationship exists between tempered fractional derivative and Riemann-Liouville fractional derivative  

$$^\alpha_0 D_\alpha u(t) = e^{-\alpha \lambda_0^\beta \mu} (e^{\beta \mu} u(t)).$$  

(1.2)

For the definition of the standard Riemann-Liouville fractional derivative and integral, we refer the reader to [1–5].

As the optimization of the Riemann-Liouville fractional derivative, the tempered fractional derivative has many advantages, which not only overcomes the defect of using the power law of the classical fractional derivative in the mathematical sense, such as the Riemann-Liouville fractional derivatives [6, 7], the Caputo fractional derivatives [8], Hadamard fractional derivatives [9–11] etc, but also brings many practical applications. It especially describes the anomalous diffusion phenomena in Brownian motion with the semi-heavy tails or semi-long range dependence, such as the limits of random walk with an exponentially tempered jump distribution [12, 13], transient super-diffusion [14], anomalous diffusions in heterogeneous systems [15], exponentially tempering Lévy flights with both the $\alpha$-stable and Gaussian trends [16], and pure jump Lévy process with the fractional derivatives in the risk management of financial derivatives traded over the counter [17].

In recent work [18], an upper and lower solutions technique has been employed to establish the existence of positive solutions for a singular tempered fractional turbulent flow model in a porous medium

$$\begin{cases}
^\alpha_0 D_\alpha (\varphi_p (^\beta_0 \mathcal{D}_\beta u(t))) = f(t, u(t)), & t \in (0, 1), \\
u(0) = 0, & ^\alpha_0 D_\alpha u(0) = 0, & u(1) = \int_0^1 e^{-\lambda_0^\beta (t^\gamma)} u(t) dt,
\end{cases}$$  

(1.3)

with $\alpha \in (0, 1], \beta \in (1, 2]$ and the nonlinearity $f$ is decreasing in the second variable. In [19], in view of the monotone iterative method, the iterative properties of positive solutions for a tempered fractional equation

$$\begin{cases}
^\alpha_0 D_\alpha (\varphi_p (^\beta_0 \mathcal{D}_\beta u(t))) = f(t, u(t)), & t \in (0, 1), \\
u(0) = 0, & ^\alpha_0 D_\beta u(0) = 0, & u(1) = \int_0^1 e^{-\lambda_0^\beta (t^\gamma)} u(t) dt,
\end{cases}$$  

(1.4)

were established, where $\theta \in (1, 2], \delta \in (0, 1)$ with $\theta - \delta > 1$, $\mu$ is a positive constant, and $f : (0, 1) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function with respect to the two space variables. Recently, by using spaces theories [20–24], smooth theories [25–27], operator method [28, 29], the method of moving sphere [30], critical point theories [31–34] and so on, some other types of fractional equations was also studied [35–46].

However, when the nonlinearity contains lower order tempered fractional derivative, the results of multiplicity of positive solutions have not yet been obtained. In order to fill this gap, by applying the reducing order technique and fixed point theorem, some new results of existence of the multiple positive solutions for the above equation are established in this paper. The interesting points are that the nonlinearity contains the lower order tempered fractional derivative and the weight function can have infinitely many singular points in $[0, 1]$.  

2. Preliminaries and lemmas

In this section, we first recall definitions and some useful properties of the Riemann-Liouville fractional derivative and integral.

**Definition 2.1 (2.1.1 on page 69 in [1])**. The Riemann-Liouville fractional integral of order \( \alpha > 0 \) of a function \( u : (0, +\infty) \to \mathbb{R} \) is given by

\[
I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds
\]

provided that the righthand side is pointwise and defined on \((0, +\infty)\).

**Definition 2.2 (2.1.5 on page 70 in [1])**. The Riemann-Liouville fractional derivative of order \( \alpha > 0 \) of a function \( u : (0, +\infty) \to \mathbb{R} \) is given by

\[
\mathcal{D}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds,
\]

where \( n = [\alpha] + 1 \), \([\alpha]\) denotes the greatest integer part of the number \( \alpha \), provided that the righthand side is pointwise and defined on \((0, +\infty)\).

The following properties of the Riemann-Liouville fractional derivative and integral can be found on pages 73–75 (Lemmas 2.3–2.5) in [1].

**Lemma 2.1.** Suppose \( u(t) \in C[0, 1] \cap L^1[0, 1] \) and \( \alpha > \beta > 0 \). Let \( n = [\alpha] + 1 \), then

(i) \[
I_0^a \mathcal{D}_t^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},
\]

where \( c_i \in \mathbb{R}, i = 1, 2, 3, \ldots, n \).

(ii) \[
I_0^a p^\beta u(t) = I_0^a I_0^a u(t) = I_0^a I_0^a u(t) = I_0^a I_0^a u(t),
\]

**Lemma 2.2.** Let \( p \in L^1([0, 1], (0, +\infty)) \), then the singular linear tempered fractional equation

\[
\begin{cases}
- \mathcal{D}_t^{\alpha-\beta} u(t) = p(t), \\
 0 = 0, \quad u(1) = 0,
\end{cases}
\]

has a unique positive solution \( u(t) \) provided that \( 1 < \alpha - \beta \leq 2 \), which can be expressed by

\[
u(t) = \int_0^t H(t, s)p(s)ds
\]

where

\[
H(t, s) = \begin{cases}
\frac{t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} e^{-\beta t} e^{\lambda s}, & 0 \leq s \leq t \leq 1; \\
\frac{t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} e^{-\beta s} e^{\lambda t}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

is the Green function of (2.2).
Proof. In fact, it follows from $1 < \alpha - \beta \leq 2$, (1.2) and (2.1) that

$$e^{\lambda t}u(t) = - \frac{1}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} e^{\lambda s} p(s) ds + b_1 t^{\alpha - \beta - 1} + b_2 t^{\alpha - \beta - 2}, \ t \in [0, 1].$$

Since $u(0) = 0$ and $u(1) = 0$, we have $b_2 = 0$ and

$$b_1 = \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{\alpha - \beta - 1} e^{\lambda s} p(s) ds.$$

Thus

$$u(t) = \frac{1}{\Gamma(\alpha - \beta)} \left[ \int_0^1 (1 - s)^{\alpha - \beta - 1} e^{-\lambda t} e^{\lambda s} p(s) ds t^{\alpha - \beta - 1} - \int_0^t (t - s)^{\alpha - \beta - 1} e^{-\lambda t} e^{\lambda s} p(s) ds \right]$$

$$= \frac{1}{\Gamma(\alpha - \beta)} \int_0^t \left[ e^{\alpha - \beta - 1}(1 - s)^{\alpha - \beta - 1} - (t - s)^{\alpha - \beta - 1} \right] e^{-\lambda t} e^{\lambda s} p(s) ds$$

$$+ \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 t^{\alpha - \beta - 1}(1 - s)^{\alpha - \beta - 1} e^{-\lambda t} e^{\lambda s} p(s) ds$$

$$= \int_0^1 H(t, s) p(s) ds, \ t \in [0, 1].$$

The following Lemma has been proven (see Lemma 2.3 of [6] or Lemma 3 of [4]).

**Lemma 2.3.** Suppose $\alpha \in (1, 2], \beta \in (0, 1)$ with $\alpha - \beta > 1$, then $H(t, s)$ is a nonnegative continuous function in $[0, 1] \times [0, 1]$ satisfying, for any $(t, s) \in [0, 1] \times [0, 1]$,

$$\frac{t^{\alpha - \beta - 1}(1 - t)e^{-\lambda t}}{\Gamma(\alpha - \beta)}(1 - s)^{\alpha - \beta - 1} e^{\lambda s} \leq H(t, s) \leq \frac{t^{\alpha - \beta - 1}(1 - t)e^{-\lambda t}}{\Gamma(\alpha - \beta)} \left( \frac{(1 - s)^{\alpha - \beta - 1} e^{\lambda s}}{\Gamma(\alpha - \beta)} \right).$$

(2.5)

Suppose $\alpha \in (1, 2], \beta \in (0, 1)$ with $\alpha - \beta > 1$. In order to use the reducing order technique, we introduce the following integral transformation

$$u(t) = e^{-\lambda t} P^\beta (e^{\lambda t} y(t)), \ t \in [0, 1]$$

(2.6)

and then consider the following reducing order problem

$$\begin{cases}
- \frac{\partial}{\partial t} \gamma^{\alpha - \beta - 1} y(t) = p(t) h(P^\beta (e^{\lambda t} y(t)), y(t)), \ t \in (0, 1), \\
y(0) = 0, \ y(1) = 0.
\end{cases}$$

(2.7)

**Lemma 2.4.** Suppose $\alpha \in (1, 2], \beta \in (0, 1)$ with $\alpha - \beta > 1$. The reducing order problem (2.7) is equivalent to the singular tempered fractional equation (1.1). In particular, if $y$ is a positive solution of the problem (2.7), then $u(t) = e^{-\lambda t} P^\beta (e^{\lambda t} y(t))$ is a positive solution of the singular tempered fractional equation (1.1).
Proof. We first suppose that \( u \) is a positive solution of the singular tempered fractional equation (1.1). Let
\[
    u(t) = e^{-\lambda t} p(t) (e^{\lambda t} y(t)).
\]
Noticing that \( \alpha \in (1,2], \beta \in (0,1) \) with \( \alpha - \beta > 1 \), let \( n = \lceil \alpha \rceil + 1 \), i.e., \( n \) is the smallest integer greater than or equal to \( \alpha \), then it follows from (1.2) and Lemma 2.1 that
\[
    R^a D_t^\beta u(t) = e^{-\lambda t} D_t^\alpha (e^{\lambda t} u(t)) = e^{-\lambda t} D_t^n (e^{\lambda t} u(t)) = e^{-\lambda t} D_t^n (e^{\lambda t} y(t)),
\]
and
\[
    R^b D_t^\beta y(t) = e^{-\lambda t} D_t^n (e^{\lambda t} y(t)) = e^{-\lambda t} D_t^n (e^{\lambda t} y(t)) = y(t).
\]
By (2.8) and (2.9), we have
\[
\begin{aligned}
    \begin{cases}
    -R^a D_t^\beta y(t) = -R^a D_t^\beta u(t) = p(t) h(e^{\lambda t} u(t), 0), \\
y(0) = R^b D_t^\beta u(0) = 0, \quad y(1) = R^b D_t^\beta u(1) = 0,
    \end{cases}
\end{aligned}
\]
thus, \( y \) solves the equation (2.7).

Conversely, suppose that \( y \) is any solution of the reducing order problem (2.7), then we have
\[
\begin{aligned}
    \begin{cases}
    -R^a D_t^\beta y(t) = p(t) h(e^{\lambda t} u(t), 0), \\
y(0) = 0, \quad y(1) = 0.
    \end{cases}
\end{aligned}
\]
Make integral transformation (2.6), similar to (2.8) and (2.9), we have
\[
    -R^a D_t^\beta u(t) = -R^a D_t^\beta y(t), \quad R^b D_t^\beta u(t) = y(t).
\]
Substituting the above formulas into (2.7), we get
\[
    -R^a D_t^\beta u(t) = p(t) h(e^{\lambda t} y(t), 0), \quad 0 < t < 1,
\]
and
\[
    R^b D_t^\beta u(0) = 0, \quad R^b D_t^\beta u(1) = 0,
\]
which implies that \( u(t) = e^{-\lambda t} p(t) (e^{\lambda t} y(t)) \) solves the singular tempered fractional equation (1.1).

Suppose \( E = C([0,1]; \mathbb{R}) \) with the norm
\[
    \|y\| = \max_{t \in [0,1]} |y(t)|.
\]
Define a cone of \( E \) and an operator \( T \), respectively:
\[
    P = \{ y \in E : y(t) \geq f^\alpha e^{-\lambda t} (1 - t) e^{-\lambda t} |y(t)| \},
\]

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and

\[(Ty)(t) = \int_0^1 H(t, s)p(s)h(\mathcal{P}(y(s)), y(s))ds. \quad (2.10)\]

In order to obtain the existence of a positive solution of the Eq (1.1), Lemma 2.4 indicates that we only consider the fixed points of the operator \(T\).

Now, we list the following hypotheses, which are used in the rest of this paper.

(C1) \(h \in C([0, +\infty) \times [0, +\infty), [0, +\infty))\) and \(p \in L^1((0, 1), (0, +\infty))\).

(C2) There is a constant \(n > 0\) such that for any \(0 \leq u + v \leq (1 + \frac{e^1}{\Gamma(\beta + 1)})n\),

\[h(u, v) < \mu n,\]

where

\[\mu = \left[ \frac{ne^1}{\Gamma(\alpha - \beta)} \int_0^1 p(s)ds \right]^{-1}.\]

(C3) There is a constant \(\rho > 0\) such that for any

\[\left( \frac{\Gamma(\alpha - \beta)\left(\frac{1}{2}\right)^{\alpha - 1}}{\Gamma(\alpha)} + \left(\frac{1}{4}\right)^{\alpha - \beta} e^{-\frac{1}{4}} \right) \rho \leq u + v \leq \left(1 + \frac{e^1}{\Gamma(\beta + 1)}\right)\rho\]

it implies

\[h(u, v) \geq \varrho \rho,\]

where

\[\varrho = 8^{\alpha - \beta} e^{\frac{1}{4} \Gamma(\alpha - \beta)} \left[ \int_{\frac{1}{2}}^1 p(s)ds \right]^{-1}.\]

**Lemma 2.5.** Assume \(\alpha \in (1, 2], \beta \in (0, 1)\) with \(\alpha - \beta > 1\) and (C1) holds, then the operator \(T : P \to P\) is completely continuous.

**Proof.** First by (C1), \(T\) is continuous on \([0, 1]\). For any \(y \in P\), there exists a constant \(M > 0\) such that \(\|y\| \leq M\), then

\[0 \leq \mathcal{P}(y(s)) = \int_0^1 (t - s)^{\beta - 1} e^{\frac{1}{4}s} \frac{y(s)}{\Gamma(\beta)} ds \leq \frac{Me^1}{\Gamma(\beta)}. \quad (2.11)\]

Let

\[N = \max_{(u, v) \in [0, M]} h(u, v),\]

then by (2.11), for any \(y \in P\), we have

\[\|Ty\| = \max_{t \in [0, 1]} \int_0^1 H(t, s)p(s)h(\mathcal{P}(y(s)), y(s))ds \leq \int_0^1 \frac{(1 - s)^{\alpha - \beta - 1}e^{\frac{1}{4}s}}{\Gamma(\alpha - \beta)} p(s)h(\mathcal{P}(y(s)), y(s))ds \]

\[\leq \int_0^1 \frac{(1 - s)^{\alpha - \beta - 1}e^{\frac{1}{4}s}}{\Gamma(\alpha - \beta)} p(s)ds \leq \frac{Ne^1}{\Gamma(\alpha - \beta)} \int_0^1 p(s)ds < +\infty, \quad (2.12)\]
which implies that $T : P \to E$ is well-defined. In addition, by (2.5) and (2.12), we have

$$
(Ty)(t) \geq \int_0^1 \frac{(1-s)^{\alpha-1}s e^{k s}}{\Gamma(\alpha-\beta)} p(s)h(p^s y(s), y(s))ds 
\times t^{\alpha-1}(1-t)e^{-\lambda t} \geq \|Ty\|t^{\alpha-1}(1-t)e^{-\lambda t},
$$

which implies that $T(P) \subset P$.

In the end, by using the standard arguments and combining the Ascoli-Arzela theorem, we know $T(P) \subset P$ is completely continuous.

Our proof of main results depends on the fixed point theorem of cone expansion and compression (see Theorem 2.3.3 on page 93 of [47]).

Lemma 2.6. [47] Suppose $P$ is a cone of real Banach space $E$, the bounded open subsets $\Omega_1, \Omega_2$ of $E$ satisfy $\emptyset \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either

1. $\|Tz\| \leq \|z\|, z \in P \cap \partial \Omega_1$ and $\|Tz\| \geq \|z\|, z \in P \cap \partial \Omega_2$, or
2. $\|Tz\| \geq \|z\|, z \in P \cap \partial \Omega_1$ and $\|Tz\| \leq \|z\|, z \in P \cap \partial \Omega_2$;

then $T$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main results

For the convenience of the proof, we first introduce the following notations whenever the limits exist or not:

$$
h_0 = \lim_{u+v \to 0} \frac{h(u,v)}{u+v}, \quad h_\infty = \lim_{u+v \to +\infty} \frac{h(u,v)}{u+v},
$$

and then state our main results as follows.

Theorem 3.1. Assume that (C1) and (C2) hold, and

$$
h_0 = +\infty, h_\infty = +\infty,
$$

then the tempered fractional equation (1.1) has at least two positive solutions $u_1, u_2$; moreover, there exist four constants $A_1, B_1, A_2, B_2 > 0$ such that

$$
A_1 t^{\alpha-1} e^{-\lambda t} \leq u_1(t) \leq B_1 t^{\alpha(1-t)}, \quad A_2 t^{\alpha-1} e^{-\lambda t} \leq u_2(t) \leq B_2 t^{\alpha(1-t)}.
$$

Proof. It follows from $h_0 = +\infty$ that there exist $0 < m < n$ and a sufficiently large constant

$$
N > \frac{32^{\alpha-\beta} \Gamma(\alpha-\beta)}{\int_0^1 p(s)ds}
$$

such that for any $0 < u + v \leq \left(1 + \frac{\epsilon}{\Gamma(\beta+1)}\right) m$,

$$
h(u,v) \geq N(u+v).
$$
Take $\Omega_m = \{y \in E : ||y|| < m\}$, and $\partial \Omega_m = \{y \in E : ||y|| = m\}$, then for any $y \in P \cap \partial \Omega_m$, one has

$$P^\beta(e^{s \beta}y(s)) = \int_0^\infty \frac{(t-s)\beta^{-1}}{\Gamma(\beta)} e^{s \beta}y(s)ds \leq \frac{e^m t^\beta}{\Gamma(\beta + 1)} \leq \frac{e^m}{\Gamma(\beta + 1)},$$

(3.1)

then

$$P^\beta(e^{s \beta}y(s)) + y(s) \leq \left(1 + \frac{e^\beta}{\Gamma(\beta + 1)}\right)m. \quad (3.2)$$

Thus,

$$||Ty|| \geq (Ty)\left(\frac{1}{2}\right) = \int_0^1 H\left(\frac{1}{2}, s\right) p(s)h(P^\beta(e^{s \beta}y(s)), y(s))ds$$

$$\geq 2^{-a+\beta}e^{s \beta} \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{a-\beta-1} se^{s \beta}p(s)N(P^\beta(e^{s \beta}y(s)) + y(s))ds$$

$$\geq \frac{2^{-a+\beta}e^{s \beta}}{\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{a-\beta-1} se^{s \beta}p(s)Ny(s)ds$$

$$\geq 2^{-a+\beta}e^{s \beta} \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{a-\beta-1} se^{s \beta}p(s)Ny(s)ds$$

$$\geq \frac{N}{8^{a-\beta} \frac{e^{s \beta}}{\Gamma(\alpha - \beta)}} \int_0^1 (1 - s)^{a-\beta-1} se^{s \beta}p(s)ds||y||$$

$$\geq \frac{N}{32^{a-\beta} e^{s \beta} \left(\frac{1}{\Gamma(\alpha - \beta)}\right)} \int_0^1 p(s)ds||y|| \geq ||y||.$$  

(3.3)

So, for any $y \in P \cap \partial \Omega_m$, we have $||Ty|| \geq ||y||$.

Next, let $\Omega_n = \{y \in E : ||y|| < n\}$ and $\partial \Omega_n = \{y \in E : ||y|| = n\}$, then it follows from (3.2) and (C2) that for any $y \in P \cap \partial \Omega_n$, one has

$$(Ty)(t) = \int_0^1 H(t, s)p(s)h(P^\beta(e^{s \beta}y(s)), y(s))ds$$

$$\leq \int_0^1 \frac{(1 - s)^{a-\beta-1} se^{s \beta}}{\Gamma(\alpha - \beta)} p(s)h(P^\beta(e^{s \beta}y(s)), y(s))ds$$

$$\leq \frac{e^\beta}{\Gamma(\alpha - \beta)} \int_0^1 p(s)h(P^\beta(e^{s \beta}y(s)), y(s))ds$$

$$\leq \frac{\mu e^\beta}{\Gamma(\alpha - \beta)} \int_0^1 p(s)ds \leq n. \quad (3.4)$$

Therefore for any $y \in P \cap \partial \Omega_n$, we have $||Ty|| \leq ||y||$.

On the other hand, it follows from $h_\infty = +\infty$ that there exists $M > 0$ and

$$\eta \geq 32^{a-\beta} e^{s \beta} \frac{1}{\Gamma(\alpha - \beta)} \left(\int_0^1 p(s)ds\right)^{-1}$$

such that for any $u + v > M$, we have

$$h(u, v) \geq \eta(u + v).$$
Take $R = n + 4^{\alpha-\beta}e^{\frac{1}{4}t}M$ and let $\Omega_R = \{y \in E : ||y|| < R\}$ and $\partial\Omega_R = \{y \in E : ||y|| = R\}$, then for any $y \in P \cap \partial\Omega_R$ and $[\frac{1}{4}, \frac{3}{4}]$, one has

$$y(t) \geq \left(\frac{1}{4}\right)^{\alpha-\beta} e^{-\frac{1}{4}t} ||y|| \geq \left(\frac{1}{4}\right)^{\alpha-\beta} (n + 4^{\alpha-\beta}e^{\frac{1}{4}t}M) \geq M.$$  

Thus, for any $y \in P \cap \partial\Omega_R$, we have

$$||Ty|| \geq (Ty) \left(\frac{1}{2}\right) = \int_0^1 H\left(\frac{1}{2}, s\right) p(s)h(f^\beta(e^{\alpha t}y(s)), y(s)) ds$$

$$\geq \frac{2^{-\alpha+\beta}e^{-\frac{1}{4}t}}{\Gamma(\alpha-\beta)} \int_\frac{1}{4}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} se^{\alpha t} p(s)\eta(f^\beta(e^{\alpha t}y(s)) + y(s)) ds$$

$$\geq \frac{2^{-\alpha+\beta}e^{-\frac{1}{4}t}}{\Gamma(\alpha-\beta)} \int_\frac{1}{4}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} se^{\alpha t} p(s)\eta y(s) ds$$

$$\geq \frac{2^{-\alpha+\beta}e^{-\frac{1}{4}t}}{\Gamma(\alpha-\beta)} \int_\frac{1}{4}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} se^{\alpha t} p(s)\eta ||y|| ds$$

$$\geq \frac{\eta}{8^{\alpha-\beta}e^{\frac{1}{4}t}\Gamma(\alpha-\beta)} \int_\frac{1}{4}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} se^{\alpha t} p(s) ds ||y||$$

$$\geq \frac{\eta}{32^{\alpha-\beta}e^{\frac{1}{4}t}\Gamma(\alpha-\beta)} \int_\frac{1}{4}^{\frac{3}{4}} p(s) ds ||y|| \geq ||y||.$$  

Therefore, for any $y \in P \cap \partial\Omega_R$, we have $||Ty|| \leq ||y||$.

According to Lemma 2.6, $T$ has two fixed points $y_1 \in P \cap (\overline{\Omega_m} \setminus \Omega_n)$ and $y_2 \in P \cap (\overline{\Omega_R} \setminus \Omega_n)$ with $m \leq ||y_1|| \leq n \leq ||y_2|| \leq R$. Thus, it follows from Lemma 2.4 that the tempered fractional equation $(1.1)$ has at least two positive solutions satisfying

$$\frac{m\Gamma(\alpha-\beta)t^{\alpha-1}e^{-\alpha t}}{\Gamma(\alpha)} \leq u_1(t) = e^{-\alpha t} f^\beta(e^{\alpha t}y_1(t)) \leq \frac{ne^{\alpha t(1-t)}}{\Gamma(\beta + 1)},$$

and

$$\frac{n\Gamma(\alpha-\beta)t^{\alpha-1}e^{-\alpha t}}{\Gamma(\alpha)} \leq u_2(t) = e^{-\alpha t} f^\beta(e^{\alpha t}y_2(t)) \leq \frac{Re^{\alpha t(1-t)}}{\Gamma(\beta + 1)}.$$  

**Theorem 3.2.** Assume that $(C1)$ and $(C3)$ hold and

$$h_0 = 0, h_\infty = 0,$$

then the tempered fractional equation $(1.1)$ has at least two positive solutions $u_3, u_4$; moreover, there exist four constants $A_3, B_3, A_4, B_4 > 0$ such that

$$A_3 t^{\alpha-1} e^{-\alpha t} \leq u_3(t) \leq B_3 e^{\alpha t(1-t)}, \quad A_4 t^{\alpha-1} e^{-\alpha t} \leq u_4(t) \leq B_4 e^{\alpha t(1-t)}.$$
Proof. First, notice that \( h_0 = 0 \), for any \( \epsilon > 0 \). Let us select \( 0 < \kappa < \rho \) such that for any \( 0 < u + v < (1 + \frac{e^t}{\Gamma(\beta+1)})\kappa \),

\[
h(u, v) \leq \epsilon(u + v).
\]

Choose \( \epsilon \) such that

\[
\frac{\epsilon e^t}{\Gamma(\alpha - \beta)} \left( 1 + \frac{e^t}{\Gamma(\beta + 1)} \right) \int_0^1 p(s)ds \leq 1.
\]

Now, let \( \Omega_\kappa = \{ y \in E : ||y|| < \kappa \} \) and \( \partial \Omega_\kappa = \{ y \in E : ||y|| = \kappa \} \), then for any \( y \in P \cap \partial \Omega_\kappa \), the same as (3.2), we also have

\[
P^\beta(e^{\lambda t}y(s)) + y(s) \leq \left( 1 + \frac{e^t}{\Gamma(\beta + 1)} \right)\kappa,
\]

which implies that for any \( y \in P \cap \partial \Omega_\kappa \), the following estimation is valid:

\[
h(P^\beta(e^{\lambda t}y(s)), y) \leq \epsilon(P^\beta(e^{\lambda t}y(s)) + y(s)).
\]

Consequently, for any \( y \in P \cap \partial \Omega_\kappa \), one gets

\[
(Ty)(t) = \int_0^1 H(t, s)p(s)h(P^\beta(e^{\lambda t}y(s)), y(s))ds
\]

\[
\leq \int_0^1 (1 - s)^{\alpha - \beta - 1}e^{\lambda s}p(s)h(P^\beta(e^{\lambda t}y(s)), y(s))ds
\]

\[
\leq \frac{e^t}{\Gamma(\alpha - \beta)} \int_0^1 p(s)h(P^\beta(e^{\lambda t}y(s)), y(s))ds
\]

\[
\leq \frac{e^t}{\Gamma(\alpha - \beta)} \int_0^1 p(s)e(P^\beta(e^{\lambda t}y(s)) + y(s))ds
\]

\[
\leq \frac{e^t}{\Gamma(\alpha - \beta)} \int_0^1 p(s)e \left( 1 + \frac{e^t}{\Gamma(\beta + 1)} \right)kd\kappa
\]

\[
\leq \frac{\epsilon e^t}{\Gamma(\alpha - \beta)} \left( 1 + \frac{e^t}{\Gamma(\beta + 1)} \right) \int_0^1 p(s)ds \kappa \leq \kappa.
\]

(3.7) implies that \( ||Ty|| \leq ||y|| \), \( y \in P \cap \partial \Omega_\kappa \).

Next, let \( \Omega_\rho = \{ y \in E : ||y|| < \rho \} \), and \( \partial \Omega_\rho = \{ y \in E : ||y|| = \rho \} \), then for any \( y \in P \cap \partial \Omega_\rho \), we have

\[
P^\beta(e^{\lambda t}s^{\alpha - \beta - 1}y(s)) = \int_0^1 (t - s)^{\beta - 1} \Gamma(\beta)s^{\alpha - \beta - 1}e^{\lambda s}y(s)ds \geq \frac{\Gamma(\alpha - \beta)\rho^{\alpha - 1}}{\Gamma(\alpha)}. \]

Thus, it follows from (3.8) that for any \( y \in P \cap \partial \Omega_\rho \) and \( \left[ \frac{1}{4}, \frac{3}{4} \right] \), one has

\[
\left( \frac{\Gamma(\alpha - \beta)\left( \frac{1}{4} \right)^{\alpha - 1}}{\Gamma(\alpha)} + \left( \frac{1}{4} \right)^{\alpha - \beta}e^{\frac{1}{4}t} \right) \rho = \frac{\Gamma(\alpha - \beta)\left( \frac{1}{4} \right)^{\alpha - 1}}{\Gamma(\alpha)}\rho + \left( \frac{1}{4} \right)^{\alpha - \beta}e^{\frac{1}{4}t}\rho.
\]

\[
P^\beta(e^{\lambda t}s^{\alpha - \beta - 1}y(s)) + y(t) \leq \left( 1 + \frac{e^t}{\Gamma(\beta + 1)} \right)\rho.
\]
So, for any \( y \in P \cap \partial \Omega_p \), we have

\[
\|Ty\| \geq (Ty) \left( \frac{1}{2} \right) = \int_0^1 H \left( \frac{1}{2}, s \right) p(s) h(I^p(e^{t_s}y(s)), y(s))ds \\
\geq \frac{2^{-\alpha-\beta}e^{-\frac{1}{2}l}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} se^{ls} p(s) h(I^p(e^{t_s}y(s)), y(s))ds \\
\geq \frac{2^{-\alpha-\beta}e^{-\frac{1}{2}l}}{\Gamma(\alpha-\beta)} \int_0^\frac{1}{2} (1-s)^{\alpha-\beta-1} se^{ls} p(s) h(I^p(e^{t_s}y(s)), y(s))ds \\
\geq \frac{2^{-\alpha-\beta}e^{-\frac{1}{2}l}}{\Gamma(\alpha-\beta)} \int_0^\frac{1}{2} (1-s)^{\alpha-\beta-1} se^{ls} p(s) dp ds \geq \frac{\theta}{8^{-\alpha-\beta}e^{\frac{1}{2}l}} \int_0^\frac{1}{2} p(s) ds \geq ||y||.
\]

which implies that \( ||Ty|| \geq ||y|| \) holds and \( y \in P \cap \partial \Omega_p \).

On the other hand, since \( h_{\infty} = 0 \), for any \( \epsilon > 0 \), there exists \( M^* > 0 \) such that for any \( u + v > M^* \)

\[
h(u, v) \leq \epsilon (u + v).
\]

For the above \( \epsilon > 0 \), choose a sufficiently small one such that

\[
\frac{\epsilon e^l}{\Gamma(\alpha-\beta)} \left( 1 + \frac{e^l}{\Gamma(\beta+1)} \right) \int_0^1 p(s) ds < 1,
\]

and take

\[
R^* = \max \left\{ \frac{\epsilon e^l}{\Gamma(\alpha-\beta)} \max_{0 \leq u+v \leq M^*} h(u, v) \int_0^1 p(s) ds, 1 - \frac{\epsilon e^l}{\Gamma(\alpha-\beta)} \left( 1 + \frac{\epsilon e^l}{\Gamma(\beta+1)} \right) \int_0^1 p(s) ds \right\}.
\]

Next, let \( \Omega_{R^*} = \{ y \in E : ||y|| < R^* \} \), and \( \partial \Omega_{R^*} = \{ y \in E : ||y|| = R^* \} \), then for any \( y \in P \cap \partial \Omega_{R^*} \), one has

\[
(Ty)(t) = \int_0^1 H(t, s) p(s) h(I^p(e^{t_s}y(s)), y(s)) ds \leq \int_0^1 H(t, s) \left( (1-s)^{\alpha-\beta-1} se^{ls} \right) p(s) h(I^p(e^{t_s}y(s)), y(s)) ds \\
\leq \frac{e^l}{\Gamma(\alpha-\beta)} \left\{ \int_{0 \leq \rho \leq M^*} (1+\frac{e^l}{\Gamma(\beta+1)}) p(s) ds \right\} \max_{0 \leq u+v \leq M^*} h(u, v) \int_0^1 p(s) ds \\
+ \int_{M^* \leq \rho \leq e^l u(y(s)) \leq 1+\frac{e^l}{\Gamma(\beta+1)}} p(s) h(I^p(e^{t_s}y(s)), y(s)) ds \\
\leq \frac{e^l}{\Gamma(\alpha-\beta)} \left\{ \int_{0 \leq u+v \leq M^*} (1+\frac{e^l}{\Gamma(\beta+1)}) p(s) ds \right\} \max_{0 \leq u+v \leq M^*} h(u, v) + \epsilon \left( 1 + \frac{e^l}{\Gamma(\beta+1)} \right) R^* \int_0^1 p(s) ds \leq R^*.
\]

Therefore, for any \( y \in P \cap \partial \Omega_{R^*} \), we have \( ||Ty|| \geq ||y|| \).
According to Lemma 2.6, $T$ has two fixed points $y_3 \in P \cap (\Omega_0 \backslash \Omega_{\rho})$ and $y_4 \in P \cap (\Omega_{R} \backslash \Omega_{\rho})$ with $\kappa \leq ||y_3|| \leq \rho \leq ||y_4|| \leq R$. Thus, it follows from Lemma 2.4 that the tempered fractional equation (1.1) has at least two positive solutions satisfying

$$\frac{\kappa \Gamma(\alpha - \beta) t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} \leq u_3(t) = e^{-\lambda t} I^\rho (e^{\lambda t} y_3(t)) \leq \frac{\rho e^{\lambda (1-t)}}{\Gamma(\beta + 1)},$$

and

$$\frac{\rho \Gamma(\alpha - \beta) t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} \leq u_4(t) = e^{-\lambda t} I^\rho (e^{\lambda t} y_4(t)) \leq \frac{R^* e^{\lambda (1-t)}}{\Gamma(\beta + 1)}.$$

### 4. Examples

The tempered fractional diffusion equation has many important applications, including the tempered fractional Langevin and Vasicek differential equations [48] and the space-time tempered fractional diffusion-wave equation [49]. The new theorems established in the present paper are very useful in the area of tempered fractional calculus. We can find the multiple positive solutions for the singular tempered fractional equations with the lower order tempered fractional derivatives using the proposed theorems.

Let us apply the main results to solve two singular tempered fractional equations with lower order tempered fractional derivative.

**Example 4.1.** Let $\alpha = \frac{5}{3}$, $\beta = \frac{1}{3}$, $\lambda = 3$, and

$$h(u, v) = \begin{cases} (u + v)^{\frac{2}{3}}, & 0 \leq u + v \leq 47, \\ (u + v)^2, & u + v > 47. \end{cases}$$

We consider the multiplicity of positive solutions for the following singular tempered fractional equation with the lower order tempered fractional derivative:

$$\begin{cases} -\frac{\kappa}{6} \mathbb{D}^{\frac{5}{3}} u(t) = \frac{1}{2} - t \left| \frac{1}{2} \right| h\left(e^{3\lambda t} u(t), \frac{\kappa}{6} \mathbb{D}^{\frac{5}{3}} u(t)\right), 0 \leq t \leq 1, \\ \frac{\kappa}{6} \mathbb{D}^{\frac{5}{3}} u(0) = 0, \frac{\kappa}{6} \mathbb{D}^{\frac{5}{3}} u(1) = 0. \end{cases}$$

(4.1)

**Conclusion.** The singular tempered fractional equation (4.1) has at least two positive solutions $u_1, u_2$: moreover, there exist four constants $A_1, B_1, A_2, B_2 > 0$ such that

$$A_1 t^\frac{3}{2} e^{-3\lambda t} \leq u_1(t) \leq B_1 e^{3(1-t)}, \quad A_2 t^\frac{3}{2} e^{-3\lambda t} \leq u_2(t) \leq B_2 e^{3(1-t)}.$$

**Proof.** Here,

$$p(t) = \left| \frac{1}{2} - t \right|^{\frac{1}{2}}, \quad h(u, v) = \begin{cases} (u + v)^{\frac{2}{3}}, & 0 \leq u + v \leq 47, \\ \frac{600}{(u + v)^2}, & u + v > 47. \end{cases}$$
There exist four constants $A$, $C_1$, $C_2$ and $C_3$ such that
\[
\lim_{u,v \to 0} \frac{h(u,v)}{u+v} = +\infty, \quad \lim_{u,v \to +\infty} \frac{h(u,v)}{u+v} = +\infty.
\]

In the following, we verify the condition (C2). In fact, take $n = 2$, then
\[
\mu = \left[ \frac{ne^\delta}{\Gamma(\alpha - \beta)} \int_0^1 p(s)ds \right]^{-1} = \left[ \frac{2e^3}{\Gamma(\frac{3}{2})} \int_0^1 \left| \frac{1}{2} - s \right|^{\frac{1}{2}} ds \right]^{-1} = 0.0079,
\]
and for any $0 \leq u + v \leq (1 + \frac{e^1}{\Gamma(2)}) n = 2 \left( 1 + \frac{e^1}{\Gamma(2)} \right) = 46.9853$, we have
\[
h(u,v) < \frac{47.5}{600} = 0.0114 < \mu n = 0.0158,
\]
which implies that (C2) holds.

Consequently, it follows from Theorem 3.1 that the tempered fractional equation (4.1) has at least two positive solutions $u_1, u_2$; moreover, there exist four constants $A_1, B_1, A_2, B_2 > 0$ such that
\[
A_1 t^2 e^{-3t} \leq u_1(t) \leq B_1 e^{3(1-t)}, \quad A_2 t^2 e^{-3t} \leq u_2(t) \leq B_2 e^{3(1-t)}.
\]

Example 4.2. Let $\alpha = \frac{5}{3}, \beta = \frac{1}{3}, \lambda = 3$, and
\[
h(u,v) = \begin{cases}
100 \sqrt{2}(u+v)^2, & 0 \leq u + v \leq \frac{1}{2}, \\
50(u+v)^{\frac{1}{2}}, & u + v > \frac{1}{2}.
\end{cases}
\]

We consider the multiplicity of positive solutions for the following singular tempered fractional equation with the lower order tempered fractional derivative
\[
\begin{align*}
-\frac{\partial}{\partial t} \, ^{\alpha}\mathcal{D}^{1,3}_t u(t) & = \left[ \frac{1}{2} - t \right]^{\frac{1}{2}} h\left( e^3 u(t), \frac{\partial}{\partial t} \, ^{\alpha}\mathcal{D}^{1,3}_t u(t) \right), 0 \leq t \leq 1, \\
\frac{\partial}{\partial t} \, ^{\alpha}\mathcal{D}^{1,3}_t u(0) & = 0, \quad \frac{\partial}{\partial t} \, ^{\alpha}\mathcal{D}^{1,3}_t u(1) = 0.
\end{align*}
\]

Conclusion. The tempered fractional equation (4.2) has at least two positive solutions $u_1, u_2$; moreover, there exist four constants $A_1, B_1, A_2, B_2 > 0$ such that
\[
A_1 t^2 e^{-3t} \leq u_1(t) \leq B_1 e^{3(1-t)}, \quad A_2 t^2 e^{-3t} \leq u_2(t) \leq B_2 e^{3(1-t)}.
\]

Proof. Here
\[
p(t) = \left| \frac{1}{2} - t \right|^{\frac{1}{2}}, \quad h(u,v) = \begin{cases}
100 \sqrt{2}(u+v)^2, & 0 \leq u + v \leq \frac{1}{2}, \\
50(u+v)^{\frac{1}{2}}, & u + v > \frac{1}{2}.
\end{cases}
\]
Clearly, \((C1)\) holds and

\[
h_0 = \lim_{u+v \to 0} \frac{h(u, v)}{u + v} = 0, \quad h_\infty = \lim_{u+v \to +\infty} \frac{h(u, v)}{u + v} = 0.
\]

Take \(\rho = 2\), and we have

\[
\varrho = 8^{\alpha-\beta}e^{\frac{1}{3}}\Gamma(\alpha - \beta)\left[\int_{\frac{1}{4}}^{1} p(s) ds\right]^{\frac{1}{3}} = 8^{\frac{4}{3}}e^{\frac{1}{3}}\Gamma\left(\frac{4}{3}\right)\left[\int_{\frac{1}{4}}^{1} \frac{1}{2 - s}^\frac{1}{2} ds\right]^{\frac{1}{3}} = 15.1238.
\]

For any

\[
0.8184 = 2 \left(\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{3}\right)}\right)^{\frac{3}{4}} + \left(\frac{1}{4}\right)^{\frac{3}{4}} e^{-\frac{3}{4}} \leq u + v \leq 2 \left(1 + \frac{e^3}{\Gamma\left(\frac{4}{3}\right)}\right) = 46.9844,
\]

one has

\[
h(u, v) \geq 50 \times 0.8184^\frac{1}{3} = 45.232 > \varrho \rho = 30.2476,
\]

which implies that \((C3)\) holds.

Consequently, it follows from Theorem 3.2 that the tempered fractional equation (4.2) has at least two positive solutions \(u_1, u_2\); moreover, there exist four constants \(A_1, B_1, A_2, B_2 > 0\) such that

\[
A_1 t^2 e^{-3t} \leq u_1(t) \leq B_1 e^{3(1-t)}, \quad A_2 t^2 e^{-3t} \leq u_2(t) \leq B_2 e^{3(1-t)}.
\]

5. Conclusions

This work studies the multiplicity of positive solutions for a class of singular tempered fractional equations with the lower order tempered fractional derivative. By applying reducing order technique and fixed point theorem, some new results of existence of the multiple positive solutions for the equation are established. The interesting points are the nonlinearity contains the lower order tempered fractional derivative and the weight function can have infinitely many singular points in \([0, 1]\). However, in this study, the conditions \(\alpha \in (1, 2], \beta \in (0, 1)\) with \(\alpha - \beta > 1\) are required; if \(0 < \alpha - \beta < 1\) or \(h\) has singularity at space variables, these interesting problems are still worth future studying.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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