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Research article

Multitudinous potential homoclinic and heteroclinic orbits seized

Haijun Wang[1], **Jun Pan**[2,*] **and Guiyao Ke**[3,4,5,*]

[1] School of Electronic and Information Engineering (School of Big Data Science), Taizhou University, Taizhou, 318000, China

[2] Department of Big Data Science, School of Science, Zhejiang University of Science and Technology, Hangzhou 310023, China

[3] School of Information, Zhejiang Guangsha Vocational and Technical University of Construction, Dongyang, Zhejiang 322100, China

[4] HUIKE Education Technology Group Co., Ltd., China

[5] School of Information Engineering, GongQing Institute of Science and Technology, Gongqingcheng 332020, China

* **Correspondence:** Email: panjun78@qq.com, guiyao.ke@zjgsdx.edu.cn.

Abstract: Revisiting a newly reported modified Chen system by both the definitions of α-limit and ω-limit set, Lyapunov function and Hamiltonian function, this paper seized a multitude of pairs of potential heteroclinic orbits to (1) E_0 and $E_\pm$, or (2) E_+ or (3) E_-, and homoclinic and heteroclinic orbits on its invariant algebraic surface $Q = z - \frac{x^2}{2a} = 0$ with cofactor $-2a$, which is not available in the existing literature to the best of our knowledge. Particularly, the theoretical conclusions were verified via numerical examples.

Keywords: new 3D modified Chen system; multi-wing chaotic attractor; homoclinic and heteroclinic orbit; Lyapunov function

1. Introduction

In this paper, we review a recently reported modified Chen system and examines its singular orbits, giving some new insights into it and extending the existing results [1]. It not only contains the classic Chen system as a special case, but also gives rise to multi-wings attractors with the higher largest Lyapunov exponent. To further understand its nature, we will consider the existence of homoclinic and heteroclinic orbits, which involve some real world applications [2–9], i.e., heart tissue, neurons, cell signalling, chemistry, biomathematics and mechanics, etc. Particularly, when planetary scientists design space missions, the heteroclinic connections between period orbits need to be studied in planar restricted circular three body problem [10–12].

For the sake of the following discussion, the concepts of homoclinic and heteroclinic orbits are introduced, i.e., a heteroclinic (resp., homoclinic) orbit is a type of orbit doubly asymptotic to two different equilibrium points or closed orbits (resp., the same equilibrium point or closed orbit) [13, 14]. In order to uncover homoclinic and heteroclinic orbits, Shilnikov et al. developed a powerful tool, a combination of contraction map and boundary problem, and categorized chaos of three-dimensional quadratic autonomous differential systems, i.e., chaos of the Shilnikov homoclinic or heteroclinic orbit type, or the hybrid type with both Shilnikov homoclinic and heteroclinic orbits, etc. [14].

With a combination of definitions of the α-limit and ω-limit set and the Lyapunov function, Li et al. revisited the Chen system and proved the existence of a pair of heteroclinic orbits of it [15]. Inspired by that example, other researchers began to consider heteroclinic orbits of other Lorenz-like systems [16–25] one after another. When studying the Tricomi problem on homoclinic and heteroclinic orbits [26], Leonov formulated the fishing principle and applied it to Lorenz-type systems [27]. Through tracing the stable and unstable manifolds of the Shimizu-Morioka model, Tigan and Turaev [28] detected a pair of homoclinic orbits to the origin. Wiggins, Feng and Hu [13, 29] also utilized the Melnikov method to study homoclinic and heteroclinic orbits and, thus, determined chaos in the sense of Smale horseshoes, such as its existence, stability, bifurcation, mutual position between the stable and the unstable manifold, and so on.

More importantly, some bifurcations of homoclinic or heteroclinic orbits may shed light on the problem of the nonlinear relationship between equilibria and the number of multi-wings/scrolls [1, 30, 31]. Recently, Gilardi-Velázquez et al. [32] collided two heteroclinic orbits to create a new square chaotic attractor, providing a mechanism to establish bistability in a new class of piecewise linear (PWL) dynamical systems. By rupturing heteroclinic-like orbits of a class of PWL systems, Escalante-González and Campos [33] revealed hidden attractors coexisting with self-excited ones.

So far as is known, scholars seldom consider homoclinic and heteroclinic orbits of the modified Chen system [1]. To achieve this target, we reinvestigate it and present the following main contribution:

1) Utilizing a combination of definitions of the α-limit and ω-limit set and the Lyapunov function to prove that there may exist many pairs of heteroclinic orbits to (1) E_0 and $E_\pm$ or (2) E_+ or (3) E_-.

2) Applying a Hamiltonian function to prove that there may exist multitudinous potential homoclinic orbits to E_0 and $E_\pm$, and heteroclinic orbits to $E_\pm$ on the invariant algebraic surface $z = \frac{x^2}{2a}$.

This paper is organized as follows. The new modified Chen system is introduced and the main results are reported in Section 2. In Section 3, one derives the existence of multitudinous potential homoclinic and heteroclinic orbits. At last, the conclusions are given in Section 4.

2. New modified Chen system and the main results

In this section, we consider the modified Chen system [1]:

$$\begin{cases} \dot{x} &= a(y - x), \\ \dot{y} &= cy - xz(1 - k\sin(k_1 z)) + (c - a)x, \\ \dot{z} &= -bz + xy, \end{cases} \tag{2.1}$$

where $a, c, k, k_1, b \in \mathbb{R}$. Apparently, for $k = 0$, system (2.1) reduces to the Chen system. Of particular interest is that the new added term $k\sin(k_1 z)$ guarantees the creation of multi-wings attractors with higher largest Lyapunov exponent coexisting with multiple equilibria, as illustrated in [1, Figures 2–4, p.2]. Next, we mainly focus on its multiple homoclinic and heteroclinic orbits.

First of all, let us present equilibria of system (2.1):

Proposition 2.1. *1) If $b = 0$, then $E_z = \{(0, 0, z)|z \in \mathbb{R}\}$ is non-isolated equilibria.*

2) When $\forall\sigma \neq 0$, $2c - a = \sigma[1 - k\sin(k_1\sigma)]$, $b\sigma > 0$, and $a \neq 0$. System (2.1) may have many pairs of equilibria: $E_\pm = \{(\pm\sqrt{b\sigma}, \pm\sqrt{b\sigma}, \sigma)|b\sigma > 0, 2c - a = \sigma[1 - k\sin(k_1\sigma)]\}$, except for $E_0 = (0, 0, 0)$.

We arrive at heteroclinic orbits in the case of non-Hamiltonian modified Chen system (2.1):

Proposition 2.2. *If $2c > a > c > 0$, $b \geq 2a > 0$, $\forall\sigma > 0$, $2c - a = \sigma[1 - k\sin(k_1\sigma)]$, then*

(i) there are no homoclinic orbits to E_+ or E_-, or heteroclinic orbits to E_+ and E_- in system (2.1);

(ii) multitudinous potential heteroclinic orbits to (1) E_0 and $E_\pm$, or (2) E_+ or (3) E_- exist in system (2.1).

Finally, we give homoclinic and heteroclinic orbits on the invariant algebraic surface $z = \frac{x^2}{2a}$ in the following statement.

Proposition 2.3. *Suppose $c = a$, $b - 2a = 0$, $\triangle = x^2 - \frac{x^4}{4a^2} + \frac{2}{a}[\frac{ak}{k_1^2}\sin(\frac{k_1x^2}{2a}) - \frac{kx^2}{2k_1}\cos(\frac{k_1x^2}{2a})]$, $2a^2 = x_*^2(1 - k\sin(\frac{k_1x_*^2}{2a}))$ and $\Gamma_* = \frac{-ax_*^2}{2} + \frac{x_*^4}{8a} - [\frac{ak}{k_1^2}\sin(\frac{k_1x_*^2}{2a}) - \frac{kx_*^2}{2k_1}\cos(\frac{k_1x_*^2}{2a})]$, then the following two assertions hold.*

1) If $\triangle \geq 0$, then system (2.1) has a pair of homoclinic orbits to E_0: $y = x \pm \sqrt{\triangle}$.

2) If $\triangle + \Gamma_* \geq 0$, then system (2.1) has a multitude of heteroclinic or homoclinic orbits to $E_\pm$: $y = x \pm \sqrt{\triangle + \Gamma_*}$, $|x| < |x_*|$.

3. Homoclinic and heteroclinic orbit and proofs of Proposition 2.2 and 2.3

In this section, we first study the existence of heteroclinic orbits when $2c > a > c > 0$, $b \geq 2a > 0$, $\forall\sigma > 0$, and $2c - a = \sigma[1 - k\sin(k_1\sigma)]$, as [15–25].

To facilitate derivation, denote by $\phi_t(q_0) = (x(t; x_0), y(t; y_0), z(t; z_0))$ a solution of system (2.1) with the initial value $q_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$. Let γ^- (resp., γ^+) be the negative (resp., positive) branch of the unstable manifold $W^u(E_0)$ corresponding to $-x_+ < 0$ (resp. $x_+ > 0$) as $t \to -\infty$, i.e., $\gamma^\mp = \{\phi_t^\mp(q_0)|\phi_t^\mp(q_0) = (\mp x_+(t; x_0), \mp y_+(t; y_0), z_+(t; z_0)) \in W^u_\mp(E_0), t \in \mathbb{R}\}$.

Put the first Lyapunov function

$$V_1(\phi_t(q_0)) = V_1(x, y, z) = \frac{1}{2}[\frac{2ab(b-2a)\sigma}{2c-a}(y - x)^2 + 2a(-bz + x^2)^2 + (b - 2a)(-b\sigma + x^2)^2]$$

with the derivative of it along trajectories of system (2.1):

$$\frac{dV_1(\phi_t(q_0))}{dt}\Big|_{(2.1)} = -\frac{2ab(b-2a)\sigma(a-c)}{2c-a}(y - x)^2 - 2ab(-bz + x^2)^2 \tag{3.1}$$

for the case of $b > 2a > 0$ and $\sin(k_1z) = \sin(k_1\sigma)$, and the second one

$$V_2(\phi_t(q_0)) = V_2(x, y, z) = \frac{1}{2}[4a^2\sigma(y - x)^2 + (2c - a)(-2a\sigma + x^2)^2]$$

with the corresponding derivative:

$$\frac{dV_2(\phi_t(q_0))}{dt}\Big|_{(2.1)} = \quad -4a^2\sigma(a-c)(y-x)^2 \tag{3.2}$$

for the case of $b = 2a > 0$, $z = \frac{x^2}{2a}$ and $\sin(\frac{k_1 x^2}{2a}) = \sin(k_1\sigma)$.

Here, one only considers the case of heteroclinic orbits to E_0 and $E_\pm$. The other case of heteroclinic orbits to E_+ or E_- is similar to the ones in [24, 25] and is omitted.

For proving Proposition 2.2, one introduces the following result.

Proposition 3.1. *When $2c > a > c > 0$, $b \geq 2a > 0$, $\forall\sigma > 0$, and $2c - a = \sigma[1 - k\sin(k_1\sigma)]$, the following two assertions are right.*

(i) If $\exists\, t_{1,2}$, such that $t_1 < t_2$ and $V_{1,2}(\phi_{t_1}(q_0)) = V_{1,2}(\phi_{t_2}(q_0))$, then q_0 is only one of the equilibria.

(ii) If $\lim\phi_t(q_0)_{t\to-\infty} = E_0$, and $\exists\, t \in \mathbb{R}$, $x(t; x_0) < 0$, then $V_{1,2}(E_0) > V_{1,2}(\phi_t(q_0))$ and $x(t; x_0) < 0$ for all $t \in \mathbb{R}$. Therefore, $q_0 \in W^u_-(E_0)$.

Proof. (i) If $2c > a > c > 0$, $b \geq 2a > 0$, $\forall\sigma > 0$, and $2c - a = \sigma[1 - k\sin(k_1\sigma)]$, then one arrives at $\frac{dV_{1,2}(\phi_t(q_0))}{dt}\Big|_{(2.1)} \leq 0$ from Eqs (3.1) and (3.2). On the basis of the hypothesis (i), one obtains $\frac{dV_{1,2}(\phi_t(q_0))}{dt}\Big|_{(2.1)} = 0$, $\forall t \in (t_1, t_2)$, and derives that q_0 is one stationary point, i.e.,

$$\dot{x}(t; x_0) \equiv \dot{y}(t; y_0) \equiv \dot{z}(t; z_0) \equiv 0 \tag{3.3}$$

namely, $\dot{x}(t; x_0) = a(y - x) = 0$ implies $x(t) = x_0$ and $\dot{y}(t, y_0) = 0$, $y(t) = y_0 = 0$, $\forall t \in \mathbb{R}$.

Since $z = \frac{x^2}{2a}$ is an invariant algebraic surface of system (2.1) with cofactor $-2a$ for $b = 2a$ [34], $\phi_t(q_0) \in \{y - x = 0\}$ results in Eq (3.3).

(ii) Due to $\lim\phi_t(q_0)_{t\to-\infty} = E_0$ and $x(t; x_0) < 0$, $\exists\, t \in \mathbb{R}$, q_0 cannot be a fixed point. Otherwise, $0 < V_{1,2}(E_0) \leq V_{1,2}(\phi_{t_0}(q_0))$, $\exists t_0 \in \mathbb{R}$. Therefore, one arrives at $V_{1,2}(\phi_{t_1}(q_0)) \leq V_{1,2}(\phi_{t_0}(q_0))$, $\exists t_1 \in \mathbb{R}$. Based on $\frac{dV_{1,2}(\phi_t(q_0))}{dt}\Big|_{(2.1)} \leq 0$, $V_{1,2}(\phi_{t_1}(q_0)) = V_{1,2}(\phi_{t_0}(q_0))$ and assertion (i), q_0 is one stationary point. The condition $\lim_{t\to-\infty}\phi_t(q_0) = E_0$ leads to $q_0 = E_0$ and $x(t; x_0) = 0$, $\forall t \in \mathbb{R}$. A contradiction happens. The fact $V_{1,2}(E_0) > V_{1,2}(\phi_t(q_0))$ holds, $\forall t \in \mathbb{R}$.

Next, let us show $x(t, x_0) < 0$, $\forall t \in \mathbb{R}$. If not, $\exists\, t' \in \mathbb{R}$, $x(t', x_0) \geq 0$. Based on the hypothesis (ii), one obtains $x(t'', x_0) < 0$, $\exists\, t'' \in \mathbb{R}$. As a result, $x(\tau, x_0) = 0$, $\exists\,\tau \in \mathbb{R}$. According to $V_{1,2}(E_0) > V_{1,2}(\phi_t(q_0))$, $\forall t \in \mathbb{R}$, one gets $\phi_\tau(q_0) \in \{(x, y, z) : V_{1,2}(E_0) > V_{1,2}(x, y, z)\} \cap \{x = 0\}$, which is contradiction. Actually, both $\{(x, y, z) : V_1(E_0) > V_1(x, y, z)\} \cap \{x = 0\} = \{(x, y, z) : \frac{1}{2}[\frac{2ab(b-2a)\sigma}{2c-a}y^2 + 2ab^2z^2 + (b - 2a)b^2\sigma^2] < \frac{(b-2a)b^2\sigma^2}{2}\}$ and $\{(x, y, z) : V_2(E_0) > V_2(x, y, z)\} \cap \{x = 0\} = \{(x, y, z) : 2a^2\sigma y^2 + 2(2c - a)a^2\sigma^2 < 2(2c - a)a^2\sigma^2\}$ are all empty. So, $\forall t \in \mathbb{R}$, $x(t, q_0) < 0$. The proof is completed.

From Proposition 3.1, one proves Proposition 2.2 as follows.

Proof of Proposition 2.2: (i) When $2c > a > c > 0$, $b \geq 2a > 0$, $\forall\sigma > 0$, and $2c-a = \sigma[1-k\sin(k_1\sigma)]$, there are neither homoclinic orbits to E_+ or E_-, nor heteroclinic orbits to E_+ and E_- in system (2.1). Otherwise, suppose $p(t) = (x, y, z)$ is a homoclinic orbit to E'_+ or E'_-, or heteroclinic orbit to $E'_\pm$, where $\forall E'_+ \in E_+$ and $\forall E'_- \in E_-$, i.e., $\lim\limits_{t\to-\infty} p(t) = e_-$, $\lim\limits_{t\to+\infty} p(t) = e_+$, where e_- and e_+ satisfy either $e_- = e_+ \in \{E'_-, E_0, E'_+\}$ or $\{e_-, e_+\} = \{E'_-, E'_+\}$.

Based on $\frac{dV_{1,2}(\phi_t(q_0))}{dt}\Big|_{(2.1)} \le 0$, one arrives at

$$V_{1,2}(e_-) \ge V_{1,2}(p(t)) \ge V_{1,2}(e_+). \tag{3.4}$$

In any case, the relation $V_{1,2}(e_-) = V_{1,2}(e_+)$ all hold, which also leads to $V_{1,2}(p(t)) \equiv V_{1,2}(e_-)$. According to the assertion (i) of Proposition 3.1, $p(t)$ is just one stationary point. In a word, homoclinic orbits to E_+ or E_-, or heteroclinic orbits to E_+ and E_- do not exist in system (2.1).

(ii) Next, one proves that $p(t)$ is a heteroclinic orbit to E_0 and $E_-^{'}$, i.e., $\lim\limits_{t\to+\infty} p(t) = E_-^{'}$. On the basis of the definition of $p(t)$ and the second assertion of Proposition 3.1, $V_{1,2}(E_0) > V_{1,2}(p(t))$ holds, which also yields $\lim\limits_{t\to+\infty} p(t) \ne E_0$, i.e., $\lim\limits_{t\to+\infty} p(t) = E_-^{'}$.

At last, let us show that, if system (2.1) has a heteroclinic orbit to E_0 and $E_-^{'}$, then this orbit is just $p(t)$.

Suppose $p_-^*(t) = (x^*(t), y^*(t), z^*(t))$ is a solution of system (2.1) such that

$$\lim_{t\to-\infty} p_-^*(t) = e_1^-, \quad \lim_{t\to+\infty} p_-^*(t) = e_1^+,$$

where e_1^- and e_1^+ satisfy $\{e_1^-, e_1^+\} = \{E_0, E_-^{'}\}$. Since $\frac{dV_{1,2}(\phi_t(q_0))}{dt}\Big|_{(2.1)} \le 0$, $\forall t \in \mathbb{R}$, $V_{1,2}(e_1^-) \ge V_{1,2}(p_-^*(t)) \ge V_{1,2}(e_1^+)$ holds. Due to $V_{1,2}(E_0) > V_{1,2}(E_-^{'})$, one gets $e_1^- = E_0$ and $e_1^+ = E_-^{'}$, i.e.,

$$\lim_{t\to-\infty} p_-^*(t) = E_0, \quad \lim_{t\to+\infty} p_-^*(t) = E_-^{'},$$

which yields $p_-^*(t) \in \gamma^-$ based on the assertion (ii) of Proposition 3.1. Since system (2.1) is symmetrical w.r.t. the z–axis, there exists a unique heteroclinic orbit $p_+^*(t) \in \gamma^+$, i.e., $\lim\limits_{t\to-\infty} p_+^*(t) = E_0$, $\lim\limits_{t\to+\infty} p_+^*(t) = E_+^{'}$.

Due to the term of $k\sin(k_1 z)$, $E_\pm$ may include many isolated stationary points $E_\pm^{'}$ and, thus, may generate many heteroclinic orbits to E_0 and $E_\pm$. The proof is over.

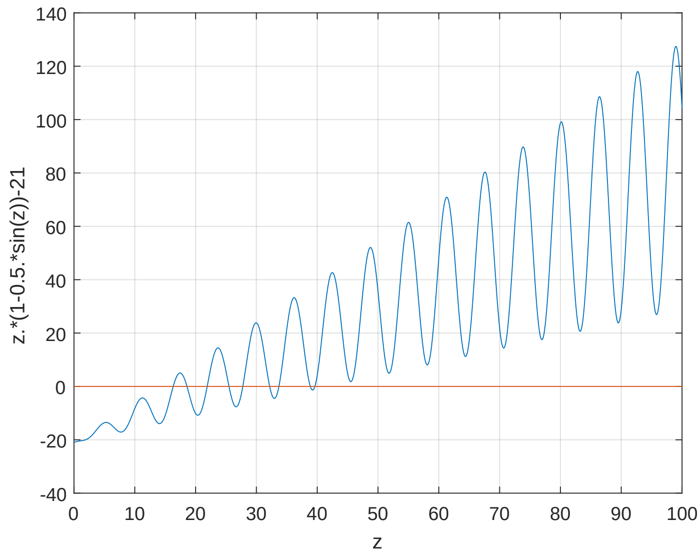


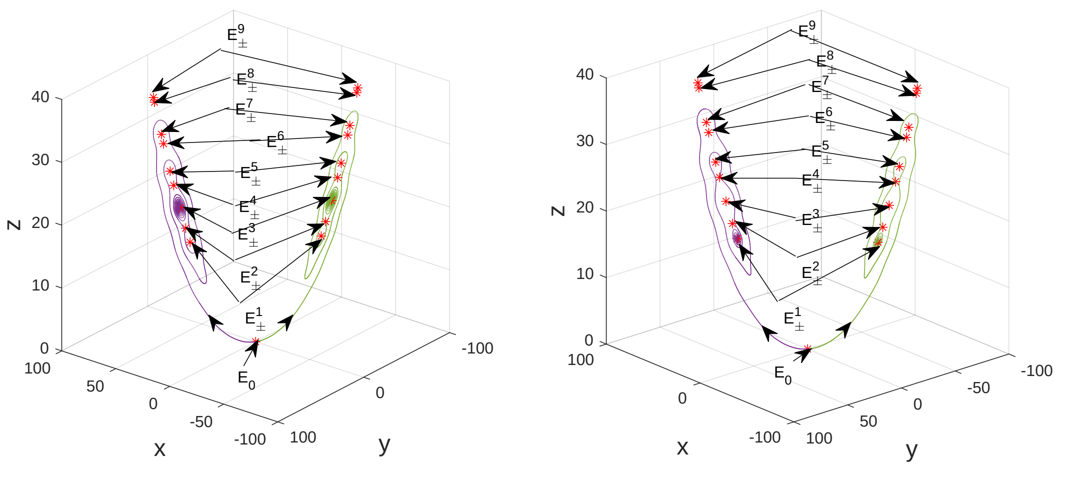
Figure 1. Nine solutions of $21 = z(1-0.5\sin z)$ with $z > 0$ when $(a, c, k, k_1) = (35, 28, 0.5, 1)$.

Table 1. The dynamics of $E_{\pm}^{i}$ with $(a, c, k, k_1, b) = (35, 28, 0.5, 1, 70)$, $i = 1, 2, \cdots, 9$.

$E_{\pm}$	Classification	Eigenvalues
$E_{\pm}^{1} = (\pm 33.7984, \pm 33.7984, 16.319)$	Stable focus	$-3.5 \pm 95.3515i, -70$
$E_{\pm}^{2} = (\pm 36.0705, \pm 36.0705, 18.5869)$	Saddle	$97.589, -104.589, -70$
$E_{\pm}^{3} = (\pm 39.1608, \pm 39.1608, 21.9081)$	Stable focus	$-3.5 \pm 134.9i, -70$
$E_{\pm}^{4} = (\pm 42.2435, \pm 42.2435, 25.493)$	Saddle	$137.312, -144.312, -70$
$E_{\pm}^{5} = (\pm 44.086, \pm 44.086, 27.7653)$	Stable focus	$-3.5 \pm 158.18i, -70$
$E_{\pm}^{6} = (\pm 47.4648, \pm 47.4648, 32.1844)$	Saddle	$153.3704, -160.3704, -70$
$E_{\pm}^{7} = (\pm 48.5721, \pm 48.5721, 33.7036)$	Stable focus	$-3.5 \pm 166.07i, -70$
$E_{\pm}^{8} = (\pm 52.1592, \pm 52.1592, 38.8655)$	Saddle	$135.592, -142.592, -70$
$E_{\pm}^{9} = (\pm 52.662, \pm 52.662, 39.6184)$	Stable focus	$-3.5 \pm 142.18i, -70$

Table 2. The dynamics of $E_{\pm}^{i}$ with $(a, c, k, k_1, b) = (35, 28, 0.5, 1, 75)$, $i = 1, 2, \cdots, 9$.

$E_{\pm}$	Classification	Eigenvalues
$E_{\pm}^{1} = (\pm 34.9846, \pm 34.9846, 16.319)$	Stable focus	$-3.5 \pm 98.7041i, -75$
$E_{\pm}^{2} = (\pm 37.3365, \pm 37.3365, 18.5869)$	Saddle	$101.1325, -108.1325, -75$
$E_{\pm}^{3} = (\pm 40.5353, \pm 40.5353, 21.9081)$	Stable focus	$-3.5 \pm 139.64i, -75$
$E_{\pm}^{4} = (\pm 43.7261, \pm 43.7261, 25.493)$	Saddle	$142.2534, -149.2534, -75$
$E_{\pm}^{5} = (\pm 45.6333, \pm 45.6333, 27.7653)$	Stable focus	$-3.5 \pm 163.73i, -75$
$E_{\pm}^{6} = (\pm 49.1307, \pm 49.1307, 32.1844)$	Saddle	$158.8772, -165.8772, -75$
$E_{\pm}^{7} = (\pm 50.2769, \pm 50.2769, 33.7036)$	Stable focus	$-3.5 \pm 171.9i, -75$
$E_{\pm}^{8} = (\pm 53.9899, \pm 53.9899, 38.8655)$	Saddle	$140.4712, -147.4712, -75$
$E_{\pm}^{9} = (\pm 54.5104, \pm 54.5104, 39.6184)$	Stable focus	$-3.5 \pm 147.19i, -75$



(a) $b = 70$ (b) $b = 75$

Figure 2. When $(a, c, k, k_1) = (35, 28, 0.5, 1)$ and $(x_0^{1,2}, y_0^{1,2}, z_0^{1}) = (\pm 0.13, \pm 1.3, 1.6) \times 10^{-7}$, (a) $b = 70$, system (2.1) has a pair of heteroclinic orbits to $E_{\pm}^{3}$ and E_0, which coexist with $E_{\pm}^{1,2,4,5,6,7,8,9}$; (b) $b = 75$, system (2.1) has a pair of heteroclinic orbits to $E_{\pm}^{1}$ and E_0, which coexist with $E_{\pm}^{2,3,4,5,6,7,8,9}$.

Set $(a, c, k, k_1) = (35, 28, 0.5, 1)$, $b = 70, 75$. The nontrivial equilibria of system (2.1) satisfies $2c - a = z(1 - k\sin(k_1 z))$, i.e., $21 = z(1 - 0.5\sin(z))$ with $z > 0$. At this time, system (2.1) has nine pairs of nontrivial equilibria: $E^i_{\pm} = (\pm\sqrt{bz_i}, \pm\sqrt{bz_i}, z_i)$, where $z_1 = 16.319$, $z_2 = 18.5869$, $z_3 = 21.9081$, $z_4 = 25.493$, $z_5 = 27.7653$, $z_6 = 32.1844$, $z_7 = 33.7036$, $z_8 = 38.8655$, and $z_9 = 39.6184$, as depicted in Figure 1 and Tables 1 and 2.

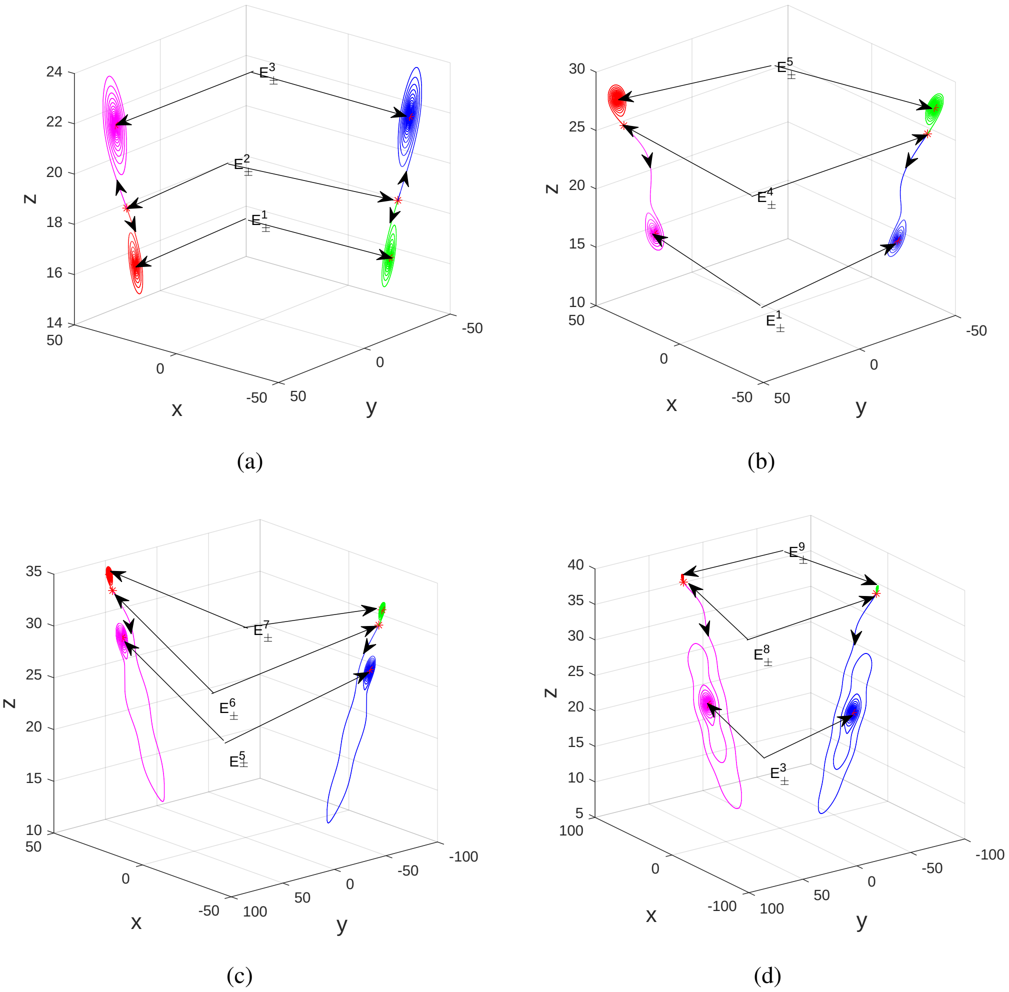


Figure 3. When $(a, c, k, k_1, b) = (35, 28, 0.5, 1, 70)$, (a) $(x_0^{3,4}, y_0^{3,4}, z_0^2) = (\pm 36.0705, \pm 36.0705, 18.5868)$ (resp., $(\pm 36.0705, \pm 36.0705, 18.587)$), a pair of heteroclinic orbits to $E^2_{\pm}$ and $E^1_{\pm}$ (resp., $E^3_{\pm}$), (b) $(x_0^{5,6}, y_0^{5,6}, z_0^3) = (\pm 42.2435, \pm 42.2435, 25.494)$ (resp., $(\pm 42.2435, \pm 42.2435, 25.492)$), a pair of heteroclinic orbits to $E^4_{\pm}$ and $E^5_{\pm}$ (resp., $E^1_{\pm}$), (c) $(x_0^{7,8}, y_0^{7,8}, z_0^4) = (\pm 47.4648, \pm 47.4648, 32.1845)$ (resp., $(\pm 47.4648, \pm 47.4648, 32.1843)$), a pair of heteroclinic orbits to $E^6_{\pm}$ and $E^7_{\pm}$ (resp., $E^5_{\pm}$), (d) $(x_0^{9,10}, y_0^{9,10}, z_0^5) = (\pm 52.1592, \pm 52.1592, 38.8656)$ (resp., $(\pm 52.1592, \pm 52.1592, 38.8654)$), a pair of heteroclinic orbits to $E^8_{\pm}$ and $E^9_{\pm}$ (resp., $E^3_{\pm}$). These figures suggest that system (2.1) has multitudinous potential heteroclinic orbits to E_+ or E_-.

In contrast to the Lorenz-like systems [19, 20, 24, 25], Figures 2–4 show multitudinous heteroclinic orbits to (1) E_0 and $E_\pm$, (2) E_+, (3) E_-.

Remark 1. *Motivated by [16], a pair of homoclinic orbits to* E_0 *may exist in the scenario of non-Hamiltonian modified Chen system (2.1) when* $(a, c, k, k_1, b) = (1, 0.8, 0.5, 1, 1.088966)$*, as shown in Figure 5.*

Remark 2. *When* $a = c$*,* V_2 *is a first integral. Proposition 2.2 includes the result on heteroclinic orbits given in [15, 16] as a special case when* $k = 0$*.*

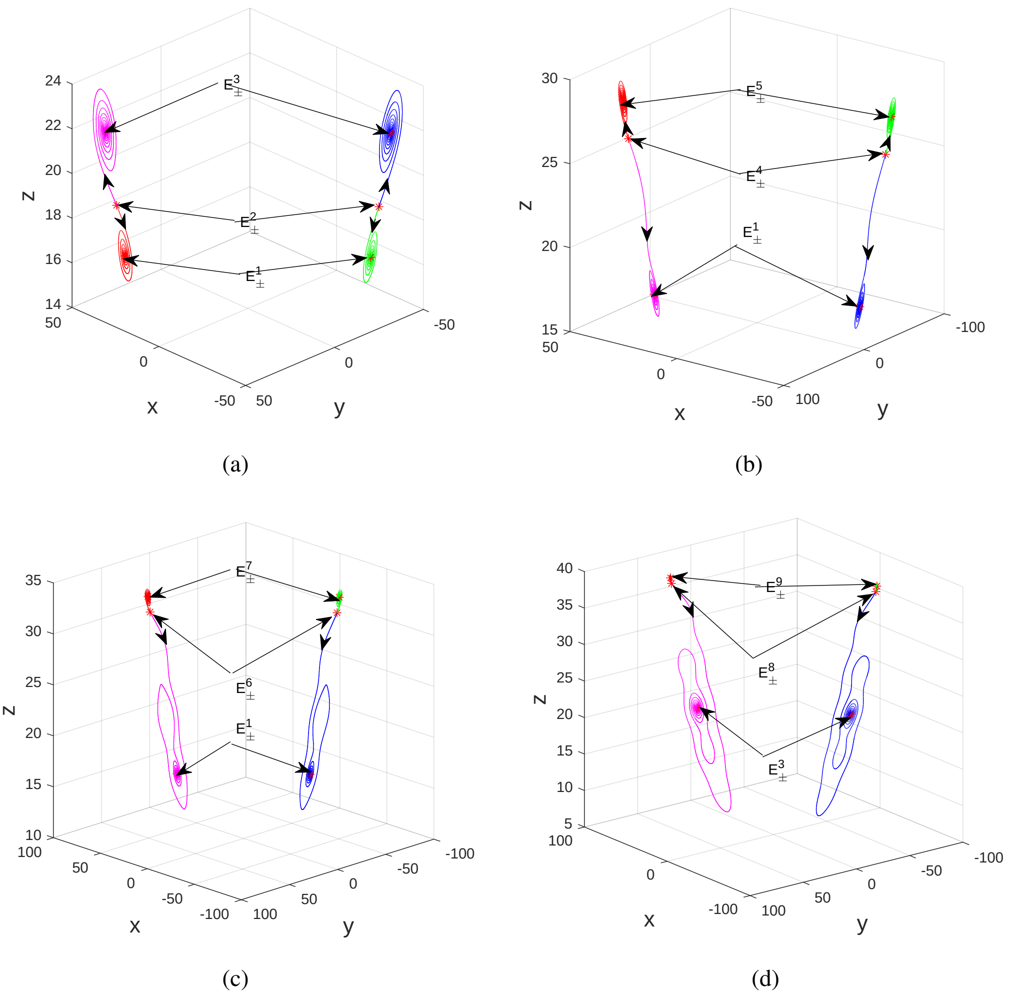


Figure 4. When $(a, c, k, k_1, b) = (35, 28, 0.5, 1, 75)$, (a) $(x_0^{11,12}, y_0^{11,12}, z_0^2) = (\pm 37.3365, \pm 37.3365, 18.5868)$ (resp., $(\pm 37.3365, \pm 37.3365, 18.587)$), a pair of heteroclinic orbits to $E_\pm^2$ and $E_\pm^1$ (resp., $E_\pm^3$), (b) $(x_0^{13,14}, y_0^{13,14}, z_0^3) = (\pm 43.7261, \pm 43.7261, 25.494)$ (resp., $(\pm 43.7261, \pm 43.7261, 25.492)$), a pair of heteroclinic orbits to $E_\pm^4$ and $E_\pm^5$ (resp., $E_\pm^1$), (c) $(x_0^{15,16}, y_0^{15,16}, z_0^4) = (\pm 49.1307, \pm 49.1307, 32.1845)$ (resp., $(\pm 49.1307, \pm 49.1307, 32.1843)$), a pair of heteroclinic orbits to $E_\pm^6$ and $E_\pm^7$ (resp., $E_\pm^1$), (d) $(x_0^{17,18}, y_0^{17,18}, z_0^5) = (\pm 53.9899, \pm 53.9899, 38.8656)$ (resp., $(\pm 53.9899, \pm 53.9899, 38.8654)$), a pair of heteroclinic orbits to $E_\pm^8$ and $E_\pm^9$ (resp., $E_\pm^3$). These figures suggest that system (2.1) has multitudinous potential heteroclinic orbits to E_+ or E_-.

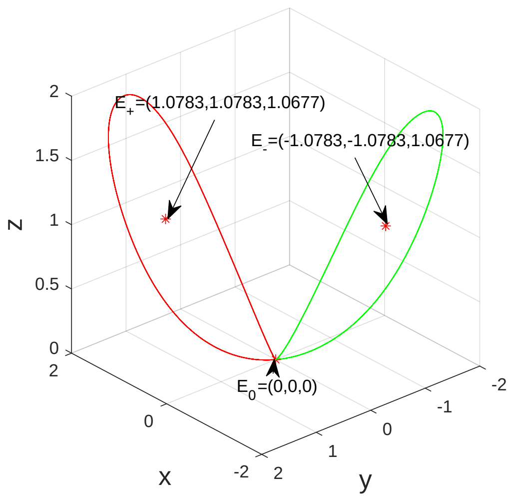
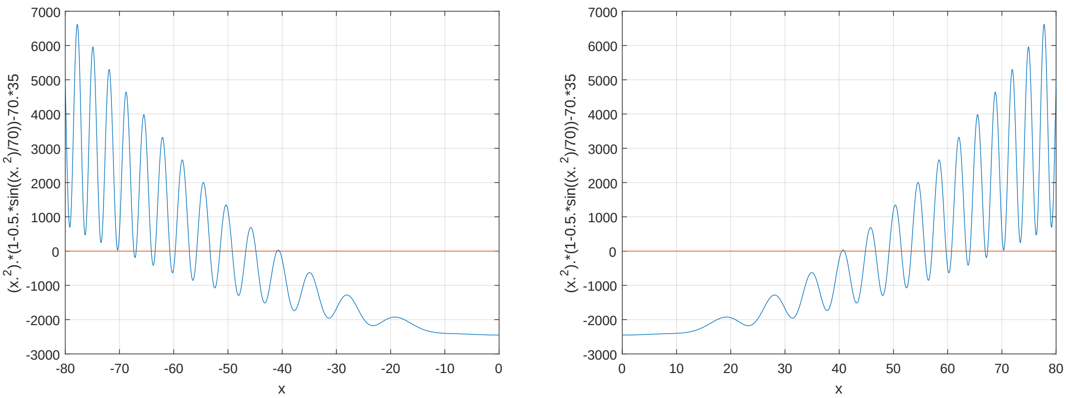
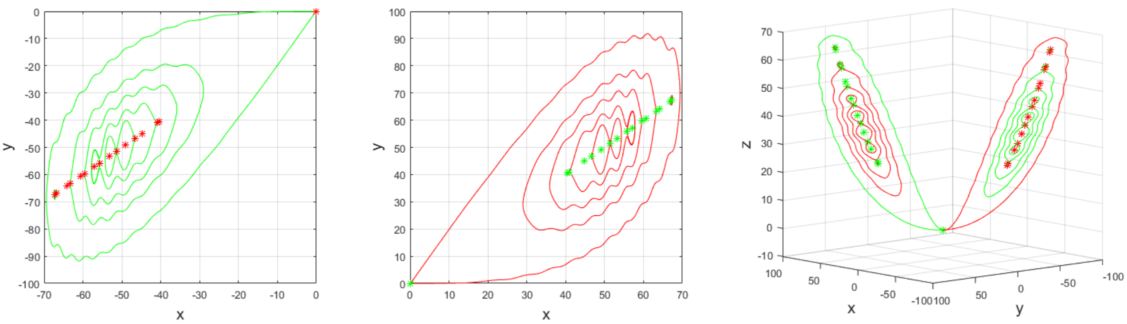


Figure 5. When $(a, c, k, k_1, b) = (1, 0.8, 0.5, 1, 1.088966)$ and $(x_0^{1,2}, y_0^{1,2}, z_0^1) = (\pm 0.13, \pm 1.3, 1.6) \times 10^{-7}$, a pair of homoclinic orbits to E_0 may exist in system (2.1).



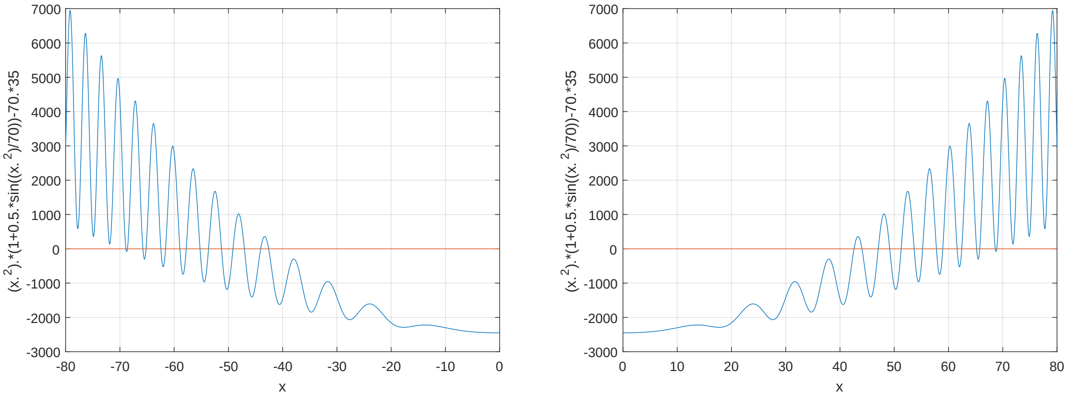
(a) (b)

Figure 6. Fifteen pairs of nontrivial equilibria $S_\pm^i$ when $(a, c, k, k_1) = (35, 35, 0.5, 1)$, $i = 1, 2, \cdots, 15$.



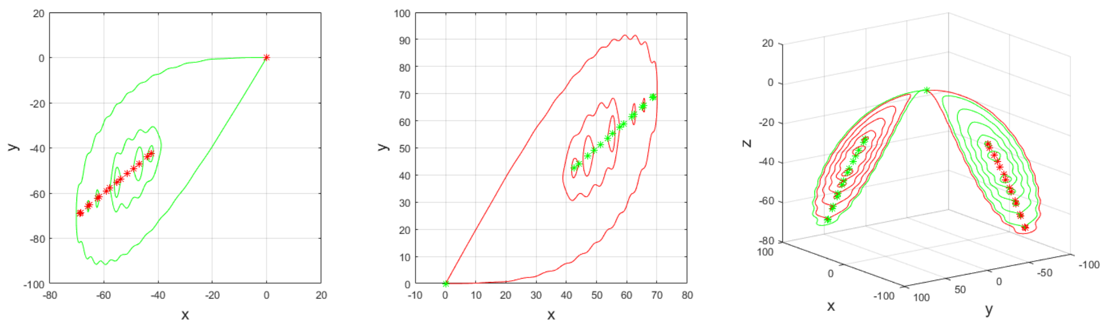
(a) (b) (c)

Figure 7. When $(a, c, k, k_1, b) = (35, 35, 0.5, 1, 70)$, (a) (b) eight pairs of homoclinic orbits to $S_\pm^i$ and S_0 of system (3.5), (c) eight pairs of homoclinic orbits to $E_\pm^i$ and E_0 of system (2.1), $i = 2, 4, 6, 8, 10, 12, 14$.



(a) (b)

Figure 8. Fifteen pairs of nontrivial equilibria $S_{\pm}^{i}$ when $(a, c, k, k_1) = (-35, -35, 0.5, 1)$, $i = 1, 2, \cdots, 15$.



(a) (b) (c)

Figure 9. When $(a, c, k, k_1, b) = (-35, -35, 0.5, 1, -70)$, (a) (b) eight pairs of homoclinic orbits to $S_{\pm}^{i}$ and S_0 of system (3.5), (c) eight pairs of homoclinic orbits to $E_{\pm}^{i}$ and E_0 of system (2.1), $i = 2, 4, 6, 8, 10, 12, 14$.

Finally, for $b = 2a$, let us consider homoclinic and heteroclinic orbits on the invariant algebraic surface $z = \frac{x^2}{2a}$ [34] of system (2.1), i.e., the ones of following system:

$$\begin{cases} \dot{x} & = a(y - x), \\ \dot{y} & = cy - \frac{x^3}{2a}[1 - k\sin(\frac{k_1 x^2}{2a})] + (c - a)x. \end{cases} \tag{3.5}$$

When $c = a$, system (3.5) reduces into a Hamiltonian system with the corresponding Hamiltonian function

$$H(x, y) = -axy + \frac{a}{2}y^2 + \frac{x^4}{8a} - [\frac{ak}{k_1^2}\sin(\frac{k_1 x^2}{2a}) - \frac{kx^2}{2k_1}\cos(\frac{k_1 x^2}{2a})]. \tag{3.6}$$

Obviously, the equilibrium points of system (3.5) are $S_0 = (0, 0)$, $S_{\pm} = (\pm x, \pm x)$ for $2a^2 = x^2(1 - k\sin(\frac{k_1 x^2}{2a}))$. Further, the proof of Proposition 2.3 easily follows and is omitted here.

Set $(a, c, k, k_1) = (\pm 35, \pm 35, 0.5, 1)$, except for S_0, Figures 6 and 8 show another fifteen pairs of nontrivial equilibria $S_{\pm}^{i}$ of system (3.5), i.e., the $E_{\pm}^{i}$ of system (2.1), $i = 1, 2, \cdots, 15$, as shown in

Tables 3 and 4. Obviously, $S_{\pm}^{i}$ are included in $E_{\pm}^{i}$ and are not listed here, $i = 1, 2, \cdots, 15$. Further, system (3.5) (resp., (2.1)) has eight pairs of homoclinic orbits to $S_{\pm}^{i}$ and S_0 (resp., $E_{\pm}^{i}$ and E_0), $i = 2, 4, 6, 8, 10, 12, 14$, as depicted in Figures 7 and 9. Being different from the ones [25, 35], the heteroclinic orbits to $S_{\pm}^{i}$ (resp., $E_{\pm}^{i}$) are not observed in numerical simulation.

Table 3. The dynamics of $E_{\pm}^{i}$ with $(a, c, k, k_1, b) = (35, 35, 0.5, 1, 70)$, $i = 1, 2, \cdots, 15$.

$E_{\pm}$	Classification	Eigenvalues
$E_{\pm}^{1} = (\pm 40.4862, \pm 40.4862, 23.4162)$	Andronov-Hopf bifurcation	$\pm 72.37i, -70$
$E_{\pm}^{2} = (\pm 40.9545, \pm 40.9545, 23.9610)$	Saddle	$\pm 73.1982, -70$
$E_{\pm}^{3} = (\pm 44.8424, \pm 44.8424, 28.7263)$	Andronov-Hopf bifurcation	$\pm 168.62i, -70$
$E_{\pm}^{4} = (\pm 46.7089, \pm 46.7089, 31.1674)$	Saddle	$\pm 174.6572, -70$
$E_{\pm}^{5} = (\pm 49.2008, \pm 49.2008, 34.5817)$	Andronov-Hopf bifurcation	$\pm 210.46i, -70$
$E_{\pm}^{6} = (\pm 51.4735, \pm 51.4735, 37.8503)$	Saddle	$\pm 217.0728, -70$
$E_{\pm}^{7} = (\pm 53.286, \pm 53.286, 40.5628)$	Andronov-Hopf bifurcation	$\pm 240.47i, -70$
$E_{\pm}^{8} = (\pm 55.7622, \pm 55.7622, 44.4203)$	Saddle	$\pm 245.134, -70$
$E_{\pm}^{9} = (\pm 57.1156, \pm 57.1156, 46.6027)$	Andronov-Hopf bifurcation	$\pm 261.48i, -70$
$E_{\pm}^{10} = (\pm 59.7153, \pm 59.7153, 50.9417)$	Saddle	$\pm 261.5179, -70$
$E_{\pm}^{11} = (\pm 60.7207, \pm 60.7207, 52.6715)$	Andronov-Hopf bifurcation	$\pm 272.84i, -70$
$E_{\pm}^{12} = (\pm 63.4129, \pm 63.4129, 57.4457)$	Saddle	$\pm 263.8527, -70$
$E_{\pm}^{13} = (\pm 64.1281, \pm 64.1281, 58.7488)$	Andronov-Hopf bifurcation	$\pm 271.2i, -70$
$E_{\pm}^{14} = (\pm 66.9144, \pm 66.9144, 63.9648)$	Saddle	$\pm 241.3816, -70$
$E_{\pm}^{15} = (\pm 67.3528, \pm 67.3528, 64.8057)$	Andronov-Hopf bifurcation	$\pm 245.16i, -70$

Table 4. The dynamics of $E_{\pm}^{i}$ with $(a, c, k, k_1, b) = (-35, -35, 0.5, 1, -70)$, $i = 1, 2, \cdots, 15$.

$E_{\pm}$	Classification	Eigenvalues
$E_{\pm}^{1} = (\pm 42.5879, \pm 42.5879, -25.9104)$	Shilnikov-Hopf bifurcation	$\pm 138.54i, 70$
$E_{\pm}^{2} = (\pm 44.0511, \pm 44.0511, -27.7214)$	Saddle	$\pm 142.9652, 70$
$E_{\pm}^{3} = (\pm 47.0547, \pm 47.0547, -31.6306)$	Shilnikov-Hopf bifurcation	$\pm 191.48i, 70$
$E_{\pm}^{4} = (\pm 49.1642, \pm 49.1642, -34.5303)$	Saddle	$\pm 198.1574, 70$
$E_{\pm}^{5} = (\pm 51.2773, \pm 51.2773, -37.5623)$	Shilnikov-Hopf bifurcation	$\pm 226.65i, 70$
$E_{\pm}^{6} = (\pm 53.6664, \pm 53.6664, -41.144)$	Saddle	$\pm 232.5816, 70$
$E_{\pm}^{7} = (\pm 55.2308, \pm 55.2308, -43.5777)$	Shilnikov-Hopf bifurcation	$\pm 252.09i, 70$
$E_{\pm}^{8} = (\pm 57.7749, \pm 57.7749, -47.6848)$	Saddle	$\pm 254.8295, 70$
$E_{\pm}^{9} = (\pm 58.9443, \pm 58.9443, -49.6347)$	Shilnikov-Hopf bifurcation	$\pm 268.49i, 70$
$E_{\pm}^{10} = (\pm 61.5922, \pm 61.5922, -54.1943)$	Saddle	$\pm 264.7765, 70$
$E_{\pm}^{11} = (\pm 62.4478, \pm 62.4478, -55.7104)$	Shilnikov-Hopf bifurcation	$\pm 2.7403i, 70$
$E_{\pm}^{12} = (\pm 65.1845, \pm 65.1845, -60.7003)$	Saddle	$\pm 257.1704, 70$
$E_{\pm}^{13} = (\pm 65.7631, \pm 65.7631, -61.7826)$	Shilnikov-Hopf bifurcation	$\pm 262.71i, 70$
$E_{\pm}^{14} = (\pm 68.6152, \pm 68.6152, -67.2578)$	Saddle	$\pm 205.6317, 70$
$E_{\pm}^{15} = (\pm 68.8911, \pm 68.8911, -67.7998)$	Shilnikov-Hopf bifurcation	$\pm 207.64i, 70$

4. Conclusions

In the present work, the existence of homoclinic orbits and heteroclinic orbits of both cases of non-Hamiltonian and Hamiltonian modified Chen system was proved based on definitions of both α-limit set and ω-limit set, the theory of the Lyapunov function, and the Hamiltonian function. In the former case, there existed multitudinous potential heteroclinic orbits to (1) E_0 and $E_\pm$, (2) E_+, or (3) E_- in that modified Chen system. In the latter case, there was a multitude of potential homoclinic orbits to E_0 and $E_\pm$ or heteroclinic orbits to $E_\pm$ on the invariant algebraic surface $z = \frac{x^2}{2a}$. Moreover, when $(a, c, k, k_1) = (35, 28, 0.5, 1)$, $b = 70, 75$ (resp., $(a, c, k, k_1, b) = (\pm 35, \pm 35, 0.5, 1, \pm 70)$), numerical simulations illustrated nine pairs of heteroclinic orbits (resp., eight pairs of homoclinic orbits). What follows is to study the multi-wing Lü system [1], Lorenz system, and other hyperchaotic Lorenz-like systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. S. Sahoo, B. K. Roy, Design of multi-wing chaotic systems with higher largest Lyapunov exponent, *Chaos, Solitons Fractals*, **157** (2022), 111926. https://doi.org/10.1016/j.chaos.2022.111926
2. E. Freire, A. J. Rodriguez-Luis, E. Gamero, E. Ponce, A case study for homoclinic chaos in an autonomous electronic circuit: A trip from Takens-Bogdanov to Hopf-Šil'nikov, *Physica D*, **62** (1993), 230–253. https://doi.org/10.1016/0167-2789(93)90284-8
3. P. Glendinning, C. Sparrow, Local and global behaviour near homoclinic orbits, *J. Stat. Phys.*, **35** (1984), 645–696. https://doi.org/10.1007/BF01010828
4. G. W. Hunt, M. A. Peletier, A. R. Champneys, P. D. Woods, M. Ahmerwaddee, C. J. Budd, et al., Cellular buckling in long structures, *Nonlinear Dyn.*, **21** (2000), 3–29. https://doi.org/10.1023/A:1008398006403

5. B. Aulbach, D. Flockerzi, The past in short hypercycles, *J. Math. Biol.*, **27** (1989), 223–231. https://doi.org/10.1007/BF00276104

6. N. J. Balmforth, Solitary waves and homoclinic orbits, *Annu. Rev. Fluid Mech.*, **27** (1995), 335–373. https://doi.org/10.1146/annurev.fl.27.010195.002003

7. R. M. May, W. Leonard, Nonlinear aspect of competition between three species, *SIAM J. Appl. Math.*, **29** (1975), 243–253. https://doi.org/10.1137/0129022

8. J. Hofbauer, K. Sigmund, On the stabilizing effect of predator and competitors on ecological communities, *J. Math. Biol.*, **27** (1975), 537–548. https://doi.org/10.1007/BF00288433

9. B. Y. Feng, The heteroclinic cycle in the model of competition between n species and its stability, *Acta Math. Appl. Sin.*, **14** (1998), 404–413. https://doi.org/10.1007/BF02683825

10. W. S. Koon, M. W. Lo, J. E. Marsden, S. D. Ross, Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics, *Chaos*, **10** (2000), 427–469. https://doi.org/10.1063/1.166509

11. D. Wilczak, P. Zgliczyński, Heteroclinic connections between periodic orbits in planar restricted circular three body problem-A computer assisted proof, *Commun. Math. Phys.*, **234** (2003), 37–75. https://doi.org/10.1007/s00220-002-0709-0

12. D. Wilczak, P. Zgliczyński, Heteroclinic connections between periodic orbits in planar restricted circular three body problem. part II, *Commun. Math. Phys.*, **259** (2005), 561–576. https://doi.org/10.1007/s00220-005-1471-x

13. S. Wiggins, *Introduction to Applied Nonlinear Dynamical System and Chaos*, 2nd edition, Springer, New York, 2003. https://doi.org/10.1007/978-1-4757-4067-7

14. L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics, Part II*, World Scientific, Singapore, 2001. https://doi.org/10.1142/4221

15. T. Li, G. Chen, G. Chen, On homoclinic and heteroclinic orbits of the Chen's system, *Int. J. Bifurcation Chaos*, **16** (2006), 3035–3041. https://doi.org/10.1142/S021812740601663X

16. G. Tigan, J. Llibre, Heteroclinic, homoclinic and closed orbits in the Chen system, *Int. J. Bifurcation Chaos*, **26** (2016), 1650072. https://doi.org/10.1142/S0218127416500723

17. H. Wang, X. Li, More dynamical properties revealed from a 3D Lorenz-like system, *Int. J. Bifurcation Chaos*, **24** (2014), 1450133. https://doi.org/10.1142/S0218127414501338

18. H. Wang, X. Li, On singular orbits and a given conjecture for a 3D Lorenz-like system, *Nonlinear Dyn.*, **80** (2015), 969–981. https://doi.org/10.1007/s11071-015-1921-8

19. H. Wang, X. Li, Infinitely many heteroclinic orbits of a complex Lorenz system, *Int. J. Bifurcation Chaos*, **27** (2017), 1750110. https://doi.org/10.1142/S0218127417501103

20. H. Wang, X. Li, A novel hyperchaotic system with infinitely many heteroclinic orbits coined, *Chaos, Solitons Fractals*, **106** (2018), 5–15. https://doi.org/10.1016/j.chaos.2017.10.029

21. H. Wang, F. Zhang, Bifurcations, ultimate boundedness and singular orbits in a unified hyperchaotic Lorenz-type system, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 1791–1820. https://doi.org/10.3934/dcdsb.2020003

22. H. Wang, H. Fan, J. Pan, Complex dynamics of a four-dimensional circuit system, *Int. J. Bifurcation Chaos*, **31** (2021), 2150208. https://doi.org/10.1142/S0218127421502084

23. H. Wang, G. Ke, J. Pan, F. Hu, H. Fan, Multitudinous potential hidden Lorenz-like attractors coined, *Eur. Phys. J. Spec. Top.*, **231** (2022), 359–368. https://doi.org/10.1140/epjs/s11734-021-00423-3

24. H. Wang, G. Ke, J. Pan, F. Hu, H. Fan, Q. Su, Two pairs of heteroclinic orbits coined in a new sub-quadratic Lorenz-like system, *Eur. Phys. J. B*, **96** (2023), 1–9. https://doi.org/10.1140/epjb/s10051-023-00491-5

25. Z. Li, G. Ke, H. Wang, J. Pan, F. Hu, Q. Su, Complex dynamics of a sub-quadratic Lorenz-like system, *Open Phys.*, **21** (2023), 20220251. https://doi.org/10.1515/phys-2022-0251

26. F. Tricomi, Integration of a differential equation presented in electrical engineering, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **2** (1933), 1–20.

27. G. A. Leonov, Fishing principle for homoclinic and heteroclinic trajectories, *Nonlinear Dyn.*, **78** (2014), 2751–2758. https://doi.org/10.1007/s11071-014-1622-8

28. G. Tigan, D. Turaev, Analytical search for homoclinic bifurcations in the Shimizu-Morioka model, *Physica D*, **240** (2011), 985–989. https://doi.org/10.1016/j.physd.2011.02.013

29. B. Feng, R. Hu, A survey on homoclinic and heteroclinic orbits, *Appl. Math. E-Notes*, **3** (2003), 16–37.

30. X. Zhang, G. Chen, Constructing an autonomous system with infinitely many chaotic attractors, *Chaos*, **27** (2017), 071101. https://doi.org/10.1063/1.4986356

31. X. Zhang, Boundedness of a class of complex Lorenz systems, *Int. J. Bifurcation Chaos*, **31** (2021), 2150101. https://doi.org/10.1142/S0218127421501017

32. H. E. Gilardi-Velázquez, R. J. Escalante-González, E. Campos, Emergence of a square chaotic attractor through the collision of heteroclinic orbits, *Eur. Phys. J. Spec. Top.*, **229** (2020), 1351–1360. https://doi.org/10.1140/epjst/e2020-900219-4

33. R. J. Escalante-González, E. Campos, Emergence of hidden attractors through the rupture of heteroclinic-like orbits of switched systems with self-excited attractors, *Complexity*, **2021** (2021), 1–24. https://doi.org/10.1155/2021/5559913

34. J. Llibre, M. Messias, P. R. D. Silva, Global dynamics in the Poincaré ball of the Chen system having invariant algebraic surface, *Int. J. Bifurcation Chaos*, **22** (2012), 1250154. https://doi.org/10.1142/S0218127412501544

35. Q. Yang, Y. Chen, Complex dynamics in the unified Lorenz-type system, *Int. J. Bifurcation Chaos*, **24** (2014), 1450055. https://doi.org/10.1142/S0218127414500552