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Research article

Positive solutions for a system of fractional q-difference equations with generalized p-Laplacian operators

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Abstract: In this paper, we consider the existence of positive solutions for a system of fractional q-difference equations with generalized p-Laplacian operators. By using Guo-Krasnosel'skii fixed point theorem, we obtain some existence results of positive solutions for this system with two parameters under some different combinations of superlinearity and sublinearity of the nonlinear terms. In the end, we give two examples to illustrate our main results.

Keywords: fixed point theorem; fractional q-difference equation; generalized p-Laplacian operator; existence; positive solution

1. Introduction

In this paper, we consider the existence of positive solutions for the following system of fractional q-difference equations with generalized p-Laplacian operators:

$$\begin{cases} -D_q^{\gamma}(\phi_1(D_q^{\alpha}x))(t) = \eta f(t, x(t), y(t)), \ \ 0 < t < 1, \\ -D_q^{\gamma}(\phi_2(D_q^{\alpha}y))(t) = \zeta g(t, x(t), y(t)), \ \ 0 < t < 1, \\ x(0) = D_q x(0) = 0, \ D_q x(1) = \beta > 0, \ D_q^{\alpha}x(0) = 0, \\ y(0) = D_q y(0) = 0, \ D_q y(1) = \beta > 0, \ D_q^{\alpha}y(0) = 0, \end{cases} \tag{1.1}$$

where $0 < q < 1,\ 2 < \alpha < 3,\ 0 < \gamma < 1,\ f,\ g : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are continuous, $\eta > 0$ and $\zeta > 0$ are two parameters, ϕ_1 and ϕ_2 are generalized p-Laplacian operators; D_q^{γ} and D_q^{α} are the fractional q-derivative of the Riemann-Liouville type, D_q is the q-derivative.

Due to the extensive application of fractional order equations, many scholars have studied the existence of nontrivial solutions of boundary value problems for fractional order differential equations. In recent years, some authors [1–5] have considered the existence of positive solutions for some Riemann-Liouville type, tempered type, Caputo type and Hadamard type fractional order differential equations. The authors [6,7] have considered the existence of nontrivial solution of Hadamard-type singular fractional differential equations. Some authors [8–12] have considered the existence of nontrivial solutions for some Riemann-Liouville type, tempered type, Caputo type and Hadamard type fractional order differential equations with p-Laplacian operator. Some authors [13,14] have considered the eigenvalue problems of fractional differential equations.

Meanwhile, after Jackson [15] introduced the q-calculus, Al-Salam [16] and Agarwal [17] developed the fractional q-calculus. Many researchers have studied the existence of nontrivial solutions for fractional q-difference equations these years. The commonly used methods include fixed point theorems, lower-upper solution method, monotone iterative technique, and so on. For example, in [18], Ferreira studied the following boundary value problem of fractional q-difference equation:

$$\begin{cases}(D_q^\alpha y)(x) = -f(x, y(x)),\ 0 < x < 1,\\ y(0) = (D_q y)(0) = 0,\ (D_q y)(1) = \beta \geq 0,\end{cases} \tag{1.2}$$

where $0 < q < 1, 2 < \alpha \leq 3, f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous; D_q^α is the fractional q-derivative of the Riemann-Liouville type, D_q is the q-derivative. The author obtained the existence of positive solutions about the boundary value problem (1.2) by using Guo-Krasnosel'skii fixed point theorem.

In [19], Zhai and Ren applied iterative algorithm and lower-upper solution method to study the following fractional q-difference equation:

$$\begin{cases}(D_q^\alpha u)(t) + f(t, u(t)) = 0,\ 0 < t < 1,\\ u(0) = (D_q u)(0) = 0,\ (D_q u)(1) = 0,\end{cases} \tag{1.3}$$

where $q \in (0,1),\ \alpha \in (2,3)$. Under some conditions, the authors obtained some existence results of positive or negative solutions for the boundary value problem (1.3).

In [20], Mao et al. used iterative technique to consider the general fractional q-difference equation of the problem (1.3) as followings:

$$\begin{cases}(D_q^\alpha u)(t) + f(t, u(t), v(t)) = 0,\ 0 < t < 1,\\ u(0) = (D_q u)(0) = 0,\ (D_q u)(1) = 0,\end{cases} \tag{1.4}$$

where $q \in (0,1),\ \alpha \in (2,3)$, f may be singular at $v = 0, t = 0,1$. The existence of a unique positive solution of the problem (1.4) has been proved.

In [21], Jiang and Zhong studied the following fractional q-difference equation with p-Laplacian operator:

$$\begin{cases}D_q^\beta(\phi_p(D_q^\alpha x)(t)) + f(t, x(t), D_q^\rho(t)) = 0,\\ x(0) = (D_q x)(0) = 0,\ (D_q^\alpha x)(0) = 0,\\ x(1) = \zeta I_q x(\eta),\end{cases} \tag{1.5}$$

where $\alpha \in (2,3),\ \beta, q, \eta, \rho \in (0,1),\ \phi_p(s) = |s|^{p-2}s$ is the p-Laplacian operator ($p > 1$). The authors used Banach's contraction principle to prove the existence and uniqueness of nontrivial solution of the problem (1.5), and also used Guo-Krasnosel'skii fixed point theorem to obtain the existence of positive solutions of the problem (1.5).

In [22], Li et al. considered the following boundary value problem of nonlinear fractional q-difference equation:

$$\begin{cases} D_q^{\gamma}(\phi(D_q^{\alpha}u)(t)) + \eta f(u(t)) = 0, \ \ 0 < t < 1, \\ u(0) = D_q u(0) = 0, \ D_q u(1) = \beta > 0, \ D_q^{\alpha} u(0) = 0, \end{cases} \tag{1.6}$$

where $0 < q < 1$, $2 < \alpha < 3$, $0 < \gamma < 1$, ϕ is the generalized p-Laplacian operator; D_q^{γ} and D_q^{α} are the fractional q-derivative of the Riemann-Liouville type, D_q is the q-derivative. The authors used the fixed point theorem to prove the existence of positive solutions of the boundary value problem (1.6).

In [23], Wang et al. investigated the following boundary value problem of fractional q-difference equation with ϕ-Laplacian:

$$\begin{cases} D_q^{\beta}(\phi(D_q^{\alpha}u(t))) = \lambda f(u(t)), \ \ 0 < t < 1, \\ u(0) = D_q u(0) = D_q u(1) = 0, \ \phi(D_q^{\alpha}u(0)) = D_q(\phi(D_q^{\alpha}u(1))) = 0, \end{cases} \tag{1.7}$$

where $0 < q < 1$, $2 < \alpha \leq 3$, $1 < \beta \leq 2$, $\lambda > 0$ is a parameter, and D_q^{β}, D_q^{α} are the standard Riemann-Liouville fractional q-derivatives. The existence and nonexistence of positive solutions of the boundary value problem (1.7) was obtained based on Guo-Krasnosel'skii fixed point theorem on cones.

Currently, many other authors have studied fractional q-difference equations. Some authors [24,25] have considered the existence of multiple positive solutions for some fractional q-difference equations. The authors [26–30] have considered the existence of nontrivial solutions for fractional q-difference equations with various boundary conditions.

Meanwhile, many authors have studied the existence of positive solutions of systems of some fractional differential equations with various boundary conditions, see [31–35] and the references therein. For example, in [31], Li et al. investigated the following system of fractional differential equations with p-Laplacian operators:

$$\begin{cases} D_{0^+}^{\alpha_1}(\varphi_{p_1}(D_{0^+}^{\beta_1}u(t))) = f(t, v(t)), 0 < t < 1, \\ D_{0^+}^{\alpha_2}(\varphi_{p_2}(D_{0^+}^{\beta_2}v(t))) = g(t, u(t)), 0 < t < 1, \\ u(0) = D_{0^+}^{\beta_1}u(0) = 0, \ \ D_{0^+}^{\gamma_1}u(1) = \sum_{j=1}^{m-2} a_{1j} D_{0^+}^{\gamma_1}u(\eta_j), \\ v(0) = D_{0^+}^{\beta_2}v(0) = 0 \ \ D_{0^+}^{\gamma_2}v(1) = \sum_{j=1}^{m-2} a_{2j} D_{0^+}^{\gamma_2}v(\eta_j), \end{cases} \tag{1.8}$$

where $\alpha_i, \gamma_i \in (0, 1], \beta_i \in (1, 2], D_{0^+}^{\alpha_i}, D_{0^+}^{\beta_i}$ and $D_{0^+}^{\gamma_i}$ are the standard Riemann-Liouville derivatives, $i = 1, 2$. The authors derived the conditions for the existence of the maximal and minimal solutions, and obtained the existence of extremal solutions of the system (1.8).

In [32], He and Song considered the following system of fractional differential equations with p-Laplacian operators and two parameters:

$$\begin{cases} D_{0^+}^{\alpha_1}(\varphi_{p_1}(D_{0^+}^{\beta_1}u(t))) = \eta f(t, v(t)), 0 < t < 1, \\ D_{0^+}^{\alpha_2}(\varphi_{p_2}(D_{0^+}^{\beta_2}v(t))) = \zeta g(t, u(t)), 0 < t < 1, \\ u(0) = 0, u(1) = a_1 u(\xi_1), D_{0^+}^{\beta_1}u(0) = 0, D_{0^+}^{\beta_1}u(1) = b_1 D_{0^+}^{\beta_1}u(\eta_1), \\ v(0) = 0, v(1) = a_2 v(\xi_2), D_{0^+}^{\beta_2}v(0) = 0, D_{0^+}^{\beta_2}v(1) = b_2 D_{0^+}^{\beta_2}v(\eta_2), \end{cases} \tag{1.9}$$

where $\alpha_i, \beta_i \in (1,2]$, $D_{0^+}^{\alpha_i}$ and $D_{0^+}^{\beta_i}$ are the standard Riemann-Liouville derivatives, $\xi_i, \eta_i \in (0,1), a_i, b_i \in [0,1], i = 1,2$. η and ζ are positive parameters. By using the Banach contraction mapping principle, the authors gave the existence and uniqueness of the solution for the system (1.9).

In [33], Hao et al. investigated the following system of fractional boundary value problems with p-Laplacian operators and two parameters:

$$\begin{cases} -D_{0^+}^{\alpha_1}(\varphi_{p_1}(D_{0^+}^{\beta_1}u(t))) = \lambda f(t, u(t), v(t)), t \in (0,1), \\ -D_{0^+}^{\alpha_2}(\varphi_{p_2}(D_{0^+}^{\beta_2}v(t))) = \mu g(t, u(t), v(t)), t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, D_{0^+}^{\beta_1}u(0) = 0, D_{0^+}^{\beta_1}u(1) = b_1 D_{0^+}^{\beta_1}p(\eta_1), \\ v(0) = v(1) = v'(0) = v'(1) = 0, D_{0^+}^{\beta_2}v(0) = 0, D_{0^+}^{\beta_2}v(1) = b_2 D_{0^+}^{\beta_2}v(\beta_2), \end{cases} \tag{1.10}$$

where $\alpha_i \in (1,2]$, $\beta_i \in (3,4]$, $D_{0^+}^{\alpha_i}$ and $D_{0^+}^{\beta_i}$ are the Riemann-Liouville derivatives, $\varphi_{p_i}(s) = |s|^{p_i-2}s$, $p_i > 1$, $f, g \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$, λ and μ are positive parameters. By means of Guo-Krasnosel'skii fixed point theorem, the authors obtained various existence results of positive solutions of the system (1.10).

To the best of our knowledge, limited attention has been devoted to the investigation of systems of fractional q-difference equations. Inspired by the above literatures, in this paper, we consider the existence of positive solutions for the system of fractional q-difference equations (1.1) with generalized p-Laplacian operators and two parameters. Under some sublinear and superlinear conditions, we establish some existence results of positive solutions for the system (1.1) by using Guo-Krasnosel'skii fixed point theorem.

2. Preliminaries

In this section, we firstly introduce Guo-Krasonsel'skill fixed point theorem. Secondly, we give some knowledge about fractional q-calculus. In the end, we give some lemmas that are used to prove the main results.

Lemma 2.1. *([36]) Let E be a Banach space, $P \subset E$ be a cone. Assume that $\Omega_1 \subset \mathrm{E}$ and $\Omega_2 \subset \mathrm{E}$ are bounded open sets with $\theta \in \Omega_1 \subset \Omega_2$, the operator $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous. If the following conditions are satisfied:*

$$\text{(i) } \| Ax \| \leq \| x \|, \forall x \in P \cap \partial\Omega_1, \| Ax \| \geq \| x \|, \forall x \in P \cap \partial\Omega_2, \textit{or}$$
$$\text{(ii) } \| Ax \| \geq \| x \|, \forall x \in P \cap \partial\Omega_1, \| Ax \| \leq \| x \|, \forall x \in P \cap \partial\Omega_2,$$

then the operator A has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

In the following, we give some definitions and lemmas about fractional q-calculus. For the detailed knowledge about fractional q-derivative and fractional q-integral, we can refer to [15–18].

Define

$$[a]_q = \frac{1-q^a}{1-q}, a \in R, \ q \in (0,1).$$

The q-analogue of the power function $(a-b)^{(\alpha)}$ with $\alpha \in R$ is

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}, \ n \in N.$$

The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, x \in R\backslash\{0, -1, -2, ...\},$$

and satisfies $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

The q-derivative of a function f is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, (D_q f)(0) = \lim_{x\to 0}(D_q f)(x),$$

and q-derivative of higher order by

$$(D_q^0 f)(x) = f(x), (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), n \in N.$$

The q-integral of a function f is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, x \in [0, b],$$

where f is defined in the interval $[0, b]$.

Definition 2.1. *([17, 18]) The fractional q-integral of the Riemann-Liouville type is defined by $(I_0^q f)(x) = f(x)$, and*

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-1)} f(t) d_q t, \alpha > 0, x \in [0, 1].$$

Definition 2.2. *([18]) The fractional q-derivative of the Riemann-Liouville type order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and*

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \alpha > 0,$$

where $m = [\alpha]$.

The paper [37] introduced the definition of a generalized p-Laplacian operator ϕ, which included two important cases $\phi(u) = u$ and $\phi(u) = |u|^{p-2}u (p \geq 1)$. In this paper, we assume that ϕ_1 and ϕ_2 are generalized p-Laplacian operators, namely, ϕ_1 and ϕ_2 satisfy the following condition:
(H_0) $\phi_i : R \to R (i = 1, 2)$ is an odd and increasing homeomorphism, and there exist increasing homeomorphisms $\psi_1, \psi_2, \psi_3, \psi_4 : (0, \infty) \to (0, \infty)$ such that

$$\psi_1(x)\phi_1(y) \leq \phi_1(xy) \leq \psi_2(x)\phi_1(y), \ \forall\ x, y > 0,$$

$$\psi_3(x)\phi_2(y) \leq \phi_2(xy) \leq \psi_4(x)\phi_2(y), \ \forall\ x, y > 0.$$

In the following, we give the other important lemmas. We list the following fractional q-difference equation with homogeneous boundary conditions:

$$\begin{cases} D_q^\gamma(\phi(D_q^\alpha v))(t) + \eta f(v(t) + \varphi(t)) = 0, \ 0 < t < 1, \\ v(0) = D_q v(0) = 0, D_q v(1) = 0, D_q^\alpha v(0) = 0, \end{cases} \tag{2.1}$$

where $\varphi(t) = \frac{\beta t^{\alpha-1}}{[\alpha-1]_q}$, and $\varphi(t)$ is the unique solution of the following fractional q-difference equation with nonhomogeneous boundary condition

$$\begin{cases} D_q^{\alpha}\varphi(t) = 0,\ 0 < t < 1, \\ \varphi(0) = D_q\varphi(0) = 0, D_q\varphi(1) = \beta > 0,\ D_q^{\alpha}\varphi(0) = 0, \end{cases} \tag{2.2}$$

where $0 < q < 1, 2 < \alpha < 3, 0 < \gamma < 1$, ϕ is a generalized p-Laplacian operator.

Lemma 2.2. *([22]) Let $v(t)$ be a solution of the boundary value problem (2.1). Then $u(t) = v(t) + \varphi(t)$ is the solution of the boundary value problem (1.6).*

Lemma 2.3. *([22]) Let $2 < \alpha < 3, 0 < \gamma < 1, y \in C[0,1]$ be a given function. Then the following boundary value problem of fractional q-difference equation*

$$\begin{cases} D_q^{\gamma}(\phi(D_q^{\alpha}x))(t) + \eta y(t) = 0, \\ x(0) = D_qx(0) = 0, D_qx(1) = 0, D_q^{\alpha}x(0) = 0 \end{cases}$$

has a unique solution

$$x(t) = \int_0^1 G(t,qs)\phi^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)}y(\tau)d_q\tau)d_qs,$$

where

$$G(t,qs) = \frac{1}{\Gamma_q(\alpha)}\begin{cases} (1-qs)^{(\alpha-2)}t^{\alpha-1} - (t-qs)^{(\alpha-1)}, 0 \le qs \le t \le 1, \\ (1-qs)^{(\alpha-2)}t^{\alpha-1}, \qquad 0 \le t \le qs \le 1. \end{cases} \tag{2.3}$$

Lemma 2.4. *([18]) The function $G(t,qs)$ defined by (2.3) has the following properties:*

$$(1)\ G(t,qs) \ge 0,\ \ G(t,qs) \le G(1,qs),\ \forall\ 0 \le t,s \le 1, \tag{2.4}$$

$$(2)\ G(t,qs) \ge t^{\alpha-1}G(1,qs),\ \forall\ 0 \le t,s \le 1. \tag{2.5}$$

Lemma 2.5. *([37]) Let (H_0) hold. Then we have*

$$\psi_2{}^{-1}(x)y \le \phi_1{}^{-1}(x\phi_1(y)) \le \psi_1{}^{-1}(x)y,\ \forall\ x,y > 0,$$

$$\psi_4{}^{-1}(x)y \le \phi_2{}^{-1}(x\phi_2(y)) \le \psi_3{}^{-1}(x)y,\ \forall\ x,y > 0.$$

Let $E = C[0,1] \times C[0,1]$ with the norm $\|(x,y)\|_E = \|x\| + \|y\|$, where $\|x\| = \max_{t\in[0,1]}|x(t)|$ and $\|y\| = \max_{t\in[0,1]}|y(t)|$. It is obvious that E is a Banach space.

Set $P = \{(x,y) \in E : x(t) \ge 0, y(t) \ge 0, \min_{t\in[\theta,1]}(x(t)+y(t)) \ge \theta^{\alpha-1}\|(x,y)\|_E\}$, where θ is a real constant and $0 < \theta < 1$.

By [22], we define the following operators A_η and A_ζ:

$$A_\eta(x,y)(t) = \int_0^1 G(t,qs)\phi_1{}^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)}f(\tau,x(\tau)+\varphi(\tau),y(\tau)+\varphi(\tau))d_q\tau)d_qs,\ t \in [0,1],$$

$$A_\zeta(x,y)(t) = \int_0^1 G(t,qs)\phi_2{}^{-1}(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)}g(\tau,x(\tau)+\varphi(\tau),y(\tau)+\varphi(\tau))d_q\tau)d_qs, t \in [0,1],$$

where $G(t, qs)$ is defined by (2.3). Let $A(x, y) = (A_\eta(x, y), A_\zeta(x, y)), (x, y) \in E$. Then by the literature [22], we easily know that the fixed points of the operator A are solutions of the system of fractional q-difference equations (1.1).

Lemma 2.6. *$A : P \to P$ is completely continuous.*

Proof. For $(x, y) \in P$, we easily have $A_\eta(x, y)(t) \geq 0, A_\zeta(x, y)(t) \geq 0, \forall t \in [0, 1]$.

By (2.5), for $t \in [\theta, 1]$, we have

$$\begin{aligned} A_\eta(x, y)(t) &= \int_0^1 G(t, qs)\phi_1^{-1}\Big(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} f(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau))d_q\tau\Big)d_q s \\ &\geq \int_0^1 t^{\alpha-1}G(1, qs)\phi_1^{-1}\Big(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} f(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau))d_q\tau\Big)d_q s \\ &\geq \theta^{\alpha-1}\int_0^1 G(1, qs)\phi_1^{-1}\Big(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} f(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau))d_q\tau\Big)d_q s \\ &= \theta^{\alpha-1}\|A_\eta(x, y)\|. \end{aligned} \tag{2.6}$$

Similar to the proof of (2.6), when $t \in [\theta, 1]$, we easily have

$$\begin{aligned} A_\zeta(x, y)(t) &= \int_0^1 G(t, qs)\phi_2^{-1}\Big(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} g(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau))d_q\tau\Big)d_q s \\ &\geq t^{\alpha-1}\int_0^1 G(1, qs)\phi_2^{-1}\Big(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} g(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau))d_q\tau\Big)d_q s \\ &\geq \theta^{\alpha-1}\|A_\zeta(x, y)\|. \end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we get

$$\min_{t\in[\theta,1]}(A_\eta(x, y)(t) + A_\zeta(x, y)(t)) \geq \theta^{\alpha-1}(\|A_\eta(x, y)\| + \|A_\zeta(x, y)\|) = \theta^{\alpha-1}\|A(x, y)\|_E. \tag{2.8}$$

From (2.8), we have $A(P) \subset P$.

In the following, we prove $A : P \to P$ is completely continuous. Firstly, we prove A is bounded.

Let $D \subset P$ be bounded. Namely, there exists $K > 0$ such that $\|(x, y)\|_E < K, \forall (x, y) \in D$. By the continuity of f and g, we know that there exists $M > 0$ such that

$$\max_{t\in[0,1],(x,y)\in D} |f(t, x(t) + \varphi(t), y(t) + \varphi(t))| < M, \tag{2.9}$$

$$\max_{t\in[0,1],(x,y)\in D} |g(t, x(t) + \varphi(t), y(t) + \varphi(t))| < M. \tag{2.10}$$

By (2.9), (2.10) and Lemma 2.4, for $(x, y) \in D$, we get

$$|A_\eta(x, y)(t)| \leq \int_0^1 G(1, qs)\phi_1^{-1}\Big(\frac{\eta M}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)}d_q\tau\Big)d_q s, \tag{2.11}$$

$$|A_\zeta(x, y)(t)| \leq \int_0^1 G(1, qs)\phi_2^{-1}\Big(\frac{\zeta M}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)}d_q\tau\Big)d_q s. \tag{2.12}$$

By (2.11) and (2.12), we easily know that $A(D)$ is bounded. Secondly, we prove A is equicontinuous on D. Namely, for each $(x,y) \in D, \forall \varepsilon > 0, \exists \delta > 0$ such that $|t_2 - t_1| < \delta$, we have

$$|A_\eta(x,y)(t_2) - A_\eta(x,y)(t_1)| < \varepsilon, \quad |A_\zeta(x,y)(t_2) - A_\zeta(x,y)(t_1)| < \varepsilon.$$

In fact, assume that $0 < t_1 < t_2 < 1$, then we have

$$\begin{aligned}
&|A_\eta(x,y)(t_2) - A_\eta(x,y)(t_1)| \\
&= |\int_0^1 (G(t_2,qs) - G(t_1,qs))\phi_1{}^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} f(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau)d_qs| \\
&\leq \int_0^1 |G(t_2,qs) - G(t_1,qs)|\phi_1^{-1}(\frac{\eta M}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)}d_q\tau)d_qs \\
&\leq \int_0^{t_1} |(1-qs)^{(\alpha-2)}(t_2{}^{\alpha-1} - t_1{}^{\alpha-1})|\frac{1}{\Gamma_q(\alpha)}\phi_1^{-1}(\frac{\eta M}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)}d_q\tau)d_qs \\
&+ \int_{t_1}^{t_2} |(1-qs)^{(\alpha-2)}(t_2{}^{\alpha-1} - t_1{}^{\alpha-1}) - (t_2-qs)^{(\alpha-1)}|\frac{1}{\Gamma_q(\alpha)}\phi_1^{-1}(\frac{\eta M}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)}d_q\tau)d_qs \\
&+ \int_{t_2}^{1} |(1-qs)^{(\alpha-2)}(t_2{}^{\alpha-1} - t_1{}^{\alpha-1})|\frac{1}{\Gamma_q(\alpha)}\phi_1^{-1}(\frac{\eta M}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)}d_q\tau)d_qs.
\end{aligned} \tag{2.13}$$

By (2.13), we have

$$|A_\eta(x,y)(t_2) - A_\eta(x,y)(t_1)| \to 0(t_1 \to t_2).$$

Similarly, we also have

$$|A_\zeta(x,y)(t_2) - A_\zeta(x,y)(t_1)| \to 0(t_1 \to t_2).$$

Hence, by Arzela-Ascoli theorem and the continuity of f and g, we have $A : P \to P$ is completely continuous. □

3. Main results

In the following, we give the denotations that we need in this section.

Let

$$f_0 = \limsup_{x+y\to 0^+} \max_{t\in[0,1]} \frac{f(t, x+\varphi(t), y+\varphi(t))}{\phi_1(x+y)}, \; g_0 = \limsup_{x+y\to 0^+} \max_{t\in[0,1]} \frac{g(t, x+\varphi(t), y+\varphi(t))}{\phi_2(x+y)},$$

$$f_\infty = \liminf_{x+y\to \infty} \min_{t\in[\theta,1]} \frac{f(t, x+\varphi(t), y+\varphi(t))}{\phi_1(x+y)}, \; g_\infty = \liminf_{x+y\to \infty} \min_{t\in[\theta,1]} \frac{g(t, x+\varphi(t), y+\varphi(t))}{\phi_2(x+y)}.$$

For $f_0, g_0, f_\infty, g_\infty \in (0,\infty)$, we denote that

$$D_1 = \frac{\psi_2(\frac{\theta^{2-2\alpha}}{2M_3})}{f_\infty}, \; D_2 = \frac{\psi_1(\frac{1}{2M_1})}{f_0},$$

$$D_3 = \frac{\psi_4(\frac{\theta^{2-2\alpha}}{2M_4})}{g_\infty}, \; D_4 = \frac{\psi_3(\frac{1}{2M_2})}{g_0}.$$

Let

$$\bar{f}_0=\liminf_{x+y\to 0^+}\min_{t\in[\theta,1]}\frac{f(t,x+\varphi(t),y+\varphi(t))}{\phi_1(x+y)},\ \bar{g}_0=\liminf_{x+y\to 0^+}\min_{t\in[\theta,1]}\frac{g(t,x+\varphi(t),y+\varphi(t))}{\phi_2(x+y)},$$

$$\bar{f}_\infty=\limsup_{x+y\to\infty}\max_{t\in[0,1]}\frac{f(t,x+\varphi(t),y+\varphi(t))}{\phi_1(x+y)},\ \bar{g}_\infty=\limsup_{x+y\to\infty}\max_{t\in[0,1]}\frac{g(t,x+\varphi(t),y+\varphi(t))}{\phi_2(x+y)}.$$

For $\bar{f}_0,\bar{g}_0,\bar{f}_\infty,\bar{g}_\infty\in(0,\infty)$, we give the following denotations:

$$Z_1=\frac{\psi_2(\frac{\theta^{2-2\alpha}}{2M_3})}{\bar{f}_0},\ \ Z_2=\frac{\psi_1(\frac{1}{2M_1})}{\bar{f}_\infty},$$

$$Z_3=\frac{\psi_4(\frac{\theta^{2-2\alpha}}{2M_4})}{\bar{g}_0},\ \ Z_4=\frac{\psi_3(\frac{1}{2M_2})}{\bar{g}_\infty},$$

where

$$M_1=\int_0^1 G(1,qs){\psi_1}^{-1}(\frac{s^\gamma}{\Gamma_q(\gamma+1)})d_qs,\ \ M_2=\int_0^1 G(1,qs){\psi_3}^{-1}(\frac{s^\gamma}{\Gamma_q(\gamma+1)})d_qs,$$

$$M_3=\int_\theta^1 G(1,qs){\psi_2}^{-1}(\frac{1}{\Gamma_q(\gamma)}\int_\theta^s(s-q\tau)^{(\gamma-1)}d_q\tau)d_qs,$$

$$M_4=\int_\theta^1 G(1,qs){\psi_4}^{-1}(\frac{1}{\Gamma_q(\gamma)}\int_\theta^s(s-q\tau)^{(\gamma-1)}d_q\tau)d_qs.$$

Theorem 3.1. *(1) Assume that $f_0,g_0,f_\infty,g_\infty\in(0,\infty),D_1<D_2,D_3<D_4$, then for each $\eta\in(D_1,D_2)$ and $\zeta\in(D_3,D_4)$, the system of fractional q-difference equations (1.1) has at least one positive solution.*

(2) Assume that $f_0=0,g_0,f_\infty,g_\infty\in(0,\infty),D_3<D_4$, then for each $\eta\in(D_1,\infty)$ and $\zeta\in(D_3,D_4)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(3) Assume that $f_0,f_\infty,g_\infty\in(0,\infty),g_0=0,D_1<D_2$, then for each $\eta\in(D_1,D_2)$ and $\zeta\in(D_3,\infty)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(4) Assume that $f_0=g_0=0,f_\infty,g_\infty\in(0,\infty)$, then for each $\eta\in(D_1,\infty)$ and $\zeta\in(D_3,\infty)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(5) Assume that $f_0,g_0\in(0,\infty),f_\infty=\infty$ or $f_0,g_0\in(0,\infty),g_\infty=\infty$, then for each $\eta\in(0,D_2)$ and $\zeta\in(0,D_4)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(6) Assume that $f_0=0,g_0\in(0,\infty),g_\infty=\infty$ or $f_0=0,g_0\in(0,\infty),f_\infty=\infty$, then for each $\eta\in(0,\infty)$ and $\zeta\in(0,D_4)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(7) Assume that $f_0\in(0,\infty),g_0=0,g_\infty=\infty$ or $f_0\in(0,\infty),g_0=0,f_\infty=\infty$, then for each $\eta\in(0,D_2)$ and $\zeta\in(0,\infty)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(8) Assume that $f_0=g_0=0,g_\infty=\infty$ or $f_0=g_0=0,f_\infty=\infty$, then for each $\eta\in(0,\infty)$ and $\zeta\in(0,\infty)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

Proof. In the following, we only prove the Cases (1) and (6).

Case (1): Since $\eta\in(D_1,D_2)$ and $\zeta\in(D_3,D_4)$, we easily know that there exists $\varepsilon>0$ such that

$$0<\frac{\psi_2(\frac{\theta^{2-2\alpha}}{2M_3})}{f_\infty-\varepsilon}\le\eta\le\frac{\psi_1(\frac{1}{2M_1})}{f_0+\varepsilon},\tag{3.1}$$

$$0 < \frac{\psi_4(\frac{\theta^{2-2\alpha}}{2M_4})}{g_\infty - \varepsilon} \le \zeta \le \frac{\psi_3(\frac{1}{2M_2})}{g_0 + \varepsilon}. \tag{3.2}$$

For the above $\varepsilon > 0$ in (3.1) and (3.2), there exists $r_1 > 0$ such that

$$f(t, x + \varphi, y + \varphi) \le (f_0 + \varepsilon)\phi_1(x + y),\ t \in [0, 1],\ 0 \le x + y \le r_1, \tag{3.3}$$

$$g(t, x + \varphi, y + \varphi) \le (g_0 + \varepsilon)\phi_2(x + y),\ t \in [0, 1],\ 0 \le x + y \le r_1. \tag{3.4}$$

Let $W_1 = \{(x, y) \in E : \|(x, y)\|_E < r_1\}$. For $(x, y) \in P \cap \partial W_1$, we have

$$0 \le x(t) + y(t) \le \|x\| + \|y\| = \|(x, y)\|_E = r_1,\ \forall t \in [0, 1].$$

By Lemmas 2.4, 2.5 and (3.1)–(3.4), we have

$$\begin{aligned}
A_\eta(x, y)(t) &= \int_0^1 G(t, qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)} \int_0^s (s - q\tau)^{(\gamma-1)} f(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau)) d_q\tau) d_q s \\
&\le \int_0^1 G(t, qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)} \int_0^s (s - q\tau)^{(\gamma-1)} (f_0 + \varepsilon)\phi_1(x(\tau) + y(\tau)) d_q\tau) d_q s \\
&\le \int_0^1 G(1, qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)} \int_0^s (s - q\tau)^{(\gamma-1)} (f_0 + \varepsilon)\phi_1(r_1) d_q\tau) d_q s \\
&\le \int_0^1 G(1, qs)\psi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)} \int_0^s (s - q\tau)^{(\gamma-1)} (f_0 + \varepsilon) d_q\tau) d_q s \cdot r_1 \\
&\le \psi_1^{-1}(\eta(f_0 + \varepsilon)) \int_0^1 G(1, qs)\psi_1^{-1}(\frac{s^\gamma}{\Gamma_q(\gamma + 1)}) d_q s \cdot r_1 \\
&\le \frac{r_1}{2} = \frac{\|(x, y)\|_E}{2}.
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
A_\zeta(x, y)(t) &= \int_0^1 G(t, qs)\phi_2^{-1}(\frac{\zeta}{\Gamma_q(\gamma)} \int_0^s (s - q\tau)^{(\gamma-1)} g(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau)) d_q\tau) d_q s \\
&\le \int_0^1 G(t, qs)\phi_2^{-1}(\frac{\zeta}{\Gamma_q(\gamma)} \int_0^s (s - q\tau)^{(\gamma-1)} (g_0 + \varepsilon)\phi_2(x(\tau) + y(\tau)) d_q\tau) d_q s \\
&\le \int_0^1 G(1, qs)\phi_2^{-1}(\frac{\zeta(g_0 + \varepsilon)}{\Gamma_q(\gamma)} \int_0^s (s - q\tau)^{(\gamma-1)} \phi_2(r_1) d_q\tau) d_q s \\
&\le \int_0^1 G(1, qs)\psi_3^{-1}(\frac{\zeta(g_0 + \varepsilon)}{\Gamma_q(\gamma)} \int_0^s (s - q\tau)^{(\gamma-1)} d_q\tau) d_q s \cdot r_1 \\
&\le \psi_3^{-1}(\zeta(g_0 + \varepsilon)) \int_0^1 G(1, qs)\psi_3^{-1}(\frac{s^\gamma}{\Gamma_q(\gamma + 1)}) d_q s \cdot r_1 \\
&\le \frac{r_1}{2} = \frac{\|(x, y)\|_E}{2}.
\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we have

$$\|A(x, y)\|_E = \|A_\eta(x, y)\| + \|A_\zeta(x, y)\| \le \|(x, y)\|_E, \forall (x, y) \in P \cap \partial W_1. \tag{3.7}$$

For $\varepsilon > 0$ in (3.1) and (3.2), from the definitions of f_∞ and g_∞, there exists $\bar{r}_2 > 0$ such that

$$f(t, x+\varphi, y+\varphi) \geq (f_\infty - \varepsilon)\phi_1(x+y), t \in [\theta, 1], x+y \geq \bar{r}_2, \tag{3.8}$$

$$g(t, x+\varphi, y+\varphi) \geq (g_\infty - \varepsilon)\phi_2(x+y), t \in [\theta, 1], x+y \geq \bar{r}_2. \tag{3.9}$$

Take $r_2 = \max\{2r_1, \theta^{1-\alpha}\bar{r}_2\}$. Let $W_2 = \{(x,y) \in E : \|(x,y)\|_E < r_2\}$. For $(x,y) \in P \cap \partial W_2$, we have $\min_{t\in[\theta,1]}(x(t)+y(t)) \geq \theta^{\alpha-1}\|(x,y)\|_E = \theta^{\alpha-1}r_2 \geq \bar{r}_2$.

By (3.8), (3.9) and Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
\|A_\eta(x,y)\| &\geq A_\eta(x,y)(\theta) \\
&= \int_0^1 G(\theta, qs)\phi_1^{-1}\Big(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} f(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau\Big)d_qs \\
&\geq \int_\theta^1 G(\theta, qs)\phi_1^{-1}\Big(\frac{\eta}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} f(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau\Big)d_qs \\
&\geq \int_\theta^1 G(\theta, qs)\phi_1^{-1}\Big(\frac{\eta}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} (f_\infty-\varepsilon)\phi_1(x(\tau)+y(\tau))d_q\tau\Big)d_qs \\
&\geq \int_\theta^1 \theta^{\alpha-1}G(1, qs)\phi_1^{-1}\Big(\frac{\eta}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} (f_\infty-\varepsilon)\phi_1(\theta^{\alpha-1}\|(x,y)\|_E)d_q\tau\Big)d_qs \\
&\geq \theta^{2\alpha-2}\int_\theta^1 G(1, qs)\psi_2^{-1}\Big(\frac{\eta}{\Gamma_q(\gamma)}(f_\infty-\varepsilon)\int_\theta^s (s-q\tau)^{(\gamma-1)}d_q\tau\Big)\|(x,y)\|_E d_qs \\
&= \theta^{2\alpha-2}\psi_2^{-1}(\eta(f_\infty-\varepsilon))\int_\theta^1 G(1, qs)\psi_2^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)}d_q\tau\Big)d_qs \cdot r_2 \\
&\geq \frac{\|(x,y)\|_E}{2},
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\|A_\zeta(x,y)\| &\geq A_\zeta(x,y)(\theta) \\
&= \int_0^1 G(\theta, qs)\phi_2^{-1}\Big(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} g(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau\Big)d_qs \\
&\geq \int_\theta^1 G(\theta, qs)\phi_2^{-1}\Big(\frac{\zeta}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} g(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau\Big)d_qs \\
&\geq \theta^{\alpha-1}\int_\theta^1 G(1, qs)\phi_2^{-1}\Big(\frac{\zeta}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} (g_\infty-\varepsilon)\phi_2(x(\tau)+y(\tau))d_q\tau\Big)d_qs \\
&\geq \theta^{\alpha-1}\int_\theta^1 G(1, qs)\phi_2^{-1}\Big(\frac{\zeta(g_\infty-\varepsilon)}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} \phi_2(\theta^{\alpha-1}\|(x,y)\|_E)d_q\tau\Big)d_qs \\
&\geq \theta^{2\alpha-2}\int_\theta^1 G(1, qs)\psi_4^{-1}\Big(\frac{\zeta(g_\infty-\varepsilon)}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)}d_q\tau\Big)d_qs \cdot \|(x,y)\|_E \\
&= \theta^{2\alpha-2}\psi_4^{-1}(\zeta(g_\infty-\varepsilon))\int_\theta^1 G(1, qs)\psi_4^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)}d_q\tau\Big)d_qs \cdot \|(x,y)\|_E \\
&= \theta^{2\alpha-2}\psi_4^{-1}(\zeta(g_\infty-\varepsilon))M_4 \cdot r_2 \\
&\geq \frac{1}{2}r_2 = \frac{\|(x,y)\|_E}{2}.
\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we have

$$\|A(x,y)\|_E = \|A_\eta(x,y)\| + \|A_\zeta(x,y)\| \geq \|(x,y)\|_E, \forall (x,y) \in P \cap \partial W_2. \tag{3.12}$$

By (3.7), (3.12) and Lemma 2.1, we know that A has at least one fixed point $(x,y) \in P \cap (\overline{W}_2 \backslash W_1)$. So the system of fractional q-difference equations (1.1) has at least one positive solution. The proof of the case (1) is completed.

Case (6): Since $\eta \in (0,\infty)$ and $\zeta \in (0, D_4)$, we easily know that there exists $\varepsilon > 0$ such that

$$0 < \eta < \psi_1(\frac{1}{2M_1})\frac{1}{\varepsilon}, \quad \psi_4(\frac{\theta^{2-2\alpha}}{M_4})\varepsilon < \zeta < \psi_3(\frac{1}{2M_2})\frac{1}{g_0+\varepsilon}. \tag{3.13}$$

Since $f_0 = 0$ and $g_0 \in (0,\infty)$, for the above $\varepsilon > 0$ in (3.13), we know that there exists $r_3 > 0$ such that

$$f(t, x+\varphi, y+\varphi) < \varepsilon\phi_1(x+y), t \in [0,1], 0 \leq x+y \leq r_3, \tag{3.14}$$

$$g(t, x+\varphi, y+\varphi) < (g_0+\varepsilon)\phi_2(x+y), t \in [0,1], 0 \leq x+y \leq r_3. \tag{3.15}$$

Let $W_3 = \{(x,y) \in E : \|(x,y)\|_E < r_3\}$. By (3.13), (3.14) and Lemma 2.5, for any $(x,y) \in P \cap \partial W_3, t \in [0,1]$, we have

$$\begin{aligned}
A_\eta(x,y)(t) &= \int_0^1 G(t,qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} f(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau)d_q s \\
&\leq \int_0^1 G(t,qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)}\varepsilon\phi_1(x(\tau)+y(\tau))d_q\tau)d_q s \\
&\leq \int_0^1 G(1,qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)}\varepsilon\phi_1(r_3)d_q\tau)d_q s \\
&\leq \int_0^1 G(1,qs)\psi_1^{-1}(\frac{\eta\varepsilon}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)}d_q\tau)d_q s \cdot r_3 \\
&= \psi_1^{-1}(\eta\varepsilon)\int_0^1 G(1,qs)\psi_1^{-1}(\frac{s^\gamma}{\Gamma_q(\gamma+1)})d_q s \cdot r_3 \\
&< \frac{r_3}{2} = \frac{\|(x,y)\|_E}{2}.
\end{aligned} \tag{3.16}$$

By (3.16), we have

$$\|A_\eta(x,y)\| \leq \frac{\|(x,y)\|_E}{2}, \forall (x,y) \in P \cap \partial W_3. \tag{3.17}$$

By (3.13), (3.15) and Lemma 2.5, similar to the proof of (3.16), we easily obtain

$$\|A_\zeta(x,y)\| \leq \frac{\|(x,y)\|_E}{2}, \forall (x,y) \in P \cap \partial W_3. \tag{3.18}$$

By (3.17) and (3.18), we have

$$\|A(x,y)\|_E = \|A_\eta(x,y)\| + \|A_\zeta(x,y)\| \leq \|(x,y)\|_E, \forall (x,y) \in P \cap \partial W_3. \tag{3.19}$$

Since $g_\infty = \infty$, for $\varepsilon > 0$ in (3.13), we know that there exists $\bar{r}_4 > 0$ such that

$$g(t, x + \varphi, y + \varphi) \geq \frac{1}{\varepsilon}\phi_2(x, y), t \in [\theta, 1], x, y \geq 0, x + y \geq \bar{r}_4. \tag{3.20}$$

Take $r_4 = \max\{3r_3, \bar{r}_4\theta^{1-\alpha}\}$. Let $W_4 = \{(x, y) \in E : \|(x, y)\|_E < r_4\}$. For any $(x, y) \in P \cap \partial W_4$, we can easily know that

$$\min_{t\in[\theta,1]}(x(t) + y(t)) \geq \theta^{\alpha-1}\|(x, y)\|_E = \theta^{\alpha-1}r_4 \geq \bar{r}_4. \tag{3.21}$$

Hence, by (3.20), (3.21) and Lemma 2.5, for any $(x, y) \in P \cap \partial W_4$, we have

$$\begin{aligned}
A_\zeta(x, y)(\theta) &= \int_0^1 G(\theta, qs)\phi_2^{-1}(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)}g(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau))d_q\tau)d_qs \\
&\geq \int_\theta^1 G(\theta, qs)\phi_2^{-1}(\frac{\zeta}{\Gamma_q(\gamma)}\int_\theta^s (s - q\tau)^{(\gamma-1)}g(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau))d_q\tau)d_qs \\
&\geq \theta^{\alpha-1}\int_\theta^1 G(1, qs)\phi_2^{-1}(\frac{\zeta}{\Gamma_q(\gamma)}\frac{1}{\varepsilon}\int_\theta^s (s - q\tau)^{(\gamma-1)}\phi_2(x(\tau) + y(\tau))d_q\tau)d_qs \\
&\geq \theta^{\alpha-1}\int_\theta^1 G(1, qs)\phi_2^{-1}(\frac{\zeta}{\Gamma_q(\gamma)}\frac{1}{\varepsilon}\int_\theta^s (s - q\tau)^{(\gamma-1)}\phi_2(\theta^{\alpha-1}\|(x, y)\|_E)d_q\tau)d_qs \\
&\geq \theta^{\alpha-1}\int_\theta^1 G(1, qs)\psi_4^{-1}(\frac{\zeta}{\varepsilon\Gamma_q(\gamma)}\int_\theta^s (s - q\tau)^{(\gamma-1)}d_q\tau \cdot \theta^{\alpha-1}\|(x, y)\|_E)d_qs \\
&= \theta^{2\alpha-2}\psi_4^{-1}(\frac{\zeta}{\varepsilon})\int_\theta^1 G(1, qs)\psi_4^{-1}(\frac{1}{\Gamma_q(\gamma)}\int_\theta^s (s - q\tau)^{(\gamma-1)}d_q\tau)d_qs \cdot r_4 \\
&= \theta^{2\alpha-2}\psi_4^{-1}(\frac{\zeta}{\varepsilon})M_4 \cdot r_4 \\
&\geq r_4 = \|(x, y)\|_E.
\end{aligned} \tag{3.22}$$

By (3.22), we have

$$\|A(x, y)\|_E \geq \|A_\zeta(x, y)\| \geq \|(x, y)\|_E, \forall (x, y) \in P \cap \partial W_4. \tag{3.23}$$

Hence, by (3.19), (3.23) and Lemma 2.1, we can obtain that A has at least one fixed point $(x, y) \in P \cap (\overline{W}_4 \backslash W_3)$. So the system of fractional q-difference equations (1.1) has at least one positive solution. □

Theorem 3.2. *(1) Assume that $\bar{f}_0, \bar{g}_0, \bar{f}_\infty, \bar{g}_\infty \in (0, \infty)$, and $Z_1 < Z_2, Z_3 < Z_4$, then for each $\eta \in (Z_1, Z_2)$ and $\zeta \in (Z_3, Z_4)$, the system of fractional q-difference equations (1.1) has at least one positive solution.*

(2) Assume that $\bar{f}_0, \bar{g}_0, \bar{f}_\infty \in (0, \infty), \bar{g}_\infty = 0$, and $Z_1 < Z_2$, then for each $\eta \in (Z_1, Z_2)$ and $\zeta \in (Z_3, \infty)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(3) Assume that $\bar{f}_0, \bar{g}_0, \bar{g}_\infty \in (0, \infty), \bar{f}_\infty = 0$, and $Z_3 < Z_4$, then for each $\eta \in (Z_1, \infty)$ and $\zeta \in (Z_3, Z_4)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(4) Assume that $\bar{f}_0, \bar{g}_0 \in (0, \infty), \bar{f}_\infty = \bar{g}_\infty = 0$, then for each $\eta \in (Z_1, \infty)$ and $\zeta \in (Z_3, \infty)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(5) Assume that $\bar{f}_\infty, \bar{g}_\infty \in (0, \infty), \bar{f}_0 = \infty$ or $\bar{f}_\infty, \bar{g}_\infty \in (0, \infty), \bar{g}_0 = \infty$, then for each $\eta \in (0, Z_2)$ and $\zeta \in (0, Z_4)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(6) Assume that $\bar{f}_0 = \infty, \bar{g}_\infty = 0, \bar{f}_\infty \in (0, \infty)$ or $\bar{f}_\infty \in (0, \infty), \bar{g}_\infty = 0, \bar{g}_0 = \infty$, then for each $\eta \in (0, Z_2)$ and $\zeta \in (0, \infty)$, the system of fractional q-difference equations (1.1) has at least one positive solution.

(7) Assume that $\bar{f}_0 = \infty, \bar{g}_\infty \in (0,\infty), \bar{f}_\infty = 0$ *or* $\bar{g}_\infty \in (0,\infty), \bar{g}_0 = \infty, \bar{f}_\infty = 0$*, then for each* $\eta \in (0,\infty)$ *and* $\zeta \in (0, Z_4)$*, the system of fractional q-difference equations (1.1) has at least one positive solution.*

(8) Assume that $\bar{f}_0 = \infty, \bar{f}_\infty = \bar{g}_\infty = 0$ *or* $\bar{f}_\infty = \bar{g}_\infty = 0, \bar{g}_0 = \infty$*, then for each* $\eta \in (0,\infty)$ *and* $\zeta \in (0,\infty)$*, the system of fractional q-difference equations (1.1) has at least one positive solution.*

Proof. We will only prove the Cases (1) and (6). Since the other proofs are similar, so we omit.

We firstly prove the Case (1). Since $\eta \in (Z_1, Z_2)$ and $\zeta \in (Z_3, Z_4)$, there exists $\varepsilon > 0$ such that

$$0 < \frac{\psi_2(\frac{\theta^{2-2\alpha}}{2M_2})}{\bar{f}_0 - \varepsilon} \leq \eta \leq \frac{\psi_1(\frac{1}{2M_1})}{\bar{f}_\infty + \varepsilon}, 0 < \frac{\psi_4(\frac{\theta^{2-2\alpha}}{2M_4})}{\bar{g}_0 - \varepsilon} \leq \zeta \leq \frac{\psi_3(\frac{1}{2M_2})}{\bar{g}_\infty + \varepsilon} \tag{3.24}$$

From the definitions of $\bar{f}_0$ and $\bar{g}_0$, we easily know that there exists $R_1 > 0$ such that

$$f(t, x+\varphi, y+\varphi) \geq (\bar{f}_0 - \varepsilon)\phi_1(x+y), t \in [\theta, 1], x, y \geq 0, x+y \leq R_1, \tag{3.25}$$

$$g(t, x+\varphi, y+\varphi) \geq (\bar{g}_0 - \varepsilon)\phi_g(x+y), t \in [\theta, 1], x, y \geq 0, x+y \leq R_1. \tag{3.26}$$

Let $W_1 = \{(x,y) \in E : \|(x,y)\|_E < R_1\}$. By (3.24), (3.25) and Lemma 2.5, for any $(x,y) \in P \cap \partial W_1$, we can get

$$\begin{aligned}
A_\eta(x,y)(\theta) &= \int_0^1 G(\theta, qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} f(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau)d_qs \\
&\geq \int_\theta^1 G(\theta, qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} f(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau)d_qs \\
&\geq \int_\theta^1 G(\theta, qs)\phi_1^{-1}(\frac{\eta(\bar{f}_0-\varepsilon)}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)}\phi_1(x(\tau)+y(\tau))d_q\tau)d_qs \\
&\geq \theta^{\alpha-1}\int_\theta^1 G(1, qs)\phi_1^{-1}(\frac{\eta(\bar{f}_0-\varepsilon)}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)}\phi_1(\theta^{\alpha-1}\|(x,y)\|_E)d_q\tau)d_qs \\
&\geq \theta^{\alpha-1}\int_\theta^1 G(1, qs)\phi_1^{-1}(\frac{\eta(\bar{f}_0-\varepsilon)}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)}dq\tau\cdot\phi_1(\theta^{\alpha-1}\|(x,y)\|_E)d_q\tau)d_qs \\
&\geq \theta^{\alpha-1}\int_\theta^1 G(1, qs)\psi_2^{-1}(\frac{\eta(\bar{f}_0-\varepsilon)}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)}d_q\tau)\theta^{\alpha-1}\|(x,y)\|_E d_qs \\
&= \theta^{2\alpha-2}\psi_2^{-1}(\eta(\bar{f}_0-\varepsilon))\int_\theta^1 G(1, qs)\psi_2^{-1}(\frac{1}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)}d_q\tau)d_qs\cdot R_1 \\
&= \theta^{2\alpha-2}\psi_2^{-1}(\eta(\bar{f}_0-\varepsilon))M_3\cdot R_1 \\
&\geq \frac{\|(x,y)\|_E}{2}.
\end{aligned} \tag{3.27}$$

By (3.24), (3.26) and Lemma 2.5, for any $(x,y) \in P \cap \partial W_1$, we have

$$
\begin{aligned}
A_\zeta(x,y)(\theta) &= \int_0^1 G(\theta,qs)\phi_2^{-1}\Big(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} g(\tau,x(\tau)+\varphi(\tau),y(\tau)+\varphi(\tau))d_q\tau\Big)d_q s \\
&\geq \int_\theta^1 G(\theta,qs)\phi_2^{-1}\Big(\frac{\zeta}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} g(\tau,x(\tau)+\varphi(\tau),y(\tau)+\varphi(\tau))d_q\tau\Big)d_q s \\
&\geq \int_\theta^1 G(\theta,qs)\phi_2^{-1}\Big(\frac{\zeta}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} (\bar{g}_0-\varepsilon)\phi_2(x(\tau)+y(\tau))d_q\tau\Big)d_q s \\
&\geq \theta^{\alpha-1}\int_\theta^1 G(1,qs)\phi_2^{-1}\Big(\frac{\zeta}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} (\bar{g}_0-\varepsilon)\phi_2(\theta^{\alpha-1}\|(x,y)\|_E)d_q\tau\Big)d_q s \\
&\geq \theta^{\alpha-1}\int_\theta^1 G(1,qs)\psi_4^{-1}\Big(\frac{\zeta(\bar{g}_0-\varepsilon)}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} d_q\tau\Big)\theta^{\alpha-1}\|(x,y)\|_E d_q s \\
&= \theta^{2\alpha-2}\cdot\psi_4^{-1}(\zeta(\bar{g}_0-\varepsilon))\int_\theta^1 G(1,qs)\psi_4^{-1}\Big(\frac{1}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} d_q\tau\Big)d_q s\cdot\|(x,y)\|_E \\
&\geq \frac{\|(x,y)\|_E}{2}.
\end{aligned}
\tag{3.28}
$$

By (3.27) and (3.28), we have

$$
\|A(x,y)\|_E = \|A_\eta(x,y)\| + \|A_\zeta(x,y)\| \geq \|(x,y)\|_E, \forall (x,y)\in P\cap\partial W_1. \tag{3.29}
$$

Let $F(t,u) = \max\limits_{0\leq x+y\leq u} f(t,x+\varphi,y+\varphi)$, $G^*(t,u) = \max\limits_{0\leq x+y\leq u} g(t,x+\varphi,y+\varphi)$. Then we have

$$
f(t,x+\varphi,y+\varphi) \leq F(t,u), t\in[0,1],\ \ x,y\geq 0,\ \ x+y\leq u,
$$

$$
g(t,x+\varphi,y+\varphi) \leq G^*(t,u), t\in[0,1],\ \ x,y\geq 0,\ \ x+y\leq u.
$$

Similar to the proof of [33], we know that

$$
\limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{F(t,u)}{\phi_1(u)} \leq \bar{f}_\infty,\ \ \limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{G^*(t,u)}{\phi_2(u)} \leq \bar{g}_\infty.
$$

Clearly, we know that there exists $\overline{R}_2 > 0$ such that

$$
\frac{F(t,u)}{\phi_1(u)} \leq \limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{F(t,u)}{\phi_1(u)} + \varepsilon \leq \bar{f}_\infty + \varepsilon, u\geq \overline{R}_2, t\in[0,1],
$$

$$
\frac{G^*(t,u)}{\phi_2(u)} \leq \limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{G^*(t,u)}{\phi_2(u)} + \varepsilon \leq \bar{g}_\infty + \varepsilon, u\geq \overline{R}_2, t\in[0,1].
$$

Hence, we have

$$
F(t,u) \leq (\bar{f}_\infty+\varepsilon)\phi_1(u),\ \ G^*(t,u) \leq (\bar{g}_\infty+\varepsilon)\phi_2(u)\ \ t\in[0,1], u\geq \overline{R}_2. \tag{3.30}
$$

Let $R_2 = \max\{2R_1, \overline{R}_2\}$, and $W_2 = \{(x,y)\in E : \|(x,y)\|_E < R_2\}$, for any $(x,y)\in P\cap\partial W_2$, we get

$$
f(t,x+\varphi,y+\varphi) \leq F(t,\|(x,y)\|_E), t\in[0,1], \tag{3.31}
$$

$$g(t, x+\varphi, y+\varphi) \leq G^*(t, \|(x,y)\|_E), t \in [0,1]. \tag{3.32}$$

By (3.30)–(3.32), for any $(x, y) \in P \cap \partial W_2$, we have

$$\begin{aligned}
A_\eta(x,y)(t) &= \int_0^1 G(t,qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} f(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau)d_qs \\
&\leq \int_0^1 G(1,qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} F(\tau, \|(x,y)\|_E)d_q\tau)d_qs \\
&\leq \int_0^1 G(1,qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} (\bar{f}_\infty+\varepsilon)\phi_1(\|(x,y)\|_E)d_q\tau)d_qs \\
&\leq \int_0^1 G(1,qs)\psi_1^{-1}(\frac{\eta(\bar{f}_\infty+\varepsilon)}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} d_q\tau)\|(x,y)\|_E d_qs \\
&= \psi_1^{-1}(\eta(\bar{f}_\infty+\varepsilon))\int_0^1 G(1,qs)\psi_1^{-1}(\frac{1}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} d_q\tau)d_qs \cdot R_2 \\
&= \psi_1^{-1}(\eta(\bar{f}_\infty+\varepsilon))\int_0^1 G(1,qs)\psi_1^{-1}(\frac{s^\gamma}{\Gamma_q(\gamma+1)})d_qs \cdot R_2 \\
&\leq \frac{\|(x,y)\|_E}{2},
\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}
A_\zeta(x,y)(t) &= \int_0^1 G(t,qs)\phi_2^{-1}(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} g(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau)d_qs \\
&\leq \int_0^1 G(1,qs)\phi_2^{-1}(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} G^*(\tau, \|(x,y)\|_E)d_q\tau)d_qs \\
&\leq \int_0^1 G(1,qs)\phi_2^{-1}(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} (\bar{g}_\infty+\varepsilon)\phi_2(\|(x,y)\|_E)d_q\tau)d_qs \\
&\leq \int_0^1 G(1,qs)\psi_3^{-1}(\frac{\zeta(\bar{g}_\infty+\varepsilon)}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} d_q\tau)\|(x,y)\|_E d_qs \\
&= \psi_3^{-1}(\zeta(\bar{g}_\infty+\varepsilon))\int_0^1 G(1,qs)\psi_3^{-1}(\frac{s^\gamma}{\Gamma_q(\gamma+1)})d_qs \cdot R_2 \\
&\leq \frac{\|(x,y)\|_E}{2}.
\end{aligned} \tag{3.34}$$

By (3.33) and (3.34), we have

$$\|A(x,y)\|_E = \|A_\eta(x,y)\| + \|A_\zeta(x,y)\| \leq \|(x,y)\|_E, \forall (x,y) \in P \cap \partial W_2. \tag{3.35}$$

By (3.29), (3.35) and Lemma 2.1, we know that A has at least one fixed point $(x, y) \in P \cap (\overline{W}_2 \setminus W_1)$, so the system of fractional q-difference equations (1.1) has at least one positive solution. The proof of the case (1) is completed.

In the following, we prove the Case (6). Since $\bar{f}_0 = \infty, \bar{f}_\infty \in (0, \infty), \bar{g}_\infty = 0$, we can easily know that there exist $\varepsilon > 0$ and $R_3 > 0$ such that

$$\psi_2(\frac{\theta^{2-2\alpha}}{M_3})\varepsilon < \eta < \frac{\psi_1(\frac{1}{2M_1})}{\bar{f}_\infty+\varepsilon}, \tag{3.36}$$

$$0 < \zeta < \psi_3(\frac{1}{2M_2})\frac{1}{\varepsilon}, \tag{3.37}$$

and

$$f(t, x+\varphi, y+\varphi) \geq \frac{1}{\varepsilon}\phi_1(x+y), t \in [\theta, 1], x, y > 0, 0 \leq x+y \leq R_3. \tag{3.38}$$

Let $W_3 = \{(x, y) \in E : \|(x, y)\|_E < R_3\}$. For $t \in [\theta, 1]$, $(x, y) \in P \cap \partial W_3$, we easily know that

$$\min_{t\in[\theta,1]}(x(t) + y(t)) \geq \theta^{\alpha-1}\|(x, y)\|_E.$$

By (3.36) and (3.38), we have

$$\begin{aligned} A_\eta(x,y)(\theta) &= \int_0^1 G(\theta, qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s-q\tau)^{(\gamma-1)} f(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau)d_qs \\ &\geq \int_\theta^1 G(\theta, qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)} f(\tau, x(\tau)+\varphi(\tau), y(\tau)+\varphi(\tau))d_q\tau)d_qs \\ &\geq \theta^{\alpha-1}\int_\theta^1 G(1, qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)\varepsilon}\int_\theta^s (s-q\tau)^{(\gamma-1)}\phi_1(x(\tau)+y(\tau))d_q\tau)d_qs \\ &\geq \theta^{\alpha-1}\int_\theta^1 G(1, qs)\phi_1^{-1}(\frac{\eta}{\Gamma_q(\gamma)\varepsilon}\int_\theta^s (s-q\tau)^{(\gamma-1)}\phi_1(\theta^{\alpha-1}\|(x,y)\|_E)d_q\tau)d_qs \\ &\geq \theta^{2\alpha-2}\psi_2^{-1}(\frac{\eta}{\varepsilon})\int_\theta^1 G(1, qs)\psi_2^{-1}(\frac{1}{\Gamma_q(\gamma)}\int_\theta^s (s-q\tau)^{(\gamma-1)}d_q\tau)d_qs \\ &= \theta^{2\alpha-2}\psi_2^{-1}(\frac{\eta}{\varepsilon})M_3 \cdot \|(x,y)\|_E \geq \|(x,y)\|_E. \end{aligned} \tag{3.39}$$

So by (3.39), we have

$$\|A(x, y)\|_E \geq \|A_\eta(x, y)\| \geq \|(x, y)\|_E, \forall (x, y) \in P \cap \partial W_3. \tag{3.40}$$

Similar to the proof of [33], we obtain

$$\limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{F(t,u)}{\phi_1(u)} \leq \bar{f}_\infty,\ \limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{G^*(t,u)}{\phi_2(u)} = 0.$$

So we know that for above $\varepsilon > 0$ in (3.36) and (3.37), there exists $\overline{R}_4 > 0$ such that

$$\frac{F(t,u)}{\phi_1(u)} \leq \limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{F(t,u)}{\phi_1(u)} + \varepsilon \leq \bar{f}_\infty + \varepsilon, \forall\ t \in [0,1], u \geq \overline{R}_4,$$

$$\frac{G^*(t,u)}{\phi_2(u)} \leq \limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{G^*(t,u)}{\phi_2(u)} + \varepsilon \leq \varepsilon, \forall\ t \in [0,1], u \geq \overline{R}_4,$$

so we have

$$F(t, u) \leq (\bar{f}_\infty + \varepsilon)\phi_1(u), \forall t \in [0, 1], u \geq \overline{R}_4,$$

$$G^*(t, u) \leq \varepsilon\phi_2(u), \forall t \in [0, 1], u \geq \overline{R}_4.$$

Let $R_4 = \max\{2R_3, \overline{R}_4\}$ and $W_4 = \{(x, y) \in E : \|(x, y)\|_E < R_4\}$. We easily have

$$f(t, x + \varphi, y + \varphi) \leq F(t, \|(x, y)\|_E), \forall\ t \in [0, 1],$$

$$g(t, x + \varphi, y + \varphi) \leq G^*(t, \|(x, y)\|_E), \forall\ t \in [0, 1].$$

Hence, for any $t \in [0, 1]$ and $(x, y) \in P \cap \partial W_4$, we get

$$\begin{aligned} A_\eta(x, y)(t) &= \int_0^1 G(t, qs)\phi_1^{-1}\left(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} f(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau))d_q\tau\right)d_q s \\ &\leq \int_0^1 G(1, qs)\phi_1^{-1}\left(\frac{\eta}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} F(\tau, \|(x, y)\|_E)d_q\tau\right)d_q s \\ &\leq \int_0^1 G(1, qs)\phi_1^{-1}\left(\frac{\eta(\bar{f}_\infty + \varepsilon)}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} \phi_1(\|(x, y)\|_E)d_q\tau\right)d_q s \\ &\leq \int_0^1 G(1, qs)\psi_1^{-1}\left(\frac{\eta(\bar{f}_\infty + \varepsilon)}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} d_q\tau\right)d_q s \cdot \|(x, y)\|_E \\ &= \psi_1^{-1}(\eta(\bar{f}_\infty + \varepsilon))\int_0^1 G(1, qs)\psi_1^{-1}\left(\frac{s^\gamma}{\Gamma_q(\gamma + 1)}\right)d_q s \cdot \|(x, y)\|_E \\ &\leq \frac{\|(x, y)\|_E}{2}, \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} A_\zeta(x, y)(t) &= \int_0^1 G(t, qs)\phi_2^{-1}\left(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} g(\tau, x(\tau) + \varphi(\tau), y(\tau) + \varphi(\tau))d_q\tau\right)d_q s \\ &\leq \int_0^1 G(1, qs)\phi_2^{-1}\left(\frac{\zeta}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} G^*(\tau, \|(x, y)\|_E)d_q\tau\right)d_q s \\ &\leq \int_0^1 G(1, qs)\phi_2^{-1}\left(\frac{\zeta\varepsilon}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} \phi_2(\|(x, y)\|_E)d_q\tau\right)d_q s \\ &\leq \int_0^1 G(1, qs)\psi_3^{-1}\left(\frac{\zeta\varepsilon}{\Gamma_q(\gamma)}\int_0^s (s - q\tau)^{(\gamma-1)} d_q\tau\right)d_q s \cdot \|(x, y)\|_E \\ &= \psi_3^{-1}(\zeta\varepsilon)\int_0^1 G(1, qs)\psi_3^{-1}\left(\frac{s^\gamma}{\Gamma_q(\gamma + 1)}\right)d_q s \cdot \|(x, y)\|_E \\ &\leq \frac{\|(x, y)\|_E}{2}. \end{aligned} \tag{3.42}$$

So by (3.41) and (3.42), we have

$$\|A(x, y)\|_E = \|A_\eta(x, y)\| + \|A_\zeta(x, y)\| \leq \|(x, y)\|_E, \forall (x, y) \in P \cap \partial W_4. \tag{3.43}$$

By (3.40), (3.43) and Lemma 2.1, we know that A has at least one fixed point $(x, y) \in P \cap (\overline{W}_4 \setminus W_3)$. Hence the system of fractional q-difference equations (1.1) has at least one positive solution. □

4. Applications

Example 4.1. *We consider the following system of fractional q-difference equations:*

$$\begin{cases} -D_q^{\frac{1}{2}}(\phi_1(D_q^{\frac{5}{2}}x))(t) = \eta f(t, x(t), y(t)), \quad 0 < t < 1, \\ -D_q^{\frac{1}{2}}(\phi_2(D_q^{\frac{5}{2}}y))(t) = \zeta g(t, x(t), y(t)), \quad 0 < t < 1, \\ x(0) = D_q x(0) = 0,\ D_q x(1) = 1,\ D_q^{\frac{5}{2}}x(0) = 0, \\ y(0) = D_q y(0) = 0,\ D_q y(1) = 1,\ D_q^{\frac{5}{2}}y(0) = 0, \end{cases} \tag{4.1}$$

where $q = \frac{1}{2}$, $\phi_1(u) = u, \phi_2(u) = |u|^{-1}u$. *Take* $f(t, x, y) = t(x + y - 2\varphi(t))^2$, $g(t, x, y) = t(x + y)$, *where* $\varphi(t) = \frac{4+\sqrt{2}}{7}t^{\frac{3}{2}}$. *By a simple calculation we get*

$$f_0 = \limsup_{x+y\to 0^+} \max_{t\in[0,1]} \frac{f(t, x+\varphi(t), y+\varphi(t))}{\phi_1(x+y)} = \limsup_{x+y\to 0^+} \max_{t\in[0,1]} t(x+y) = 0,$$

$$g_0 = \limsup_{x+y\to 0^+} \max_{t\in[0,1]} \frac{g(t, x+\varphi(t), y+\varphi(t))}{\phi_2(x+y)} = \limsup_{x+y\to 0^+} \max_{t\in[0,1]} t(x+y+2\varphi(t)) = \frac{2}{7}(4+\sqrt{2}),$$

$$g_\infty = \liminf_{x+y\to\infty} \min_{t\in[\theta,1]} \frac{g(t, x+\varphi(t), y+\varphi(t))}{\phi_2(x+y)} = \liminf_{x+y\to\infty} \min_{t\in[\theta,1]} t(x+y+2\varphi(t)) = \infty;$$

and $\psi_1(x) = \psi_2(x) = x$, $\psi_3(x) = \psi_4(x) = 1$, $D_4 \approx 0.6465$.

Then, for each $\eta \in (0, \infty)$ *and* $\zeta \in (0, 0.6465)$, *by Theorem 3.1 Case (6) we obtain that the system (4.1) has at least one positive solution.*

Example 4.2. *We consider the following system of fractional q-difference equations:*

$$\begin{cases} -D_q^{\frac{1}{2}}(\phi_1(D_q^{\frac{5}{2}}x))(t) = \eta f(t, x(t), y(t)), \quad 0 < t < 1, \\ -D_q^{\frac{1}{2}}(\phi_2(D_q^{\frac{5}{2}}y))(t) = \zeta g(t, x(t), y(t)), \quad 0 < t < 1, \\ x(0) = D_q x(0) = 0,\ D_q x(1) = 1,\ D_q^{\frac{5}{2}}x(0) = 0, \\ y(0) = D_q y(0) = 0,\ D_q y(1) = 1,\ D_q^{\frac{5}{2}}y(0) = 0, \end{cases} \tag{4.2}$$

where $q = \frac{1}{2}$, $\phi_1(u) = u, \phi_2(u) = |u|^{-1}u$. *Take* $f(t, x, y) = \frac{t(x+y-2\varphi(t))}{\arctan(x+y-2\varphi(t))}$, $g(t, x, y) = \frac{t}{x+y}$, *where* $\varphi(t) = \frac{4+\sqrt{2}}{7}t^{\frac{3}{2}}$. *By a simple calculation we get*

$$\bar{f}_0 = \liminf_{x+y\to 0^+} \min_{t\in[\theta,1]} \frac{f(t, x+\varphi(t), y+\varphi(t))}{\phi_1(x+y)} = \liminf_{x+y\to 0^+} \min_{t\in[\theta,1]} \frac{t}{\arctan(x+y)} = \infty,$$

$$\bar{g}_\infty = \limsup_{x+y\to\infty} \max_{t\in[0,1]} \frac{g(t, x+\varphi(t), y+\varphi(t))}{\phi_2(x+y)} = \limsup_{x+y\to\infty} \max_{t\in[0,1]} \frac{t}{x+y+2\varphi(t)} = 0,$$

$$\bar{f}_\infty = \limsup_{x+y\to\infty} \max_{t\in[0,1]} \frac{f(t, x+\varphi(t), y+\varphi(t))}{\phi_1(x+y)} = \limsup_{x+y\to\infty} \max_{t\in[0,1]} \frac{t}{\arctan(x+y)} = \frac{2}{\pi};$$

and $\psi_1(x) = \psi_2(x) = x$, $\psi_3(x) = \psi_4(x) = 1$, $\Gamma_{\frac{1}{2}}(\frac{5}{2}) \approx 1.1906$, $\Gamma_{\frac{1}{2}}(\frac{3}{2}) \approx 0.9209$, $M_1 \leq 0.2991$, $Z_2 \geq 2.6259$.

Then, for each $\eta \in (0, 2.6259)$ *and* $\zeta \in (0, \infty)$, *by Theorem 3.2(6) we obtain that the system (4.2) has at least one positive solution.*

5. Conclusions

The system of fractional q-difference equations plays an important role in the study of many fields, such as quantum mechanics, mathematical physics equations and so on, for example, see [16,17,24,35] and the references therein. In [35], by using some classical fixed point theorems, the authors studied the existence of nontrivial solutions of a system of fractional q-difference equations with Riemann-Stieltjes integrals conditions. In this paper, we investigate the existence of positive solutions for a system of fractional q-difference equations with generalized p-Laplacian operators and two parameters. The system in this paper is different from that of [35]. We give some assumptions which are combinations of superlinearity and sublinearity of the nonlinear terms f and g. Under those assumptions, by using Guo-Krasnosel'skii fixed point theorem, we obtain some existence results of positive solutions in terms of different values of the parameters η and ζ. In fact, since the system studied in this paper contains generalized p-Laplacian operators, the obtained results in this paper can enrich the relevant knowledge of theories for the system of fractional q-difference equations and expand the range of the possible applications. However, this study still has certain limitations, as we only investigated the existence of positive solutions. In the future, some further work can continue to be considered such as the uniqueness and multiplicity of positive solutions and iterative sequences of positive solutions, the case where the nonlinear terms may be changing sign or the generalized p-Laplacian operator becomes a $p(t)$ -Laplacian operator, etc.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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