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Research article

Existence, uniqueness and numerical solution of stochastic fractional differential equations with integer and non-integer orders

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Abstract: The parametrized approach is extended in this study to find solutions to differential equations with fractal, fractional, fractal-fractional, and piecewise derivatives with the inclusion of a stochastic component. The existence and uniqueness of the solution to the stochastic Atangana-Baleanu fractional differential equation are established using Caratheodory's existence theorem. For the solution of differential equations using piecewise differential operators, which take into account combining deterministic and stochastic processes utilizing certain significant mathematical tools such as fractal and fractal-fractional derivatives, the applicability of the parametrized technique is being examined. We discuss the crossover behaviors of the model obtained by including these operators and we present some illustrative examples for some problems with piecewise differential operators.

Keywords: Caratheodory's conditions; fractal-fractional differentiation; piecewise calculus; parametrized method

1. Introduction

Fractional analysis is a theory that started with Leibniz asking if there is a derivative of order $1/2$ of a function. This theory interested many researchers when different types of fractional derivatives were introduced. One well-known definition is the Riemann-Liouville fractional derivative where the power-law kernel is incorporated. Caputo [1] introduced a derivative with a modification on the

Riemann-Liouville fractional derivative [2] because it was useful in theory but not appropriate for solving real-life problems. These operators, which are used to model power law processes, have behavior that is both nonlocal and singular. Even though some processes are unique, another form of math is needed to describe processes that behave differently. Caputo and Fabrizio [3] have created a mathematical concept called a derivative with fading memory, which uses an exponential pattern. This derivative deals with processes that behave predictably and within a small area. However, we needed a derivative that is predictable but acts over a larger area. The Atangana-Baleanu fractional derivative [4] is a mathematical tool that meets this requirement, and it utilizes the Mittag-Leffler function. The fractal derivative or Hausdorff derivative [5] is a different kind of derivative used for measuring fractals in fractal geometry. Fractal derivatives were made to study how things spread in a strange way when normal ways of studying do not consider the fractal shape of the thing that things are spreading through. A fractal measure t changes its size in relation to t raised to the power of β. This type of derivative is only used in a specific area, unlike the fractional derivative, which is used in a similar way. Later, Atangana introduced fractal-fractional derivatives [6] by combining the concepts of fractal and fractional derivatives. Although there is no doubt that fractional differential operators are useful in modeling relevant processes [7–11], these operators cannot be used to model crossover processes such as from stochastic to power-law or from fading memory to stochastic [12–15]. Concluding that a new class of differential operators was needed for this, Atangana and Araz introduced piecewise differential operators [16], which can be created by including various differential operators to model such processes. These operators, which can be used to describe many processes, from modeling the different rates (or even stopping) of an individual's heartbeat over a period of time, to modeling the spread of a virus, first cumulatively and then daily, have become focus of attention for researchers.

In order to better understand and analyze the processes discussed, it is necessary to solve the equations that represent these processes. Because it is difficult to solve these equations using analytic methods when the operators mentioned above and the nonlinearity of the associated equations are involved, we have to use numerical methods to obtain solutions to such equations. The parametrized method, which deals with the approximation of a function with constants depending on a parameter, is one of the well-known numerical methods. While the parameterized method is presented in the literature [8–10] for classical differential equations, Atangana and Araz [17] extended this method to solve fractional and fractal-fractional differential equations. The parametrized method was compared with existing methods in the literature in [17] and it was shown that the method is more effective than other methods, especially when the parameter is close to 1.

However, in [17], the application of the relevant method to stochastic differential equations with fractional, fractal-fractional and piecewise derivatives [16] is not taken into account. Therefore, in this study, we present the derivation of this method for stochastic differential equations with fractional, fractal-fractional and piecewise derivatives. We employ the parametrized method to solve different types of equations obtained by incorporating these mathematical tools into differential equations. Before presenting the associated method, first the definitions of the above-mentioned fractional, fractal fractional and piecewise derivatives will be presented. In the following section, with the help of Carathéodory conditions [15,16], the existence and uniqueness of the solution of Atangana-Baleanu stochastic differential equations [18] will be investigated. In the remaining sections, in addition to the derivation of the parametrized method with these derivatives, some illustrative examples will be included.

2. Preliminaries

In this section, the definitions of fractional derivatives with power law behavior, fading memory and exhibiting power law behavior after fading memory, fractal-fractional derivatives and piecewise derivatives, which can be represented in different ways by including fractional and fractal-fractional derivatives, will be discussed.

The Caputo-Fabrizio fractional derivative [3] of the function $f(t) \in H^1(0,T)$ is defined by

$$ {}_0^{CF}D_t^{\alpha} f(t) = \frac{1}{1-\alpha}\int_0^t f'(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau, \tag{2.1}$$

where $0 < \alpha < 1$ and $H^1(0,T)$ describes the Hilbert space. The associated integral is given as

$$ {}_0^{CF}J_t^{\alpha} f(t) = (1-\alpha)f(t) + \alpha\int_0^t f(\tau)\,d\tau. \tag{2.2}$$

The Caputo fractional derivative [1] of the function $f(t) \in H^1(0,T)$ is defined by

$$ {}_0^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t f'(\tau)(t-\tau)^{-\alpha}\,d\tau, \tag{2.3}$$

where $0 < \alpha \leq 1$ and the Riemann-Liouville fractional derivative of the function $f(t) \in C(0,T)$ is defined by

$$ {}_0^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{-\alpha}\,d\tau. \tag{2.4}$$

The integral with power-law kernel [2] is given by

$$ {}_0^{RL}J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t f(\tau)(t-\tau)^{\alpha-1}\,d\tau. \tag{2.5}$$

The following formulas describe the Atangana-Baleanu fractional derivative [4], which has the crossover behavior from stretched exponential to power-law,

$$ {}_0^{ABC}D_t^{\alpha} f(t) = \frac{1}{1-\alpha}\int_0^t f'(\tau)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right]d\tau, \tag{2.6}$$

and

$$ {}_0^{ABR}D_t^{\alpha} f(t) = \frac{1}{1-\alpha}\frac{d}{dt}\int_0^t f(\tau)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right]d\tau. \tag{2.7}$$

The above operators are called Atangana-Baleanu fractional derivative in the Caputo sense and Atangana-Baleanu fractional derivative in the Riemann-Liouville sense [4], respectively. The associated integral is given by

$$ {}_0^{AB}J_t^{\alpha} f(t) = (1-\alpha)f(t) + \frac{\alpha}{\Gamma(\alpha)}\int_0^t f(\tau)(t-\tau)^{\alpha-1}\,d\tau. \tag{2.8}$$

The concept of fractal-fractional differentiation and integration has appeared previously with the idea of combining the fractal and fractional derivatives. The fractal-fractional derivative [6] with power-law kernel is defined by

$$ {}_{0}^{FFP}D_t^{\alpha,\beta} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt^{\beta}} \int_0^t f(\tau)(t-\tau)^{-\alpha} d\tau, \tag{2.9} $$

where the definition of fractal derivative [5] is

$$ \frac{d}{dt^{\beta}} f(t) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{\beta} - t_1^{\beta}}. \tag{2.10} $$

The associated fractal-fractional integral [6] with power-law kernel is given by

$$ {}_{0}^{FFP}J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \beta \tau^{\beta-1} f(\tau)(t-\tau)^{\alpha-1} d\tau. \tag{2.11} $$

The fractal-fractional derivative with Mittag-Leffler kernel [6] is defined by

$$ {}_{0}^{FFM}D_t^{\alpha,\beta} f(t) = \frac{1}{1-\alpha} \frac{d}{dt^{\beta}} \int_0^t f(\tau) E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right] d\tau \tag{2.12} $$

and the associated fractal-fractional integral is given by

$$ {}_{0}^{FFM}J_t^{\alpha} f(t) = (1-\alpha)\beta t^{\beta-1} f(t) + \frac{\alpha}{\Gamma(\alpha)} \int_0^t \beta \tau^{\beta-1} f(\tau)(t-\tau)^{\alpha-1} d\tau. \tag{2.13} $$

The fractal-fractional derivative with exponential decay kernel [6] is defined by

$$ {}_{0}^{FFE}D_t^{\alpha,\beta} f(t) = \frac{1}{1-\alpha} \frac{d}{dt^{\beta}} \int_0^t f(\tau) \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau \tag{2.14} $$

and the associated fractal-fractional integral is given by

$$ {}_{0}^{FFE}J_t^{\alpha} f(t) = (1-\alpha)\beta t^{\beta-1} f(t) + \alpha \int_0^t \beta \tau^{\beta-1} f(\tau) d\tau. \tag{2.15} $$

We now present the definitions of the piecewise derivative and integral operators, which made significant contribution to literature [16].

The piecewise derivative with classical and fractional derivative with power-law kernel such that it can be taken as [16]

$$ {}_{0}^{PRL}D_t^{\alpha} y(t) = \begin{cases} y'(t) \text{ if } 0 \le t \le t_0 \\ {}_{t_0}^{RL}D_t^{\alpha} y(t) \text{ if } t_0 \le t \le T \end{cases} \tag{2.16} $$

where ${}_{0}^{PRL}D_t^{\alpha}$ represents the classical derivative within $0 \le t \le t_0$ and the Riemann-Liouville fractional derivative within $t_0 \le t \le T$.

The piecewise with Caputo derivative is given as [16]

$$ {}_{0}^{PC}D_t^{\alpha} y(t) = \begin{cases} y'(t) \text{ if } 0 \le t \le t_0 \\ {}_{t_0}^{C}D_t^{\alpha} y(t) \text{ if } t_0 \le t \le T \end{cases} \tag{2.17} $$

where the function $y(t)$ is continuous but not necessarily differentiable in $[t_0, T]$. Here, ${}_{0}^{PRL}D_t^{\alpha}$ represents the classical derivative on $0 \le t \le t_0$ and the Caputo fractional derivative [1] on $t_0 \le t \le T$. The associated piecewise integral of y is given as [16]

$$ {}^{PPL}I_t y\ (t) = \begin{cases} \int_0^{t_0} y\ (\tau)\, d\tau \text{ if } 0 \le t \le t_0 \\ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} y\ (\tau)(t-\tau)^{\alpha-1}\, d\tau \text{ if } t_0 \le t \le T \end{cases} \tag{2.18} $$

where ${}_0^{PPL}I_t^{\alpha}$ represents the classical integral on $0 \le t \le t_0$ and the integral with power-law kernel on $t_0 \le t \le T$.

The piecewise derivative with classical derivative and exponential decay kernel is given as [16]

$$ {}_0^{PCF}D_t^{\alpha} y\ (t) = \begin{cases} y'(t) \text{ if } 0 \le t \le t_0 \\ {}_{t_0}^{CF}D_t^{\alpha} y\ (t) \text{ if } t_0 \le t \le T \end{cases} \tag{2.19} $$

where ${}_0^{PCF}D_t^{\alpha}$ is the classical derivative on $0 \le t \le t_0$ and the Caputo-Fabrizio fractional derivative [3] on $t_0 \le t \le T$. Here, it is assumed that the function $y(t)$ is differentiable. A piecewise integral is given as [16]

$$ {}^{PCF}I_t y\ (t) = \begin{cases} \int_0^{t_0} y\ (\tau)\, d\tau \text{ if } 0 \le t \le t_0 \\ \frac{1-\alpha}{M(\alpha)} y\ (t) + \frac{\alpha}{M(\alpha)} \int_{t_0}^{t} y\ (\tau)\, d\tau \text{ if } t_0 \le t \le T \end{cases}. \tag{2.20} $$

The piecewise derivative with classical derivative and Mittag-Leffler kernel is defined by [16]

$$ {}_0^{PAB}D_t^{\alpha} y\ (t) = \begin{cases} y'(t) \text{ if } 0 \le t \le t_0 \\ {}_{t_0}^{ABC}D_t^{\alpha} y\ (t) \text{ if } t_0 \le t \le T \end{cases} \tag{2.21} $$

where ${}_0^{PAB}D_t^{\alpha}$ represents the classical derivative on $0 \le t \le t_0$ and the Atangana-Baleanu fractional derivative [4] on $t_0 \le t \le T$. The associated piecewise integral is given as [16]

$$ {}^{PAB}I_t y\ (t) = \begin{cases} \int_0^{t_0} y\ (\tau)\, d\tau \text{ if } 0 \le t \le t_0 \\ (1-\alpha) y\ (t) + \frac{\alpha}{\Gamma(\alpha)} \int_{t_0}^{t} y\ (\tau)(t-\tau)^{\alpha-1}\, d\tau \text{ if } t_0 \le t \le T \end{cases}. \tag{2.22} $$

Lemma 1. *(The generalization of the Gronwall inequality) Assume that* $b \ge 0, \alpha > 0$, *and* $x(t)$ *is a nonnegative function locally integrable on* $0 \le t < T$, *and assume that* $y(t)$ *is nonnegative and locally integrable on* $0 \le t < T$ *with*

$$ y\ (t) \le x(t) + b \int_0^t y\ (\tau)(t-\tau)^{\alpha-1}\, d\tau. \tag{2.23} $$

Then,

$$ y\ (t) \le x(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} y(\tau)(t-\tau)^{n\alpha-1} x(\tau) \right] d\tau. \tag{2.24} $$

Definition 1. *(Stirling formula) The Stirling formula for the Gamma function is formulated by*

$$ \Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}}. \tag{2.25} $$

3. Caratheodory's theory for existence and uniqueness for a general Cauchy problem with stochastic Atangana-Baleanu fractional derivative

In this section, we prove the existence and uniqueness of the solution for the Atangana-Baleanu stochastic differential equation [18] by employing Carathéodory's existence theory [19, 20], which is a more general version of Peano's existence theorem. It is worth noting that the existence and uniqeness of the solution for stochastic differential equations with the Caputo fractional derivative is presented in [21]. Here, we will examine the existence and uniqueness of the stochastic differential equation with Atangana-Baleanu fractional derivative. The differential equation under investigation is represented in the form:

$$\begin{aligned} {}_0^{AB}D_t^{\alpha} y(t) &= f_1(t,y)\,dt + \sigma y(t)\,dB(t), \ t \geq 0 \\ y(t_0) &= y_0 \end{aligned} \tag{3.1}$$

under the conditions

E1) For all $y, \bar{y} \in H$, there is a constant $k > 0$ such that

$$|f_1(t,y) - f_1(t,\bar{y})|^2, \ |f_2(t,y) - f_2(t,\bar{y})|^2 \leq k\,|y - \bar{y}|^2, \ t \geq 0. \tag{3.2}$$

E2) For all $y \in H$, there is a constant $\bar{k} > 0$ such that

$$|f_1(t,y)|^2, \ |f_2(t,y)|^2 \leq \bar{k}\left(1 + |y|^2\right), \ t \geq 0 \tag{3.3}$$

where H is a Banach space. Note that conditions E1 and E2 are known as the Lipschitz condition and the growth condition, respectively.

Theorem 1. *For each $y_0 \in L_2(\Omega, H)$, Eq (26) has a unique mild solution $y \in C\left([0,T], L^2(\Omega,H)\right) = S$ such that*

$$\sup_{0 \leq t \leq T} E\,|y|^2 < \infty.$$

Proof. For the proof, we will use the contraction mapping principle. Before proceeding with the proof, we define the norm

$$\|\eta\|_{\gamma}^2 = \sup_{0 \leq t \leq T} E\,|\eta(t)|^2 \tag{3.4}$$

where E denotes the expectation.

For any $t \in [0,T]$ and $y \in S$, we define the mapping subject to $\Omega = C\left([0,T], L^2(\Omega,H)\right) \to C\left([0,T], L^2(\Omega,H)\right)$

$$\begin{aligned} (\Lambda y)(t) &= y_0 + (1-\alpha) f_1(t,y) + (1-\alpha)\sigma y(t) B'(t) \\ &\quad + \frac{\alpha}{\Gamma(\alpha)} \int_0^t f_1(s,y)(t-s)^{\alpha-1}\,ds + \frac{\alpha\sigma}{\Gamma(\alpha)} \int_0^t y(s)(t-s)^{\alpha-1}\,dB(s). \end{aligned} \tag{3.5}$$

Thus, we write

$$E\,|(\Lambda y)(t) - (\Lambda \bar{y})(t)|^2 = E \left| \begin{array}{c} (1-\alpha)(f_1(t,y) - f_1(t,\bar{y})) \\ +(1-\alpha)\,\sigma\,(y(t) - \bar{y}(t))\,B'(t) \\ +\frac{\alpha}{\Gamma(\alpha)} \int_0^t (f_1(s,y) - f_1(s,\bar{y}))(t-s)^{\alpha-1}\,ds \\ +\frac{\alpha\sigma}{\Gamma(\alpha)} \int_0^t (y(s) - \bar{y}(s))(t-s)^{\alpha-1}\,dB(s) \end{array} \right|^2. \tag{3.6}$$

Taking $2\alpha - 1 > 0$, by the Cauchy-Schwartz inequality, Ito's isometry formula and the Lipschitz condition [22], we have

$$\begin{aligned} E\left|(\Lambda y)(t) - (\Lambda \bar{y})(t)\right|^2 &\leq 4(1-\alpha)^2 k\sigma\left(1 + |B'|^2\right) E\left|y - \bar{y}\right|^2 \\ &\quad + (T+1)\frac{4\alpha^2 k}{\Gamma^2(\alpha)}\int_0^t E\left|y - \bar{y}\right|^2 (t-s)^{2\alpha-2}\,ds \\ &\leq 4(1-\alpha)^2 \sigma k\left(1 + |B'|^2\right)\|y - \bar{y}\|_\gamma \\ &\quad + (T+1)\frac{4\alpha^2 k}{\Gamma^2(\alpha)}\frac{t^{2\alpha-1}}{(2\alpha-1)}\|y - \bar{y}\|_\gamma \\ &\leq 4\sigma(1-\alpha)^2 k\left(1 + \sup_{t\in[0,T]}|B'|^2\right)\|y - \bar{y}\|_\gamma \\ &\quad + (T+1)\frac{4\alpha^2 k}{\Gamma^2(\alpha)}\frac{t^{2\alpha-1}}{(2\alpha-1)}\|y - \bar{y}\|_\gamma \\ &\leq 4\sigma(1-\alpha)^2 k\left(1 + \|B'\|_\infty\right)\|y - \bar{y}\|_\gamma \\ &\quad + (T+1)\frac{4\sigma\alpha^2 k}{\Gamma^2(\alpha)}\frac{t^{2\alpha-1}}{(2\alpha-1)}\|y - \bar{y}\|_\gamma \\ &\leq \tilde{k}\|y - \bar{y}\|_\gamma, \end{aligned} \tag{3.7}$$

where

$$\tilde{k} = 4\sigma(1-\alpha)^2 k\left(1 + \|B'\|_\infty\right) + (T+1)\frac{4\sigma\alpha^2 k}{\Gamma^2(\alpha)}\frac{t^{2\alpha-1}}{(2\alpha-1)}. \tag{3.8}$$

Using the generalized Gronwall inequality [23], we write

$$\begin{aligned} E\left|\left(\Lambda^2 y\right)(t) - \left(\Lambda^2 \bar{y}\right)(t)\right| &\leq 4\sigma(1-\alpha)^2 k\left(1 + \|B'\|_\infty\right) E\left|\Lambda y - \Lambda \bar{y}\right|^2 \\ &\quad + \frac{4\sigma\alpha^2 k(T+1)}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2} E\left|\Lambda y - \Lambda \bar{y}\right|^2 ds \end{aligned} \tag{3.9}$$

$$\leq 4\sigma(1-\alpha)^2 k\left(1 + \|B'\|_\infty\right)\left[\begin{array}{c} 4(1-\alpha)^2 k\left(1 + \|B'\|_\infty\right) \\ + (T+1)\frac{4\alpha^2 k}{\Gamma^2(\alpha)}\frac{t^{2\alpha-1}}{(2\alpha-1)} \end{array}\right] \tag{3.10}$$

$$+ \frac{4\sigma\alpha^2 k(T+1)}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\left[\begin{array}{c} 4(1-\alpha)^2 k\left(1 + \|B'\|_\infty\right) \\ + (T+1)\frac{4\alpha^2 k}{\Gamma^2(\alpha)}\frac{s^{2\alpha-1}}{(2\alpha-1)} \end{array}\right] ds \tag{3.11}$$

$$\leq \left[\begin{array}{c} \left[\begin{array}{c} \left(4\sigma(1-\alpha)^2 k\left(1 + \|B'\|_\infty\right)\right)^2 \\ + (T+1)\frac{4\sigma\alpha^2 k}{\Gamma^2(\alpha)}\frac{T^{2\alpha-1}}{(2\alpha-1)}\left(4(1-\alpha)^2\left(1 + \|B'\|_\infty\right)\right) \end{array}\right] \\ + \left[\begin{array}{c} \left(4\sigma(1-\alpha)^2 k\left(1 + \|B'\|_\infty\right)\right)\left(\frac{4\alpha^2 k(T+1)}{\Gamma^2(\alpha)}\right)\frac{T^{2\alpha-1}}{(2\alpha-1)} \\ + \left(\frac{4\sigma\alpha^2 k(T+1)}{\Gamma^2(\alpha)}\right)^2\frac{\Gamma^2(2\alpha-1)}{\Gamma(4\alpha-2)}\frac{T^{4\alpha-2}}{(2\alpha-1)} \end{array}\right] \end{array}\right]\|y - \bar{y}\|_\gamma.$$

By the induction formula for n, we can then write

$$E\left|(\Lambda^n y)(t) - (\Lambda^n \bar{y})(t)\right| \leq \left[\begin{array}{c} \left(4\sigma(1-\alpha)^2\left(1 + \|B'\|_\infty\right)\right)^n \\ + \left(4\sigma(1-\alpha)^2\left(1 + \|B'\|_\infty\right)(T+1)\frac{4\alpha^2 k}{\Gamma^2(\alpha)}\frac{T^{2\alpha-1}}{(2\alpha-1)}\right)^{n-1} \\ + \left(4\sigma(1-\alpha)^2\left(1 + \|B'\|_\infty\right)\right)^{n-1}\left(\frac{4\sigma\alpha^2 k(T+1)}{\Gamma^2(\alpha)}\right)\frac{T^{n(2\alpha-1)}}{\Gamma(n(2\alpha-1))} \\ + \left(\frac{4\sigma\alpha^2 k(T+1)}{\Gamma^2(\alpha)}\right)^n\frac{T^{n(2\alpha-1)}}{(2\alpha-1)}\frac{\Gamma^n(2\alpha-1)}{\Gamma(n(2\alpha-1))} \end{array}\right]\|y - \bar{y}\|_\gamma \tag{3.12}$$

$$\leq \quad L\,\|y-\bar{y}\|_{\gamma}$$

where

$$L=\begin{bmatrix} \left(4\sigma(1-\alpha)^2(1+\|B'\|_\infty)\right)^n \\ +\left(4\sigma(1-\alpha)^2(1+\|B'\|_\infty)(T+1)\frac{4\alpha^2k}{\Gamma^2(\alpha)}\frac{T^{2\alpha-1}}{(2\alpha-1)}\right)^{n-1} \\ +\left(4\sigma(1-\alpha)^2(1+\|B'\|_\infty)\right)^{n-1}\left(\frac{4\sigma\alpha^2k(T+1)}{\Gamma^2(\alpha)}\right)\frac{T^{n(2\alpha-1)}}{\Gamma(n(2\alpha-1))} \\ +\left(\frac{4\sigma\alpha^2k(T+1)}{\Gamma^2(\alpha)}\right)^n\frac{T^{n(2\alpha-1)}}{(2\alpha-1)}\frac{\Gamma^n(2\alpha-1)}{\Gamma(n(2\alpha-1))} \end{bmatrix}. \tag{3.13}$$

To prove the theorem holds, we will show that $L<1$ for sufficient large n. Let us consider the following series of positive terms

$$\sum_{n=1}^{\infty}\begin{bmatrix} \left(4\sigma(1-\alpha)^2(1+\|B'\|_\infty)\right)^n \\ +\left(4\sigma(1-\alpha)^2(1+\|B'\|_\infty)(T+1)\frac{4\alpha^2k}{\Gamma^2(\alpha)}\frac{T^{2\alpha-1}}{(2\alpha-1)}\right)^{n-1} \\ +\left(4\sigma(1-\alpha)^2(1+\|B'\|_\infty)\right)^{n-1}\left(\frac{4\sigma\alpha^2k(T+1)}{\Gamma^2(\alpha)}\right)\frac{T^{n(2\alpha-1)}}{\Gamma(n(2\alpha-1))} \\ +\left(\frac{4\sigma\alpha^2k(T+1)}{\Gamma^2(\alpha)}\right)^n\frac{T^{n(2\alpha-1)}}{(2\alpha-1)}\frac{\Gamma^n(2\alpha-1)}{\Gamma(n(2\alpha-1))} \end{bmatrix}. \tag{3.14}$$

Using the d'Alembert discriminant method

$$\lim_{n\to\infty}\frac{\left(\frac{4\alpha^2k(T+1)}{\Gamma^2(\alpha)}\right)^{n+1}\frac{T^{(n+1)(2\alpha-1)}}{(2\alpha-1)}\frac{\Gamma^{n+1}(2\alpha-1)}{\Gamma((n+1)(2\alpha-1))}}{\left(\frac{4\sigma\alpha^2k(T+1)}{\Gamma^2(\alpha)}\right)^n\frac{T^{n(2\alpha-1)}}{(2\alpha-1)}\frac{\Gamma^n(2\alpha-1)}{\Gamma(n(2\alpha-1))}}<1 \tag{3.15}$$

which is equivalent to

$$\lim_{n\to\infty}\frac{\left(\frac{4\sigma\alpha^2k(T+1)}{\Gamma^2(\alpha)}\right)T^{(2\alpha-1)}\Gamma(2\alpha-1)\,\Gamma(n(2\alpha-1))}{\Gamma((n+1)(2\alpha-1))}<1. \tag{3.16}$$

Using the Stirling formula [21], we have the following for last term

$$\lim_{n\to\infty}\begin{bmatrix} \left(\frac{4\sigma\alpha^2k(T+1)}{\Gamma^2(\alpha)}\right)\Gamma(2\alpha-1)\,T^{(2\alpha-1)}e^{(2\alpha-1)} \\ \frac{\sqrt{n+1}}{\sqrt{n}}\left(\frac{n}{n+1}\right)^{n(2\alpha-1)}\frac{1}{((n+1)(2\alpha-1))^{(2\alpha-1)}} \end{bmatrix}=0 \tag{3.17}$$

and knowing that $\alpha<1$, we can have

$$\lim_{n\to\infty}\begin{bmatrix} \left(4\sigma(1-\alpha)^2(1+\|B'\|_\infty)\right)^n \\ +\left(4\sigma(1-\alpha)^2(1+\|B'\|_\infty)(T+1)\frac{4\alpha^2k}{\Gamma^2(\alpha)}\frac{T^{2\alpha-1}}{(2\alpha-1)}\right)^{n-1} \\ +\left(4\sigma(1-\alpha)^2(1+\|B'\|_\infty)\right)^{n-1}\left(\frac{4\sigma\alpha^2k(T+1)}{\Gamma^2(\alpha)}\right)\frac{T^{n(2\alpha-1)}}{\Gamma(n(2\alpha-1))} \\ +\left(\frac{4\sigma\alpha^2k(T+1)}{\Gamma^2(\alpha)}\right)^n\frac{T^{n(2\alpha-1)}}{(2\alpha-1)}\frac{\Gamma^n(2\alpha-1)}{\Gamma(n(2\alpha-1))} \end{bmatrix}=0. \tag{3.18}$$

This guarantees that $L<1$ holds. This proves that $\Lambda y(t)$ is a contraction mapping, which completes the proof.

4. Parametrized method for a general Cauchy problem with stochastic fractional derivatives

In this section, we develop the parametrized approach to numerically solving differential equations with fractional derivatives that incorporate stochastic components. Before presenting the extension of the method to the solutions of different differential equations, we shall recall the formulation of the parametrized approach [17, 24–26]. The approach is formulated by the following:

$$\varphi_1(t,y) \approx \left[\left(1-\frac{1}{2\xi}\right)\varphi_1\left(t_k,y^k\right)+\frac{1}{2\xi}\varphi_1\left(t_{k+1},\tilde{y}^{k+1}\right)\right]. \tag{4.1}$$

4.1. Parametrized method for a general Cauchy problem with stochastic component

To derive the associated method, in this subsection, we consider a general Cauchy problem with stochastic component given by

$$dy(t) = \varphi_1(t,y)\,dt + \sigma y(t)\,dB(t). \tag{4.2}$$

We convert the above into an integral equation, by applying on both sides the classical integral

$$y(t) = y(0) + \int_0^t \varphi_1(\tau,y)\,d\tau + \int_0^t \sigma y(\tau)\,dB(\tau). \tag{4.3}$$

At $t = t_{k+1}$, we write

$$y(t_{k+1}) = y(0) + \int_0^{t_{k+1}} \varphi_1(\tau,y)\,d\tau + \int_0^{t_{k+1}} \sigma y(\tau)\,dB(\tau) \tag{4.4}$$

and at $t = t_k$

$$y(t_k) = y(0) + \int_0^{t_k} \varphi_1(\tau,y)\,d\tau + \int_0^{t_k} \sigma y(\tau)\,dB(\tau). \tag{4.5}$$

Substracting these two equalities gives

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} \varphi_1(\tau,y)\,d\tau + \int_{t_k}^{t_{k+1}} \sigma y(\tau)\,dB(\tau). \tag{4.6}$$

The function $\varphi_1(\tau,y)$ can be approximated by using the parametrized approach [17, 24–26] presented earlier. After simplification, we have the predictor-corrector formula [27]

$$\begin{aligned} y^{k+1} &= y^k + h\left[\left(1-\frac{1}{2\xi}\right)\varphi_1\left(t_k,y^k\right)+\frac{1}{2\xi}\varphi_1\left(t_{k+1},\tilde{y}^{k+1}\right)\right] \\ &\quad +\sigma y(c_k)\left(B(t_{k+1})-B(t_k)\right), \end{aligned} \tag{4.7}$$

where $c_k \in [t_k, t_{k+1}]$ and the predictor term

$$\tilde{y}^{k+1} = y^k + h\varphi_1\left(t_k,y^k\right). \tag{4.8}$$

4.2. Parametrized method for a general Cauchy problem with stochastic Caputo-Fabrizio fractional derivative

In this part, we will present the derivation of the parametrized method for a general nonlinear problem whose differential operator is the Caputo-Fabrizio derivative [3]. This case of nonlinear differential equations is of practical importance as it allows us to understand memory decay processes in various fields of science, technology and engineering. A stochastic version of such a differential equation is provided by

$$\begin{cases} {}_0^{CF}D_t^{\alpha} y(t) = \varphi_1(t,y) + \sigma y(t)\,dB(t) \\ y(0) = y_0 \end{cases} \tag{4.9}$$

where $B(t)$ is the Brownian function and σ is the stochastic constant. The aforementioned equation will then be transformed into an integral equation by applying on both sides the integral associated with the Caputo-Fabrizio derivative [3] in order to obtain

$$\begin{aligned} y(t) &= y(0) + (1-\alpha)\varphi_1(t,y) + (1-\alpha)\sigma y(t)\,dB(t) \\ &\quad +\alpha\int_0^t \varphi_1(\tau,y)\,d\tau + \alpha\int_0^t \sigma y(\tau)\,dB(\tau). \end{aligned} \tag{4.10}$$

At $t = t_k$ and $t = t_{k+1}$, we have

$$\begin{aligned} y(t_{k+1}) &= y(t_k) + (1-\alpha)\left(\varphi_1\left(t_{k+1}, y^{k+1}\right) - \varphi_1\left(t_k, y^k\right)\right) \\ &\quad + (1-\alpha)\sigma y(c_{k+1})(B(t_{k+1}) - B(t_k)) \\ &\quad +\alpha\int_{t_k}^{t_{k+1}} \varphi_1(\tau,y)\,d\tau + \alpha\int_{t_k}^{t_{k+1}} \sigma y(\tau)\,dB(\tau), \end{aligned} \tag{4.11}$$

where $c_k \in [t_k, t_{k+1}]$. Since the function $\varphi_1(\tau, y)$ is nonlinear, the component with integral can be approximated using the parametrized approach [17, 24–26] as follows:

$$\begin{aligned} y^{k+1} &= y^k + (1-\alpha)\left(\varphi_1\left(t_{k+1}, y^{k+1}\right) - \varphi_1\left(t_k, y^k\right)\right) \\ &\quad + (1-\alpha)\sigma y(c_{k+1})(B(t_{k+1}) - B(t_k)) \\ &\quad +\alpha\int_{t_k}^{t_{k+1}} \left[\left(1 - \frac{1}{2\xi}\right)\varphi_1\left(t_k, y^k\right) + \frac{1}{2\xi}\varphi_1\left(t_{k+1}, \tilde{y}^{k+1}\right)\right] d\tau \\ &\quad +\alpha\int_{t_k}^{t_{k+1}} \sigma y(\tau)\,dB(\tau). \end{aligned} \tag{4.12}$$

We know that the above is implicit, and thus we replace the term y^{k+1} with $\tilde{y}^{k+1}$ to obtain

$$\begin{aligned} y^{k+1} &= y^k + (1-\alpha)\left(\varphi_1\left(t_{k+1}, \tilde{y}^{k+1}\right) - \varphi_1\left(t_k, y^k\right)\right) \\ &\quad + (1-\alpha)\sigma y(c_{k+1})(B(t_{k+1}) - B(t_k)) \\ &\quad +\alpha h\left[\left(1 - \frac{1}{2\xi}\right)\varphi_1\left(t_k, y^k\right) + \frac{1}{2\xi}\varphi_1\left(t_{k+1}, \tilde{y}^{k+1}\right)\right] \\ &\quad +\alpha\sigma y(c_k)(B(t_{k+1}) - B(t_k)). \end{aligned} \tag{4.13}$$

The predictor formula is determined by the following:

$$\tilde{y}^{k+1} = y^0 + {}_{t_0}^{CF}I_{t_{k+1}}^{\alpha}\left[\varphi_1(t,y)\right] \tag{4.14}$$

$$= y^0 + (1-\alpha)\varphi_1(t_k, y_k) + \alpha\int_{t_0}^{t_{k+1}} \varphi_1(\tau, y)\, d\tau$$
$$= y^0 + (1-\alpha)\varphi_1\left(t_k, y^k\right) + \alpha\sum_{n=0}^{k}\int_{t_n}^{t_{n+1}} \varphi_1(\tau, y)\, d\tau$$
$$= y^0 + (1-\alpha)\varphi_1\left(t_k, y^k\right) + \alpha h\sum_{n=0}^{k} \varphi_1(t_n, y^n).$$

We need to emphasize that the predictor-corrector technique [27] is required because of both the parameterized techniques and the first part of the Caputo-Fabrizio fractional integral [3].

4.3. Parametrized method for a general Cauchy problem with stochastic Caputo fractional derivative

In this section, we deal with the numerical solution of a general Cauchy problem with stochastic Caputo derivative [1]

$$\begin{cases} {}_0^C D_t^{\alpha} y(t) = \varphi_1(t, y) + \sigma y(t)\, dB(t), \\ y(0) = y_0 \end{cases}. \tag{4.15}$$

Applying the integral with power-law kernel [2], we have

$$\begin{aligned} y(t) &= y(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \varphi_1(\tau, y)(t-\tau)^{\alpha-1}\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t \sigma y(\tau)(t-\tau)^{\alpha-1}\, dB(\tau). \end{aligned} \tag{4.16}$$

At $t = t_{k+1}$ we have

$$\begin{aligned} y(t_{k+1}) &= y(0) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}} \varphi_1(\tau, y)(t_{k+1}-\tau)^{\alpha-1}\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}} \sigma y(\tau)(t_{k+1}-\tau)^{\alpha-1}\, dB(\tau). \end{aligned} \tag{4.17}$$

The above can be arranged as follows:

$$\begin{aligned} y(t_{k+1}) &= y(0) + \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{k}\int_{t_n}^{t_{n+1}} \varphi_1(\tau, y)(t_{k+1}-\tau)^{\alpha-1}\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{k}\int_{t_n}^{t_{n+1}} \sigma y(\tau)(t_{k+1}-\tau)^{\alpha-1}\, dB(\tau). \end{aligned} \tag{4.18}$$

The function $\varphi_1(\tau, y)$ can be approximated by using the parametrized approach [17, 24–26] within $[t_k, t_{k+1}]$, and we have the following corrector formula with predictor term:

$$y^{k+1} = y^0 + \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{k}\int_{t_n}^{t_{n+1}} \left[\left(1-\frac{1}{2\xi}\right)\varphi_1(t_n, y^n) + \frac{1}{2\xi}\varphi_1\left(t_{n+1}, \tilde{y}^{n+1}\right)\right](t_{k+1}-\tau)^{\alpha-1}\, d\tau \tag{4.19}$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{k}\sigma y(c_n)(B(t_{n+1})-B(t_n))\int_{t_n}^{t_{n+1}}(t_{k+1}-\tau)^{\alpha-1}d\tau.$$

Then, we have

$$\begin{aligned} y^{k+1} &= y^0+\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{k}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1(t_n,y^n)+\frac{1}{2\xi}\varphi_1\left(t_{n+1},\tilde{y}^{n+1}\right)\right] \\ &\times\int_{t_n}^{t_{n+1}}(t_{k+1}-\tau)^{\alpha-1}d\tau \\ &+\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{k}\sigma y(c_n)(B(t_{n+1})-B(t_n))\int_{t_n}^{t_{n+1}}(t_{k+1}-\tau)^{\alpha-1}d\tau, \end{aligned} \tag{4.20}$$

and from here, we write

$$\begin{aligned} y^{k+1} &= y^0+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{n=0}^{k}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1(t_n,y^n)+\frac{1}{2\xi}\varphi_1\left(t_{n+1},\tilde{y}^{n+1}\right)\right] \\ &\times\left[(k-n+1)^\alpha-(k-n)^\alpha\right] \\ &+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{n=0}^{k}\sigma y(c_n)(B(t_{n+1})-B(t_n))\left[(k-n+1)^\alpha-(k-n)^\alpha\right]. \end{aligned} \tag{4.21}$$

The predictor component on the right side of the equation is calculated using the Euler approximation as follows:

$$\begin{aligned} \tilde{y}^{k+1} &= y^0+{}_0^C I_{t_{k+1}}^\alpha\left[\varphi_1(t,y)\right] \\ &= y^0+\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}}\varphi_1(\tau,y)(t_{k+1}-\tau)^{\alpha-1}d\tau \\ &= y^0+\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{k}\int_{t_n}^{t_{n+1}}\varphi_1(\tau,y)(t_{k+1}-\tau)^{\alpha-1}d\tau \\ &= y^0+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{n=0}^{k}\varphi_1(t_n,y^n)\left[(k-n+1)^\alpha-(k-n)^\alpha\right]. \end{aligned} \tag{4.22}$$

4.4. Parametrized method for a general Cauchy problem with stochastic Atangana-Baleanu fractional derivative

In this section, we devote our attention to the derivation of the parametrized method for solving the stochastic Cauchy problem with Atangana–Baleanu fractional derivative [4]

$$\begin{cases} {}_0^{AB}D_t^\alpha y(t)=\varphi_1(t,y)+\sigma y(t)\,dB(t), \\ y(0)=y_0 \end{cases}. \tag{4.23}$$

We convert the above problem into an integral equation by applying on both sides the Atangana–Baleanu fractional integral [4]

$$y(t) = (1-\alpha)\varphi_1(t,y)+\frac{\alpha}{\Gamma(\alpha)}\int_0^t\varphi_1(\tau,y)(t-\tau)^{\alpha-1}d\tau \tag{4.24}$$

$$+ (1-\alpha)\varphi_2(t,y)\,dB(t) + \frac{\alpha}{\Gamma(\alpha)}\int_0^t \sigma y(\tau) B'(\tau)(t-\tau)^{\alpha-1}\,d\tau.$$

At $t = t_{k+1}$, we write the following:

$$\begin{aligned} y(t_{k+1}) &= y(0) + (1-\alpha)\varphi_1\left(t_{k+1}, y^{k+1}\right) + \frac{\alpha}{\Gamma(\alpha)}\int_0^{t_{k+1}} \varphi_1(\tau,y)(t_{k+1}-\tau)^{\alpha-1}\,d\tau \\ &+ (1-\alpha)\sigma y(c_{k+1})(B(t_{k+1}) - B(t_k)) \\ &+ \frac{\alpha}{\Gamma(\alpha)}\int_0^{t_{k+1}} \sigma y(\tau)(t_{k+1}-\tau)^{\alpha-1}\,dB(\tau). \end{aligned} \tag{4.25}$$

The above can be arranged as follows:

$$\begin{aligned} y(t_{k+1}) &= y(0) + (1-\alpha)\varphi_1\left(t_{k+1}, y^{k+1}\right) + \frac{\alpha}{\Gamma(\alpha)}\sum_{n=0}^{k}\int_{t_n}^{t_{n+1}} \varphi_1(\tau,y)(t_{k+1}-\tau)^{\alpha-1}\,d\tau \\ &+ (1-\alpha)\sigma y(c_{k+1})(B(t_{k+1}) - B(t_k)) \\ &+ \frac{\alpha}{\Gamma(\alpha)}\sum_{n=0}^{k}\int_{t_n}^{t_{n+1}} \sigma y(\tau)(t_{k+1}-\tau)^{\alpha-1}\,dB(\tau). \end{aligned} \tag{4.26}$$

Based on the idea of approximating the right-hand side of the equation by the parametrized approach [17, 24–26], the above can be arranged as

$$\begin{aligned} y^{k+1} &= y^0 + (1-\alpha)\varphi_1\left(t_{k+1}, y^{k+1}\right) \\ &+ \frac{\alpha}{\Gamma(\alpha)}\sum_{n=0}^{k}\int_{t_n}^{t_{n+1}} \left[\left(1-\frac{1}{2\xi}\right)\varphi_1(t_n,y^n) + \frac{1}{2\xi}\varphi_1\left(t_{n+1},\tilde{y}^{n+1}\right)\right](t_{k+1}-\tau)^{\alpha-1}\,d\tau \\ &+ (1-\alpha)\sigma y(c_{k+1})(B(t_{k+1}) - B(t_k)) \\ &+ \frac{\alpha}{\Gamma(\alpha)}\sum_{n=0}^{k} \sigma y(c_k)(B(t_{k+1}) - B(t_k))\int_{t_n}^{t_{n+1}} (t_{k+1}-\tau)^{\alpha-1}\,d\tau. \end{aligned} \tag{4.27}$$

Then, we have

$$\begin{aligned} y^{k+1} &= y^0 + (1-\alpha)\varphi_1\left(t_{k+1}, y^{k+1}\right) \\ &+ \frac{\alpha}{\Gamma(\alpha)}\sum_{n=0}^{k}\left[\begin{array}{c} \left(1-\frac{1}{2\xi}\right)\varphi_1(t_n,y^n) \\ +\frac{1}{2\xi}\varphi_1\left(t_{n+1},\tilde{y}^{n+1}\right) \end{array}\right]\int_{t_n}^{t_{n+1}} (t_{k+1}-\tau)^{\alpha-1}\,d\tau \\ &+ (1-\alpha)\sigma y(c_{k+1})(B(t_{k+1}) - B(t_k)) \\ &+ \frac{\alpha}{\Gamma(\alpha)}\sum_{n=0}^{k} \sigma y(c_k)(B(t_{k+1}) - B(t_k)) \\ &\times \int_{t_n}^{t_{n+1}} (t_{k+1}-\tau)^{\alpha-1}\,d\tau \end{aligned} \tag{4.28}$$

and from here we write

$$y^{k+1} = y^0 + (1-\alpha)\varphi_1\left(t_{k+1}, y^{k+1}\right) \tag{4.29}$$

$$
\begin{aligned}
&+\frac{\alpha}{\Gamma(\alpha)}\sum_{n=0}^{k}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1(t_n,y^n)+\frac{1}{2\xi}\varphi_1\left(t_{n+1},\tilde{y}^{n+1}\right)\right]\\
&\times\frac{(t_{k+1}-t_n)^{\alpha}-(t_{k+1}-t_{n+1})^{\alpha}}{\alpha}\\
&+(1-\alpha)\sigma y(c_{k+1})(B(t_{k+1})-B(t_k))\\
&+\frac{h^{\alpha-1}}{\Gamma(\alpha)}\sum_{n=0}^{k}\sigma y(c_n)(B(t_{n+1})-B(t_n))\left[(k-n+1)^{\alpha}-(k-n)^{\alpha}\right].
\end{aligned}
$$

After the simplification, the above is arranged as:

$$
\begin{aligned}
y^{k+1} &= y^0+(1-\alpha)\varphi_1\left(t_{k+1},y^{k+1}\right) \qquad (4.30)\\
&+\frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{n=0}^{k}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1(t_n,y^n)+\frac{1}{2\xi}\varphi_1\left(t_{n+1},\tilde{y}^{n+1}\right)\right]\\
&\times\left[(k-n+1)^{\alpha}-(k-n)^{\alpha}\right]\\
&+(1-\alpha)\sigma y(c_{k+1})(B(t_{k+1})-B(t_k))\\
&+\frac{h^{\alpha-1}}{\Gamma(\alpha)}\sum_{n=0}^{k}\sigma y(c_n)(B(t_{n+1})-B(t_n))\left[(k-n+1)^{\alpha}-(k-n)^{\alpha}\right].
\end{aligned}
$$

The term $\tilde{y}^{k+1}$ is predicted by the following:

$$
\begin{aligned}
\tilde{y}^{k+1} &= y^0+{}_0^{AB}I_{t_{k+1}}^{\alpha}\left[\varphi_1(t,y)\right] \qquad (4.31)\\
&= y^0+(1-\alpha)\varphi_1(t,y)+\frac{\alpha}{\Gamma(\alpha)}\int_0^{t_{k+1}}\varphi_1(\tau,y)(t_{k+1}-\tau)^{\alpha-1}d\tau\\
&= y^0+(1-\alpha)\varphi_1(t,y)+\frac{\alpha}{\Gamma(\alpha)}\sum_{n=0}^{k}\int_{t_n}^{t_{n+1}}\varphi_1(\tau,y)(t_{k+1}-\tau)^{\alpha-1}d\tau\\
&= y^0+(1-\alpha)\varphi_1\left(t_{k+1},y^{k+1}\right)+\frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{n=0}^{k}\varphi_1(t_n,y^n)\left[(k-n+1)^{\alpha}-(k-n)^{\alpha}\right].
\end{aligned}
$$

We should be aware that the predictor-corrector technique has been developed not only from the use of the parametrized technique, but also due to the first part of the Atangana-Baleanu fractional integral [4].

5. Parametrized method for a general Cauchy problem with stochastic fractal-fractional derivatives

5.1. Parametrized method for a general Cauchy problem with stochastic fractal derivative

In this section, we consider a general Cauchy problem with stochastic component

$$
\begin{cases} {}_0^F D_t^{\alpha} y(t)=\varphi_1(t,y)+\sigma y(t)\,dB(t)\\ y(0)=y_0 \end{cases} \qquad (5.1)
$$

where ${}_0^F D_t^{\alpha}$ is the fractal derivative [5]. Note that using the relation between classical and fractal derivative [5], the above equation can be rewritten as

$$
dy(t)=\beta t^{\beta-1}\varphi_1(t,y)\,dt+\beta t^{\beta-1}\sigma y(t)\,dB(t). \qquad (5.2)
$$

By integrating the above, we obtain the following integral equation:

$$y(t) = y(0) + \beta \int_0^t \tau^{\beta-1} \varphi_1(\tau, y)\, d\tau + \beta \int_0^t \tau^{\beta-1} \sigma y(\tau)\, dB(\tau). \tag{5.3}$$

In this case, we have the following scheme [24] for the considered problem at $t = t_k$ and $t = t_{k+1}$

$$\begin{aligned} y^{k+1} &= y^k + h^\beta \left[\left(1 - \frac{1}{2\xi}\right) t_k^{\beta-1} \varphi_1\left(t_k, y^k\right) + \frac{1}{2\xi} t_{k+1}^{\beta-1} \varphi_1\left(t_{k+1}, \tilde{y}^{k+1}\right) \right] \\ &\quad \times \left((k+1)^\beta - k^\beta\right) + \beta\sigma c_k^{\beta-1} y(c_k) B(t_{k+1}) - B(t_k). \end{aligned} \tag{5.4}$$

here,

$$\tilde{y}^{k+1} = y^0 + h^\beta \sum_{n=0}^{k} \varphi_1(t_n, y^n) \left((n+1)^\beta - n^\beta\right). \tag{5.5}$$

5.2. *Parametrized method for a general Cauchy problem with stochastic fractal-fractional derivative with exponential decay kernel*

In this section, we consider a general Cauchy problem with stochastic fractal-fractional derivative [6] with exponential decay kernel

$$\begin{cases} {}_0^{FFE}D_t^\alpha y(t) = \varphi_1(t, y) + \sigma y(t)\, dB(t), & \text{if } t > 0, \\ y(0) = y_0, & \text{if } t = 0 \end{cases}. \tag{5.6}$$

Applying the fractal-fractional integral [6] with the exponential decay kernel, we obtain

$$\begin{aligned} y(t) &= \beta t^{\beta-1} (1-\alpha) \varphi_1(t, y) + \beta t^{\beta-1} (1-\alpha) \sigma y(t) B'(t) \\ &\quad + \alpha\beta \int_0^t \tau^{\beta-1} \varphi_1(\tau, y)\, d\tau + \alpha\beta \int_0^t \tau^{\beta-1} \sigma y(\tau)\, dB(\tau). \end{aligned} \tag{5.7}$$

Based on the idea of approximating the right-hand side of the equation by the parametrized approach [17, 24–26], the above problem is solved by the following:

$$\begin{aligned} y^{k+1} &= y^k + (1-\alpha)\left(\beta t_{k+1}^{\beta-1} \varphi_1\left(t_{k+1}, \tilde{y}^{k+1}\right) - \beta t_k^{\beta-1} \varphi_1\left(t_k, y^k\right)\right) \\ &\quad + \beta c_k^{\beta-1} y(c_k) \left(B(t_{k+1}) - B(t_k)\right) \\ &\quad + \alpha h^\beta \left[\left(1 - \frac{1}{2\xi}\right) \varphi_1\left(t_k, y^k\right) + \frac{1}{2\xi} \varphi_1\left(t_{k+1}, \tilde{y}^{k+1}\right) \right] \\ &\quad \times \left((k+1)^\beta - k^\beta\right). \end{aligned} \tag{5.8, 5.9}$$

Note that the predictor formula is obtained by

$$\tilde{y}^{k+1} = (1-\alpha)\beta t_k^{\beta-1} \varphi_1\left(t_k, y^k\right) + \alpha h^\beta \sum_{n=0}^{k} \varphi_1(t_n, y^n) \left((n+1)^\beta - n^\beta\right). \tag{5.10}$$

5.3. Parametrized method for a general Cauchy problem with stochastic fractal-fractional derivative with power-law kernel

In this section, we obtain the numerical solution of a general Cauchy problem with stochastic fractal-fractional derivative [18] with power-law kernel by using the parametrized method [17]. The associated problem under consideration is represented by

$$\begin{cases} {}_0^{FFP}D_t^{\alpha} y(t) = \varphi_1(t,y) + \sigma y(t)\, dB(t), \text{ if } t>0, \\ y(0) = y_0, \text{ if } t=0 \end{cases}. \tag{5.11}$$

Applying the fractal-fractional derivative [6] with power-law kernel, we have

$$y(t) = \frac{\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1} \varphi_1(\tau, y)(t-\tau)^{\alpha-1} d\tau + \frac{\beta}{\Gamma(\alpha)} \int_0^t \sigma y(\tau)\tau^{\beta-1}(t-\tau)^{\alpha-1} dB(\tau). \tag{5.12}$$

At $t = t_{k+1}$, we have

$$\begin{aligned} y(t_{k+1}) &= \frac{\beta}{\Gamma(\alpha)} \sum_{n=0}^{k} \int_{t_n}^{t_{n+1}} \tau^{\beta-1}\varphi_1(\tau,y)(t_{k+1}-\tau)^{\alpha-1} d\tau \\ &\quad + \frac{\beta}{\Gamma(\alpha)} \sum_{n=0}^{k} \int_{t_n}^{t_{n+1}} \tau^{\beta-1}\sigma y(\tau)(t_{k+1}-\tau)^{\alpha-1} B'(\tau)\, d\tau. \end{aligned} \tag{5.13}$$

Replacing the function $\varphi_1(\tau, y)$ by its parametrized approximation, we have

$$\begin{aligned} y^{k+1} &= \frac{\beta}{\Gamma(\alpha)} \sum_{n=0}^{k} \left[\left(1-\frac{1}{2\xi}\right)\varphi_1(t_n, y^n) + \frac{1}{2\xi}\varphi_1\left(t_{n+1}, \tilde{y}^{n+1}\right)\right] \\ &\quad \times \int_{t_n}^{t_{n+1}} \tau^{\beta-1}(t_{k+1}-\tau)^{\alpha-1} d\tau \\ &\quad + \frac{\beta}{h\Gamma(\alpha)} \sum_{n=0}^{k} \sigma y(c_n)(B(t_{n+1}) - B(t_n)) \\ &\quad \times \int_{t_n}^{t_{n+1}} \tau^{\beta-1}(t_{k+1}-\tau)^{\alpha-1} d\tau. \end{aligned} \tag{5.14}$$

The integral on the right hand side of the above equation is calculated by using the change of variables $\tau = t_{k+1}u$ and $d\tau = t_{k+1}du$ as follows:

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \tau^{\beta-1}(t_{k+1}-\tau)^{\alpha-1} d\tau &= t_{k+1}^{\alpha+\beta-1} \int_{t_n}^{t_{n+1}} u^{\beta-1}(1-u)^{\alpha-1} du \\ &= t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{n+1}}{t_{k+1}}, \beta, \alpha\right) - B\left(\frac{t_n}{t_{k+1}}, \beta, \alpha\right)\right), \end{aligned}$$

where the function $B(\cdot,\cdot,\cdot)$ is the incomplete Beta function. By calculation of these integrals, the following numerical scheme is obtained:

$$y^{k+1} = \frac{\beta}{\Gamma(\alpha)} \sum_{n=0}^{k} \left[\left(1-\frac{1}{2\xi}\right)\varphi_1(t_n, y^n) + \frac{1}{2\xi}\varphi_1\left(t_{n+1}, \tilde{y}^{n+1}\right)\right] \tag{5.15}$$

$$\times t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{n+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_n}{t_{k+1}},\beta,\alpha\right)\right)$$
$$+\frac{\beta}{h\Gamma(\alpha)}\sum_{n=0}^{k}\sigma y(c_n)(B(t_{n+1})-B(t_n))$$
$$\times t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{n+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_n}{t_{k+1}},\beta,\alpha\right)\right).$$

We know that the term $\tilde{y}^{n+1}$ is predicted by the following:

$$\tilde{y}^{n+1}=y^0+\frac{\beta}{\Gamma(\alpha)}\sum_{n=0}^{k}\varphi_1(t_n,y^n)\,t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{n+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_n}{t_{k+1}},\beta,\alpha\right)\right). \tag{5.16}$$

5.4. Parametrized method for a general Cauchy problem with stochastic fractal-fractional derivative with Mittag-Leffler kernel

To examine the solution of a general Cauchy problem with stochastic fractal-fractional derivative with Mittag-Leffler kernel [18], we consider the following problem:

$$\begin{cases} {}_0^{FFM}D_t^{\alpha}y(t)=\varphi_1(t,y)+\sigma y(t)\,dB(t), & \text{if } t>0,\\ y(0)=y_0, & \text{if } t=0\end{cases} \tag{5.17}$$

After taking the associated integral, the above can be arranged as follows:

$$\begin{aligned} y(t) &= (1-\alpha)\varphi_1(t,y)+(1-\alpha)\sigma y(t)\,dB(t) \\ &+\frac{\alpha\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1}\varphi_1(\tau,y)(t-\tau)^{\alpha-1}\,d\tau \\ &+\frac{\alpha\beta}{\Gamma(\alpha)}\int_0^t \sigma y(\tau)\,\tau^{\beta-1}(t-\tau)^{\alpha-1}B'(\tau)\,d\tau. \end{aligned} \tag{5.18}$$

At $t=t_{k+1}$, we have

$$\begin{aligned} y(t) &= (1-\alpha)\varphi_1\left(t_{k+1},y^{k+1}\right)+(1-\alpha)\sigma y(t_{k+1})\,dB(t_{k+1}) \\ &+\frac{\alpha\beta}{\Gamma(\alpha)}\int_0^{t_{k+1}} \tau^{\beta-1}\varphi_1(\tau,y)(t_{k+1}-\tau)^{\alpha-1}\,d\tau \\ &+\frac{\alpha\beta}{\Gamma(\alpha)}\int_0^{t_{k+1}} \sigma y(\tau)\,\tau^{\beta-1}(t_{k+1}-\tau)^{\alpha-1}\,dB(\tau). \end{aligned} \tag{5.19}$$

Using the $\varphi_1(\tau,y)$ approximations, we have

$$\begin{aligned} y^{k+1} &= (1-\alpha)\varphi_1\left(t_{k+1},y^{k+1}\right)+(1-\alpha)\sigma y(c_{k+1})(B(t_{k+1})-B(t_k)) \\ &+\frac{\alpha\beta}{\Gamma(\alpha)}\sum_{n=0}^{k}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1(t_n,y^n)+\frac{1}{2\xi}\varphi_1\left(t_{n+1},\tilde{y}^{n+1}\right)\right] \\ &\times\int_{t_n}^{t_{n+1}} \tau^{\beta-1}(t_{k+1}-\tau)^{\alpha-1}\,d\tau \end{aligned} \tag{5.20}$$

$$
+\frac{\alpha\beta}{h\Gamma(\alpha)}\sum_{n=0}^{k}\sigma y(c_n)(B(t_{n+1})-B(t_n))
\times\int_{t_n}^{t_{n+1}}\tau^{\beta-1}(t_{k+1}-\tau)^{\alpha-1}d\tau.
$$

Using the calculations for these integrals and arranging the above, we have

$$
\begin{aligned}
y^{k+1} &= (1-\alpha)\varphi_1\left(t_{k+1},y^{k+1}\right)+(1-\alpha)\sigma y(c_{k+1})\left(\frac{B(t_{k+1})-B(t_k)}{h}\right) \\
&+\frac{\alpha\beta}{\Gamma(\alpha)}\sum_{n=0}^{k}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1(t_n,y^n)+\frac{1}{2\xi}\varphi_1\left(t_{n+1},\tilde{y}^{n+1}\right)\right] \\
&\times t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{n+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_n}{t_{k+1}},\beta,\alpha\right)\right) \\
&+\frac{\alpha\beta}{h\Gamma(\alpha)}\sum_{n=0}^{k}\sigma y(c_n)(B(t_{n+1})-B(t_n)) \\
&\times t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{n+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_n}{t_{k+1}},\beta,\alpha\right)\right),
\end{aligned}
\tag{5.21}
$$

where the predictor formula is stated as:

$$
\begin{aligned}
\tilde{y}^{k+1} &= y^0+(1-\alpha)\beta t_{k+1}^{\beta-1}\varphi_1\left(t_{k+1},y^{k+1}\right)+\frac{\alpha\beta}{\Gamma(\alpha)}\sum_{n=0}^{k}\varphi_1(t_n,y^n) \\
&\times t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{n+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_n}{t_{k+1}},\beta,\alpha\right)\right).
\end{aligned}
\tag{5.22}
$$

6. Parametrized method for a general Cauchy problem with piecewise derivative

In this section, we derive the parametrized method [17] for some versions of nonlinear differential equations with piecewise differentiation. We shall start with the version of nonlinear differential equations with piecewise derivative [16], in which classical processes can be used in the first time interval, processes with power-law after fading memory in the second time interval, and stochastic processes can be used in the third time interval. The associated model is represented by the following:

$$
\begin{cases}
\frac{dy}{dt}=\varphi(t,y)\,,\ \text{if } 0\le t\le t_1 \\
y(0)=y^0, \\
{}_{t_1}^{ABC}D_t^{\alpha}y=\varphi(t,y)\,,\ \text{if } t_1\le t\le t_2 \\
y(t_1)=y^1, \\
dy(t)=\varphi(t,y)\,dt+\sigma y dB(t)\ ,\ \text{if } t_2\le t\le T \\
y(t_2)=y^2
\end{cases}.
\tag{6.1}
$$

The function $\varphi(t,y)$ can be approximated by employing the parametrized formulation [17], thus integrating within $[t_n,t_{n+1}]$, we have the following corrector formula with predictor term:

$$y^{k+1}=\left\{\begin{cases} y^0+h^{k_1}_{j_1=0}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1\left(t_{j_1},y^{j_1}\right)+\frac{1}{2\xi}\varphi_1\left(t_{j_1+1},\tilde{y}^{j_1+1}\right)\right], \\ \text{if } 0\le t\le t_1 \end{cases}\right., \tag{6.2}$$

$$\begin{cases} y^1+(1-\alpha)\varphi_1\left(t_{k_2+1},\tilde{y}^{k_2+1}\right) \\ +(1-\alpha)\sigma y\left(c_{k_2+1}\right)\left(B\left(t_{k_2+1}\right)-B\left(t_{k_2}\right)\right) \\ +\frac{h^{\alpha}}{\Gamma(\alpha)}{}^{k_2}_{j_2=k_1+1}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1\left(t_{j_2},y^{j_2}\right)+\frac{1}{2\xi}\varphi_1\left(t_{j_2+1},\tilde{y}^{j_2+1}\right)\right] \\ \times\left[(k_2-j_2+1)^{\alpha}-(k_2-j_2)^{\alpha}\right] \\ +\frac{h^{\alpha-1}}{\Gamma(\alpha)}{}^{k_2}_{j_2=k_1+1}\sigma y\left(c_{j_2}\right)\left(B\left(t_{j_2+1}\right)-B\left(t_{j_2}\right)\right) \\ \times\left[(k_2-j_2+1)^{\alpha}-(k_2-j_2)^{\alpha}\right], \\ \text{if } t_1\le t\le t_2, \end{cases}$$

$$\begin{cases} y^2+h^{k}_{j_3=k_2+1}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1\left(t_{j_3},y^{j_3}\right)+\frac{1}{2\xi}\varphi_1\left(t_{j_3+1},\tilde{y}^{j_3+1}\right)\right] \\ +\sigma y\left(c_k\right)\left(B\left(t_{k+1}\right)-B\left(t_k\right)\right), \\ \text{if } t_2\le t\le T. \end{cases}$$

The predictor components for each interval are calculated as

$$\begin{cases} \left\{\tilde{y}^{k_1+1}=y^0+h^{k_1}_{j_1=0}\varphi_1\left(t_{j_1},y^{j_1}\right), \text{if } 0\le t\le t_1,\right. \\ \begin{cases} \tilde{y}^{k_2+1}=y^1+(1-\alpha)\varphi_1\left(t_{k_2},y^{k_2}\right)+\frac{h^{\alpha}}{\Gamma(\alpha)}{}^{k_2}_{j_2=k_1+1}\varphi_1\left(t_{j_2},y^{j_2}\right) \\ \times\left[(k_2-j_2+1)^{\alpha}-(k_2-j_2)^{\alpha}\right], \text{if } t_1\le t\le t_2, \end{cases} \\ \left\{\tilde{y}^{k_3+1}=y^2+h^{k}_{j_3=k_2+1}\varphi_1\left(t_{j_3},y^{j_3}\right), \text{if } t_2\le t\le T\right. . \end{cases} \tag{6.3}$$

Now, we proceed with an another version of nonlinear differential equations with piecewise derivatives [16]. In the first time interval, fading memory processes can be utilized, while stochastic processes can be used in the second time interval. For the third time interval, processes that deal with power-law behaviors having fractal properties can be employed. The model that explains the process presented here is shown as follows:

$$\begin{cases} {}^{CF}_{0}D^{\alpha}_{t}y=\varphi(t,y), \text{if } 0\le t\le t_1 \\ y(0)=y^0, \\ dy(t)=\varphi(t,y)\,dt+\sigma y dB(t)\ , \text{ if } t_1\le t\le t_2 \\ y(t_1)=y^1, \\ {}^{FFP}_{t_2}D^{\alpha}_{t}y=\varphi(t,y), \text{if } t_2\le t\le T \\ y(t_2)=y^2. \end{cases} \tag{6.4}$$

Using the aforementioned concept of numerical scheme, the numerical scheme for the Cauchy problem in the framework of piecewise derivative [16] is achieved as

$$y^{k+1}=\left\{\begin{cases} y^0+(1-\alpha)\varphi_1\left(t_{k_1+1},y^{k_1+1}\right) \\ +\alpha h^{k_1}_{j_1=0}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1\left(t_{j_1},y^{j_1}\right)+\frac{1}{2\xi}\varphi_1\left(t_{j_1+1},\tilde{y}^{j_1+1}\right)\right], \\ \text{if } 0\le t\le t_1 \end{cases}\right. \tag{6.5}$$

$$\begin{cases} y^1+h^{k_2}_{j_2=k_1+1}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1\left(t_{j_2},y^{j_2}\right)+\frac{1}{2\xi}\varphi_1\left(t_{j_2+1},\tilde{y}^{j_2+1}\right)\right] \\ +\sigma y\left(c_{k_2}\right)\left(B\left(t_{k_2+1}\right)-B\left(t_{k_2}\right)\right), \\ \text{if } t_1\le t\le t_2 \end{cases}$$

$$\left\{\begin{array}{c} \frac{\beta}{\Gamma(\alpha)}{}_{j_3=k_2+1}^{k}\left[\left(1-\frac{1}{2\xi}\right)\varphi_1\left(t_{j_3},y^{j_3}\right)+\frac{1}{2\xi}\varphi_1\left(t_{j_3+1},\tilde{y}^{j_3+1}\right)\right] \\ \times t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{j_3+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_{j_3}}{t_{k+1}},\beta,\alpha\right)\right) \\ +\frac{\beta}{h\Gamma(\alpha)}{}_{j_3=k_2+1}^{k}\sigma y\left(c_{j_3}\right)\left(B\left(t_{j_3+1}\right)-B\left(t_{j_3}\right)\right) \\ \times t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{j_3+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_{j_3}}{t_{k+1}},\beta,\alpha\right)\right), \\ \text{if } t_2\le t\le T. \end{array}\right.$$

The predictor components for each interval are determined as

$$\left\{\begin{array}{c} \left\{\tilde{y}^{k_1+1}=y^0+h_{j_1=0}^{k_1}\varphi_1\left(t_{j_1},y^{j_1}\right),\text{if } 0\le t\le t_1\right., \\ \left\{\tilde{y}^{k_2+1}=y^1+h_{j_2=k_1+1}^{k_2}\varphi_1\left(t_{j_2},y^{j_2}\right),\text{if } t_1\le t\le t_2\right., \\ \left\{\begin{array}{c} \tilde{y}^{k_1+1}=(1-\alpha)\beta t_{k_1}^{\beta-1}\varphi_1\left(t_{k_1},y^{k_1}\right)+\frac{\alpha\beta}{\Gamma(\alpha)}{}_{j_3=k_2+1}^{k}\varphi_1\left(t_{j_3},y^{j_3}\right) \\ \times t_{k_1+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{j_3+1}}{t_{k_1+1}},\beta,\alpha\right)-B\left(\frac{t_{j_3}}{t_{k_1+1}},\beta,\alpha\right)\right),\text{if } t_2\le t\le T \end{array}\right. \end{array}\right. . \tag{6.6}$$

7. Illustrative examples

In this section, we will investigate the applicability of the parametrized method to differential equations with piecewise derivatives with the help of some illustrative examples. This will be performed with the combination of deterministic and stochastic processes where the concepts of classical, stochastic, fractional, and fractal-fractional are added. We will start with a simple piecewise Cauchy problem in which the first part is with classical deterministic, the second part is with Atangana-Baleanu derivative and last part is with the classical stochastic. Another simple scenario will be presented with classical deterministic, Caputo fractional derivative and the classical stochastic. Finally, we will consider an anxiety model [28] employing the different versions of the piecewise derivative.

Example 1. We consider a general Cauchy problem with piecewise derivative

$$\left\{\begin{array}{c} \frac{dy}{dt}=-t,\text{ if } 0\le t\le t_1 \\ y(0)=0, \\ {}_{t_1}^{ABC}D_t^{\alpha}y=-t,\ \text{if } t_1\le t\le t_2 \\ y(t_1)=y_1, \\ dy(t)=-tdt+\sigma y dB(t)\ ,\text{if } t_2\le t\le T \\ y(t_2)=y_2. \end{array}\right. \tag{7.1}$$

The numerical solution of above problem is represented by

$$y^{k+1}=\left\{\left\{\begin{array}{c} y^0+h_{j_1=0}^{k_1}\left[-\left(1-\frac{1}{2\xi}\right)t_{j_1}-\frac{1}{2\xi}t_{j_1+1}\right], \\ \text{if } 0\le t\le t_1 \end{array}\right.\right. , \tag{7.2}$$

$$\left\{\begin{array}{c} y^1-(1-\alpha)t_{k_2+1}+(1-\alpha)\sigma y\left(c_{k_2+1}\right)\left(B\left(t_{k_2+1}\right)-B\left(t_{k_2}\right)\right) \\ +\frac{h^{\alpha}}{\Gamma(\alpha)}{}_{j_2=k_1+1}^{k_2}\left[-\left(1-\frac{1}{2\xi}\right)t_{j_2}-\frac{1}{2\xi}t_{j_2+1}\right] \\ \times\left[(k_2-j_2+1)^{\alpha}-(k_2-j_2)^{\alpha}\right] \\ +\frac{h^{\alpha-1}}{\Gamma(\alpha)}{}_{j_2=k_1+1}^{k_2}\sigma y\left(c_{j_2}\right)\left(B\left(t_{j_2+1}\right)-B\left(t_{j_2}\right)\right) \\ \times\left[(k_2-j_2+1)^{\alpha}-(k_2-j_2)^{\alpha}\right], \\ \text{if } t_1\le t\le t_2, \end{array}\right.$$

$$\begin{cases} y^2 + h^k_{j_3=k_2+1}\left[-\left(1-\frac{1}{2\xi}\right)t_{j_3} - \frac{1}{2\xi}t_{j_3+1}\right] \\ +\sigma y\left(c_k\right)\left(B\left(t_{k+1}\right) - B\left(t_k\right)\right), \\ \text{if } t_2 \leq t \leq T. \end{cases}$$

The predictor terms are as follows:

$$\begin{cases} \left\{\tilde{y}^{k_1+1} = y^0 + h^{k_1}_{j_1=0} - t_{j_1}, \text{if } 0 \leq t \leq t_1, \right. \\ \left\{ \begin{array}{c} \tilde{y}^{k_2+1} = y^1 - (1-\alpha)\, t_{k_2+1} - \frac{h^\alpha}{\Gamma(\alpha)}{}^{k_2}_{j_2=k_1+1} t_{j_2} \\ \times\left[(k_2 - j_2 + 1)^\alpha - (k_2 - j_2)^\alpha\right], \text{if } t_1 \leq t \leq t_2, \end{array} \right. \\ \left\{\tilde{y}^{k_3+1} = y^2 - h^k_{j_3=k_2+1} t_{j_3}, \text{if } t_2 \leq t \leq t\ . \right. \end{cases} \tag{7.3}$$

Noting that the stochastic constant σ is taken as 0.1, the following initial conditions are as follows:

$$y(0) = 0, \tag{7.4}$$

$$\begin{cases} y(t_1) = -45 \text{ if } \alpha = 0.9 \\ y(t_1) = -47.9 \text{ if } \alpha = 0.8 \\ y(t_1) = -48.49 \text{ if } \alpha = 0.6 \\ y(t_1) = -48.8 \text{ if } \alpha = 0.4 \\ y(t_1) = -48.2 \text{ if } \alpha = 0.2 \end{cases},$$

$$\begin{cases} y(t_2) = -145 \text{ if } \alpha = 0.9 \\ y(t_2) = -123.6 \text{ if } \alpha = 0.8 \\ y(t_2) = -85.2 \text{ if } \alpha = 0.6 \\ y(t_2) = -76 \text{ if } \alpha = 0.4 \\ y(t_2) = -60.2 \text{ if } \alpha = 0.2 \end{cases}.$$

In Figure 1, the numerical simulation for the considered problem with piecewise derivative is performed by considering different values of fractional orders.

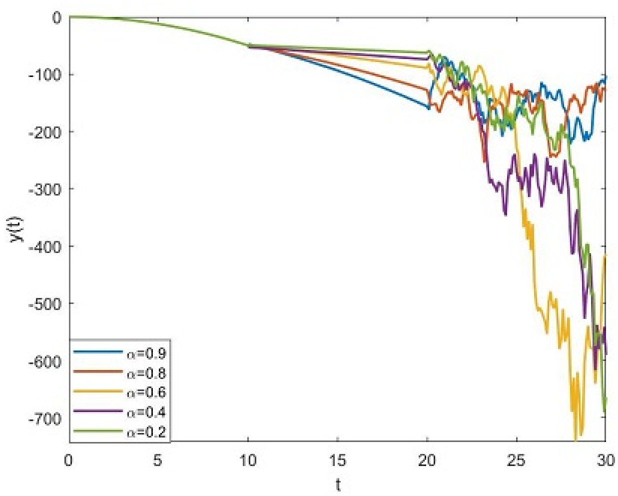

Figure 1. The graphical visualization for the piecewise Cauchy problem.

Example 2. We consider a general Cauchy problem with piecewise derivative

$$\begin{cases} \frac{dy}{dt} = \sin t, \text{ if } 0 \le t \le t_1 \\ y(0) = 1, \\ {}_{t_1}^{ABC}D_t^{\alpha} y = \sin t, \text{ if } t_1 \le t \le t_2 \\ y(t_1) = y_1, \\ dy(t) = \sin t dt + \sigma y dB(t) \text{ , if } t_2 \le t \le T \\ y(t_2) = y_2. \end{cases} \tag{7.5}$$

The numerical solution of above problem is represented by

$$y^{k+1} = \left\{ \left\{ \begin{array}{c} y^0 + h_{j_1=0}^{k_1} \left[\left(1 - \frac{1}{2\xi}\right) \sin\left(t_{j_1}\right) + \frac{1}{2\xi} \sin\left(t_{j_1+1}\right) \right], \\ \text{if } 0 \le t \le t_1 \end{array} \right. \right. , \tag{7.6}$$

$$\begin{cases} y^1 + (1-\alpha)\sin(t_{k_2+1}) + (1-\alpha)\sigma y(c_{k_2+1})(B(t_{k_2+1}) - B(t_{k_2})) \\ + \frac{h^{\alpha}}{\Gamma(\alpha)}{}_{j_2=k_1+1}^{k_2} \left[\left(1 - \frac{1}{2\xi}\right) \sin\left(t_{j_2}\right) + \frac{1}{2\xi} \sin\left(t_{j_2+1}\right) \right] \\ \times \left[(k_2 - j_2 + 1)^{\alpha} - (k_2 - j_2)^{\alpha} \right] \\ + \frac{h^{\alpha-1}}{\Gamma(\alpha)}{}_{j_2=k_1+1}^{k_2} \sigma y\left(c_{j_2}\right)\left(B\left(t_{j_2+1}\right) - B\left(t_{j_2}\right)\right) \\ \times \left[(k_2 - j_2 + 1)^{\alpha} - (k_2 - j_2)^{\alpha} \right], \\ \text{if } t_1 \le t \le t_2, \end{cases}$$

$$\begin{cases} y^2 + h_{j_3=k_2+1}^{k} \left[\left(1 - \frac{1}{2\xi}\right) \sin\left(t_{j_3}\right) + \frac{1}{2\xi} \sin\left(t_{j_3+1}\right) \right] \\ + \sigma y(c_k)(B(t_{k+1}) - B(t_k)), \\ \text{if } t_2 \le t \le T. \end{cases}$$

The predictor components for each interval are calculated as

$$\begin{cases} \left\{ \tilde{y}^{k_1+1} = y^0 + h_{j_1=0}^{k_1} \sin\left(t_{j_1}\right), \text{if } 0 \le t \le t_1, \right. \\ \left\{ \begin{array}{c} \tilde{y}^{k_2+1} = y^1 + (1-\alpha)\sin(t_{k_2+1}) + \frac{h^{\alpha}}{\Gamma(\alpha)}{}_{j_2=k_1+1}^{k_2} \sin\left(t_{j_2}\right) \\ \times \left[(k_2 - j_2 + 1)^{\alpha} - (k_2 - j_2)^{\alpha} \right], \text{if } t_1 \le t \le t_2, \end{array} \right. \\ \left\{ \tilde{y}^{k_3+1} = y^2 + h_{j_3=k_2+1}^{k} \sin\left(t_{j_3}\right), \text{if } t_2 \le t \le T \right. . \end{cases} \tag{7.7}$$

Noting that the stochastic constant σ is taken as 0.1, the initial conditions are as follows:

$$y(0) = 1, \tag{7.8}$$

$$\begin{cases} y(t_1) = 1.6 \text{ if } \alpha = 0.9 \\ y(t_1) = 2 \text{ if } \alpha = 0.8 \\ y(t_1) = 2.5 \text{ if } \alpha = 0.6 \\ y(t_1) = 3.2 \text{ if } \alpha = 0.4 \\ y(t_1) = 3.4 \text{ if } \alpha = 0.2 \end{cases},$$

$$\begin{cases} y(t_2) = 2 \text{ if } \alpha = 0.9 \\ y(t_2) = 2.48 \text{ if } \alpha = 0.8 \\ y(t_2) = 3.36 \text{ if } \alpha = 0.6 \\ y(t_2) = 4.7 \text{ if } \alpha = 0.4 \\ y(t_2) = 4.27 \text{ if } \alpha = 0.2 \end{cases}.$$

In Figure 2, the numerical simulation for the considered problem with piecewise derivative is performed by considering different values of fractional orders.

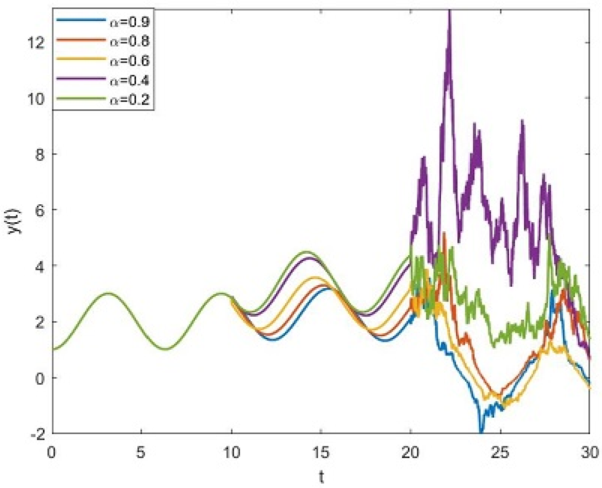

Figure 2. The graphical visualization for the piecewise Cauchy problem.

Example 3. (Mathematical modeling of anxiety of mathematics) Instructional and social psychological environment are some of the attitude attribute of students and possible factors affecting the students' disliking or liking of mathematics and mathematics anxiety is closely related to a broad spectrum of cognitive, psychological, and behavioral problems [28, 29]. We next consider a mathematical model associated with the anxiety of mathematics [28]. The mathematical model under investigation is presented by the following:

$$\begin{aligned} \frac{dS}{dt} &= (1-\varepsilon)\chi + \omega R + \rho(1-\eta)P - \left(\frac{\theta}{N}(A+\varphi Q) + \kappa\right)S \\ \frac{dP}{dt} &= \varepsilon\chi - (\kappa + (1-\eta)\rho)P \\ \frac{dE}{dt} &= \frac{\theta}{N}(A+\varphi Q)S - (\kappa+\upsilon)E \\ \frac{dA}{dt} &= (1-\varrho)\upsilon E - (\kappa+\delta+\varsigma)A \\ \frac{dQ}{dt} &= \delta A - \kappa Q \\ \frac{dR}{dt} &= \varsigma A + \varrho\upsilon E - (\kappa+\omega)R \end{aligned} \tag{7.9}$$

and the initial conditions are taken as

$$S(0) \geq 0,\ P(0) \geq 0,\ E(0) \geq 0,\ A(0) \geq 0,\ Q(0) \geq 0,\ R(0) \geq 0. \tag{7.10}$$

Here, S: anxiety towards mathematics susceptible students; P: anxiety towards mathematics protected students; E: anxiety towards mathematics exposed students; A: students who have anxiety towards mathematics; Q: students who have permanent anxiety towards mathematics; R: students recovered from anxiety towards mathematics.

Replacing the classical derivative by the piecewise differential operators and simplifying the model with piecewise derivative, we get the following modified model of anxiety:

$$\begin{cases} \frac{dU}{dt} = \psi(t, U), \text{ if } 0 \leq t \leq t_1 \\ U(0) = U^0, \\ {}_{t_1}^{C}D_t^{\alpha} U = \psi(t, U), \text{ if } t_1 \leq t \leq t_2 \\ U(t_i) = U^1, \\ dU(t) = \psi(t, U)\, dt + \sigma_i U dB_i(t) \text{ , if } t_2 \leq t \leq T \\ U(t_2) = U^2, \end{cases} \tag{7.11}$$

where

$$U = \begin{bmatrix} S \\ P \\ E \\ A \\ Q \\ R \end{bmatrix}, \ \psi(t, U) = \begin{bmatrix} (1-\varepsilon)\chi + \omega R + \rho(1-\eta) P - \left(\frac{\theta}{N}(A + \varphi Q) + \kappa\right) S \\ \varepsilon\chi - (\kappa + (1-\eta)\rho) P \\ \frac{\theta}{N}(A + \varphi Q) S - (\kappa + \upsilon) E \\ (1-\varrho)\upsilon E - (\kappa + \delta + \varsigma) A \\ \delta A - \kappa Q \\ \varsigma A + \varrho\upsilon E - (\kappa + \omega) R \end{bmatrix}. \tag{7.12}$$

Using the suggested method for each interval, the numerical solution can be obtained as

$$U^{k+1} = \left\{\left\{ \begin{array}{c} U^0 + h_{j_1=0}^{k_1} \left[\left(1 - \frac{1}{2\xi}\right)\psi\left(t_{j_1}, U^{j_1}\right) + \frac{1}{2\xi}\psi\left(t_{j_1+1}, \widetilde{U}^{j_1+1}\right)\right], \\ \text{if } 0 \leq t \leq t_1 \end{array} \right.\right. , \tag{7.13}$$

$$\left\{ \begin{array}{c} U^1 + \frac{h^{\alpha}}{\Gamma(\alpha+1)}{}_{j_2=k_1+1}^{k_2} \left[\left(1 - \frac{1}{2\xi}\right)\psi\left(t_{j_2}, U^{j_2}\right) + \frac{1}{2\xi}\psi\left(t_{j_2+1}, \widetilde{U}^{j_2+1}\right)\right] \\ \times\left[(k_2 - j_2 + 1)^{\alpha} - (k_2 - j_2)^{\alpha}\right] \\ + \frac{h^{\alpha-1}}{\Gamma(\alpha+1)}{}_{j_2=k_1+1}^{k_2} \sigma_i U\left(c_{j_2}\right)\left(B_i\left(t_{j_2+1}\right) - B_i\left(t_{j_2}\right)\right) \\ \times\left[(k_2 - j_2 + 1)^{\alpha} - (k_2 - j_2)^{\alpha}\right], \\ \text{if } t_1 \leq t \leq t_2, \end{array} \right.$$

$$\left\{ \begin{array}{c} U^2 + h_{j_3=k_2+1}^{k} \left[\left(1 - \frac{1}{2\xi}\right)\psi\left(t_{j_3}, U^{j_3}\right) + \frac{1}{2\xi}\psi\left(t_{j_3+1}, \widetilde{U}^{j_3+1}\right)\right] \\ +\sigma y(c_k)\left(B(t_{k+1}) - B(t_k)\right), \\ \text{if } t_2 \leq t \leq T. \end{array} \right.$$

The predictor components for each interval are calculated as

$$\left\{ \begin{array}{c} \left\{\widetilde{U}^{k_1+1} = U^0 + h_{j_1=0}^{k_1}\psi\left(t_{j_1}, U^{j_1}\right), \text{if } 0 \leq t \leq t_1,\right. \\ \left\{ \begin{array}{c} \widetilde{U}^{k_2+1} = U^1 + \frac{h^{\alpha}}{\Gamma(\alpha+1)}{}_{j_2=k_1+1}^{k_2}\psi\left(t_{j_2}, U^{j_2}\right)\begin{bmatrix} (k_2 - j_2 + 1)^{\alpha} \\ -(k_2 - j_2)^{\alpha} \end{bmatrix}, \\ \text{if } t_1 \leq t \leq t_2, \end{array} \right. \\ \left\{\widetilde{U}^{k_3+1} = U^2 + h_{j_3=k_2+1}^{k}\psi\left(t_{j_3}, U^{j_3}\right), \text{if } t_2 \leq t \leq T\right. . \end{array} \right. \tag{7.14}$$

In Figure 3, we simulate the numerical solution of the anxiety model with piecewise derivative for $\alpha = 0.9$.

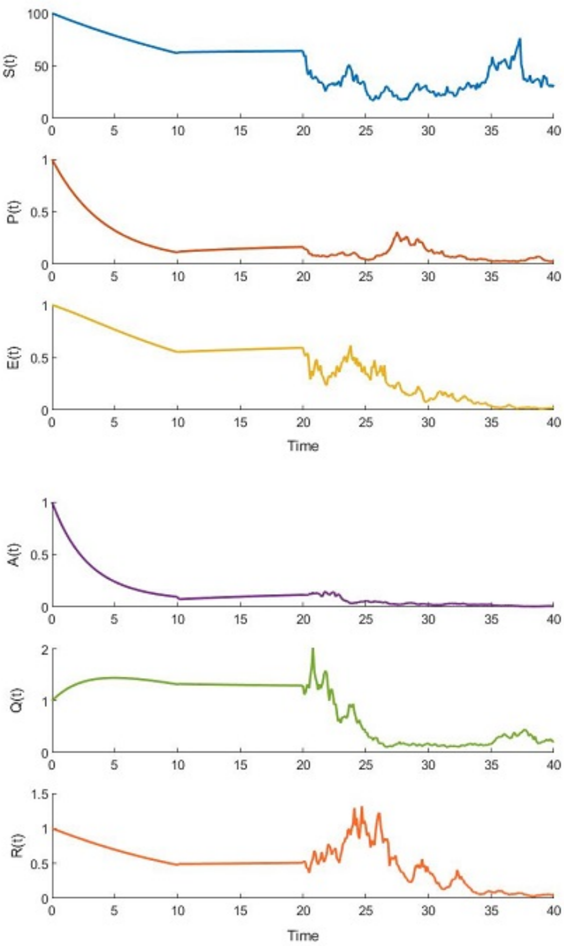

Figure 3. The graphical visualization for the anxiety model with piecewise setting.

We present another case for our model since we know that the model can be modified with different derivatives in each intervals. For another case of our model, it can be written as follows:

$$\begin{cases} {}_0^{CF}D_t^{\alpha}U = \psi(t,U), \text{ if } 0 \le t \le t_1 \\ U(0) = U^0, \\ dU(t) = \psi(t,U)\,dt + \sigma U dB(t)\ , \text{ if } t_1 \le t \le t_2 \\ U(t_1) = U^1, \\ {}_{t_2}^{FFP}D_t^{\alpha,\beta}U = \psi(t,U), \text{ if } t_2 \le t \le T \\ U(t_2) = U^2. \end{cases} \tag{7.15}$$

For such a model, we obtain

$$U^{k+1} = \left\{\left\{ \begin{array}{c} U^0 + (1-\alpha)\psi\left(t_{k_1+1}, U^{k_1+1}\right) \\ +\alpha h_{j_1=0}^{k_1}\left[\left(1-\frac{1}{2\xi}\right)\psi\left(t_{j_1}, U^{j_1}\right) + \frac{1}{2\xi}\psi\left(t_{j_1+1}, \widetilde{U}^{j_1+1}\right)\right], \\ \text{if } 0 \le t \le t_1 \end{array}\right.\right. \tag{7.16}$$

$$\begin{cases} U^1 + h_{j_2=k_1+1}^{k_2} \left[\left(1-\frac{1}{2\xi}\right)\psi\left(t_{j_2}, U^{j_2}\right) + \frac{1}{2\xi}\psi\left(t_{j_2+1}, \widetilde{U}^{j_2+1}\right)\right] \\ +\sigma U\left(c_{k_2}\right)\left(B\left(t_{k_2+1}\right) - B\left(t_{k_2}\right)\right), \\ \text{if } t_1 \le t \le t_2 \end{cases}$$

$$\begin{cases} \frac{\beta}{\Gamma(\alpha)} {}_{j_3=k_2+1}^{k} \left[\left(1-\frac{1}{2\xi}\right)\psi\left(t_{j_3}, U^{j_3}\right) + \frac{1}{2\xi}\psi\left(t_{j_3+1}, \widetilde{U}^{j_3+1}\right)\right] \\ \times t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{j_3+1}}{t_{k+1}}, \beta, \alpha\right) - B\left(\frac{t_{j_3}}{t_{k+1}}, \beta, \alpha\right)\right) \\ +\frac{\beta}{h\Gamma(\alpha)} {}_{j_3=k_2+1}^{k} \sigma U\left(c_{j_3}\right)\left(B\left(t_{j_3+1}\right) - B\left(t_{j_3}\right)\right) \\ \times t_{k+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{j_3+1}}{t_{k+1}}, \beta, \alpha\right) - B\left(\frac{t_{j_3}}{t_{k+1}}, \beta, \alpha\right)\right), \\ \text{if } t_2 \le t \le T. \end{cases}$$

The predictor components for each interval are determined as

$$\begin{cases} \left\{\widetilde{U}^{k_1+1} = U^0 + h_{j_1=0}^{k_1}\psi\left(t_{j_1}, U^{j_1}\right), \text{if } 0 \le t \le t_1\right., \\ \left\{\widetilde{U}^{k_2+1} = U^1 + h_{j_2=k_1+1}^{k_2}\psi\left(t_{j_2}, U^{j_2}\right), \text{if } t_1 \le t \le t_2\right., \\ \begin{cases} \widetilde{U}^{k_1+1} = (1-\alpha)\beta t_{k_1}^{\beta-1}\psi\left(t_{k_1}, U^{k_1}\right) + \frac{\alpha\beta}{\Gamma(\alpha)} {}_{j_3=k_2+1}^{k}\psi\left(t_{j_3}, U^{j_3}\right) \\ \times t_{k_1+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{j_3+1}}{t_{k_1+1}}, \beta, \alpha\right) - B\left(\frac{t_{j_3}}{t_{k_1+1}}, \beta, \alpha\right)\right), \text{if } t_2 \le t \le T \end{cases} \end{cases} . \tag{7.17}$$

In Figure 4, we perform the numerical simulation for anxiety model with piecewsie derivative for $\alpha = 0.95, \beta = 0.8$.

8. Conclusions

This study is based on the use of the parametrized method for the numerical solution of fractional, fractal-fractional and piecewise derivative initial value problems where the stochastic component is added. After presenting the definition of these differential operators, we demonstrate the condition under which the nonlinear ordinary differential equations with stochastic Atangana-Baleanu fractional derivative admit a unique solution using the Carathéodory conditions. Since piecewise derivative allows fractal, fractal-fractal and stochastic situations to be addressed together, the piecewise derivative was used in the models discussed for the illustrative examples presented. We provide the graphical representations for the solutions of these models, which are simple piecewise Cauchy proplems and the anxiety model. The presented models with piecewise derivatives, which are separated into three intervals and involved different differential operators in each interval, exhibit different behaviors during simulations, ranging from deterministic to stochastic. When looking at the graphical representation provided for the fragmented anxiety model, for example, for class A, which is a class of anxious people, it is observed that this problem that the person is exposed to may change in some time intervals and that there is a possibility of encountering this situation again while attempting to overcome anxiety. It is argued that piecewise derivatives, by virtue of displaying such distinctive characteristics, have an advantage over other operators. Our future work will focus on the existence-uniqueness proofs of stochastic equations with fractal-fractional derivatives and the application of the relevant method to numerical solutions of different models.

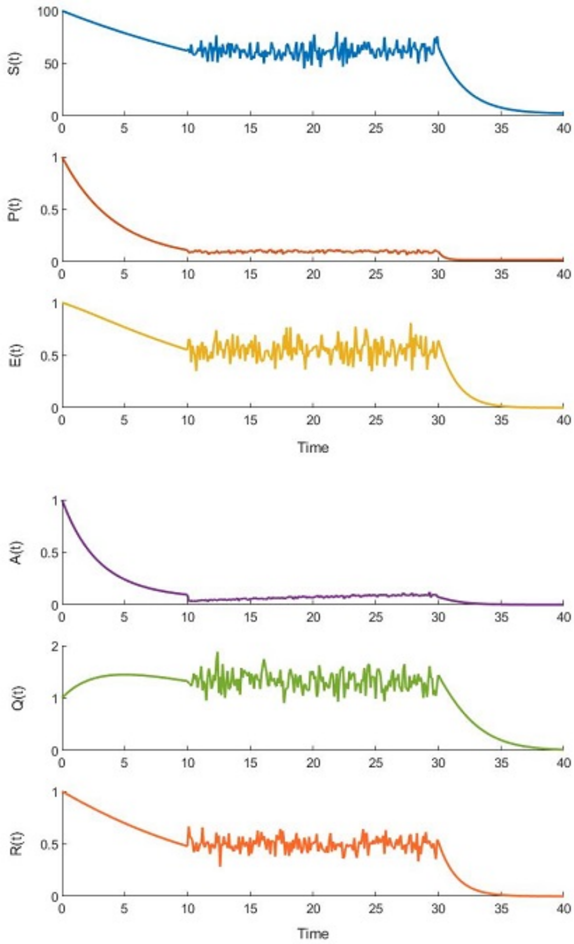

Figure 4. The graphical visualization for anxiety model with piecewise setting.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

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