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Research article

Exponential projective synchronization analysis for quaternion-valued memristor-based neural networks with time delays

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Abstract: The issues of exponential projective synchronization and adaptive exponential projective synchronization are analyzed for quaternion-valued memristor-based neural networks (QVMNNs) with time delays. Different from the results of existing decomposition techniques, a direct analytical approach is used to discuss the projection synchronization problem. First, in the framework of measurable selection and differential inclusion, the QVMNNs is transformed into a system with parametric uncertainty. Next, the sign function related to quaternion is introduced. Different proper control schemes are designed and several criteria for ascertaining exponential projective synchronization and adaptive exponential projective synchronization are derived based on Lyapunov theory and the properties of sign function. Furthermore, several corollaries about global projective synchronization are proposed. Finally, the reliability and validity of our results are substantiated by two numerical examples and its corresponding simulation.

Keywords: quaternion-valued memristor-based neural networks; time delays; global exponential projective synchronization; adaptive exponential projective synchronization

1. Introduction

Memristor, was predicted theoretically by Chua in 1971 [1], did not attract much scholars' attention until Hewllet Packard laboratories created the first nanometer-sized memristor successfully in 2008 [2, 3]. As the fourth fundamental circuit element after resistor, capacitor, and inductor, compared with the traditional circuit element, it cannot only change its own properties when the external electrical signal flows but also remember the latest value between the voltage shutdown and the next opening.

These advantages help the memristor to be used to simulate synapses in the biological nervous system; thereby, human brains can be better simulated. By replacing the synapse in traditional neural networks (NNs) with memristor, the memristor-based neural networks(MNNs) can be created, which may help us build artificial NNs better than other NNs. Therefore, research concerning MNNs has become a hot spot and some remarkable research results about MNNs have been reported [4, 5].

Quaternion, as a special case of the Clifford algebra, initially was introduced and proposed by Hamilton [6]. For decades, since the commutativity law of multiplication was no longer available to quaternion, the research on quaternion was not widely exciting until its advantages in image processing [7] were discovered. Since then, the quaternion was introduced into neural networks, and the quaternion-valued neural networks (QVNNs) model was created. At present, QVNNs have exhibited great prospects for utilization in the color image compression, night vision at color low light level, posture control for satellite [8, 9] and other fields [10, 11]. Additionally, the dynamics of QVNNs have triggered the research interest of excellent scholars from domestic and foreign countries. However, due to the non-exchangeability of quaternion multiplication, the research method adopted in real-valued NNs and complex-valued NNs cannot be directly applied to QVNNs, which makes it more difficult to explore the dynamics of QVNNs. In addition, compared with the real-valued NNs and complex-valued NNs, the dynamics behavior of QVNNs is more complex since a quaternion consists of a real part and three imaginary parts. Therefore, studying the dynamics characteristics of QVNNs is a meaningful and challenging topic. Recently, some interesting results about QVNNs have been reported [12–21].

Synchronization, as a crucial dynamical behavior of NNs, has attracted the attention of researchers due to its promising prospects for utilization in information science, image processing, and secure communication. So far, scholars have proposed several synchronization patterns including anti-synchronization control [22, 23], fixed-time synchronization control [24–28], projection synchronization control [29], and so on. Projective synchronization (PS) means that master-slave systems are synchronized by a specific proportional factor. For the control systems, PS is an extremely important dynamics behavior, which extends complete synchronization and anti-synchronization control. Different from PS of real-valued NNs, PS of QVNNs considers the quaternion proportional factor, which improves the complexity and diversity of synchronization. Additionally, the PS issues related to real-valued NNs or complex-valued NNs are special cases of PS problems for QVNNs. Therefore, studying PS of QVNNs has important theoretical and practical value. So far, some interesting results about PS have been reported [30, 31].

Adaptive control is a significant synchronization control method. If adaptive laws are designed appropriately, they can automatically adjust controller parameters in line with the states of systems so that the master-slave system can achieve synchronization. Currently, some meaningful results about the adaptive synchronization of NNs are mostly concentrated on real domains and plural domains [32, 33], while the exploration in quaternion domains is relatively rare. Fortunately, some scholars began considering this problem in the quaternion field. In [34], the adaptive PS of fractional-order delayed QVNNs was successfully explored. Nevertheless, as far as the authors know, there are no previous reports concerning the exploration of adaptive exponential projective synchronization for QVMNNs with time delays.

Motivated by the above discussions, we aim to investigate the controls of exponential projective synchronization for QVMNNs with time delays. The distinctive contributions of this work are reflected as follows:

1) It is the first study to explore the adaptive exponential projective synchronization and exponential projective synchronization for QVMNNs with time delays.

2) Using the one-norm method, the measurable selection and differential inclusion, combined with the sign function of quaternion, two different control schemes that are easy to implement are designed, which can achieve exponential projective synchronization and adaptive exponential projective synchronization, respectively.

3) The theoretical results proposed in this paper can be easily extended to control synchronization problems of other QVNNs, such as complete synchronization and anti-synchronization. It is obvious that it will enrich the literature on exploring control synchronization problems for QVNNs.

The remaining contents of this work are outlined as follows. In Section 2, we introduce the model and its initial information, the definitions and lemmas needed to discuss. In Section 3, we design different proper control schemes and obtain criteria about exponential projective synchronization and adaptive exponential projective synchronization. In Section 4, the reliability and validity of the theoretical results proposed in this work are tested and verified by two numerical examples.

Notations: $\mathbb{Q}$ and $\mathbb{R}$ denote quaternion field and real field, respectively. A quaternion $x = x^{(r)} + x^{(i)}\mathbf{i} + x^{(j)}\mathbf{j} + x^{(k)}\mathbf{k} \in \mathbb{Q}$, $x^* = x^{(r)} - x^{(i)}\mathbf{i} - x^{(j)}\mathbf{j} - x^{(k)}\mathbf{k}$ denotes conjugate. $\|x\|_1 = |x^{(r)}| + |x^{(i)}| + |x^{(j)}| + |x^{(k)}|$ denotes the one-norm. The one-norm of $y = (y_1, ..., y_n)^T \in \mathbb{Q}^n$ is written as $\|y\|_1 = \sum_{p=1}^{n} \|y_p\|_1$.

2. Problem description and preliminaries

In this work, we will consider the QVMNNs model with time delays through the following:

$$\dot{x}_p(t) = -a_p x_p(t) + \sum_{q=1}^{n} b_{pq}(x_p(t)) f_q(x_q(t)) + \sum_{q=1}^{n} c_{pq}(x_p(t)) g_q(x_q(t-\upsilon)) + I_p(t) \tag{2.1}$$

where $p = 1, 2, ..., n$. At time t, $x_p(t) \in \mathbb{Q}$ denotes the state of the pth neuron. $f_q(\cdot), g_q(\cdot) \in \mathbb{Q}$ are the activation functions of the qth neuron. $a_p > 0$ is the self-feedback connection weight. $b_{pq}(\cdot), c_{pq}(\cdot) \in \mathbb{Q}$ denote the memristive connection weights without and with delays, respectively. υ is the transmission time delays with $\upsilon > 0$. $I_p(t) \in \mathbb{Q}$ indicates the external input.

From the current-votage characteristics and the nature of memristor, we have:

$$b_{pq}(x_p(t)) = \begin{cases} b_{pq}^{TT}, & \|x_p(t)\|_1 \le H_p, \\ b_{pq}^{T}, & \|x_p(t)\|_1 > H_p, \end{cases} \qquad c_{pq}(x_p(t)) = \begin{cases} c_{pq}^{TT}, & \|x_p(t)\|_1 \le H_p, \\ c_{pq}^{T}, & \|x_p(t)\|_1 > H_p, \end{cases} \tag{2.2}$$

where the switching jump $H_p > 0$, and b_{pq}^{TT}, b_{pq}^{T}, c_{pq}^{TT}, $c_{pq}^{T} \in \mathbb{Q}$, $p, q = 1, 2, ..., n$, are known constants about memristances.

The initial values for system (1) are $x_p(s) = \psi_p(s) \in C([t_0 - \upsilon, t_0], \mathbb{Q})$, $p = 1, 2, ...n$. Moreover, $x(t) = (x_1(t), ..., x_n(t))$, let $x(t) \in C([t_0 - \upsilon, t_0], \mathbb{Q}^n)$.

Let $\tilde{b}_{pq} = \frac{1}{2}(b_{pq}^{TT} + b_{pq}^{T})$, $\tilde{\tilde{b}}_{pq} = \frac{1}{2}(b_{pq}^{TT} - b_{pq}^{T})$, $\tilde{c}_{pq} = \frac{1}{2}(c_{pq}^{TT} + c_{pq}^{T})$, $\tilde{\tilde{c}}_{pq} = \frac{1}{2}(c_{pq}^{TT} - c_{pq}^{T})$, system (2.1) can be rewritten as

$$\begin{aligned} \dot{x}_p(t) &= -a_p x_p(t) + \sum_{q=1}^{n} (\tilde{b}_{pq} + \Delta b_{pq}(x_p(t))) f_q(x_q(t)) \\ &\quad + \sum_{q=1}^{n} (\tilde{c}_{pq} + \Delta c_{pq}(x_p(t))) g_q(x_q(t-\upsilon)) + I_p(t), \end{aligned} \tag{2.3}$$

where

$$\Delta b_{pq}(x_p(t)) = \begin{cases} \tilde{\tilde{b}}_{pq}, & \|x_p(t)\|_1 \le H_p, \\ -\tilde{\tilde{b}}_{pq}, & \|x_p(t)\|_1 > H_p, \end{cases} \quad \Delta c_{pq}(x_p(t)) = \begin{cases} \tilde{\tilde{c}}_{pq}, & \|x_p(t)\|_1 \le H_p, \\ -\tilde{\tilde{c}}_{pq}, & \|x_p(t)\|_1 > H_p. \end{cases} \tag{2.4}$$

Next, recall that the use of differential inclusion theory, system (2.3) is equivalent to the following differential inclusion:

$$\begin{aligned} \dot{x}_p(t) \in & -a_p x_p(t) + \sum_{q=1}^{n} (\tilde{b}_{pq} + co[-\tilde{\tilde{b}}_{pq}, \tilde{\tilde{b}}_{pq}]) f_q(x_q(t)) \\ & + \sum_{q=1}^{n} (\tilde{c}_{pq} + co[-\tilde{\tilde{c}}_{pq} \tilde{\tilde{c}}_{pq}]) g_q(x_q(t-\upsilon)) + I_p(t). \end{aligned} \tag{2.5}$$

According to the measurable selection theory, there exist two measurable functions $\pi_{pq}(t) = \pi^1_{pq}(t) \cdot (b^{TT}_{pq} - b^T_{pq}) \in co[-\tilde{\tilde{b}}_{pq}, \tilde{\tilde{b}}_{pq}]$ and $w_{pq}(t) = w^1_{pq}(t) \cdot (c^{TT}_{pq} - c^T_{pq}) \in co[-\tilde{\tilde{c}}_{pq}, \tilde{\tilde{c}}_{pq}]$ such that

$$\dot{x}_p(t) = -a_p x_p(t) + \sum_{q=1}^{n} (\tilde{b}_{pq} + \pi_{pq}(t)) f_q(x_q(t)) + \sum_{q=1}^{n} (\tilde{c}_{pq} + w_{pq}(t)) g_q(x_q(t-\upsilon)) + I_p(t), \tag{2.6}$$

where $\pi^1_{pq}(t), w^1_{pq}(t) \in co[-\frac{1}{2}, \frac{1}{2}]$.

For master system (2.1), we design the slave system as follows:

$$\dot{y}_p(t) = -a_p y_p(t) + \sum_{q=1}^{n} b_{pq}(y_p(t)) f_q(y_q(t)) + \sum_{q=1}^{n} c_{pq}(y_p(t)) g_q(y_q(t-\upsilon)) + I_p(t) + u_p(t), \tag{2.7}$$

where $y_p(t) \in \mathbb{Q}$ denotes the state of the pth neuron and $u_p(t)$ denotes the controller to be designed. The initial values for system (7) are $y_p(s) = \phi_p(s) \in C([t_0 - \upsilon, t_0], \mathbb{Q})$, $p = 1, 2, ...n$. Moreover, $x(t) = (x_1(t), ..., x_n(t))$, let $x(t) \in C([t_0 - \upsilon, t_0], \mathbb{Q}^n)$.

Analogously, system (2.7) can be can be rewritten as

$$\begin{aligned} \dot{y}_p(t) = & -a_p y_p(t) + \sum_{q=1}^{n} (\tilde{b}_{pq} + \Delta b_{pq}(y_p(t))) f_q(y_q(t)) \\ & + \sum_{q=1}^{n} (\tilde{c}_{pq} + \Delta c_{pq}(y_p(t))) g_q(y_q(t-\upsilon)) + I_p(t) + u_p(t), \end{aligned} \tag{2.8}$$

where

$$\Delta b_{pq}(y_p(t)) = \begin{cases} \tilde{\tilde{b}}_{pq}, & \|y_p(t)\|_1 \le H_p, \\ -\tilde{\tilde{b}}_{pq}, & \|y_p(t)\|_1 > H_p, \end{cases} \quad \Delta c_{pq}(y_p(t)) = \begin{cases} \tilde{\tilde{c}}_{pq}, & \|y_p(t)\|_1 \le H_p, \\ -\tilde{\tilde{c}}_{pq}, & \|y_p(t)\|_1 > H_p. \end{cases} \tag{2.9}$$

Analogously, system (2.8) can be converted as the following differential inclusion:

$$\begin{aligned}\dot{y}_p(t) \in & -a_p y_p(t) + \sum_{q=1}^{n}(\tilde{b}_{pq} + co[-\tilde{\tilde{b}}_{pq}, \tilde{\tilde{b}}_{pq}])f_q(y_q(t)) \\ & + \sum_{q=1}^{n}(\tilde{c}^*_{pq} + co[-\tilde{\tilde{c}}_{pq}, \tilde{\tilde{c}}_{pq}])g_q(y_q(t-\upsilon)) + I_p(t) + u_p(t). \end{aligned} \tag{2.10}$$

Next, by applying the similar method, there exist two measurable functions $\gamma_{pq}(t) = \gamma^1_{pq}(t) \cdot (b^{TT}_{pq} - b^T_{pq}) \in co[-\tilde{\tilde{b}}_{pq}, \tilde{\tilde{b}}_{pq}]$ and $\delta_{pq}(t) = \delta^1_{pq}(t) \cdot (c^{TT}_{pq} - c^T_{pq}) \in co[-\tilde{\tilde{c}}_{pq}, \tilde{\tilde{c}}_{pq}]$ such that

$$\begin{aligned}\dot{y}_p(t) = & -a_p y_p(t) + \sum_{q=1}^{n}(\tilde{b}_{pq} + \gamma_{pq}(t))f_q(y_q(t)) + \sum_{q=1}^{n}(\tilde{c}_{pq} + \delta_{pq}(t))g_q(y_q(t-\upsilon)) \\ & + I_p(t) + u_p(t), \end{aligned} \tag{2.11}$$

where $\gamma^1_{pq}(t), \delta^1_{pq}(t) \in co[-\frac{1}{2}, \frac{1}{2}]$.

Letting $\sigma_p(t) = y_p(t) - \alpha x_p(t)$ as projective synchronization error signal, then error systems between (2.1) and (2.7) can be expressed as:

$$\begin{aligned}\dot{\sigma}_p(t) = & -a_p\sigma_p(t) + \sum_{q=1}^{n}(\tilde{b}_{pq} + \gamma_{pq}(t))(f_q(y_q(t)) - f_q(\alpha x_q(t))) + \sum_{q=1}^{n}(\tilde{b}_{pq} + \gamma_{pq}(t))f_q(\alpha x_q(t)) \\ & - \sum_{q=1}^{n}\alpha(\tilde{b}_{pq} + \gamma_{pq}(t))f_q(\alpha x_q(t)) + \sum_{q=1}^{n}\alpha(\gamma_{pq}(t) - \pi_{pq}(t))f_q(x_q(t)) + u_p(t) \\ & + (1-\alpha)I_p(t) + \sum_{q=1}^{n}(\tilde{c}_{pq} + \delta_{pq}(t))(g_q(y_q(t-\upsilon)) - g_q(\alpha_q x_q(t-\upsilon))) \\ & + \sum_{q=1}^{n}(\tilde{c}_{pq} + \delta_{pq}(t))g_q(\alpha_q x_q(t-\upsilon)) - \sum_{q=1}^{n}\alpha(\tilde{c}_{pq} + \delta_{pq}(t))g_q(x_q(t-\upsilon)) \\ & + \sum_{q=1}^{n}\alpha(\delta_{pq}(t) - w_{pq}(t))g_q(x_q(t-\upsilon)). \end{aligned} \tag{2.12}$$

Before going further, we introduce the following hypotheses, lemmas and definitions.

Hypothesis 1. For any $p = 1, 2, ..., n$, the activation function $f_p(\cdot)$ and $g_p(\cdot)$ satisfy the Lipschitz condition. Additionally, for $\forall v_1, v_2 \in \mathbb{Q}$, there exist positive constants l_p and m_p such that

$$\|f_p(v_1) - f_p(v_2)\|_1 \leq l_p\|v_1 - v_2\|_1, \quad \|g_p(v_1) - g_p(v_2)\|_1 \leq m_p\|v_1 - v_2\|_1.$$

Hypothesis 2. For any $p = 1, 2, ..., n$, the activation function $f_p(\cdot)$ and $g_p(\cdot)$ satisfy:

$$\|f_p(\cdot)\|_1 \leq L_p, \quad \|g_p(\cdot)\|_1 \leq M_p,$$

where $L_p, M_p \in \mathbb{R}$ are positive constants.

Lemma 1 [35]. Let any $x(t) = (x_1(t), ..., x_n(t)), y(t) = (y_1(t), ..., y_n(t)) \in \mathbb{Q}^n$, $p > 0$, then

$i)$ $x^*(t)sgn(y(t)) + sgn(y(t))^*x(t) \leq 2\|x(t)\|_1$;

$ii)$ $D^+(x^*(t)sgn(x(t)) + sgn(x(t))^*x(t)) = \dot{x}^*(t)sgn(x(t)) + sgn(x(t))^*\dot{x}(t)$,
$\|x(t)\|_1 \neq 0$;
$iii)$ $\|x(t)y(t)\|_1 \leq \|x(t)\|_1\|y(t)\|_1$.

Lemma 2 [24]. Assume that function $Z(t)$ is nonnegative when $t \in (t-c, \infty)$ and satisfies the following inequality:

$$D^+Z(t) \leq -\alpha Z(t) + \beta z(t), \quad t > t_0$$

where α, β are positive constants $\alpha > \beta$ and $z(t) = \sup_{t-c<s<t} Z(s)$. Then,

$$Z(t) \leq z(t)e^{-r(t-t_0)},$$

holds, in which, r is the positive solution of the equation

$$\alpha - \beta e^{-rc} = 0.$$

Lemma 3 [36]. Assume that functions $f(t)$ and $g(t)$ are continuous on $[t_1, t_2]$, $f(t) \geq 0, \alpha \geq 0, \beta \geq 0$ are constants. If

$$g(t) \leq \alpha + \int_{t_1}^{t} (f(z)g(z) + \beta)dz,$$

then

$$g(t) \leq (\alpha + \beta T)e^{\int_{t_1}^{t} f(z)dz},$$

where $t \in [t_1, t_2]$, $T = t_2 - t_1$.

Definition 1. Systems (2.1) and (2.7) are considered to achieve globally exponential projective synchronization, if there exist a projective coefficient $\alpha \in \mathbb{Q}$ and two constants $\beta, M > 0$ such that $\lim_{t\to\infty}\|y(t) - \alpha x(t)\|_1 \leq M \sup_{s\in[-\upsilon,t_0]} \|\phi(s) - \alpha\psi(s)\|_1 e^{-\beta(t-t_0)}$.

Definition 2 [34]. Systems (2.1) and (2.7) are considered to achieve globally projective synchronization, if there exists a projective coefficient $\alpha \in \mathbb{Q}$ such that $\lim_{t\to\infty}\|y(t) - \alpha x(t)\|_1 = 0$.

Definition 3. A sign function for quaternion $q = q^{(r)} + q^{(i)}\mathbf{i} + q^{(j)}\mathbf{j} + q^{(k)}\mathbf{k} \in \mathbb{Q}$ can be defined as

$$sgn(q) = sgn(q^{(r)}) + sgn(q^{(i)})\mathbf{i} + sgn(q^{(j)})\mathbf{j} + sgn(q^{(k)})\mathbf{k}.$$

3. Main results

In this chapter, via quaternion analysis technique and appropriate Lyapunov functional, the criteria that ensure exponential projective synchronization and adaptive exponential projective synchronization are obtained.

3.1. Exponential projective synchronization analysis

To achieve the exponential projective synchronization, the following controller is designed:

$$u_p(t) = -d_p\sigma_p(t) - h_p\frac{\sigma_p(t)}{\|\sigma_p(t)\|_1} - (I_p(t) - \alpha I_p(t)), \tag{3.1}$$

where $d_p, h_p > 0$, and $p = 1, 2, 3, ..., n$.

Theorem 3.1 If Hypotheses 1–2 hold, there exist positive constants d_p, h_p such that the following conditions are satisfied:

$$\begin{aligned}
&2(a_p + d_p) - \sum_{q=1}^{n}(\|b_{qp}^{TT} + b_{qp}^{T}\|_1 + \|b_{qp}^{TT} - b_{qp}^{T}\|_1)l_p > 0,\\
&(1 + \|\alpha\|_1)\sum_{q=1}^{n}(\|b_{pq}^{TT} + b_{pq}^{T}\|_1 + \|b_{pq}^{TT} - b_{pq}^{T}\|_1)L_q + 2\|\alpha\|_1\sum_{q=1}^{n}\|b_{pq}^{TT} - b_{pq}^{T}\|_1 L_q\\
&\quad +(1 + \|\alpha\|_1)\sum_{q=1}^{n}(\|c_{pq}^{TT} + c_{pq}^{T}\|_1 + \|c_{pq}^{TT} - c_{pq}^{T}\|_1)M_q + 2\|\alpha\|_1\sum_{q=1}^{n}\|c_{pq}^{TT} - c_{pq}^{T}\|_1 m_q - 2h_p < 0,\\
&\xi - \eta > 0,
\end{aligned} \tag{3.2}$$

where $p = 1, 2, ...n$, and

$$\xi = \min_{p}\{2(a_p + d_p) - \sum_{q=1}^{n}(\|b_{qp}^{TT} + b_{qp}^{T}\|_1 + \|b_{qp}^{TT} - b_{qp}^{T}\|_1)l_p\},$$
$$\eta = \max_{1\le p\le n}\{\sum_{q=1}^{n}(\|c_{qp}^{TT} + c_{qp}^{T}\|_1 + \|c_{qp}^{TT} - c_{qp}^{T}\|_1)m_p\}.$$

Then, systems (2.1) and (2.7) can achieve the global exponential projective synchronization under the controller (3.1).

Proof. Consider a Lyapunov functional as follows:

$$V(t) = \sum_{p=1}^{n}\sigma_p^*(t)sgn(\sigma_p(t)) + \sum_{p=1}^{n}sgn(\sigma_p(t))^*\sigma_p(t). \tag{3.3}$$

Then, calculate the derivative of $V(t)$ with respect to t along the solutions of system (2.12), one has:

$$\begin{aligned}
\dot{V}(t) &= \sum_{p=1}^{n}\dot{\sigma}_p^*(t)sgn(\sigma_p(t)) + \sum_{p=1}^{n}sgn(\sigma_p(t))^*\dot{\sigma}_p(t)\\
&= -\sum_{p=1}^{n}(a_p + d_p)(sgn(\sigma_p(t))^*\sigma_p(t) + \overline{\sigma}_p(t)sgn(\sigma_p(t)))\\
&\quad + \sum_{p=1}^{n}\sum_{q=1}^{n}\{(f_q^*(y_q(t)) - f_q^*(\alpha x_q(t)))(\tilde{b}_{pq}^* + \gamma_{pq}^*(t))sgn(\sigma_p(t))\\
&\qquad\qquad + sgn(\sigma_p(t))^*(\tilde{b}_{pq} + \gamma_{pq}(t))(f_q(y_q(t)) - f_q(\alpha x_q(t)))\}\\
&\quad + \sum_{p=1}^{n}\sum_{q=1}^{n}\{f_q^*(\alpha x_q(t))(\tilde{b}_{pq}^* + \gamma_{pq}^*(t))sgn(\sigma_p(t)) + sgn(\sigma_p(t))^*(\tilde{b}_{pq} + \gamma_{pq}(t))f_q(\alpha x_q(t))\}\\
&\quad - \sum_{p=1}^{n}\sum_{q=1}^{n}\{f_q^*(x_q(t))(\tilde{b}_{pq}^* + \gamma_{pq}^*(t))\alpha_p^* sgn(\sigma_p(t)) + sgn(\sigma_p(t))^*\alpha(\tilde{b}_{pq} + \gamma_{pq}(t))f_q(x_q(t))\}
\end{aligned}$$

$$+\sum_{p=1}^{n}\sum_{q=1}^{n}\{f_q^*(x_q(t))(\gamma_{pq}^*(t)-\pi_{pq}^*(t))\alpha_p^* sgn(\sigma_p(t))$$
$$+sgn(\sigma_p(t))^*\alpha(\gamma_{pq}(t)-\pi_{pq}(t))f_q(x_q(t))\}$$
$$+\sum_{p=1}^{n}\sum_{q=1}^{n}\{(g_q^*(y_q(t-\upsilon))-g_q^*(\alpha x_q(t-\upsilon)))(\tilde{c}_{pq}^*+\delta_{pq}^*(t))sgn(\sigma_p(t))$$
$$+sgn(\sigma_p(t))^*(\tilde{c}_{pq}+\delta_{pq}(t))(g_q(y_q(t-\upsilon))-g_q(\alpha_q x_q(t-\upsilon)))\}$$
$$+\sum_{p=1}^{n}\sum_{q=1}^{n}\{sgn(\sigma_p(t))^*(\tilde{c}_{pq}+\delta_{pq}(t))g_q(\alpha_q x_q(t-\upsilon))+g_q^*(\alpha_q x_q(t-\upsilon))(\tilde{c}_{pq}^*+\delta_{pq}^*(t))sgn(\sigma_p(t))\}$$
$$-\sum_{p=1}^{n}\sum_{q=1}^{n}\{g_q^*(x_q(t-\upsilon))(\tilde{c}_{pq}^*+\delta_{pq}^*(t))\alpha_p^* sgn(\sigma_p(t))+sgn(\sigma_p(t))^*\alpha(\tilde{c}_{pq}+\delta_{pq}(t))g(x_q(t-\upsilon))\}$$
$$+\sum_{p=1}^{n}\sum_{q=1}^{n}\{sgn(\sigma_p(t))^*\alpha(\delta_{pq}(t)-w_{pq}(t))g(x_q(t-\upsilon))$$
$$+g^*(x_q(t-\upsilon))(\delta_{pq}^*(t)-w_{pq}^*(t))\alpha_p^* sgn(\sigma_p(t))\}$$
$$-\sum_{p=1}^{n}(h_p\frac{\sigma_p^*(t)}{\|\sigma_p(t)\|_1}sgn(\sigma_p(t))+sgn(\sigma_p(t))^*h_p\frac{\sigma_p^*(t)}{\|\sigma_p(t)\|_1}). \tag{3.4}$$

From Lemma 1, Hypothesis 1–2, the following inequalities can be obtained:

$$\sum_{p=1}^{n}\sum_{q=1}^{n}(f_q^*(y_q(t))-f_q^*(\alpha x_q(t)))(\tilde{b}_{pq}^*+\gamma_{pq}^*(t))sgn(\sigma_p(t))$$
$$+\sum_{p=1}^{n}\sum_{q=1}^{n}sgn(\sigma_p(t))^*(\tilde{b}_{pq}+\gamma_{pq}(t))\times(f_q(y_q(t))-f_q(\alpha x_q(t)))$$
$$\leq \sum_{p=1}^{n}\sum_{q=1}^{n}2\|(\tilde{b}_{pq}+\gamma_{pq}(t))(f_q(y_q(t))-f_q(\alpha x_q(t)))\|_1$$
$$\leq \sum_{p=1}^{n}\sum_{q=1}^{n}\|b_{pq}^{TT}+b_{pq}^{T}+2\pi_{pq}^{1}(t)(b_{pq}^{TT}-b_{pq}^{T})\|_1 l_q\|\sigma_q(t)\|_1$$
$$\leq \sum_{p=1}^{n}\sum_{q=1}^{n}(\|b_{pq}^{TT}+b_{pq}^{T}\|_1+\|b_{pq}^{TT}-b_{pq}^{T}\|_1)l_q\|\sigma_q(t)\|_1,$$

$$\sum_{p=1}^{n}\sum_{q=1}^{n}(g_q^*(y_q(t-\upsilon))-g_q^*(\alpha x_q(t-\upsilon)))(\tilde{c}_{pq}^*+\delta_{pq}^*(t))sgn(\sigma_p(t))$$
$$+\sum_{p=1}^{n}\sum_{q=1}^{n}sgn(\sigma_p(t))^*(\tilde{c}_{pq}+\delta_{pq}(t))(g_q(y_q(t-\upsilon))-g_q(\alpha x_q(t-\upsilon)))$$
$$\leq \sum_{p=1}^{n}\sum_{q=1}^{n}2\|(\tilde{c}_{pq}+\delta_{pq}(t))(g_q(y_q(t-\upsilon))-g_q(\alpha x_q(t-\upsilon)))\|_1$$

$$
\begin{aligned}
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n}\|c_{pq}^{TT}+c_{pq}^{T}+2\gamma_{pq}^{1}(t)(c_{pq}^{TT}-c_{pq}^{T})\|_1 m_q\|\sigma_q(t-\upsilon)\|_1\\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n}(\|c_{pq}^{TT}+c_{pq}^{T}\|_1+\|c_{pq}^{TT}-c_{pq}^{T}\|_1)m_q\|\sigma_q(t-\upsilon)\|_1,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{p=1}^{n}\sum_{q=1}^{n} f_q^{*}(\alpha x_q(t))(\tilde{b}_{pq}^{*}+\gamma_{pq}^{*}(t))sgn(\sigma_p(t))\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n} sgn(\sigma_p(t))^{*}(\tilde{b}_{pq}+\gamma_{pq}(t))f_q(\alpha x_q(t))\\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n} 2\|(\tilde{b}_{pq}+\gamma_{pq}(t))f_q(\alpha x_q(t))\|_1\\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n}(\|b_{pq}^{TT}+b_{pq}^{T}\|_1+\|b_{pq}^{TT}-b_{pq}^{T}\|_1)L_q,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{p=1}^{n}\sum_{q=1}^{n} -f_q^{*}(x_q(t))(\tilde{b}_{pq}^{*}+\gamma_{pq}^{*}(t))\alpha_p^{*}sgn(\sigma_p(t))-sgn(\sigma_p(t))^{*}\alpha(\tilde{b}_{pq}+\gamma_{pq}(t))f_q(x_q(t))\\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n} 2\|\alpha\|_1\|\tilde{b}_{pq}+\gamma_{pq}(t)\|_1\|f_q(x_q(t))\|_1\\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n} \|\alpha\|_1(\|b_{pq}^{TT}+b_{pq}^{T}\|_1+\|b_{pq}^{TT}-b_{pq}^{T}\|_1)L_q,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{p=1}^{n}\sum_{q=1}^{n} f_q^{*}(x_q(t))(\gamma_{pq}^{*}(t)-\pi_{pq}^{*}(t))\alpha_p^{*}sgn(\sigma_p(t))+sgn(\sigma_p(t))^{*}\alpha(\gamma_{pq}(t)-\pi_{pq}(t))\times f_q(x_q(t))\\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n} 2\|\alpha\|_1\|\gamma_{pq}(t)-\pi_{pq}(t)\|_1\|f_q(x_q(t))\|_1\\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n} 2\|\alpha\|_1\|b_{pq}^{TT}-b_{pq}^{T}\|_1 L_q,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{p=1}^{n}\sum_{q=1}^{n} sgn(\sigma_p(t))^{*}(\tilde{c}_{pq}+\delta_{pq}(t))g_q(\alpha_q x_q(t-\upsilon))+\sum_{p=1}^{n}\sum_{q=1}^{n} g_q^{*}(\alpha_q x_q(t-\upsilon))(\tilde{c}_{pq}^{*}+\delta_{pq}^{*}(t))\times sgn(\sigma_p(t))\\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n} 2\|(\tilde{c}_{pq}+\delta_{pq}(t))g_q(\alpha_q x_q(t-\upsilon))\|_1\\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n}(\|c_{pq}^{TT}+c_{pq}^{T}\|_1+\|c_{pq}^{TT}-c_{pq}^{T}\|_1)M_q,
\end{aligned}
$$

$$
\sum_{p=1}^{n}\sum_{q=1}^{n} -g_q^{*}(x_q(t-\upsilon))(\tilde{c}_{pq}^{*}+\delta_{pq}^{*}(t))\alpha_p^{*}sgn(\sigma_p(t))-\sum_{p=1}^{n}\sum_{q=1}^{n} sgn(\sigma_p(t))^{*}\alpha(\tilde{c}_{pq}+\delta_{pq}(t))\times g(x_q(t-\upsilon))
$$

$$
\begin{aligned}
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n} 2\|\alpha\|_1\|\tilde{c}_{pq}+\delta_{pq}(t)\|_1\|g(x_q(t-\upsilon))\|_1 \\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n} \|\alpha\|_1(\|c_{pq}^{TT}+c_{pq}^{T}\|_1+\|c_{pq}^{TT}-c_{pq}^{T}\|_1)M_q, \\
&\sum_{p=1}^{n}\sum_{q=1}^{n} sgn(\sigma_p(t))^{*}\alpha(\delta_{pq}(t)-w_{pq}(t))g(x_q(t-\upsilon)) \\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n} g^{*}(x_q(t-\upsilon))(\delta_{pq}^{*}(t)-w_{pq}^{*}(t))\alpha_p^{*}\times sgn(\sigma_p(t)) \\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n} 2\|\alpha\|_1\|\delta_{pq}(t)-w_{pq}(t)\|_1\|g(x_q(t-\upsilon))\|_1 \\
&\leq \sum_{p=1}^{n}\sum_{q=1}^{n} 2\|\alpha\|_1\|c_{pq}^{TT}-c_{pq}^{T}\|_1 M_q. \qquad (3.5)
\end{aligned}
$$

Combine with inequalities (Eq 3.4) and (Eq 3.5), one can get

$$
\begin{aligned}
\dot{V}(t) \leq\ & -2(a_p+d_p)\sum_{p=1}^{n}\|\sigma_p(t)\|_1+\sum_{p=1}^{n}\sum_{q=1}^{n}\Big(\|b_{pq}^{TT}+b_{pq}^{T}\|_1+\|b_{pq}^{TT}-b_{pq}^{T}\|_1\Big)l_q\|\sigma_q(t)\|_1 \\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\Big(\|c_{pq}^{TT}+c_{pq}^{T}\|_1+\|c_{pq}^{TT}-c_{pq}^{T}\|_1\Big)m_q\|\sigma_q(t-\upsilon)\|_1 \\
&+\sum_{p=1}^{n}\{(1+\|\alpha\|_1)\sum_{q=1}^{n}(\|b_{pq}^{TT}+b_{pq}^{T}\|_1+\|b_{pq}^{TT}-b_{pq}^{T}\|_1)L_q \\
&\qquad+2\|\alpha\|_1\sum_{q=1}^{n}\|b_{pq}^{TT}-b_{pq}^{T}\|_1L_q+(1+\|\alpha\|_1)\sum_{q=1}^{n}(\|c_{pq}^{TT}+c_{pq}^{T}\|_1+\|c_{pq}^{TT}-c_{pq}^{T}\|_1)M_q \\
&\qquad+2\|\alpha\|_1\sum_{q=1}^{n}\|c_{pq}^{TT}-c_{pq}^{T}\|_1M_q-2h_p\} \\
\leq\ & \sum_{p=1}^{n}\{-2(a_p+d_p)+\sum_{q=1}^{n}(\|b_{qp}^{TT}+b_{qp}^{T}\|_1+\|b_{qp}^{TT}-b_{qp}^{T}\|_1)l_p\}\|\sigma_p(t)\|_1 \\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}(\|c_{qp}^{TT}+c_{qp}^{T}\|_1+\|c_{qp}^{TT}-c_{qp}^{T}\|_1)m_p\|\sigma_p(t-\upsilon)\|_1 \\
\leq\ & -\min_{1\leq p\leq n}\{2(a_p+d_p)-\sum_{q=1}^{n}(\|b_{qp}^{TT}+b_{qp}^{T}\|_1+\|b_{qp}^{TT}-b_{qp}^{T}\|_1)l_p\}\|\sum_{p=1}^{n}\sigma_p(t)\|_1 \\
&+\max_{1\leq p\leq n}\{\sum_{q=1}^{n}(\|c_{qp}^{TT}+c_{qp}^{T}\|_1+\|c_{qp}^{TT}-c_{qp}^{T}\|_1)m_p\}\sum_{p=1}^{n}\|\sigma_p(t-\upsilon)\|_1 \\
=\ & -\frac{\xi}{2}V(\sigma(t))+\frac{\eta}{2}V(\sigma(t-\upsilon)). \qquad (3.6)
\end{aligned}
$$

Thus, from Lemma 2, we have

$$V(t) < \sup_{-\upsilon<s<0} V(s)e^{-rt},$$

where $\frac{\xi}{2} - \frac{\eta}{2}e^{-r\upsilon} = 0$.

Now, based on Definition 1, a conclusion that systems (2.1) and (2.7) can reach the global exponential synchronization via the given controller (3.1) can be safely obtained. This completes the proof.

Corollary 3.1 If Hypotheses 1–2 hold, there exist positive constants d_p, h_p, k_p such that the following conditions are satisfied:

$$\begin{aligned}
&2(a_p + d_p) - \sum_{q=1}^{n}(\|b_{qp}^{TT} + b_{qp}^{T}\|_1 + \|b_{qp}^{TT} - b_{qp}^{T}\|_1)l_p - 2k_p > 0,\\
&(1 + \|\alpha\|_1)\sum_{q=1}^{n}(\|b_{pq}^{TT} + b_{pq}^{T}\|_1 + \|b_{pq}^{TT} - b_{pq}^{T}\|_1)L_q + 2\|\alpha\|_1\sum_{q=1}^{n}\|b_{pq}^{TT} - b_{pq}^{T}\|_1 L_q\\
&\quad +(1 + \|\alpha\|_1)\sum_{q=1}^{n}(\|c_{pq}^{TT} + c_{pq}^{T}\|_1 + \|c_{pq}^{TT} - c_{pq}^{T}\|_1)M_q\\
&\quad +2\|\alpha\|_1\sum_{q=1}^{n}\|c_{pq}^{TT} - c_{pq}^{T}\|_1 M_q - 2h_p < 0,\\
&\sum_{q=1}^{n}(\|c_{qp}^{TT} + c_{qp}^{T}\|_1 + \|c_{qp}^{TT} - c_{qp}^{T}\|_1)m_p - 2k_p < 0,
\end{aligned} \tag{3.7}$$

where $p = 1, 2, ..., n$.

Then, systems (2.1) and (2.7) can realize global projective synchronization under the controller (3.1).

Proof. Consider the following Lyapunov function:

$$\begin{aligned}
V(t) \;=\;& \sum_{p=1}^{n}\sigma_p^*(t)sgn(\sigma_p(t)) + \sum_{p=1}^{n} sgn(\sigma_p(t))^*\sigma_p(t) + k_p\sum_{p=1}^{n}\int_{t-\upsilon}^{t}\sigma_p^*(s)sgn(\sigma_p(s))\\
&+\sum_{p=1}^{n} sgn(\sigma_p(s))^*\sigma_p(s)dz.
\end{aligned} \tag{3.8}$$

Calculate the derivative of $V(t)$, one has:

$$\begin{aligned}
\dot{V}(t) \;=\;& \sum_{p=1}^{n}\dot{\sigma}_p^*(t)sgn(\sigma_p(t)) + \sum_{p=1}^{n} sgn(\sigma_p(t))^*\dot{\sigma}_p(t) + 2\sum_{p=1}^{n}k_p\|\sigma_p(t)\|_1 - 2\sum_{p=1}^{n}k_p\|\sigma_p(t-\upsilon)\|_1\\
\le\;& \sum_{p=1}^{n}\{-2(a_p + d_p) + \sum_{q=1}^{n}(\|b_{qp}^{TT} + b_{qp}^{T}\|_1 + \|b_{qp}^{TT} - b_{qp}^{T}\|_1)l_p + 2k_p\}\|\sigma_p(t)\|_1\\
&+\sum_{p=1}^{n}\{\sum_{q=1}^{n}(\|c_{qp}^{TT} + c_{qp}^{T}\|_1 + \|c_{qp}^{TT} - c_{qp}^{T}\|_1)m_p - 2k_p\}\|\sigma_p(t-\upsilon)\|_1\\
<\;& 0.
\end{aligned}$$

Thus, based on Definition 2, systems (2.1) and (2.7) are globally synchronized. This completes the proof.

Furthermore, it is worth mentioning that when the projective coefficient in controller satisfy $\alpha = 1$ and $\alpha = -1$, from Corollary 3.1, systems (2.1) and (2.7) can achieve global synchronization and anti-synchronization in complete synchronization sense, respectively.

3.2. Adaptive exponential projective synchronization analysis

To achieve the adaptive exponential projective synchronization, the following adaptive control scheme is proposed:

$$\begin{cases} u_p(t) = u_{p1}(t) + u_{p2}(t), \\ u_{p1}(t) = -\kappa_p(t)\sigma_p(t), \\ u_{p2}(t) = -\lambda_p \frac{\sigma_p(t)}{\|\sigma_p(t)\|_1} - (I_p(t) - \alpha I_p(t)), \\ \dot{\kappa}_p(t) = D_p\|\sigma_p(t)\|_1 \end{cases} \tag{3.9}$$

where $D_p > 0$, $\lambda_p > 0$ and $p = 1, 2, 3, ..., n$.

Theorem 3.2 If Hypotheses 1–2 hold, there exist positive constants D_p, λ_p and k_p such that the following conditions are satisfied:

$$\begin{aligned} &(a_p + D_p - k_p) - \frac{1}{2}\sum_{q=1}^{n}(\|b_{qp}^{TT} + b_{qp}^{T}\|_1 + \|b_{qp}^{TT} - b_{qp}^{T}\|_1)l_p > 0, \\ &-\lambda_p + \frac{1}{2}(1 + \|\alpha\|_1)\sum_{q=1}^{n}(\|b_{pq}^{TT} + b_{pq}^{T}\|_1 + \|b_{pq}^{TT} - b_{pq}^{T}\|_1)L_q \\ &+\frac{1}{2}(1 + \|\alpha\|_1)\sum_{q=1}^{n}(\|c_{pq}^{TT} + c_{pq}^{T}\|_1 + \|c_{pq}^{TT} - c_{pq}^{T}\|_1)M_q + \|\alpha\|_1\sum_{q=1}^{n}\|b_{pq}^{TT} - b_{pq}^{T}\|_1 L_q \\ &+\|\alpha\|_1\sum_{q=1}^{n}\|c_{pq}^{TT} - c_{pq}^{T}\|_1 M_q \\ &< 0, \\ &-k_p + \frac{1}{2}\sum_{q=1}^{n}(\|c_{qp}^{TT} + c_{qp}^{T}\|_1 + \|c_{qp}^{TT} - c_{qp}^{T}\|_1)m_p < 0, \end{aligned} \tag{3.10}$$

where $p = 1, 2, ..., n$.

Then, systems (2.1) and (2.7) can achieve global exponential projective synchronization under the adaptive controller (3.9).

Proof. Consider a Lyapunov functional as follows:

$$\begin{aligned} V(t) \;=\; & \frac{1}{2}\sum_{p=1}^{n}\sigma_p^*(t)sgn(\sigma_p(t)) + \frac{1}{2}\sum_{p=1}^{n}sgn(\sigma_p(t))^*\sigma_p(t) + \frac{1}{2}\sum_{p=1}^{n}\frac{1}{D_p}(\kappa_p(t) - D_p)^2 \\ & +\frac{1}{2}\sum_{p=1}^{n}k_p\int_{t-\upsilon}^{t}\sigma_p^*(s)sgn(\sigma_p(s)) + sgn(\sigma_p(s))^*\sigma_p(s)dz, \end{aligned} \tag{3.11}$$

where

$$D_p \quad > \quad -a_p + k_p + \frac{1}{2}\sum_{q=1}^{n}(\|b_{qp}^{TT} + b_{qp}^{T}\|_1 + \|b_{qp}^{TT} - b_{qp}^{T}\|_1)L_p.$$

Then, calculate the derivative of $V(t)$ with respect to t along the solutions of system (2.12), we can get that

$$\begin{aligned}
\dot{V}(t) \;=\; & \frac{1}{2}\sum_{p=1}^{n}\dot{\sigma}_p^*(t)sgn(\sigma_p(t)) + \frac{1}{2}\sum_{p=1}^{n}sgn(\sigma_p(t))^*\dot{\sigma}_p(t) + \sum_{p=1}^{n}\frac{1}{D_p}(\kappa_p(t) - D_p)\dot{\kappa}_p(t) \\
& + \sum_{p=1}^{n}k_p\|\sigma_p(t)\|_1 - \sum_{p=1}^{n}k_p\|\sigma_p(t-\upsilon)\|_1 \\
=\; & -\frac{1}{2}\sum_{p=1}^{n}a_p(sgn(\sigma_p(t))^*\sigma_p(t) + \sigma_p^*(t)sgn(\sigma_p(t))) \\
& +\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\{(f_q^*(y_q(t)) - f_q^*(\alpha x_q(t)))(\tilde{b}_{pq}^* + \gamma_{pq}^*(t))sgn(\sigma_p(t)) \\
& \qquad +sgn(\sigma_p(t))^*(\tilde{b}_{pq} + \gamma_{pq}(t))(f_q(y_q(t)) - f_q(\alpha x_q(t)))\} \\
& +\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\{f_q^*(\alpha x_q(t))(\tilde{b}_{pq}^* + \gamma_{pq}^*(t))sgn(\sigma_p(t)) \\
& \qquad +sgn(\sigma_p(t))^*(\tilde{b}_{pq} + \gamma_{pq}(t))f_q(\alpha x_q(t))\} \\
& -\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\{f_q^*(x_q(t))(\tilde{b}_{pq}^* + \gamma_{pq}^*(t))\alpha_p^* sgn(\sigma_p(t)) + sgn(\sigma_p(t))^*\alpha(\tilde{b}_{pq} + \gamma_{pq}(t))f_q(x_q(t))\} \\
& +\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\{f_q^*(x_q(t))(\gamma_{pq}^*(t) - \pi_{pq}^*(t))\alpha_p^* sgn(\sigma_p(t)) \\
& \qquad +sgn(\sigma_p(t))^*\alpha(\gamma_{pq}(t) - \pi_{pq}(t))f_q(x_q(t))\} \\
& +\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\{(g_q^*(y_q(t-\upsilon)) - g_q^*(\alpha x_q(t-\upsilon)))(\tilde{c}_{pq}^* + \delta_{pq}^*(t))sgn(\sigma_p(t)) \\
& \qquad +sgn(\sigma_p(t))^*(\tilde{c}_{pq} + \delta_{pq}(t))(g_q(y_q(t-\upsilon)) - g_q(\alpha_q x_q(t-\upsilon)))\} \\
& +\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\{sgn(\sigma_p(t))^*(\tilde{c}_{pq} + \delta_{pq}(t))g_q(\alpha_q x_q(t-\upsilon)) \\
& \qquad +g_q^*(\alpha_q x_q(t-\upsilon))(\tilde{c}_{pq}^* + \delta_{pq}^*(t))sgn(\sigma_p(t))\} \\
& -\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\{g_q^*(x_q(t-\upsilon))(\tilde{c}_{pq}^* + \delta_{pq}^*(t))\alpha_p^* sgn(\sigma_p(t)) \\
& \qquad +sgn(\sigma_p(t))^*\alpha(\tilde{c}_{pq} + \delta_{pq}(t))g(x_q(t-\upsilon))\} \\
& +\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\{sgn(\sigma_p(t))^*\alpha(\delta_{pq}(t) - w_{pq}(t))g(x_q(t-\upsilon)) \\
& \qquad +g^*(x_q(t-\upsilon))(\delta_{pq}^*(t) - w_{pq}^*(t))\alpha_p^* sgn(\sigma_p(t))\}
\end{aligned}$$

$$-\sum_{p=1}^{n}\lambda_p-\sum_{p=1}^{n}D_p\|\sigma_p(t)\|_1+\sum_{p=1}^{n}k_p\|\sigma_p(t)\|_1-\sum_{p=1}^{n}k_p\|\sigma_p(t-\upsilon)\|_1. \tag{3.12}$$

Combined with the above inequality (Eq 3.5), one has:

$$\begin{aligned}
\dot{V}(t) \leq\ & -(a_p+D_p-k_p)\sum_{p=1}^{n}\|\sigma_p(t)\|_1+\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}(\|b_{pq}^{TT}+b_{pq}^{T}\|_1+\|b_{pq}^{TT}-b_{pq}^{T}\|_1)l_q\|\sigma_q(t)\|_1\\
&+\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\{(\|c_{qp}^{TT}+c_{qp}^{T}\|_1+\|c_{qp}^{TT}-c_{qp}^{T}\|_1)m_p-2k_p\}\|\sigma_p(t-\upsilon)\|_1\\
&+\frac{1}{2}\sum_{p=1}^{n}\{(1+\|\alpha\|_1)\sum_{q=1}^{n}(\|b_{pq}^{TT}+b_{pq}^{T}\|_1+\|b_{pq}^{TT}-b_{pq}^{T}\|_1)L_q+2\|\alpha\|_1\sum_{q=1}^{n}\|b_{pq}^{TT}-b_{pq}^{T}\|_1L_q\\
&\quad+(1+\|\alpha\|_1)\sum_{q=1}^{n}(\|c_{pq}^{TT}+c_{pq}^{T}\|_1+\|c_{pq}^{TT}-c_{pq}^{T}\|_1)M_q\\
&\quad+2\|\alpha\|_1\sum_{q=1}^{n}\|c_{pq}^{TT}-c_{pq}^{T}\|_1M_q-2\lambda_p\}\\
\leq\ & \sum_{p=1}^{n}\{-(a_p+D_p-k_p)+\frac{1}{2}\sum_{q=1}^{n}(\|b_{qp}^{TT}+b_{qp}^{T}\|_1+\|b_{qp}^{TT}-b_{qp}^{T}\|_1)l_p\}\|\sigma_p(t)\|_1\\
\leq\ & -\xi\sum_{p=1}^{n}\|\sigma_p(t)\|_1,
\end{aligned} \tag{3.13}$$

where $\xi=\underset{p}{min}\{(a_p+D_p-k_p)-\frac{1}{2}\sum_{q=1}^{n}(\|b_{qp}^{TT}+b_{qp}^{T}\|_1+\|b_{qp}^{TT}-b_{qp}^{T}\|_1)l_p\}>0$. Obviously, $\dot{V}(t)+\xi\sum_{p=1}^{n}\|\sigma_p(t)\|_1<0$.

Hence,

$$\sum_{p=1}^{n}\|\sigma_p(t)\|_1\leq V(0)-\xi\int_0^t\sum_{p=1}^{n}\|\sigma_p(s)\|_1dz. \tag{3.14}$$

From Lemma 3, we have

$$\sum_{p=1}^{n}\|\sigma_p(t)\|_1\leq V(0)e^{-\xi t}, \tag{3.15}$$

According to Lyapunov function (3.12), one can obtain

$$\begin{aligned}
V(0) =\ & \sum_{p=1}^{n}\|\sigma_p(0)\|_1+\sum_{p=1}^{n}\frac{1}{d_p}(\kappa_p(0)-D_p)^2\\
&+\frac{1}{2}k_p\sum_{p=1}^{n}\int_{t-\upsilon}^{t}\sigma_p^*(s)sgn(\sigma_p(s))+sgn(\sigma_p(s))^*\sigma_p(s)dz,
\end{aligned} \tag{3.16}$$

where $\kappa_p(0)$ is the initial value of $\kappa_p(t)$.

Then,

$$\frac{1}{2}\sum_{p=1}^{n}k_p\int_{-\upsilon}^{0}\sigma_p^*(s)sgn(\sigma_p(s))+sgn(\sigma_p(s))^*\sigma_p(s)dz$$
$$\leq\sum_{p=1}^{n}k_p\upsilon\sup_{-\upsilon<s<0}\{\|\sigma_p(s)\|_1\}. \tag{3.17}$$

Furthermore, we can find a positive constant M such that

$$\sum_{p=1}^{n}\frac{1}{d_p}(\kappa_p(0)-D_p)^2\leq M\sup_{-\upsilon<s<0}\{\sum_{p=1}^{n}\|\sigma_p(0)\|_1\}. \tag{3.18}$$

From inequalities (Eq 3.15)–(Eq 3.18), we can obtain

$$\sum_{p=1}^{n}\|\sigma_p(t)\|_1<(M+H+1)e^{-\xi t}\sup_{-\upsilon<s<0}\{\sum_{p=1}^{n}\|\sigma_p(s)\|_1\},$$

where $H=\max_{p}\{k_p\upsilon\}$.

Equivalently,

$$\|\sigma(t)\|_1<\sqrt{M+H+1}e^{-\xi t}\sup_{-\upsilon<s<0}\{\|\sigma(s)\|_1\}$$

can be easily obtained.

Therefore, the conclusion that systems (2.1) and (2.7) can reach the global exponential synchronization under the given adaptive control scheme (3.9) can be safely obtained. This completes the proof.

Corollary 3.2 If Hypotheses 1–2 hold, there exist positive constants D_p, λ_p and k_p such that the following conditions are satisfied:

$$(a_p+D_p-k_p)-\frac{1}{2}\sum_{q=1}^{n}(\|b_{qp}^{TT}+b_{qp}^{T}\|_1+\|b_{qp}^{TT}-b_{qp}^{T}\|_1)l_p>0,$$
$$\frac{1}{2}(1+\|\alpha\|_1)\sum_{q=1}^{n}(\|b_{pq}^{TT}+b_{pq}^{T}\|_1+\|b_{pq}^{TT}-b_{pq}^{T}\|_1)L_q$$
$$+\frac{1}{2}(1+\|\alpha\|_1)\sum_{q=1}^{n}(\|c_{pq}^{TT}+c_{pq}^{T}\|_1+\|c_{pq}^{TT}-c_{pq}^{T}\|_1)M_q$$
$$+\|\alpha\|_1\sum_{q=1}^{n}\|b_{pq}^{TT}-b_{pq}^{T}\|_1L_q+\|\alpha\|_1\sum_{q=1}^{n}\|c_{pq}^{TT}-c_{pq}^{T}\|_1M_q-\lambda_p<0,$$
$$-k_p+\sum_{q=1}^{n}\frac{1}{2}(\|c_{qp}^{TT}+c_{qp}^{T}\|_1+\|c_{qp}^{TT}-c_{qp}^{T}\|_1)m_p<0,$$

where $p=1,2,...,n$.

Then, systems (2.1) and (2.7) can realize global projective synchronization under the adaptive controller (3.9).

Proof. Consider a Lyapunov function as:

$$\begin{aligned} V(t) &= \sum_{p=1}^{n}\sigma_p^*(t)sgn(\sigma_p(t)) + \sum_{p=1}^{n} sgn(\sigma_p(t))^*\sigma_p(t) + \sum_{p=1}^{n}\frac{1}{D_p}(\kappa_p(t)-D_p)^2 \\ &\quad + \sum_{p=1}^{n} k_p\int_{t-\upsilon}^{t}\sigma_p^*(s)sgn(\sigma_p(s)) + sgn(\sigma_p(s))^*\sigma_p(s)dz, \end{aligned} \tag{3.19}$$

where

$$2D_p > -2a_p + 2k_p + \sum_{q=1}^{n}(\|b_{qp}^{TT} + b_{qp}^{T}\|_1 + \|b_{qp}^{TT} - b_{qp}^{T}\|_1)L_p. \tag{3.20}$$

Calculate the derivative of $V(t)$, we have

$$\begin{aligned} \dot{V}(t) &= \sum_{p=1}^{n}\dot{\sigma}_p^*(t)sgn(\sigma_p(t)) + \sum_{p=1}^{n} sgn(\sigma_p(t))^*\dot{\sigma}_p(t) + \sum_{p=1}^{n}\frac{2}{D_p}(\kappa_p(t)-D_p)\dot{\kappa}_p(t) \\ &\quad +2\sum_{p=1}^{n}k_p\|\sigma_p(t)\|_1 - 2\sum_{p=1}^{n}k_p\|\sigma_p(t-\upsilon)\|_1 \\ &\leq -2(a_p + D_p - k_p)\sum_{p=1}^{n}\|\sigma_p(t)\|_1 + \sum_{p=1}^{n}\sum_{q=1}^{n}(\|b_{pq}^{TT} + b_{pq}^{T}\|_1 + \|b_{pq}^{TT} - b_{pq}^{T}\|_1)L_q\|\sigma_q(t)\|_1 \\ &\quad + \sum_{p=1}^{n}\sum_{q=1}^{n}\{(\|c_{qp}^{TT} + c_{qp}^{T}\|_1 + \|c_{qp}^{TT} - c_{qp}^{T}\|_1)M_p - 2k_p\}\|\sigma_p(t-\upsilon)\|_1 \\ &\quad + \sum_{p=1}^{n}\{(1+\|\alpha\|_1)\sum_{q=1}^{n}(\|b_{pq}^{TT} + b_{pq}^{T}\|_1 + \|b_{pq}^{TT} - b_{pq}^{T}\|_1)L_q + \|\alpha\|_1\sum_{q=1}^{n}\|b_{pq}^{TT} - b_{pq}^{T}\|_1 L_q \\ &\quad +(1+\|\alpha\|_1)\sum_{q=1}^{n}(\|c_{pq}^{TT} + c_{pq}^{T}\|_1 + \|c_{pq}^{TT} - c_{pq}^{T}\|_1)M_q \\ &\quad +\|\alpha\|_1\sum_{q=1}^{n}\|c_{pq}^{TT} - c_{pq}^{T}\|_1 M_q - \lambda_p\} \\ &< 0, \end{aligned} \tag{3.21}$$

hold.

Then, based on Definition 2, we can conclude that systems (2.1) and (2.7) can reach the global synchronization under the given adaptive scheme (3.9). This completes the proof.

Remark 1. It is worth mentioning that complete synchronization is a special case of projective synchronization, so projective synchronization criteria proposed by this paper can be applied to the problem of synchronization for other QVNNs in complete synchronization sense [37]. In addition, since QVNNs is considered as a generalization of real value NNs (RVNNs) and complex value NNs

(CVNNs), so the conclusions mentioned above can also be applied to RVNNs and CVNNs [24, 38]. These manifests that the theoretical results presented in our paper are more general.

Remark 2. Different from the technique taken in [20, 37, 39], the QVNNs are transformed into equivalent four RVNNs or two CVNNs. In our work, the QVNNs was treated as an entirety without any decomposition directly, the advantage is that it can be applied to the situation that activation functions cannot be decomposed into real-imaginary parts. To some extent, it decreases conservativeness.

4. Numerical examples

Example 1. Consider the system (2.1) as master system, then the slave system with controller is designed as:

$$\begin{aligned}\dot{y}_p(t) &= -a_p y_p(t) + \sum_{q=1}^{n} b_{pq}(y_p(t)) f_q(y_q(t)) + \sum_{q=1}^{n} c_{pq}(y_p(t)) g_q(y_q(t-\upsilon)) - d_p\sigma_p(t) \\ &\quad -h_p \frac{\sigma_p(t)}{\|\sigma_p(t)\|_1} + \alpha I_p(t) \quad p = 1, 2, \end{aligned} \tag{4.1}$$

where $a_p = 2$, time delays $\upsilon = 0.50$. Choose the activation functions $f_p(x_p(t)) = tanh(x_p(t))$, $g_p(x_p(t)) = tanh(x_p(t))$, the external inputs $I_1(t) = 0.10 + 0.25\mathbf{i} - 0.10\mathbf{j} + 0.30\mathbf{k}$, $I_2(t) = 0.80 + 0.10\mathbf{i} - 0.20\mathbf{j} + 0.20\mathbf{k}$, the memristive connection weights as

$$b_{11}(x_1(t)) = \begin{cases} 0.40 + 0.40\mathbf{i} + 0.20\mathbf{j} + 0.30\mathbf{k}, & \|x_1(t)\|_1 > 1.5, \\ -0.30 - 0.40\mathbf{i} - 0.20\mathbf{j} - 0.20\mathbf{k}, & \|x_1(t)\|_1 \le 1.5, \end{cases}$$

$$b_{12}(x_1(t)) = \begin{cases} -0.45 + 0.25\mathbf{i} - 0.15\mathbf{j} + 0.18\mathbf{k}, & \|x_1(t)\|_1 > 1.5, \\ -0.55 + 0.12\mathbf{i} - 0.25\mathbf{j} + 0.11\mathbf{k}, & \|x_1(t)\|_1 \le 1.5, \end{cases}$$

$$b_{21}(x_2(t)) = \begin{cases} 0.50 - 0.30\mathbf{i} + 0.25\mathbf{j} + 0.30\mathbf{k}, & \|x_2(t)\|_1 > 1.5, \\ 0.20 - 0.40\mathbf{i} - 0.25\mathbf{j} - 0.40\mathbf{k}, & \|x_2(t)\|_1 \le 1.5, \end{cases}$$

$$b_{22}(x_2(t)) = \begin{cases} 0.50 - 0.30\mathbf{i} + 0.30\mathbf{j} + 0.40\mathbf{k}, & \|x_2(t)\|_1 > 1.5, \\ 0.30 - 0.50\mathbf{i} + 0.15\mathbf{j} + 0.18\mathbf{k}, & \|x_2(t)\|_1 \le 1.5, \end{cases}$$

$$c_{11}(x_1(t)) = \begin{cases} 0.35 - 0.12\mathbf{i} + 0.22\mathbf{j} + 0.10\mathbf{k}, & \|x_1(t)\|_1 > 1.5, \\ 0.30 - 0.30\mathbf{i} + 0.20\mathbf{j} + 0.10\mathbf{k}, & \|x_1(t)\|_1 \le 1.5, \end{cases}$$

$$c_{12}(x_1(t)) = \begin{cases} 0.42 + 0.12\mathbf{i} - 0.12\mathbf{j} - 0.13\mathbf{k}, & \|x_1(t)\|_1 > 1.5, \\ 0.30 + 0.12\mathbf{i} - 0.21\mathbf{j} - 0.33\mathbf{k}, & \|x_1(t)\|_1 \le 1.5, \end{cases}$$

$$c_{21}(x_2(t)) = \begin{cases} -0.10 + 0.11\mathbf{i} - 0.13\mathbf{j} - 0.12\mathbf{k}, & \|x_2(t)\|_1 > 1.5, \\ -0.30 - 0.20\mathbf{i} - 0.20\mathbf{j} - 0.20\mathbf{k}, & \|x_2(t)\|_1 \le 1.5, \end{cases}$$

$$c_{22}(x_2(t)) = \begin{cases} 0.50 + 0.30\mathbf{i} - 0.20\mathbf{j} - 0.30\mathbf{k}, & \|x_2(t)\|_1 > 1.5, \\ 0.20 + 0.30\mathbf{i} - 0.30\mathbf{j} - 0.20\mathbf{k}, & \|x_2(t)\|_1 \le 1.5. \end{cases}$$

Obviously, $l_i = 2$, $m_i = 2$, $L_i = 2$, $M_i = 2$, $i = 1, 2$.

Let projective coefficient $\alpha = 0.50 + 0.50\mathbf{i} + 0.50\mathbf{j} - 0.50\mathbf{k}$, $h_1 = h_2 = 5$, $d_1 = d_2 = 8$, then the conditions in Theorem 3.1 is satisfied. Therefore, under the controller (3.1), master system (2.1) and slave system (4.1) can achieve the global exponential projective synchronization. Under the initial conditions $x_1(0) = -0.50 - 0.50\mathbf{i} + 0.50\mathbf{j} - 0.50\mathbf{k}$, $x_2(0) = 0.60 - 0.70\mathbf{i} - 0.70\mathbf{j} - 0.50\mathbf{k}$, $y_1(0) = 0.50 + 0.50\mathbf{i} - 0.80\mathbf{j} + 0.50\mathbf{k}$, $y_2(0) = -0.60 - 0.10\mathbf{i} + 0.30\mathbf{j} + 0.30\mathbf{k}$, the trajectories of error are shown in Figure 1, which verify the validity of conclusion proposed in Theorem 3.1.

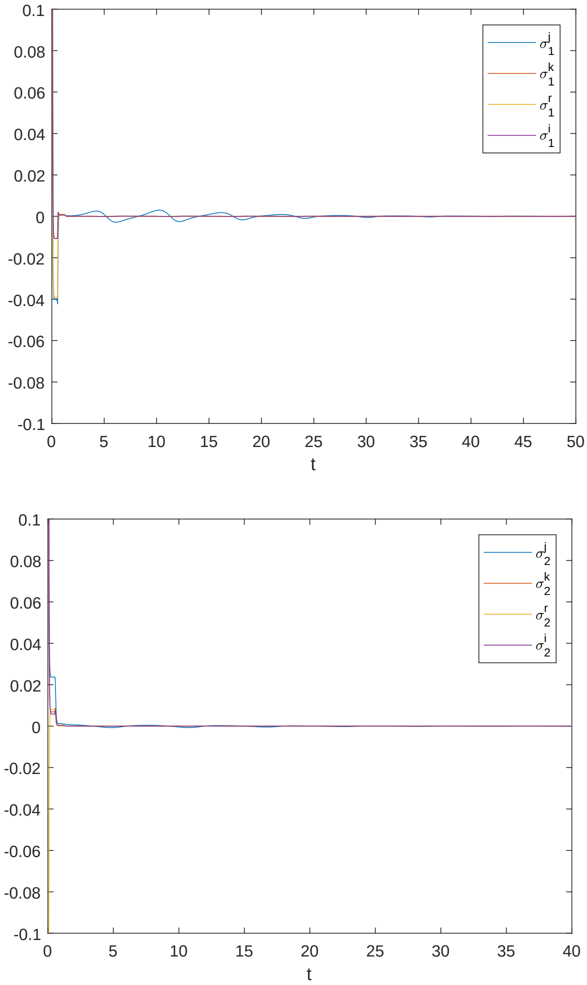


Figure 1. The trajectories of error $\sigma_p^{(r)}(t)$, $\sigma_p^{(i)}(t)$, $\sigma_p^{(j)}(t)$, $\sigma_p^{(k)}(t)$ in Example 1.

Remark 3. As writer knows, the decomposition method can not handle the situation that activation functions cannot be decomposed. Hence, when $tanh(x_p(t))$, which is not easily decomposed, is chose as activation functions, [31] cannot be dealt. However, here, we can easily deal this situation, which proves our method is less conservative.

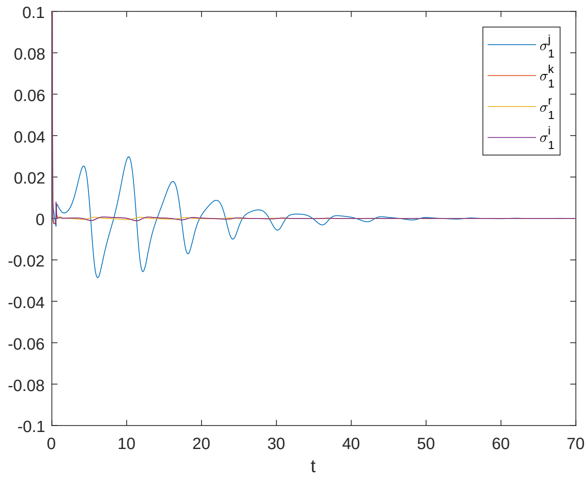
Example 2. Consider the system (2.1) as master system, then the slave system with adaptive controller is designed as

$$\begin{aligned} \dot{y}_p(t) &= -a_p y_p(t) + \sum_{q=1}^{n} b_{pq}(y_p(t)) f_q(y_q(t)) + \sum_{q=1}^{n} c_{pq}(y_p(t)) g_q(y_q(t-\upsilon)) \\ &\quad -\lambda_p \frac{\sigma_p(t)}{\|\sigma_p(t)\|_1} + \alpha I_p(t) \quad p = 1, 2. \end{aligned} \tag{4.2}$$

where $a_p = 2$, time delays $\upsilon = 1$, the external inputs $I_1(t) = -0.70 + 0.30\mathbf{i} + 0.50\mathbf{j} + 0.25\mathbf{k}$, $I_2(t) = -0.40 + 0.20\mathbf{i} + 0.30\mathbf{j} + 0.60\mathbf{k}$, the memristive connection weights as same as value in Example 1, choose the activation functions as

$$\begin{aligned}
f_1(x_1(t)) &= \frac{1}{1+e^{x_1^{(r)}(t)}} + \frac{1}{1+e^{x_1^{(i)}(t)}}\mathbf{i} + \frac{1}{1+e^{x_1^{(j)}(t)}}\mathbf{j} + \frac{1}{1+e^{x_1^{(k)}(t)}}\mathbf{k},\\
f_2(x_2(t)) &= \frac{1}{1+e^{x_2^{(r)}(t)}} + \frac{1}{1+e^{x_2^{(i)}(t)}}\mathbf{i} + \frac{1}{1+e^{x_2^{(j)}(t)}}\mathbf{j} + \frac{1}{1+e^{x_2^{(k)}(t)}}\mathbf{k},\\
g_1(x_1(t)) &= \frac{1}{1+e^{x_1^{(r)}(t)}} + \frac{1}{1+e^{x_1^{(i)}(t)}}\mathbf{i} + \frac{1}{1+e^{x_1^{(j)}(t)}}\mathbf{j} + \frac{1}{1+e^{x_1^{(k)}(t)}}\mathbf{k},\\
g_2(x_2(t)) &= \frac{1}{1+e^{x_2^{(r)}(t)}} + \frac{1}{1+e^{x_2^{(i)}(t)}}\mathbf{i} + \frac{1}{1+e^{x_2^{(j)}(t)}}\mathbf{j} + \frac{1}{1+e^{x_2^{(k)}(t)}}\mathbf{k}.
\end{aligned}$$

Hence, $l_i = 2$, $m_i = 2$, $L_i = 2$, $M_i = 2$, $i = 1, 2$.



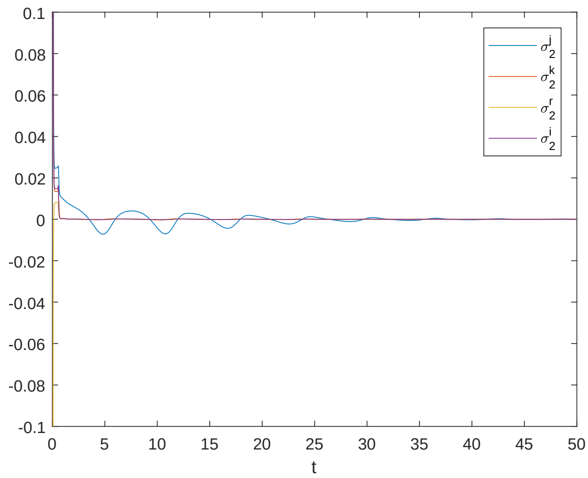


Figure 2. The trajectories of error $\sigma_p^{(r)}(t)$, $\sigma_p^{(i)}(t)$, $\sigma_p^{(j)}(t)$, $\sigma_p^{(k)}(t)$ in Example 2.

Then, for projective coefficient $\alpha = 0.50$, choose parameters $\lambda_1 = \lambda_2 = 25$, $D_1 = 17$, $D_2 = 8$, $k_1 = k_2 = 5$, which satisfy the condition of Theorem 3.2. Therefore, under the adaptive scheme (3.1), master system (2.1) and slave system (4.2) can achieve adaptive exponential projective synchronization. Under the initial conditions $x_1(0) = -1.10 - 1.30\mathbf{i} - 1.50\mathbf{j} - 1.10\mathbf{k}$, $x_2(0) = 0.30 - 0.10\mathbf{i} - 0.30\mathbf{j} - 0.50\mathbf{k}$, $y_1(0) = 0.15 + 0.15\mathbf{i} - 0.18\mathbf{j} + 0.15\mathbf{k}$, $y_2(0) = -1.20 - 1.10\mathbf{i} + 1.30\mathbf{j} + 1.30\mathbf{k}$, $\kappa_1(0) = \kappa_2(0) = 0.15$, the trajectories of error are depicted in Figure 2, which verify the validity of theoretical analysis proposed in Theorem 3.2.

5. Conclusions

In this work, the issues of exponential projective synchronization and adaptive exponential projective synchronization were addressed for QVMNNs with time delays. The results proposed in this paper are general and cover other dynamic behaviors such as complete synchronization, complete anti-synchronization and so on. On the basis of converting QVMNNs into a system with parametric uncertainty, by utilizing the sign function related to quaternion, we designed different control schemes and proposed the corresponding criteria to guarantee the exponential projection synchronization and adaptive exponential projection synchronization of the discussed model, respectively. In addition, we have given two numerical examples and corresponding simulations to verify the reliability and validity of the theoretical analysis.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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