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Research article

A fully discrete local discontinuous Galerkin method based on generalized numerical fluxes to variable-order time-fractional reaction-diffusion problem with the Caputo fractional derivative

Lijie Liu, Xiaojing Wei and Leilei Wei*

College of Science, Henan University of Technology, Zhengzhou 450001, P.R. China

* **Correspondence:** Email: leileiwei@haut.edu.cn.

Abstract: In this paper, an effective numerical method for solving the variable-order(VO) fractional reaction diffusion equation with the Caputo fractional derivative is constructed and analyzed. Based on the generalized alternating numerical flux, we get a fully discrete local discontinuous Galerkin scheme for the problem. From a practical standpoint, the generalized alternating numerical flux, which is distinct from the purely alternating numerical flux, has a more extensive scope. For $0 < \alpha(t) < 1$, we prove that the method is unconditionally stable and the errors attain $(k + 1)$-th order of accuracy for piecewise P^k polynomials. Finally, some numerical experiments are performed to show the effectiveness and verify the accuracy of the method.

Keywords: Caputo fractional derivative; generalized alternating numerical flux; stability; error estimate

1. Introduction

Fractional order partial differential equations are a generalization of classical partial differential equations [1–3]. The theory of fractional order calculus has a wide range of applications in mathematical physics equations, electrochemical processes, mechanics, anomalous diffusion, processing of signals and finance [4–8]. Since the analytical solutions of many fractional order differential equations cannot be solved exactly [9], numerical methods for fractional order differential equations have attracted a great deal of attention from an increasing number of scholars. Many researchers in domestic and foreign countries have solved many different types of fractional order partial differential equations by different methods. There are spectral methods [10–15], finite difference method [16–19], finite element method [20–25], local discontinuous Galerkin methods (LDG) [26–30]. etc.

In recent years, many scholars have studied the Caputo-type reaction diffusion equation. Imran and Shah etal [31] investigated unsteady time fractional natural convection flow of incompressible viscous fluid, and obtained some exact solutions for temperature and velocity fields by Caputo time fractional derivatives in dimensionless form. Mahsud, Shah and Vieru [32] studied unsteady flows of an upper-convected Maxwell fluid which is described by the fractional differential equations with time-fractional Caputo-Fabrizio derivatives. Shah etal [33] studied natural convective flows of Prabhakar-like fractional viscoelastic fluids by introducing the generalized fractional constitutive equations, and used the time-fractional Prabhakar derivative to describe the generalized memory effects. J. Shu et al. [34] studied the asymptotic behavior of the solution of the non-autonomous fractional-order stochastic reaction-diffusion equation with multiplicative noise in R. M. Stynes et al. [35] studied the reaction-diffusion problem with Caputo time derivatives, giving a new analysis of the standard finite difference method for this problem. C.B. Huang et al. [36] presented a fully discrete numerical method for computing an approximate solution of fractional reaction-diffusion initial-boundary value problems based on $L1$ discretization in time and direct discontinuous Galerkin (DDG) finite element in space. V.K. Baranwal et al. [37] proposed a new analytical algorithm for solving a system of highly nonlinear time-fractional order reaction-diffusion equations, a fusion of the variational iterative method and the Adomian decomposition method. S. Ali et al. [38] obtained an approximate solution of the fractional order Cauchy reaction diffusion equation using the optimal homotopy asymptotic method. New numerical schemes for solving nonlinear fractional convection-diffusion equations of order $\beta \in [1, 2]$ were developed by H. Safdari et al. [39]. They proposed locally discontinuous Galerkin methods by adopting linear, quadratic and cubic B-spline basis functions.

The Discontinuous Galerkin (DG) method is between a finite element and a finite volume method, and uses a discontinuous solution space and has high accuracy for any order of accuracy. The main idea of the LDG method is to transform the original higher-order partial differential equations into several equivalent systems of first-order equations by introducing auxiliary variables and then discretize the obtained systems of first-order equations using the DG method [40]. In this paper, we will consider the LDG method based on the generalized numerical flux to solve the time-fractional reaction-diffusion equation with Caputo fractional order derivatives.

$$\begin{aligned} &{}_0^C D_t^{\alpha(t)} u + \sigma u - u_{xx} = F(x,t), \qquad (x,t) \in (a,b) \times (0,T], \\ &u(x,0) = u_0(x), \qquad x \in [a,b], \end{aligned} \tag{1.1}$$

in which $0 < \alpha(t) \le \dot{\alpha} < 1$, $\sigma \ge 0$. The F, u_0 are smooth functions. In this paper, the solution is considered to be periodic or compactly supported.

The Caputo fractional derivative operator [41] is defined by

$${}_0^C D_t^{\alpha(t)} u(x,t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^{\alpha(t)}} ds. \tag{1.2}$$

In Section 2, some notations and projections are given. In Section 3, we will propose a fully discrete LDG method for the equation (1.1), and prove that the scheme is unconditionally stable and convergent with $O(h^{k+1} + (\Delta t)^{2-\dot{\alpha}})$. The correctness of the theoretical analysis is shown in Section 4 with numerical examples. Finally, the conclusion is given in Section 5.

2. Notations and auxiliary results

Let $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$ be partition of $\Omega = [a,b]$, denote $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, \cdots N$, and $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $1 \le j \le N$, $h = \max\limits_{1 \le j \le N} h_j$.

We denote $u^+_{j+\frac{1}{2}} = \lim_{t\to 0^+} u(x_{j+\frac{1}{2}} + t)$ and $u^-_{j+\frac{1}{2}} = \lim_{t\to 0^+} u(x_{j+\frac{1}{2}} - t)$.

The piecewise-polynomial space V_h^k is defined as

$$V_h^k = \{\vartheta : \vartheta \in P^k(I_j),\ x \in I_j,\ j = 1, 2, \cdots N\},$$

where k is the order of piecewise polynomial.

For any periodic function ϖ, the following projection is used to prove the error estimate. Let $\omega^e = \mathcal{P}_\delta \omega - \omega$, that is $\mathcal{P}$,

$$\int_{I_j} \omega^e \vartheta dx = 0,\ \ \forall\, \vartheta \in P^{k-1}(I_j),\ \ (\omega^e)^{(\delta)}_{j+\frac{1}{2}} = 0. \tag{2.1}$$

Lemma 2.1. *Let $\delta \neq \frac{1}{2}$. If $\omega \in H^{s+1}[a,b]$, there holds*

$$\|\omega^e\| + h^{\frac{1}{2}} \|\omega^e\|_{L^2(\Gamma_h)} \leq C h^{min(k+1, s+1)} \|\omega\|_{s+1}, \tag{2.2}$$

where the bounding constant $C > 0$ is independent of h and ω. Here Γ_h denotes the set of boundary points of all elements I_j, and

$$\|\omega^e\|_{L^2(\Gamma_h)} = \left(\frac{1}{2} \sum_{i=1}^{N} [((\omega^e)^+)^2_{i-\frac{1}{2}} + ((\omega^e)^-)^2_{i+\frac{1}{2}}] \right)^{\frac{1}{2}}. \tag{2.3}$$

Let the scalar inner product on $L^2(E)$ be denoted by $(\cdot,\cdot)_E$, and the associated norm by $\|\cdot\|_E$. If $E = \Omega$, we drop E. In the paper, C is a positive number that may have different values in different places.

3. The scheme

We first construct the fully discrete LDG method for the equation (1.1).

We can rewrite the equation (1.1) into the following form:

$$p = u_x, \qquad {}_0^C D_t^{\alpha(t)} u + \sigma u - p_x = F(x,t). \tag{3.1}$$

Let $t_n = n\Delta t = \frac{n}{M} T$, $\Delta t = t_n - t_{n-1}$, we approximate the Caputo fractional derivative ${}_0^C D_t^{\alpha(t_n)} u(x, t_n)$:

$$\begin{aligned}
{}_0^C D_t^{\alpha(t_n)} u(x,t_n) &= \frac{1}{\Gamma(1-\alpha(t_n))} \int_0^{t_n} \frac{\partial u(x,s)}{\partial s} (t_n - s)^{-\alpha(t_n)} ds \\
&= \frac{1}{\Gamma(1-\alpha(t_n))} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\partial u(x,s)}{\partial s} (t_n - s)^{-\alpha(t_n)} ds \\
&= \frac{1}{\Gamma(1-\alpha(t_n))} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{u(x,t_{k+1}) - u(x,t_k)}{\Delta t} (t_n - s)^{-\alpha(t_n)} ds + r^n(x) \\
&= \frac{1}{\Gamma(1-\alpha(t_n))} \sum_{k=0}^{n-1} \frac{u(x,t_{k+1}) - u(x,t_k)}{\Delta t} (\frac{1}{\alpha(t_n) - 1}) \\
&\quad \cdot ((t_n - t_{k+1})^{1-\alpha(t_n)} - (t_n - t_k)^{1-\alpha(t_n)}) + r^n(x)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}\sum_{k=1}^{n}(u(x,t_k)-u(x,t_{k-1}))\\
&\cdot((n-k+1)^{1-\alpha(t_n)}-(n-k)^{1-\alpha(t_n)})+r^n(x)\\
&= \frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}\sum_{k=1}^{n}b^n_{n-k}(u(x,t_k)-u(x,t_{k-1}))+r^n(x)\\
&= \frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}(u(x,t_n)+\sum_{k=1}^{n-1}(b^n_{n-k}-b^n_{n-k-1})u(x,t_k)\\
&- b^n_{n-1}u(x,t_0))+r^n(x),
\end{aligned}
\tag{3.2}
$$

where γ^n is the truncation error in the time direction and $\|\gamma^n\| \le C(\Delta t)^{2-\dot{\alpha}}$.

In which, $b^n_k = (k+1)^{1-\alpha(t_n)} - (k)^{1-\alpha(t_n)}$, and it can be seen that the coefficients have the following properties [35]

$$b_k > 0, \quad k = 0,1,2,\cdots,n,$$

$$1 = b_0 > b_1 > b_2 > \cdots > b_n, \quad b_n \to 0 \quad as \quad n \to \infty.$$

Let $u^n_h,\ p^n_h \in V^k_h$ be the approximations of $u(\cdot,t_n),\ p(\cdot,t_n)$, respectively, $F^n(x) = F(x,t_n)$. Find $u^n_h,\ p^n_h \in V^k_h$, such that for all test functions $v, w \in V^k_h$,

$$
\begin{aligned}
&(\sigma+\frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))})\int_\Omega u^n_h v dx + \int_\Omega p^n_h v_x dx - \sum_{j=1}^{N}((\widehat{p^n_h}v^-)_{j+\frac12} - (\widehat{p^n_h}v^+)_{j-\frac12})\\
&= \frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\int_\Omega u^k_h v dx\\
&+ \frac{b_{n-1}}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}\int_\Omega u^0_h v dx + \int_\Omega F^n v dx,\\
&\int_\Omega p^n_h w dx + \int_\Omega u^n_h w_x dx - \sum_{j=1}^{N}((\widehat{u^n_h}w^-)_{j+\frac12} - (\widehat{u^n_h}w^+)_{j-\frac12}) = 0.
\end{aligned}
\tag{3.3}
$$

Selecting the appropriate numerical flux will play a key role in theoretical analysis for the LDG scheme. From the practical aspect, the generalized alternating numerical fluxes have more application than the traditional numerical fluxes [42]. We consider the following generalized alternating numerical fluxes

$$\widehat{u^n_h} = \delta(u^n_h)^- + (1-\delta)(u^n_h)^+, \quad \widehat{p^n_h} = (1-\delta)(p^n_h)^- + \delta(p^n_h)^+, \tag{3.4}$$

here we consider the case $\delta \ne \frac{1}{2}$. For $\delta = \frac{1}{2}$, the property about unique existence and approximation of the generalized Gauss-Radau projection will become complicated [43].

For the sake of convenience, we denote

$$\Phi_\Omega(u_h^n, p_h^n; w, v) = \int_\Omega u_h^n w_x dx - \sum_{j=1}^{N}\Big((\widehat{u_h^n} w^-)_{j+\frac{1}{2}} - (\widehat{u_h^n} w^+)_{j-\frac{1}{2}}\Big) + \int_\Omega p_h^n v_x dx - \sum_{j=1}^{N}\Big((\widehat{p_h^n} v^-)_{j+\frac{1}{2}} - (\widehat{p_h^n} v^+)_{j-\frac{1}{2}}\Big). \tag{3.5}$$

Next, we give the stability analysis of the scheme (3.3).

3.1. Stability analysis

Without loss of generality, we take $F = 0$ in the theoretical analysis. The following stability result for the scheme (3.3) can be obtained.

Theorem 3.1. *For periodic or compactly supported boundary conditions, the fully-discrete LDG scheme (3.3) is unconditionally stable, and the numerical solution u_h^n satisfies*

$$\|u_h^n\| \le \|u_h^0\|, \quad n = 1, 2, \cdots, M. \tag{3.6}$$

Proof. Taking the test functions $v = u_h^n,\ w = p_h^n$ in the scheme (3.3), and with the fluxes choice (3.4), we obtain

$$\begin{aligned}
&(\sigma + \frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))})\|u_h^n\|^2 + \|p_h^n\|^2 + \Phi_\Omega(u_h^n, p_h^n; p_h^n, u_h^n) \\
&= \frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}\sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\int_\Omega u_h^k u_h^n dx \\
&+ \frac{b_{n-1}}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}\int_\Omega u_h^0 u_h^n dx.
\end{aligned} \tag{3.7}$$

In $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, we obtain

$$\begin{aligned}
\Phi_{I_j}(u_h^n, p_h^n; p_h^n, u_h^n) &= \int_\Omega u_h^n (p_h^n)_x dx - \Big(((\delta(u_h^n)^- + (1-\delta)(u_h^n)^+)(p_h^n)^-)_{j+\frac{1}{2}} \\
&+ ((\delta(u_h^n)^- + (1-\delta)(u_h^n)^+)(p_h^n)^+)_{j-\frac{1}{2}}\Big) \\
&+ \int_\Omega p_h^n (u_h^n)_x dx - \Big((((1-\delta)(p_h^n)^- + \delta(p_h^n)^+)(u_h^n)^-)_{j+\frac{1}{2}} \\
&+ (((1-\delta)(p_h^n)^- + \delta(p_h^n)^+)(u_h^n)^+)_{j-\frac{1}{2}}\Big) \\
&= ((p_h^n)^-(u_h^n)^-)_{j+\frac{1}{2}} - ((p_h^n)^+(u_h^n)^+)_{j-\frac{1}{2}} - (\delta(u_h^n)^-(p_h^n)^-)_{j+\frac{1}{2}} \\
&- ((u_h^n)^+(p_h^n)^-)_{j+\frac{1}{2}} + (\delta(u_h^n)^+(p_h^n)^-)_{j+\frac{1}{2}} + (\delta(u_h^n)^-(p_h^n)^+)_{j-\frac{1}{2}} \\
&+ ((u_h^n)^+(p_h^n)^+)_{j-\frac{1}{2}} - (\delta(u_h^n)^+(p_h^n)^+)_{j-\frac{1}{2}} - ((p_h^n)^-(u_h^n)^-)_{j+\frac{1}{2}} \\
&+ (\delta(p_h^n)^-(u_h^n)^-)_{j+\frac{1}{2}} - (\delta(p_h^n)^+(u_h^n)^-)_{j+\frac{1}{2}} + ((u_h^n)^+(p_h^n)^-)_{j-\frac{1}{2}} \\
&- (\delta(p_h^n)^-(u_h^n)^+)_{j-\frac{1}{2}} + (\delta(p_h^n)^+(u_h^n)^+)_{j-\frac{1}{2}}.
\end{aligned} \tag{3.8}$$

After some computations and summing (3.8) from 1 to N over j, we get

$$\Phi_\Omega(u_h^n, p_h^n; p_h^n, u_h^n) = 0. \tag{3.9}$$

Combine (3.9) and the Cauchy-Schwarz inequality, (3.7) becomes

$$\|u_h^n\| \leq \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\|u_h^k\| + b_{n-1}\|u_h^0\|. \tag{3.10}$$

Use mathematical induction to prove Theorem 3.1. Let $n = 1$ in (3.10), and we can obtain

$$\|u_h^1\| \leq b_{n-1}\|u_h^0\| \leq \|u_h^0\|. \tag{3.11}$$

Now we assume that the following inequality holds

$$\|u_h^m\| \leq \|u_h^0\|, \ m = 1, 2, 3, \cdots, n-1. \tag{3.12}$$

We need to prove

$$\|u_h^n\| \leq \|u_h^0\|,$$

it follows from (3.10) and (3.12) that

$$\begin{aligned}\|u_h^n\| &\leq (b_{n-2} - b_{n-1})\|u_h^1\| + (b_{n-3} - b_{n-2})\|u_h^2\| \\ &\quad + \cdots + (b_0 - b_1)\|u_h^{n-1}\| + b_{n-1}\|u_h^0\| \\ &\leq \big((b_{n-2} - b_{n-1}) + (b_{n-3} - b_{n-2}) + \cdots + (b_0 - b_1) + b_{n-1})\|u_h^0\| \\ &\leq b_0\|u_h^0\| \\ &= \|u_h^0\|.\end{aligned} \tag{3.13}$$

3.2. Error estimate

Theorem 3.2. *Let $u(x, t_n)$ be the exact solution of the problem (1.1), which is sufficiently smooth with bounded derivatives. Let u_h^n be the numerical solution of the fully discrete LDG scheme (3.3) with flux (3.4),and then the following error estimates holds*

$$\|u(x, t_n) - u_h^n\| \leq C(h^{k+1} + (\Delta t)^{2-\dot{\alpha}}), \ \ 0 < \alpha(t_n) \leq \dot{\alpha} < 1,$$

where C is a constant depending on u, T.

Proof.

$$\begin{aligned}&e_u^n = u(x, t_n) - u_h^n = \xi_u^n - \eta_u^n, \quad \xi_u^n = \mathcal{P}_\delta e_u^n, \quad \eta_u^n = \mathcal{P}_\delta u(x, t_n) - u(x, t_n), \\ &e_p^n = p(x, t_n) - p_h^n = \xi_p^n - \eta_p^n, \quad \xi_p^n = \mathcal{P}_{1-\delta} e_p^n, \quad \eta_p^n = \mathcal{P}_{1-\delta} p(x, t_n) - p(x, t_n).\end{aligned} \tag{3.14}$$

Here η_u^n and η_p^n have been estimated by the inequality (2.2).

Taking the flux (3.4), we can get the following error equation.

$$
\begin{aligned}
&(\sigma+\frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))})\int_\Omega e_u^n v dx+\int_\Omega e_p^n w dx+\Phi_\Omega(e_u^n,e_p^n;w,v)\\
&+\int_\Omega \gamma^n v dx-\frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\\
&\times\int_\Omega e_u^k v dx-\frac{b_{n-1}}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}\int_\Omega e_u^0 v dx=0.
\end{aligned}
\tag{3.15}
$$

Take the test function $v=\xi_u^n,\ w=\xi_p^n$, and use (3.14) in the error equation (3.15), we can get

$$
\begin{aligned}
&(\sigma+\frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))})||\xi_u^n||^2+||\xi_p^n||^2+\Phi_\Omega(\xi_u^n,\xi_p^n;\xi_p^n,\xi_u^n)\\
&=(\sigma+\frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))})\int_\Omega \eta_u^n\xi_u^n dx+\int_\Omega \eta_p^n\xi_p^n dx\\
&+\Phi_\Omega(\eta_u^n,\eta_p^n;\xi_p^n,\xi_u^n)-\int_\Omega \gamma^n\xi_u^n dx+\frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}\\
&\times\Big(\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\int_\Omega \xi_u^k\xi_u^n dx+b_{n-1}\int_\Omega \xi_u^0\xi_u^n dx\\
&-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\int_\Omega \eta_u^k\xi_u^n dx+b_{n-1}\int_\Omega \eta_u^0\xi_u^n dx\Big),
\end{aligned}
\tag{3.16}
$$

using the projection (2.1), and after a certain amount of analysis, we can obtain

$$
\Phi_\Omega(\eta_u^n,\eta_p^n;\xi_p^n,\xi_u^n)=0.
$$

Using the Cauchy-Schwarz inequality, we could have the following equality

$$
\begin{aligned}
&(\sigma+\frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))})||\xi_u^n||^2+||\xi_p^n||^2\\
&\leq(\sigma+\frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))})||\eta_u^n||||\xi_u^n||+||\eta_p^n||||\xi_p^n||+||\gamma^n||||\xi_u^n||\\
&+\frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}(\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})||\xi_u^k||||\xi_u^n||+b_{n-1}||\xi_u^0||||\xi_u^n||)\\
&+\frac{1}{(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))}(\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})||\eta_u^k||||\xi_u^n||+b_{n-1}||\eta_u^0||||\xi_u^n||),
\end{aligned}
\tag{3.17}
$$

let $\varpi=(\Delta t)^{\alpha(t_n)}\Gamma(2-\alpha(t_n))$, and $\varpi<\Gamma(1-\alpha(t_n))T^{\alpha(t_n)}b_{n-1}$, we have

$$
\begin{aligned}
&(\sigma\varpi+1)||\xi_u^n||^2+\varpi||\xi_p^n||^2\\
&\leq(\sigma\varpi+1)||\eta_u^n||||\xi_u^n||+\varpi||\eta_p^n||||\xi_p^n||+\varpi||\gamma^n||||\xi_u^n||\\
&+\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})||\xi_u^k||||\xi_u^n||+b_{n-1}||\xi_u^0||||\xi_u^n||\\
&+\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})||\eta_u^k||||\xi_u^n||+b_{n-1}||\eta_u^0||||\xi_u^n||.
\end{aligned}
\tag{3.18}
$$

Using the nature of projection, and noticing the fact that

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2,$$

we can get

$$\begin{aligned}
&\|\xi_u^n\|^2 + \varpi \|\xi_p^n\|^2 \\
&\leq \frac{b_{n-1}}{4}\|\xi_u^n\|^2 + \frac{\varpi}{4}\|\xi_p^n\|^2 + \frac{b_{n-1}}{4}\|\xi_u^n\|^2 + Cb_{n-1}(h^{2k+2} + (\Delta t)^{4-2\alpha(t_n)}) \\
&+ \frac{1}{2}\sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\|\xi_u^k\|^2 + \frac{1}{2}(b_0 - b_{n-1})\|\xi_u^n\|^2,
\end{aligned} \tag{3.19}$$

so

$$\|\xi_u^n\|^2 \leq Cb_{n-1}(h^{2k+2} + (\Delta t)^{4-2\alpha(t_n)}) + \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\|\xi_u^k\|^2. \tag{3.20}$$

We use mathematical induction to prove the theorem. When $n = 1$, from (3.20), it follows that

$$\|\xi_u^1\|^2 \leq Cb_0(h^{2k+2} + (\Delta t)^{4-2\alpha(t_1)}) = C(h^{2k+2} + (\Delta t)^{4-2\alpha(t_1)}), \tag{3.21}$$

we assume that inequality holds

$$\|\xi_u^m\|^2 \leq C(h^{2k+2} + (\Delta t)^{4-2\alpha(t_m)}), m = 1, 2, \cdots, n-1. \tag{3.22}$$

We only need to prove that the following results holds

$$\|\xi_u^n\|^2 \leq C(h^{2k+2} + (\Delta t)^{4-2\alpha(t_n)}). \tag{3.23}$$

Similar to the proof of stability, we can get the following formula

$$\begin{aligned}
\|\xi_u^n\|^2 &\leq Cb_{n-1}(h^{2k+2} + (\Delta t)^{4-2\alpha(t_n)}) + (b_0 - b_{n-1})C(h^{2k+2} + (\Delta t)^{4-2\alpha(t_n)}) \\
&= C(h^{2k+2} + (\Delta t)^{4-2\alpha(t_n)}),
\end{aligned} \tag{3.24}$$

so

$$\|\xi_u^n\| \leq C(h^{k+1} + (\Delta t)^{2-\alpha(t_n)}).$$

Let $\dot{\alpha} = \max\{\alpha(t_n)\}$, we have

$$\|\xi_u^n\| \leq C(h^{k+1} + (\Delta t)^{2-\dot{\alpha}}), \quad 0 < \alpha(t_n) \leq \dot{\alpha} < 1.$$

Theorem 3.2 could be proved by using the triangle inequality and the interpolation property (2.2).

4. Numerical experiment

Consider the following equation (1.1)

$$ {}_0^C D_t^{\alpha(t)} u + \sigma u - u_{xx} = F(x,t), \qquad (x,t) \in (0,1) \times (0,1], $$

with $u(x,0) = 0$ for $x \in (0,1)$, the function

$$ F(x,t) = \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} cos(2\pi x) + \sigma t^2 cos(2\pi x) + 4\pi^2 t^2 cos(2\pi x) $$

is chosen such that the exact solution of the equation is $u(x,t) = t^2 cos(2\pi x)$.

Choosing a fixed small time step $\Delta t = \frac{1}{1000}$ to avoid contamination of the temporal error, and dividing the space into N elements to form the uniform mesh. The errors in L^2-norm and L^∞-norm for different values $\alpha(t)$ and δs are demonstrated in Tables 1 and 2 where $N = 5, 10, 20, 40$. A uniform $(k+1)$-th order of accuracy for piecewise P^k polynomials can be seen.

Finally, the temporal convergence rate of the scheme (3.3) for $\alpha(t) = \frac{1+t}{2}$ by piecewise P^1 polynomials are provided. Taking the spatial mesh size $h = \frac{1}{300}$, and the temporal meshes $\Delta t = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, respectively. One can find that the temporal convergence rate is first order in Figure 1, which is also consistent with the theoretical results.

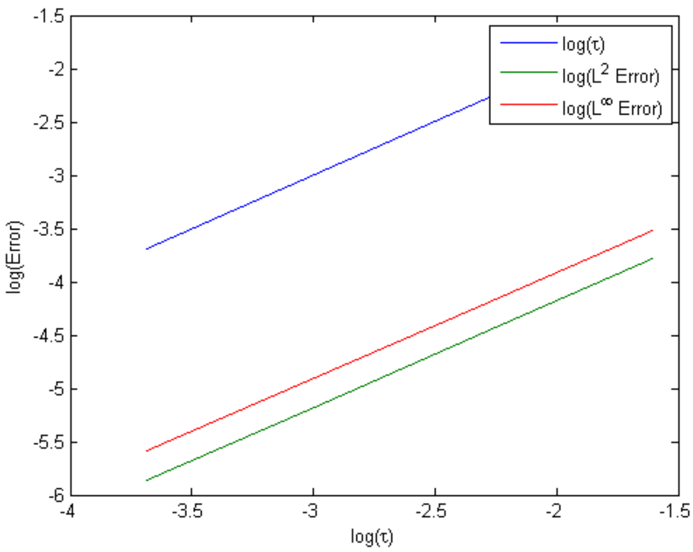

Figure 1. L^2 errors and L^∞ errors versus Δt, order for $\alpha(t) = \frac{1+t}{2}$, $k = 1$.

5. Conclusions

In this paper, a numerical method is investigated to solve the variable-order (VO) fractional reaction-diffusion equation. We obtained the scheme by using the finite difference method in time and the LDG method in space. Based on the generalized alternating numerical fluxes, we prove that the method is unconditionally stable and convergent with $O(h^{k+1} + (\Delta t)^{2-\dot{\alpha}})$. Numerical examples demonstrate the accuracy of our theoretical proofs. In the future, we will use this method to solve models for different kinds of partial differential equations.

Table 1. Accuracy test with generalized numerical fluxes on uniform meshes, $\delta = 0.3, \sigma = 1, M = 10^3, T = 1$.

δ	α	P^k	N	L^2-error	order	L^∞-error	order
$\delta = 0.3$	$\alpha(t) = \frac{4+\sin t}{7}$	P^0	5	5.578658767855278e-02	-	6.128758525527529e-02	-
			10	2.912746349468579e-02	0.93	3.261013486537069e-02	0.91
			20	1.532406061344210e-02	0.92	1.721987475896526e-02	0.92
			40	8.023054900501436e-03	0.93	8.851734358466090e-03	0.96
		P^1	5	8.755852868525588e-03	-	3.485542157254418e-02	-
			10	2.634983954506528e-03	1.73	9.926365548517203e-03	1.81
			20	7.152583393197991e-04	1.88	2.625659485212274e-03	1.91
			40	1.924896345465520e-04	1.89	6.867806235140887e-04	1.93
		P^2	5	1.156456345461556e-03	-	6.654565344565436e-03	-
			10	1.460419033487401e-04	2.98	8.491602777739223e-04	2.97
			20	1.873226320096783e-05	2.96	1.069820711279879e-04	2.98
			40	2.462096734502598e-06	2.93	1.401762242509496e-05	2.93
	$\alpha(t) = \frac{2+5t}{9}$	P^0	5	6.378678578278588e-02	-	7.527525727525726e-02	-
			10	3.149280457002207e-02	1.01	3.943375722563799e-02	0.93
			20	1.611540902599133e-02	0.96	2.067927230184625e-02	0.93
			40	8.437372744802952e-03	0.93	1.068171154099091e-02	0.95
		P^1	5	8.024572578127553e-03	-	4.222752754287276e-02	-
			10	2.414912665006871e-03	1.73	1.186028099755766e-02	1.83
			20	6.555206605418006e-04	1.88	3.253987555342441e-03	1.86
			40	1.764130880394934e-04	1.89	8.649105606198329e-04	1.91
		P^2	5	2.425741475258327e-03	-	6.045042142782754e-03	-
			10	3.223663868243479e-04	2.91	7.821497471580686e-04	2.95
			20	4.121229369412104e-05	2.96	9.691010952895744e-05	3.01
			40	5.423199418904111e-06	2.92	1.288214584710851e-05	2.91

Table 2. Accuracy test with generalized numerical fluxes on uniform meshes, $\delta = 0.8, \sigma = 2, M = 10^3, T = 1$.

δ	α	P^k	N	L^2-error	order	L^∞-error	order
$\delta = 0.8$	$\alpha(t) = \frac{1+e^t}{5}$	P^0	5	5.724552445622547e-02	-	6.478574714578859e-02	-
			10	3.105465392410161e-02	0.88	3.753485419634707e-02	0.78
			20	1.633796222783144e-02	0.92	2.042951724210881e-02	0.87
			40	8.367917577802272e-03	0.96	1.070467258951066e-02	0.93
		P^1	5	7.986854252542785e-03	-	4.012546855587588e-02	-
			10	2.256508374144700e-03	1.82	1.109538313543729e-02	1.85
			20	6.125223000282825e-04	1.88	3.002914119156936e-03	1.88
			40	1.545341652853308e-04	1.92	7.832852791128404e-04	1.93
		P^2	5	3.454554522421105e-03	-	7.274557527427825e-03	-
			10	4.908118904168216e-04	2.81	1.008788583303766e-03	2.85
			20	6.599887988462009e-05	2.89	1.415134542803046e-04	2.83
			40	8.929047525567814e-06	2.88	1.907381031282733e-05	2.89
	$\alpha(t) = \frac{1+3e^t \sin t}{12}$	P^0	5	3.174178652782924e-02	-	5.141417278572186e-02	-
			10	1.711211657764372e-02	0.89	2.790664054651039e-02	0.88
			20	9.086347468429946e-03	0.91	1.518906111189187e-02	0.87
			40	4.653812124107611e-03	0.96	8.200072385789113e-03	0.89
		P^1	5	7.287518525752729e-03	-	4.278571278572178e-02	-
			10	2.000104115036816e-03	1.86	1.183098648229600e-02	1.85
			20	5.429221477198127e-04	1.88	3.142824952164123e-03	1.91
			40	1.424140696543573e-04	1.93	8.148052969109547e-04	1.94
		P^2	5	4.127257142758528e-03	-	8.575152845100579e-03	-
			10	5.793173107659160e-04	2.83	1.131162289348755e-03	2.92
			20	7.790009646231345e-05	2.89	1.565917554275250e-04	2.85
			40	1.101400203543099e-05	2.82	2.116252962036842e-05	2.88

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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