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Research article

Existence of normalized solutions for a Sobolev supercritical Schrödinger equation

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Abstract: This paper studies the existence of normalized solutions for the following Schrödinger equation with Sobolev supercritical growth:

$$\begin{cases} -\Delta u + V(x)u + \lambda u = f(u) + \mu |u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases}$$

where $p > 2^* := \frac{2N}{N-2}$, $N \geq 3$, $a > 0$, $\lambda \in \mathbb{R}$ is an unknown Lagrange multiplier, $V \in C(\mathbb{R}^N, \mathbb{R})$, f satisfies weak mass subcritical conditions. By employing the truncation technique, we establish the existence of normalized solutions to this Sobolev supercritical problem. Our primary contribution lies in our initial exploration of the case $p > 2^*$, which represents an unfixed frequency problem.

Keywords: normalized solution; truncation technique; Sobolev supercritical growth

1. Introduction and main results

In this paper, we study the existence of L^2-normalized solutions to the following equation:

$$\begin{cases} -\Delta u + V(x)u + \lambda u = f(u) + \mu |u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.1}$$

where $p > 2^* := \frac{2N}{N-2}$, $N \geq 3$, $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function, that is not necessarily of the power type, and the mass $a > 0$ is given, while a pair $(\lambda, u) \in \mathbb{R} \times H^1\left(\mathbb{R}^N\right)$ is unknown.

Problem (1.1) appears when searching for standing waves to the following nonlinear Schrödinger equation

$$i\hbar \partial_t \psi = -\hbar^2 \Delta \psi + V(x)\psi - f(\psi) - \mu |\psi|^{p-2}\psi, \quad (t, x) \in (0, T) \times \mathbb{R}^N \tag{1.2}$$

i.e. solutions to (1.2) of the form $\psi(t,x) = e^{i\lambda t}u(x)$ for some $\lambda \in \mathbb{R}$. The constraint $\int_{\mathbb{R}^N} |u|^2 dx = a^2$ comes from the fact that, under suitable assumptions on f, a solution ψ of (1.2) preserves such a quantity in time $\int_{\mathbb{R}^N} |\psi(t,x)|^2 dx = \int_{\mathbb{R}^N} |\psi(0,x)|^2 dx$ for all $t \in (0,T)$.

To date, two distinct perspectives have emerged regarding the frequency λ in Eq (1.1). One approach views λ as a predetermined constant, leading to the so-called fixed frequency problem. Extensive research has been conducted on the existence, multiplicity, and concentration of nontrivial solutions to (1.1) under various assumptions on the nonlinearity f and the potential V, encompassing Sobolev subcritical, critical, and supercritical growth regimes (see [1–3]).

Alternatively, λ can be considered an unknown quantity within Eq (1.1). In this context, it is natural to prescribe the value of the mass, allowing λ to be interpreted as a Lagrange multiplier. Notably, the mass possesses a clear physical significance. For instance, in Bose-Einstein condensates, λ corresponds to the chemical potential, representing the energy required to add one particle to the system. In nonlinear optics, λ reflects the refractive index shift induced by nonlinear effects. Thus, λ is not merely a Lagrange multiplier in the mathematical model but is also closely tied to the system's physical properties, offering insights into its dynamics and characteristics.

Furthermore, such solutions offer valuable insights into the dynamical properties of the system, including orbital stability or instability, and can effectively describe attractive Bose-Einstein condensates. In the mathematical literature, these types of solutions are commonly referred to as prescribed L^2-norm solutions or normalized solutions. We refer to [4–6] and references therein.

In recent decades, the existence of L^2-normalized solutions has been a topic of active research, resulting in a substantial body of literature. The L^2-critical exponent p, defined as

$$p = 2 + \frac{4}{N} \tag{1.3}$$

plays a key role in analyzing (1.1) and the stability properties of (1.2), as discussed in [7]. Note that $p \in (2, 2^*)$, where, as usual, $2^* = \frac{2N}{N-2}$ for $N \geq 3$, and $2^* = \infty$ for $N = 2$.

To illustrate the importance of the L^2-critical exponent p, consider $q \in (2, 2^*)$ and $\lambda > 0$. Recall that the equation

$$-\Delta Q + \lambda Q = |Q|^{q-2}Q \quad \text{in } \mathbb{R}^N,$$

has a unique positive solution up to translations (see Kwong [8]). Its unique positive radial solution is denoted by $Q_{q,\lambda}$. It is easily observed that

$$Q_{q,\lambda} = \lambda^{\frac{1}{q-2}} Q_{q,1}\left(\lambda^{\frac{1}{2}} \cdot\right)$$

where $Q_{q,1}$ is the unique positive radial solution of

$$-\Delta Q + Q = |Q|^{q-2}Q \quad \text{in } \mathbb{R}^N.$$

Setting the mass

$$\mathcal{M}(u) = \frac{1}{2}\int_{\mathbb{R}^N} u^2 dx,$$

and the energy

$$\mathcal{E}_q(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx,$$

it follows from the Pohozaev identity that

$$\mathcal{M}\left(Q_{q,\lambda}\right) = \lambda^{\frac{N}{2(q-2)}(p-q)}\mathcal{M}\left(Q_{q,1}\right),$$

$$\mathcal{E}_q\left(Q_{q,\lambda}\right) = \lambda^{\frac{2}{q-2}-\frac{N}{2}+1}\mathcal{E}_q\left(Q_{q,1}\right),$$

and

$$\mathcal{E}_q\left(Q_{q,1}\right) = \frac{N(q-p)}{(N+2)-(N-2)(q-1)}\mathcal{M}\left(Q_{q,1}\right).$$

Thus we see that $p = 2 + \frac{4}{N}$ plays a special role. Indeed, considering three cases

$$(1)\ q \in (2, p),\quad (2)\ q \in (p, 2^*),\quad (3)\ q = p,$$

then the energy $\mathcal{E}_q(u)$ of a solution (λ, u) for (1.1) satisfies

$$\mathcal{E}_q(u)\begin{cases} < 0 & \text{for Case (1)},\\ > 0 & \text{for Case (2)},\\ = 0 & \text{for Case (3)}.\end{cases}$$

Recent studies have concentrated on normalized solutions of the Schrödinger equation, with particular attention to the Sobolev subcritical case. Soave [9] examined the existence and properties of ground states for the nonlinear Schrödinger equation involving combined power nonlinearities:

$$-\Delta u + \lambda u = \mu|u|^{q-2}u + |u|^{p-2}u, \quad x \in \mathbb{R}^N$$

on

$$S(a) := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 dx = a^2\right\},$$

where $N \geq 1$, $2 < q \leq 2 + \frac{4}{N} \leq p < 2^*$. Yang et al. [10] obtained the existence and multiplicity of normalized solutions to the following Schrödinger equations with potentials and non-autonomous nonlinearities:

$$\begin{cases} -\Delta u + V(x)u + \lambda u = f(x, u), & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N} u^2 dx = a^2, \end{cases}$$

where $f(x, s)$ satisfies Berestycki-Lions type conditions with mass subcritical growth. Claudianor, Alves and Thin [11, 12] investigated the existence of multiple normalized solutions to the following class of elliptic problems:

$$\begin{cases} -\Delta u + V(x)u + \lambda u = f(u), & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N} u^2 dx = a^2, \end{cases}$$

under different assumptions about potentials, but f verifies weak mass subcritical growth. For Sobolev critical case, Soave [13] considered the Sobolev critical problem

$$\begin{cases} -\Delta u + \lambda u = \mu|u|^{q-2}u + |u|^{2^*-2}u, & x \in \mathbb{R}^N,\\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.4}$$

where $2 + \frac{4}{N} < q < 2^*$, $N \geq 3$, but the upper boundedness of $\mu \cdot a^{(1-\gamma_q)q}$ is essential. Li and Zou [14] obtained the existence of ground states for (1.4) that does not depend on the range of $\mu \cdot a^{(1-\gamma_q)q}$, which improves and extends the result in [13]. Bieganowski and Mederski [15] proposed a simple minimization method based on the direct minimization of the energy functional on the linear combination of Nehari and Pohozaev constraints to prove the existence of normalized ground states to

$$-\Delta u + \lambda u = g(u)$$

for the Sobolev subcritical equation.

Existing research predominantly concentrates on normalized solutions within the context of Sobolev subcritical or critical problems. To the best of our knowledge, the Sobolev supercritical case has not been explored in relation to normalized solutions. In this paper, we delve into this intriguing subject and introduce the following assumptions:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $0 < V_0 := \inf_{x \in \mathbb{R}^N} V(x) \leq V(x) \leq V_\infty := \lim_{|x| \to \infty} V(x) < +\infty$ for all $x \in \mathbb{R}^N$.

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd, and there exist $m \in (2, 2 + \frac{4}{N})$ and $\alpha \in (0, +\infty)$ such that

$$\lim_{t \to 0} \frac{|f(t)|}{|t|^{m-1}} = \alpha.$$

(f_2) There exist two constants $C_1,\ C_2 > 0$ and $q \in (m, 2 + \frac{4}{N})$ such that

$$|f(t)| \leq C_1 + C_2|t|^{q-1},\ \forall t \in R.$$

(f_3) $\frac{f(t)}{t^{q-1}}$ is an increasing function of t on $(0, +\infty)$.

Theorem 1.1. *Suppose that* (V) *and* (f_1)-(f_3) *are satisfied. Then there exists some* $\mu_0 > 0$ *such that for* $\mu \in (0, \mu_0]$*, problem* (1.1) *admits a couple of weak solutions* $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}^+$ *with* $\int_{\mathbb{R}^N} |u|^2 dx = a^2$.

2. Some lemmas and proof of Theorem 1.1

Define a function

$$\phi(t) = \begin{cases} |t|^{p-2}t, & |t| \leq M, \\ M^{p-q}|t|^{q-2}t, & |t| > M, \end{cases}$$

where $M > 0$. Then

$$\phi \in C(\mathbb{R}, \mathbb{R}),$$

$$\phi(t)t \geq q\Phi(t) := q\int_0^t \phi(s)ds \geq 0$$

and

$$|\phi(t)| \leq M^{p-q}|t|^{q-1} \text{ for all } t \in \mathbb{R}.$$

Set $h_\mu(t) = \mu\phi(t) + f(t)$ for all $t \in R$. Then $h_\mu(t)$ possesses the following properties:

(h_1) $h_\mu \in C(\mathbb{R}, \mathbb{R})$ is odd, and $\lim_{t \to 0} \frac{|h_\mu(t)|}{|t|^{m-1}} = \alpha$.

(h_2) $|h_\mu(t)| \leq \mu M^{p-q}|t|^{q-1} + C_1 + C_2|t|^{q-1}$ for all $t \in \mathbb{R}$.

(h_3) $\frac{h_\mu(t)}{t^{q-1}}$ is an increasing function of t on $(0, +\infty)$.

(h_4) $h_\mu(t)t \geq qH_\mu(t) := q\int_0^t h_\mu(s)ds \geq 0$ for all $t \in \mathbb{R}$.

By (V) and (h_1)-(h_3) and [11], the following problem

$$\begin{cases} -\Delta u + V(x)u + \lambda u = h_\mu(u), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases}$$

admits a couple $(u_\mu, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}^+$ of weak solutions with $\int_{\mathbb{R}^N} |u_\mu|^2 dx = a^2$. Let

$$J_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} H_\mu(u)dx.$$

Then

$$J_\mu|'_{S(a)}(u_\mu) = 0$$

and

$$J_\mu(u_\mu) = \gamma_{0,a,\mu} := \inf_{u\in S(a)} J_{0,\mu}(u),$$

where

$$J_{0,\mu}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} H_\mu(u)dx.$$

Consequently, there exists $\lambda \in \mathbb{R}$ such that

$$-\Delta u_\mu + V(x)u_\mu + \lambda u_\mu = h_\mu(u_\mu). \tag{2.1}$$

Moreover, set

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(u)dx$$

and

$$\gamma_{0,a} := \inf_{u\in S(a)} J_0(u),$$

where

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u)dx.$$

Then, $\gamma_{0,a,\mu} \leq \gamma_{0,a}$.

Lemma 2.1. *The solution u_μ satisfies $\|\nabla u_\mu\|_2^2 \leq \frac{2(q\gamma_{0,a}+\lambda a^2)}{q-2}$.*

Proof. By (2.1), we have

$$0 = \int_{\mathbb{R}^N} |\nabla u_\mu|^2 dx + \int_{\mathbb{R}^N} V(x)|u_\mu|^2 dx + \lambda\int_{\mathbb{R}^N} |u_\mu|^2 dx - \int_{\mathbb{R}^N} h_\mu(u_\mu)u_\mu dx.$$

Therefore,

$$\begin{aligned} q\gamma_{0,a,\mu} =& \frac{q-2}{2}\int_{\mathbb{R}^N} |\nabla u_\mu|^2 dx + \frac{q-2}{2}\int_{\mathbb{R}^N} V(x)|u_\mu|^2 dx + \mu\int_{\mathbb{R}^N} [\phi(u_\mu)u_\mu - q\Phi(u_\mu)]dx \\ &+ \int_{\mathbb{R}^N} [f(u_\mu)u_\mu - qF(u_\mu)]dx - \lambda\int_{\mathbb{R}^N} |u_\mu|^2 dx \\ \geq& \frac{q-2}{2}\int_{\mathbb{R}^N} |\nabla u_\mu|^2 dx - \lambda\int_{\mathbb{R}^N} |u_\mu|^2 dx. \end{aligned}$$

Which implies that the lemma holds. □

Lemma 2.2. *There exist two constants $B,\ D > 0$ independent of μ such that $\|u_\mu\|_{L^\infty} \leq B(1+\mu)^D$.*

Proof. Let $b : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying:

- $b(s) = s$ for $|s| \leq T-1$;
- $b(-s) = -b(s)$;
- $b'(s) = 0$ for $s \geq T$;
- $b'(s)$ is decreasing in $[T-1, T]$.

Set $T > 2$, $r > 0$, and $\tilde{u}_\mu^T £ = b(u_\mu)$. It is easy to see that

$$\tilde{u}_\mu^T = u_\mu \ \text{ for } \ |u_\mu| \leq T-1;$$

$$|\tilde{u}_\mu^T| = |b(u_\mu)| \leq |u_\mu| \ \text{ for } \ T-1 \leq |u_\mu| \leq T;$$
$$|\tilde{u}_\mu^T| = C_T > 0 \ \text{ for } \ |u_\mu| \geq T,$$

where $T-1 \leq C_T \leq T$. Moreover, $0 \leq \frac{sb'(s)}{b(s)} \leq 1, \forall\ s \neq 0$. Let $\psi = u_\mu |\tilde{u}_\mu^T|^{2r}$. Then $\psi \in H^1(\mathbb{R}^N)$. Taking ψ as the test function, there holds

$$\begin{aligned}
&\int_{\mathbb{R}^N} h_\mu(u_\mu)\psi dx \\
&= \int_{\mathbb{R}^N} \nabla u_\mu \cdot \nabla \psi dx + \int_{\mathbb{R}^N} V(x) u_\mu \psi dx + \lambda \int_{\mathbb{R}^N} u_\mu \psi dx \\
&\geq \int_{\mathbb{R}^N} \nabla u_\mu \cdot \nabla [u_\mu |\tilde{u}_\mu^T|^{2r}] dx + \int_{\mathbb{R}^N} V(x) u_\mu^2 |\tilde{u}_\mu^T|^{2r} dx \\
&\geq \int_{|u_\mu| \leq T-1} (1+r) |\tilde{u}_\mu^T|^{2r} |\nabla u_\mu|^2 dx + \int_{|u_\mu| \geq T} |\tilde{u}_\mu^T|^{2r} |\nabla u_\mu|^2 dx + \int_{\mathbb{R}^N} V(x) |u_\mu|^2 |\tilde{u}_\mu^T|^{2r} dx \\
&\quad + \int_{T-1<|u_\mu|<T} [|\tilde{u}_\mu^T|^{2r} + 2r u_\mu b(u_\mu) b'(u_\mu) |\tilde{u}_\mu^T|^{2r-2}] |\nabla u_\mu|^2 dx \\
\geq &\int_{|u_\mu| \leq T-1} |\tilde{u}_\mu^T|^{2r} |\nabla u_\mu|^2 dx + \int_{|u_\mu| \geq T} |\tilde{u}_\mu^T|^{2r} |\nabla u_\mu|^2 dx + \int_{\mathbb{R}^N} V(x) |u_\mu|^2 |\tilde{u}_\mu^T|^{2r} dx \\
&\quad + \int_{T-1<|u_\mu|<T} [|\tilde{u}_\mu^T|^{2r} + 2r u_\mu b(u_\mu) b'(u_\mu) |\tilde{u}_\mu^T|^{2r-2}] |\nabla u_\mu|^2 dx \\
\geq &\frac{1}{(r+1)^2} \int_{|u_\mu| \leq T-1} |\nabla [u_\mu (\tilde{u}_\mu^T)^r]|^2 dx + \int_{|u_\mu| \geq T} |\nabla [u_\mu (\tilde{u}_\mu^T)^r]|^2 dx + \int_{\mathbb{R}^N} V(x) |u_\mu|^2 |\tilde{u}_\mu^T|^{2r} dx \\
&\quad + \int_{T-1<|u_\mu|<T} [|\tilde{u}_\mu^T|^{2r} + 2r u_\mu^2 (b'(u_\mu))^2 |\tilde{u}_\mu^T|^{2r-2}] |\nabla u_\mu|^2 dx \\
\geq &\frac{1}{(r+1)^2} \int_{|u_\mu| \leq T-1} |\nabla [u_\mu (\tilde{u}_\mu^T)^r]|^2 dx + \int_{|u_\mu| \geq T} |\nabla [u_\mu (\tilde{u}_\mu^T)^r]|^2 dx + \int_{\mathbb{R}^N} V(x) |u_\mu|^2 |\tilde{u}_\mu^T|^{2r} dx \\
&\quad + \int_{T-1<|u_\mu|<T} [\frac{1}{(r+1)^2} |\tilde{u}_\mu^T|^{2r} + \frac{r}{(r+1)^2} \cdot 2r u_\mu^2 (b'(u_\mu))^2 |\tilde{u}_\mu^T|^{2r-2}] |\nabla u_\mu|^2 dx \\
= &\frac{1}{(r+1)^2} \int_{|u_\mu| \leq T-1} |\nabla [u_\mu (\tilde{u}_\mu^T)^r]|^2 dx + \int_{|u_\mu| \geq T} |\nabla [u_\mu (\tilde{u}_\mu^T)^r]|^2 dx + \int_{\mathbb{R}^N} V(x) |u_\mu|^2 |\tilde{u}_\mu^T|^{2r} dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{T-1<|u_\mu|<T}[\frac{1}{(r+1)^2}b^{2r}(u_\mu)|\nabla u_\mu|^2+\frac{2}{(r+1)^2}u_\mu^2|\nabla b^r(u_\mu)|^2]dx\\
\geq&\frac{1}{(r+1)^2}\int_{|u_\mu|\leq T-1}|\nabla[u_\mu(\tilde{u}_\mu^T)^r]|^2dx+\int_{|u_\mu|\geq T}|\nabla[u_\mu(\tilde{u}_\mu^T)^r]|^2dx+\int_{\mathbb{R}^N}V(x)|u_\mu|^2|\tilde{u}_\mu^T|^{2r}dx\\
&+\frac{2C_1}{(r+1)^2}\int_{T-1<|u_\mu|<T}[b^{2r}(u_\mu)|\nabla u_\mu|^2+u_\mu^2|\nabla b^r(u_\mu)|^2]dx\\
\geq&\frac{1}{(r+1)^2}\int_{|u_\mu|\leq T-1}|\nabla[u_\mu(\tilde{u}_\mu^T)^r]|^2dx+\int_{|u_\mu|\geq T}|\nabla[u_\mu(\tilde{u}_\mu^T)^r]|^2dx+\int_{\mathbb{R}^N}V(x)|u_\mu|^2|\tilde{u}_\mu^T|^{2r}dx\\
&+\frac{C_1}{(r+1)^2}\int_{T-1<|u_\mu|<T}|\nabla[u_\mu(\tilde{u}_\mu^T)^r]|^2dx\\
\geq&\frac{C_1}{(r+1)^2}\int_{\mathbb{R}^N}|\nabla[u_\mu(\tilde{u}_\mu^T)^r]|^2dx+\int_{\mathbb{R}^N}V(x)|u_\mu|^2|\tilde{u}_\mu^T|^{2r}dx.
\end{aligned}
$$

As a consequence, from (h_1) and (h_2), for fixed $\mu>0$ and small $\varepsilon>0$,

$$|h_\mu(t)|\leq V_0|t|+(1+\mu)C|t|^{2^*-1},\ \forall t\in\mathbb{R}.$$

Combining with the above equations, we obtain

$$\frac{C_1}{(r+1)^2}\int_{\mathbb{R}^N}|\nabla[u_\mu(\tilde{u}_\mu^T)^r]|^2dx\leq(1+\mu)C\int_{\mathbb{R}^N}|u_\mu|^{2^*}|\tilde{u}_\mu^T|^{2r}dx.$$

By the Sobolev embedding theorem,

$$\frac{C_2}{(r+1)^2}[\int_{\mathbb{R}^N}|u_\mu|^{2^*}|\tilde{u}_\mu^T|^{2r\cdot\frac{2^*}{2}}dx]^{\frac{2}{2^*}}\leq\frac{C_1}{(r+1)^2}\int_{\mathbb{R}^N}|\nabla[u_\mu(\tilde{u}_\mu^T)^r]|^2dx.$$

As a result,

$$[\int_{\mathbb{R}^N}|u_\mu|^{2^*}|\tilde{u}_\mu^T|^{2r\cdot\frac{2^*}{2}}dx]^{\frac{2}{2^*}}\leq(1+\mu)C(r+1)^2\int_{\mathbb{R}^N}|u_\mu|^{2^*}|\tilde{u}_\mu^T|^{2r}dx.$$

Take $r_0>0$ and $r_k=r_0(\frac{2^*}{2})^k=r_{k-1}\cdot\frac{2^*}{2}$. Then

$$
\begin{aligned}
&[\int_{\mathbb{R}^N}|u_\mu|^{2^*}|\tilde{u}_\mu^T|^{2r_k}dx]^{\frac{1}{2r_k}}\\
\leq&[\sqrt{1+\mu}\sqrt{C}(r_{k-1}+1)]^{\frac{1}{r_{k-1}}}[\int_{\mathbb{R}^N}|u_\mu|^{2^*}|\tilde{u}_\mu^T|^{2r_{k-1}}dx]^{\frac{1}{2r_{k-1}}}\\
\leq&\prod_{i=0}^{k-1}[\sqrt{1+\mu}\sqrt{C}(r_i+1)]^{\frac{1}{r_i}}[\int_{\mathbb{R}^N}|u_\mu|^{2^*}|\tilde{u}_\mu^T|^{2r_0}dx]^{\frac{1}{2r_0}}\\
=&\prod_{i=0}^{k-1}(1+\mu)^{\frac{1}{2r_i}}\prod_{i=0}^{k-1}[\sqrt{C}(r_i+1)]^{\frac{1}{r_i}}[\int_{\mathbb{R}^N}|u_\mu|^{2^*}|\tilde{u}_\mu^T|^{2r_0}dx]^{\frac{1}{2r_0}}\\
=&\prod_{i=0}^{k-1}(1+\mu)^{\frac{1}{2r_i}}exp\{\sum_{i=0}^{k-1}\frac{1}{r_i}\ln[\sqrt{C}(r_i+1)]\}[\int_{\mathbb{R}^N}|u_\mu|^{2^*}|\tilde{u}_\mu^T|^{2r_0}dx]^{\frac{1}{2r_0}}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&[\int_{R^N} |u_\mu|^{2^*} |\tilde{u}_\mu^T|^{2r_0\cdot\frac{N}{N-2}} dx]^{\frac{N-2}{N}} \\
\le& C(r_0+1)^2 \int_{\mathbb{R}^N} |u_\mu|^{2^*} |\tilde{u}_\mu^T|^{2r_0} dx \\
\le& C(r_0+1)^2 \int\limits_{|u_\mu(x)|<\rho} |u_\mu|^{2^*} |\tilde{u}_\mu^T|^{2r_0} dx \\
&+ C(r_0+1)^2 (\int\limits_{|u_\mu(x)|\ge\rho} |u_\mu|^{2^*} dx)^{\frac{2}{N}} (\int_{R^N} |u_\mu|^{2^*} |\tilde{u}_\mu^T|^{2r_0\cdot\frac{N}{N-2}} dx)^{\frac{N-2}{N}}.
\end{aligned}
$$

Take $\rho > 0$ be such that $C(r_0+1)^2(\int\limits_{|u_\mu(x)|\ge\rho} |u_\mu|^{2^*} dx)^{\frac{2}{N}} < \frac{1}{2}$. Then

$$
[\int_{R^N} |u_\mu|^{2^*} |\tilde{u}_\mu^T|^{2r_0\cdot\frac{N}{N-2}} dx]^{\frac{N-2}{N}} \le C(r_0+1)^2 \int\limits_{|u_\mu(x)|<\rho} |u_\mu|^{2^*} |\tilde{u}_\mu^T|^{2r_0} dx \le C.
$$

Set

$$
d_k = exp\{\sum_{i=0}^{k-1} \frac{1}{r_i} \ln[\sqrt{C}(r_i+1)]\}
$$

and

$$
e_k = \prod_{i=0}^{k-1} (1+\mu)^{\frac{1}{2r_i}} = (1+\mu)^{\frac{2^*}{(2^*-2)2r_0}[1-(\frac{2}{2^*})^k]}.
$$

Then $d_k \to d_\infty$ as $k \to \infty$ and $e_k \to e_\infty = (1+\mu)^{\frac{2^*}{(2^*-2)2r_0}}$ as $k \to \infty$. By Lemma 2.1, we have

$$
\begin{aligned}
&[\int_{\mathbb{R}^N} |u_\mu|^{2^*} |\tilde{u}_\mu^T|^{2r_k} dx]^{\frac{1}{2r_k}} \\
\le& d_k e_k [\int_{\mathbb{R}^N} |u_\mu|^{2^*} |\tilde{u}_\mu^T|^{2r_0} dx]^{\frac{1}{2r_0}} \\
\le& d_k e_k [(\int_{\mathbb{R}^N} |u_\mu|^{2^*})^{\frac{2}{N}} (\int_{\mathbb{R}^N} |u_\mu|^{2^*} |\tilde{u}_\mu^T|^{2r_0\cdot\frac{N}{N-2}} dx)^{\frac{N-2}{N}}]^{\frac{1}{2r_0}} \\
\le& C d_k e_k (\int_{\mathbb{R}^N} |u_\mu|^{2^*})^{\frac{1}{Nr_0}} \\
\le& C d_k e_k.
\end{aligned}
$$

Using Fatous lemma in $T \to +\infty$, we obtain

$$
\|u_\mu\|_{2^*+2r_k}^{\frac{2^*+2r_k}{2r_k}} \le C d_k e_k.
$$

Consequently, let $k \to \infty$, we obtain

$$
\|u_\mu\|_{L^\infty} \le C d_\infty e_\infty = C d_\infty (1+\mu)^{\frac{2^*}{(2^*-2)2r_0}} := B(1+\mu)^D.
$$

This completes the proof. □

Proof of Theorem 1.1: For large $M > 0$, we can choose small $\mu_0 > 0$ such that $\|u_\mu\|_{L^\infty} \leq B(1+\mu)^D \leq M$ for all $\mu \in (0, \mu_0]$. Which indicates $h_\mu(u_\mu) = \mu|u_\mu|^{p-2}u_\mu + f(u_\mu)$. Consequently, problem (1.1) admits a couple of weak solutions $(u_\mu, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}^+$ with $\int_{\mathbb{R}^N} |u_\mu|^2 dx = a^2$. This completes the proof.

3. Conclusions

In this paper, we study the existence of normalized solutions for the following Schrödinger equation with Sobolev supercritical growth:

$$\begin{cases} -\Delta u + V(x)u + \lambda u = f(u) + \mu |u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{3.1}$$

where $p > 2^* := \frac{2N}{N-2}$, $N \geq 3$, $a > 0$, $\lambda \in \mathbb{R}$ is an unknown Lagrange multiplier, $V \in C(\mathbb{R}^N, \mathbb{R})$, f satisfies the following assumptions:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $0 < V_0 := \inf_{x \in \mathbb{R}^N} V(x) \leq V(x) \leq V_\infty := \lim_{|x| \to \infty} V(x) < +\infty$ for all $x \in \mathbb{R}^N$.

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd, and there exist $m \in (2, 2 + \frac{4}{N})$ and $\alpha \in (0, +\infty)$ such that

$$\lim_{t \to 0} \frac{|f(t)|}{|t|^{m-1}} = \alpha.$$

(f_2) There exist two constants $C_1,\ C_2 > 0$ and $q \in (m, 2 + \frac{4}{N})$ such that

$$|f(t)| \leq C_1 + C_2|t|^{q-1},\ \forall t \in R.$$

(f_3) $\frac{f(t)}{t^{q-1}}$ is an increasing function of t on $(0, +\infty)$.

By employing the truncation technique, we establish the existence of normalized solutions to this Sobolev supercritical problem. In particular, there exists some $\mu_0 > 0$ such that for $\mu \in (0, \mu_0]$, problem (3.1) admits a couple of weak solutions $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}^+$ with $\int_{\mathbb{R}^N} |u|^2 dx = a^2$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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