



Research article

Optimal harvesting strategy of a stochastic n -species marine food chain model driven by Lévy noises

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Abstract: A stochastic n -species marine food chain model with harvesting and Lévy noises is proposed. First, the criterion on the asymptotic stability in distribution is established. Second, the criterion on the existence of optimal harvesting strategy (OHS) and the maximum of expectation of sustainable yield (MESY) are derived. Furthermore, the numerical simulations are presented to verify the theoretical results. Our results show that (i) noises intensity can easily affect the dynamics of marine populations, leading to the imbalances of marine ecology, (ii) the establishment of an optimal harvesting strategy should fully consider the impact of noises intensity for better managing and protecting marine resources.

Keywords: stochastic marine food chain model; Lévy noises; stability in distribution; optimal harvesting strategy.

1. Introduction

Overfishing occurs when fish are harvested at a rate faster than the resource can replenish itself. This practice has consequences that extend beyond the depletion of single-species fish populations, potentially leading to global food insecurity, ecological extinction, and other devastating effects [1–4]. Marine ecosystems and their associated livelihoods worldwide are threatened by overfishing. Nicholas et al. [5] have shown that overfishing is pushing over one-third of all sharks and rays towards global extinction. They suggest that marine life could face a future similar to that of land biodiversity, where human pressures have resulted in the loss of numerous species, possibly triggering a sixth mass extinction. According to IUCN Red List criteria [6], they estimate that one-quarter of 1,041 chondrichthyan fishes are threatened due to overfishing (both targeted and incidental). Therefore, it is urgent to establish the optimal harvesting strategy (OHS) for overfishing to prevent extinctions and promote popula-

tion recovery. Consequently, investigating OHS has significant implications for marine ecosystems.

Mathematical models are widely utilized to analyze the OHS of populations [7–13]. Clark [7] proposes several deterministic population systems with harvesting and establishes the OHS. Martin and Ruan [8] investigate the predator-prey models with delay and prey harvesting. However, nature is full of random phenomena, and various populations in the ecosystem experience different forms of random perturbations. May [9] indicates that environmental noise affects competition coefficient, environmental capacity, the growth rate and other parameters of population systems. Lande et al. [10] demonstrate that the extinction of numerous species caused by over-harvesting may be largely due to the ignorance of randomness. Therefore, it is crucial to investigate the OHS of population system in random environment. Li and Wang [11] establish a general stochastic logistic population model with white noises and explore the optimal harvesting policy. Wang et al. [12] propose a stochastic predator-prey model with time delays and shows how randomness influences the extinction and OHS. Therefore, exploring the OHS of stochastic marine population systems is meaningful.

Lévy noise, another type of environmental perturbations, is considered below. During the process of species evolution, population growth is often affected by random disturbances in the environment, such as toxic pollutants, epidemics, earthquakes, floods and other factors [14–17]. For example, the Sandoz chemical accident in 1986 resulted in the death of over 600,000 fish in the Rhine [14], while the 2010 Gulf of Mexico oil spill caused a devastating impact on marine life in the Gulf of Mexico [15]. Bao et al. [16] propose that non-Gaussian Lévy noise is suitable for concluding these phenomena and revealed that Lévy noise can inhibit the explosive increase of population. Zhang et al. [17] also proposed to use Lévy noise to describe the random perturbations.

A lot of work has been done to study the stochastic food chain population dynamics model so far. Our previous work has investigated the dynamics of a stochastic three species food chain model in [18], the dynamics of a four-species stochastic food chain model with delays in [19] and the dynamic behaviors of a stochastic n -species food chain model with Lévy noises and white noises in [20] as follows:

$$\left\{ \begin{array}{l} dx_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t)]dt + \sigma_1 x_1(t)dB_1(t) \\ \quad + \int_{\mathcal{Z}} x_1(t^-)\gamma_1(u)\tilde{N}(dt, du), \\ dx_i(t) = x_i(t)[r_i - a_{ii}x_i(t) + a_{i-1}x_{i-1}(t) - a_{i+1}x_{i+1}(t)]dt + \sigma_i x_i(t)dB_i(t) \\ \quad + \int_{\mathcal{Z}} x_i(t^-)\gamma_i(u)\tilde{N}(dt, du), \quad 2 \leq i \leq n-1, \\ dx_n(t) = x_n(t)[r_n - a_{nn}x_n(t) + a_{n-1}x_{n-1}(t)]dt + \sigma_n x_n(t)dB_n(t) \\ \quad + \int_{\mathcal{Z}} x_n(t^-)\gamma_n(u)\tilde{N}(dt, du). \end{array} \right. \quad (1.1)$$

By introducing a new research method, that is algebraic approach, the sufficient and necessary criteria for the stochastic extinction, persistence in mean and global stability are established in [20]. These conclusions directly generalize some research results on three and four species stochastic food chain dynamics models in [18, 19]. However, due to the high dimensionality of model (1.1) and technical difficulties, the stability in distribution and optimal harvesting problems have not been discussed until now. In this paper, we consider the effects of harvesting and Lévy noises, then address these issues by

proposing a stochastic n -species food chain model as follows:

$$\left\{ \begin{array}{l} dx_1(t) = x_1(t)[r_1 - h_1 - a_{11}x_1(t) - a_{12}x_2(t)]dt + \sigma_1x_1(t)dB_1(t) \\ \quad + \int_Z x_1(t^-)\gamma_1(u)\tilde{N}(dt, du), \\ dx_i(t) = x_i(t)[r_i - h_i - a_{ii}x_i(t) + a_{ii-1}x_{i-1}(t) - a_{ii+1}x_{i+1}(t)]dt + \sigma_ix_i(t)dB_i(t) \\ \quad + \int_Z x_i(t^-)\gamma_i(u)\tilde{N}(dt, du), \quad 2 \leq i \leq n-1, \\ dx_n(t) = x_n(t)[r_n - h_n - a_{nn}x_n(t) + a_{nn-1}x_{n-1}(t)]dt + \sigma_nx_n(t)dB_n(t) \\ \quad + \int_Z x_n(t^-)\gamma_n(u)\tilde{N}(dt, du). \end{array} \right. \quad (1.2)$$

On the basis of the research results in [20], our primary objective is to establish criteria for the stability in distribution of model (1.2). Additionally, we aim to establish criteria for the optimal harvesting strategy in the sense of sustainable yield. Lastly, we will verify the theoretical results through numerical examples and propose some open questions and propositions.

This paper is organized as follows. In Section 2, we provide the basic assumption, and further introduce several important lemmas that will be used to prove the major results. In Section 3, we state and prove the stability in distribution of model (1.2). In Section 4, we outline the non-existence and existence criteria for OHS and MESY. Finally, in Section 5, we illustrate our major results through numerical examples.

2. Preliminaries

Denote by $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, P)$ the complete probability space, and its filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions. In this paper, we assume that model (1.2) is defined on the probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, P)$. In addition, for an integrable function $f(t)$ defined for $t \in [0, +\infty)$, we define the mean value of $f(t)$ on interval $[0, t]$ for any $t > 0$ by $\langle f(t) \rangle = \frac{1}{t} \int_0^t f(s)ds$.

In models (1.1) and (1.2), parameter $r_1 > 0$ is the intrinsic growth rate of species x_1 , $r_i \leq 0$ represents the death rate of species x_i for $i = 2, 3, \dots, n$, $h_i \geq 0$ represents the harvesting rate of human for the species x_i for $i = 1, 2, \dots, n$, $x_i(t^-)$ is the left limit of $x_i(t)$ at time t , $a_{ii} > 0$ ($i = 1, 2, \dots, n$) is the density dependent coefficient of species x_i , $a_{ii+1} \geq 0$ ($i = 2, 3, \dots, n-1$) is the capture rate of which species x_{i+1} preys on species x_i , $a_{ii-1} \geq 0$ ($i = 2, 3, \dots, n$) stands for efficiency of food conversion from species x_{i-1} to species x_i . $B_i(t)$ ($i = 1, 2, \dots, n$) are independent standard Brownian motion, and $\sigma_i^2 > 0$ is the intensity of $B_i(t)$. Lévy noises $\tilde{N}(dt, du) = N(dt, du) - \lambda(du)dt$, where $N(dt, du)$ is a poisson counting measure with characteristic measure λ on a Borel-measurable subset Z of $(0, \infty)$ satisfying $\lambda(Z) < +\infty$. Functions $\gamma_i(u) : Z \rightarrow R$ ($i = 1, 2, \dots, n$) represent the effects of random jumps for species x_i that are assumed to be continuous on Z . In this paper we further assume that Lévy noises $\tilde{N}(dt, du)$ and Brownian motion $B_i(t)$ ($i = 1, 2, \dots, n$) are also independent.

For any solution $(x_1(t), \dots, x_n(t))$ of model (1.2), the initial value is given as follows.

$$x_i(0) = x_{i0}, \quad i = 1, 2, \dots, n, \quad (2.1)$$

where x_{i0} ($i = 1, 2, \dots, n$) are positive constants.

Denote $b_i = r_i + \int_Z [\ln(1 + \gamma_i(u)) - \gamma_i(u)] \lambda(du) - \frac{\sigma_i^2}{2}$ and $c_i = b_i - h_i$ for $i = 1, 2, \dots, n$, and $\alpha_k = (c_1, c_2, \dots, c_k)^T$ for $k = 1, 2, \dots, n$. Furthermore, we denote

$$A_k = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & 0 & 0 \\ -a_{21} & a_{22} & a_{23} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -a_{k-1k-2} & a_{k-1k-1} & a_{k-1k} \\ 0 & 0 & 0 & \cdots & 0 & -a_{kk-1} & a_{kk} \end{bmatrix}.$$

Particularly, we have

$$A_1 = [a_{11}], \quad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{bmatrix}.$$

Denote by H_k the determinant of A_k . Let $A_k^{(i)}$ be the matrix gained from matrix A_k by changing its i -th column to α_k , and $D_k^{(i)}$ is the determinant of $A_k^{(i)}$. Furthermore, for any square matrix P we denote by $P(i, j)$ the algebraic complement corresponding to the element of i -th row and j -th column of P .

For the Lévy noises in models (1.1) and (1.2), we always assume that the following condition is satisfied.

(H) For $i = 1, 2, \dots, n$ and $u \in Z$, $1 + \gamma_i(u) > 0$ holds, and a constant $z > 0$ exists such that

$$\int_Z [\ln(1 + \gamma_i(u))]^2 \lambda(du) < z.$$

Using the same method as in [20], the following conclusions can be established for model (1.2).

Lemma 1. $H_k > 0$ for every $k = 1, 2, \dots, n$.

Lemma 2. For any initial values $(x_{10}, x_{20}, \dots, x_{n0}) \in \mathbb{R}_+^n$, model (1.2) possesses a globally unique positive solution $x(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}_+^n$ a.s. for all $t \geq 0$ satisfying the initial condition (2.1). Furthermore, for any constant $q > 0$, there are constants $C_i(q) > 0$ ($i = 1, 2, \dots, n$) such that

$$\limsup_{t \rightarrow \infty} E[x_i^q(t)] \leq C_i(q), \quad i = 1, 2, \dots, n.$$

Similar to [20], we define

$$N_1 = \frac{a_{12} A_k^{(2)}(3, 2)}{a_{11} H_k(3, k)}, \quad N_i = \frac{a_{ii-1}}{a_{ii}} N_{i-1} + \frac{a_{ii+1} A_k^{(i+1)}(i+2, i+1)}{a_{ii} H_k(i+2, k)}, \quad i = 1, 2, \dots, k-2,$$

$$N_{k-1} = \frac{a_{k-1k-2}}{a_{k-1k-1}} N_{k-2}.$$

Lemma 3. For any positive solution $(x_1(t), \dots, x_n(t))$ of model (1.2), assume that $D_n^{(n)} > 0$ and $(1 + \frac{a_{nn-1}}{a_{nn}} N_{n-1}) > 0$ if $n \geq 3$, then $\lim_{t \rightarrow \infty} \langle x_i(t) \rangle = \frac{D_n^{(i)}}{H_n}$ a.s., $i = 1, 2, \dots, n$.

A simple calculation we can obtain that for any integers $k, i, j \in \{1, 2, \dots, n\}$ with $i, j \leq k$

$$A_k^{(i)}(j, i) = A_k(j, i).$$

Therefore, we further have

$$\begin{aligned} N_1 &= \frac{a_{12} A_k(3, 2)}{a_{11} H_k(3, k)}, \\ N_i &= \frac{a_{ii-1}}{a_{ii}} N_{i-1} + \frac{a_{ii+1}}{a_{ii}} \frac{A_k(i+2, i+1)}{H_k(i+2, k)}, \quad i = 1, 2, \dots, k-2, \\ N_{k-1} &= \frac{a_{k-1k-2}}{a_{k-1k-1}} N_{k-2}. \end{aligned} \quad (2.2)$$

3. Stability in distribution

Theorem 1. Let $(x_1(t; \psi), x_2(t; \psi), \dots, x_n(t; \psi))$ and $(z_1(t; \psi^*), z_2(t; \psi^*), \dots, z_n(t; \psi^*))$ be two solutions of model (1.2) with initial values $\psi, \psi^* \in \mathbb{R}_+^n$. Assume that there are positive constants w_i ($i = 1, 2, \dots, n$) such that

$$\begin{aligned} w_1 a_{11} - w_2 a_{21} > 0, \quad w_i a_{ii} - w_{i-1} a_{i-1i} - w_{i+1} a_{i+1i} > 0 \quad (i = 2, 3, \dots, n-1), \\ w_n a_{nn} - w_{n-1} a_{n-1n} > 0. \end{aligned}$$

Then we have

$$\lim_{t \rightarrow \infty} E \left(\sum_{i=1}^n |x_i(t, \psi) - z_i(t, \psi^*)|^2 \right)^{\frac{1}{2}} = 0.$$

Proof. We only need to show

$$\lim_{t \rightarrow \infty} E |x_i(t; \psi) - z_i(t; \psi^*)| = 0, \quad i = 1, 2, \dots, n. \quad (3.1)$$

Define the functions as follows

$$V_i(x_i) = |\ln x_i(t; \psi) - \ln z_i(t; \psi^*)|, \quad i = 1, 2, \dots, n.$$

Applying Itô formula, then

$$\begin{aligned} \mathcal{L}V_i(x_i) &\leq -a_{ii}|x_i(t; \psi) - z_i(t; \psi^*)| + a_{ii-1}|x_{i-1}(t; \psi) - z_{i-1}(t; \psi^*)| \\ &\quad + a_{ii+1}|x_{i+1}(t; \psi) - z_{i+1}(t; \psi^*)|, \quad i = 1, 2, \dots, n. \end{aligned} \quad (3.2)$$

where $a_{10} = a_{n+1} = 0$ are stipulated. Define the following function

$$V(t) = \sum_{i=1}^n w_i V_i(x_i). \quad (3.3)$$

From (3.1) we obtain

$$\begin{aligned} \mathcal{L}V(t) &= \sum_{i=1}^n w_i \mathcal{L}V_i(x_i) \\ &\leq - \sum_{i=1}^n (w_i a_{ii} - w_{i-1} a_{i-1i} - w_{i+1} a_{i+1i}) |x_i(t; \psi) - z_i(t; \psi^*)|, \end{aligned}$$

where $a_{01} = a_{n+1n} = 0$ and $\omega_0 = \omega_{n+1} = 0$ are stipulated. Therefore,

$$E[V(t)] \leq E[V(0)] - \sum_{i=1}^n (w_i a_{ii} - w_{i-1} a_{i,i-1} - w_{i+1} a_{i+1,i}) \int_0^t E[|x_i(s; \psi) - z_i(s; \psi^*)|] ds.$$

which implies

$$\int_0^{\infty} E[|x_i(s; \psi) - z_i(s; \psi^*)|] ds < +\infty, \quad i = 1, 2, \dots, n. \quad (3.4)$$

Define the functions

$$F_i(t) = E[|x_i(t; \psi) - z_i(t; \psi^*)|], \quad i = 1, 2, \dots, n.$$

Then, for any $t_1, t_2 \in [0, +\infty)$ and each $i = 1, 2, \dots, n$,

$$\begin{aligned} |F_i(t_2) - F_i(t_1)| &= |E[|x_i(t_2; \psi) - z_i(t_2; \psi^*)| - |x_i(t_1; \psi) - z_i(t_1; \psi^*)|]| \\ &\leq E[|x_i(t_2; \psi) - x_i(t_1; \psi)|] + E[|z_i(t_2; \psi^*) - z_i(t_1; \psi^*)|]. \end{aligned} \quad (3.5)$$

Applying Itô formula to model (1.2), then

$$\begin{aligned} x_i(t_2; \phi) - x_i(t_1; \phi) &= \int_{t_1}^{t_2} x_i(s; \phi) [r_i - h_i + a_{ii-1} x_{i-1}(s; \phi) - a_{ii} x_i(s) \\ &\quad - a_{ii+1} x_{i+1}(s; \phi)] ds + \int_{t_1}^{t_2} \sigma_i x_i(s; \phi) dB_i(s) \\ &\quad + \int_{t_1}^{t_2} \int_Z x_i(s^-) \gamma_i(u) \tilde{N}(ds, du), \quad i = 1, 2, \dots, n. \end{aligned} \quad (3.6)$$

For any $t_2 > t_1$ and $q > 1$, from the first equation of (3.6) and Hölder's inequality, then

$$\begin{aligned} &(E[|x_1(t_2; \psi) - x_1(t_1; \psi)|])^q \\ &\leq E[(\int_{t_1}^{t_2} x_1(s; \psi) |r_1 - h_1 - a_{11} x_1(s; \psi) - a_{12} x_2(s; \psi)| ds \\ &\quad + |\int_{t_1}^{t_2} \sigma_1 x_1(s; \psi) dB_1(s)|)^q] + |\int_{t_1}^{t_2} \int_Z x_1(s^-) \gamma_1(u) \tilde{N}(ds, du)| \\ &\leq 3^q E[(\int_{t_1}^{t_2} x_1(s; \psi) |r_1 - h_1 - a_{11} x_1(s; \psi) - a_{12} x_2(s; \psi)| ds)^q] \\ &\quad + 3^q E[|\int_{t_1}^{t_2} \sigma_1 x_1(s; \psi) dB_1(s)|^q] + 3^q E[|\int_{t_1}^{t_2} \int_Z x_1(s^-) \gamma_1(u) \tilde{N}(ds, du)|^q]. \end{aligned} \quad (3.7)$$

Similarly,

$$\begin{aligned} &E[(\int_{t_1}^{t_2} x_1(s; \psi) |r_1 - h_1 - a_{11} x_1(s; \psi) - a_{12} x_2(s; \psi)| ds)^q] \\ &\leq (t_2 - t_1)^{q-1} E[\int_{t_1}^{t_2} (|r_1 - h_1| x_1(s; \psi) + a_{11} x_1^2(s; \psi) + a_{12} x_1(s; \psi) x_2(s; \psi))^q ds] \\ &\leq 3^q (t_2 - t_1)^{q-1} |r_1 - h_1|^q \int_{t_1}^{t_2} E[x_1^q(s; \psi)] ds + 3^q a_{11}^q (t_2 - t_1)^{q-1} \int_{t_1}^{t_2} E[x_1^{2q}(s; \psi)] ds \\ &\quad + 3^q (t_2 - t_1)^{q-1} E[\int_{t_1}^{t_2} (a_{12} x_1(s; \psi) x_2(s; \psi))^q ds]. \end{aligned} \quad (3.8)$$

and

$$\begin{aligned}
 & E\left[\int_{t_1}^{t_2} (a_{12}x_1(s; \psi)x_2(s; \psi))^q ds\right] \\
 & \leq E\left[\int_{t_1}^{t_2} \left(\frac{1}{2}a_{12}x_1^2(s; \psi) + \frac{1}{2}a_{12}x_2^2(s; \psi)\right)^q ds\right] \\
 & \leq a_{12}^q \int_{t_1}^{t_2} E[x_1^{2q}(s; \psi)] ds + a_{12}^q \int_{t_1}^{t_2} E[x_2^{2q}(s; \psi)] ds.
 \end{aligned} \tag{3.9}$$

In view of Theorem 7.1 in [21] on the Burkholder-Davis-Gundy inequality, for $1 < q \leq 2$ and any $t_2 > t_1$, then

$$\begin{aligned}
 E\left[\left|\int_{t_1}^{t_2} \sigma_1 x_1(s; \psi) dB_1(s)\right|^q\right] & \leq D_q E\left[\int_{t_1}^{t_2} |\sigma_1 x_1(s; \psi)|^2 ds\right]^{q/2} \\
 & \leq D_q \sigma_1^q (t_2 - t_1)^{\frac{q-2}{2}} \int_{t_1}^{t_2} E[x_1^q(s; \psi)] ds.
 \end{aligned} \tag{3.10}$$

Using the Kunita first inequality in [21], there exists a constant $K_q > 0$ and from assumption (H) we can obtain

$$\begin{aligned}
 & E\left[\left|\int_{t_1}^{t_2} \int_Z x_1(s; \psi) \gamma_1(u) \tilde{N}(ds, du)\right|^q\right] \\
 & \leq K_q E\left[\int_{t_1}^{t_2} \int_Z |x_1(s) \gamma_1(u)|^2 \lambda(du) ds\right]^{q/2} + E\left[\int_{t_1}^{t_2} \int_Z |x_1(s) \gamma_1(u)|^q \lambda(du) ds\right] \\
 & \leq K_q L^{q/2} (t_2 - t_1)^{\frac{q-2}{2}} \int_{t_1}^{t_2} E[x_1^q(s; \psi)] ds + K_q L^{q/2} \int_{t_1}^{t_2} E[x_1^q(s; \psi)] ds.
 \end{aligned} \tag{3.11}$$

From Lemma 2, there exists $C_i^{**}(q) > 0$ for $i = 1, 2, \dots, n$ such that $\sup_{t \geq -\gamma} E[x_i^q(t)] \leq C_i^{**}(q)$. Therefore, from (3.7)–(3.10) there is a $\delta > 0$ satisfies that for $1 < q \leq 2$ and any $t_1 \geq 0, t_2 \geq 0$ with $|t_2 - t_1| \leq \delta$, such that

$$\begin{aligned}
 & (E[|x_1(t_2; \psi) - x_1(t_1; \psi)|])^q \\
 & \leq 3^q [D_q \sigma_1^q (t_2 - t_1)^{\frac{q}{2}} C_1^{**}(q)] + (t_2 - t_1)^{\frac{q}{2}} 3^q [K_q L^{q/2} C_1^{**}(q) + K_q L^{q/2} (t_2 - t_1)^{1-q/2} C_1^{**}(q)] \\
 & \quad + 3^q [3^q (t_2 - t_1)^q |r_1 - h_1|^q C_1^{**}(q) + 3^q a_{11}^q (t_2 - t_1)^q C_1^{**}(2q)] \\
 & \quad + 3^{2q} (t_2 - t_1)^q a_{12}^q [C_1^{**}(2q) + C_2^{**}(2q)] \\
 & \leq M_1^{**} |t_2 - t_1|^{\frac{q}{2}},
 \end{aligned}$$

where

$$\begin{aligned}
 M_1^{**} & = 3^q D_q \sigma_1^q C_1^{**}(q) + 3^q K_q L^{q/2} [1 + \delta^{\frac{2-q}{2}}] C_1^{**}(q) + [81\delta]^{\frac{q}{2}} [|r_1 - h_1|^q C_1^{**}(q) + a_{11}^q C_1^{**}(2q)] \\
 & \quad + [81\delta]^{\frac{q}{2}} a_{12}^q [C_1^{**}(2q) + C_2^{**}(2q)].
 \end{aligned}$$

Similarly, we also obtain

$$(E[|z_1(t_2; \psi^*) - z_1(t_1; \psi^*)|])^q \leq M_1^{**} |t_2 - t_1|^{\frac{q}{2}}.$$

for any $t_1 \geq 0, t_2 \geq 0$ with $|t_2 - t_1| \leq \delta$ and $1 < q \leq 2$. Thus, from (3.5), we obtain

$$\begin{aligned}
 |F_1(t_2) - F_1(t_1)| & \leq E[|x_1(t_2; \psi) - x_1(t_1; \psi)|] + E[|z_1(t_2; \psi^*) - z_1(t_1; \psi^*)|] \\
 & \leq 2(M_1^{**})^{\frac{1}{q}} \sqrt{|t_2 - t_1|}.
 \end{aligned} \tag{3.12}$$

Using similar argument, for $F_i(t)$ ($i = 2, 3, \dots, n$), we also can obtain that there is a $\delta > 0$ for any $t_1 \geq 0, t_2 \geq 0$ with $|t_2 - t_1| \leq \delta$ and $1 < q \leq 2$

$$|F_i(t_2) - F_i(t_1)| \leq 2(M_i^{**})^{\frac{1}{q}} \sqrt{|t_2 - t_1|}, \quad i = 2, 3, \dots, n, \quad (3.13)$$

where

$$\begin{aligned} M_i^{**} = & 3^q D_q \sigma_i^q C_i^{**}(q) + 3^q K_q L^{q/2} [1 + \delta^{\frac{2-q}{2}}] C_i^{**}(q) \\ & + [144\delta]^{\frac{q}{2}} [r_i + h_i]^q C_i^{**}(q) + a_{ii}^q C_i^{**}(2q) + [144\delta]^{\frac{q}{2}} a_{ii+1}^q [C_i^{**}(2q) + C_{i+1}^{**}(2q)] \\ & + [144\delta]^{\frac{q}{2}} a_{ii-1}^q [C_{i-1}^{**}(2q) + C_{i+1}^{**}(2q)], \quad i = 2, 3, \dots, n. \end{aligned}$$

From (3.12) and (3.13), we obtain that $F_i(t)$, $i = 1, 2, \dots, n$ for $t \in (0, \infty)$ are uniformly continuous. Therefore, from (3.4) and Barbalat lemma in [22] we finally obtain that (3.1) holds. This completes the proof. \square

Let $\mathcal{P}(R_+^n)$ be the space of all probability measures on $C(R_+^n)$. For $Q_1, Q_2 \in \mathcal{P}(R_+^n)$, we define

$$d_{LB}(Q_1, Q_2) = \sup_{f \in LB} \left| \int_{R_+^n} f(y) Q_1(dy) - \int_{R_+^n} f(y) Q_2(dy) \right|, \quad (3.14)$$

where

$$LB = \{f : R_+^n \rightarrow R : |f(y_1) - f(y_2)| \leq \|y_1 - y_2\|, |f(\cdot)| \leq 1\}. \quad (3.15)$$

Denote by $p(t, \phi, dx)$ the transition probability of process $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$. We have the following results.

Theorem 2. Assume that there exists constants $w_i > 0$ ($i = 1, 2, \dots, n$) such that

$$\begin{aligned} w_1 a_{11} - w_2 a_{21} &> 0, \quad w_i a_{ii} - w_{i-1} a_{i-1i} - w_{i+1} a_{i+1i} > 0 \quad (i = 2, 3, \dots, n-1), \\ w_n a_{nn} - w_{n-1} a_{n-1n} &> 0. \end{aligned}$$

Then model (1.2) is asymptotically stable in distribution, that is, there is a probability measure $\nu(\cdot)$ satisfies that for any initial function $\psi \in R_+^n$, the transition probability $p(t, \psi, \cdot)$ of solution $(x_1(t, \psi), x_2(t, \psi), \dots, x_n(t, \psi))$ such that

$$\lim_{t \rightarrow \infty} d_{LB}(p(t, \psi, \cdot), \nu(\cdot)) = 0.$$

Proof. For any initial data $\psi \in R_+^n$, the family probability $p(t, \psi, \cdot)$ is tight which obtained from Theorem 1 and Chebyshev's inequality. For any $f \in LB$ and $t, s > 0$, we compute

$$\begin{aligned} & |Ef(y(t+s; \psi))p(t+s, \psi, dy) - Ef(y(t; \psi))p(t, \psi, dy)| \\ &= |E[E(f(y(t+s; \psi)) | \mathcal{F}_s)] - E[f(y(t; \psi))]| \\ &= \left| \int_{R_+^n} E[f(y(t; \phi))]p(s, \psi, d\phi) - E[f(y(t; \psi))] \right| \leq \int_{R_+^n} |Ef(y(t; \phi)) - Ef(y(t; \psi))| p(s, \psi, d\phi). \end{aligned}$$

It is clear that there exists a constant $T > 0$ from (3.1), such that for all $t \geq T$,

$$\sup_{f \in LB} |Ef(y(t; \phi)) - Ef(y(t; \psi))| \leq \varepsilon.$$

Consequently

$$|Ef(y(t+s; \psi)) - Ef(y(t; \psi))| \leq \varepsilon.$$

Since f is arbitrary, we gain

$$\sup_{f \in LB} |Ef(y(t+s; \psi)) - Ef(y(t; \psi))| \leq \varepsilon.$$

That is,

$$d_{LB}(p(t+s, \psi, \cdot), p(t, \psi, \cdot)) \leq \varepsilon.$$

Therefore $\{p(t, \psi, \cdot) : t \geq 0\}$ is Cauchy in the space $\mathcal{P}(R_+^n)$. Hence, there is a unique $v(\cdot) \in \mathcal{P}(R_+^n)$ such that

$$\lim_{t \rightarrow \infty} d_{LB}(p(t, \psi_0, \cdot), v(\cdot)) = 0.$$

where $\psi_0 = (c_{10}, c_{20}, \dots, c_{n0})^T \in R_+^n$, and c_{10}, \dots, c_{n0} are positive constants. It then follows from (3.1) that

$$\lim_{t \rightarrow \infty} d_{LB}(p(t, \psi, \cdot), p(t, \psi_0, \cdot)) = 0.$$

Hence,

$$\lim_{t \rightarrow \infty} d_{LB}(p(t, \psi, \cdot), v(\cdot)) \leq \lim_{t \rightarrow \infty} d_{LB}(p(t, \psi, \cdot), p(t, \psi_0, \cdot)) + \lim_{t \rightarrow \infty} d_{LB}(p(t, \psi_0, \cdot), v(\cdot)) = 0.$$

This completes the proof. \square

4. Optimal harvesting strategy

Since $H_n > 0$ in Lemma 1, the inverse matrix A_n^{-1} exists. Let $B = (b_1, b_2, \dots, b_n)^T$ and $\mathcal{H} = (h_1, h_2, \dots, h_n)^T$. Moreover, let E be the unit matrix, and $\mathcal{H}^* = (A_n(A_n^{-1})^T + E)^{-1}B \triangleq (h_1^*, h_2^*, \dots, h_n^*)^T$.

Theorem 3. Assume that the following conditions hold

- (1) $(1 + \frac{a_{m-1}}{a_m} N_{n-1}) > 0$ if $n \geq 3$;
- (2) there exists constants $w_i > 0$ ($i = 1, 2, \dots, n$) such that

$$w_1 a_{11} - w_2 a_{21} > 0, w_i a_{ii} - w_{i-1} a_{i-1i} - w_{i+1} a_{i+1i} > 0 \quad (i = 2, 3, \dots, n-1),$$

$$w_n a_{nn} - w_{n-1} a_{n-1n} > 0.$$

Then the following conclusions hold.

(\mathcal{A}_1) If $D_n^{(n)}|_{h_i=h_i^*} > 0$, $A_n^{-1} + (A_n^{-1})^T$ is positive semi-definite and $h_i^* \geq 0$ ($i = 1, 2, \dots, n$), then model (1.2) has the OHS $H = H^*$, and

$$MESY \triangleq S(\mathcal{H}^*) = (\mathcal{H}^*)^T A_n^{-1} (B - \mathcal{H}^*). \quad (4.1)$$

(\mathcal{A}_2) If any of the following conditions stands, then the OHS for model (1.2) does not exist.

- (\mathcal{B}_1) $D_n^{(n)}|_{h_i=h_i^*, i=1,2,\dots,n} \leq 0$;
- (\mathcal{B}_2) There is a h_i^* with $h_i^* < 0$;
- (\mathcal{B}_3) $A_n^{-1} + (A_n^{-1})^T$ is not positive semi-definite.

Proof. Let $\mathcal{Y} = \{\mathcal{H} = (h_1, h_2, \dots, h_n)^T \in \mathbb{R}^n : D_n^{(n)} > 0, h_i \geq 0, i = 1, 2, \dots, n\}$, which means that for any $\mathcal{H} \in \mathcal{Y}$, the conclusion of Lemma 3 stands.

First, we prove the conclusion (\mathcal{A}_1) . Since $\mathcal{H}^* \in \mathcal{Y}$, the set \mathcal{Y} is not empty. A unique probability measure $\nu(\cdot)$ is gained from Theorem 2. Then, $\nu(\cdot)$ is also strong mixing by Corollary 3.4.3 [23]. From Theorem 3.2.6 [23], we gain that the measure $\nu(\cdot)$ is ergodic. For any initial value $\psi \in \mathbb{R}_+^n$, model (1.2) possess a positive gloabl solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$. It yields by Theorem 3.3.1 [23] that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathcal{H}^T x(s) ds = \int_{\mathbb{R}_+^n} \mathcal{H}^T x \nu(dx), \quad (4.2)$$

where $\mathcal{H} = (h_1, h_2, \dots, h_n)^T \in \mathcal{Y}$. Let $\varrho(z)$ be the stationary probability density of model (1.2), then we have

$$S(\mathcal{H}) = \lim_{t \rightarrow \infty} E\left[\sum_{i=1}^n h_i x_i(t)\right] = \lim_{t \rightarrow \infty} E[\mathcal{H}^T x(t)] = \int_{\mathbb{R}_+^n} \mathcal{H}^T x \varrho(x) dx. \quad (4.3)$$

In view of the invariant measure is sole, there also exists a one-to-one correspondence among $\varrho(z)$ and its corresponding invariant measure. Hence, we have

$$\int_{\mathbb{R}_+^n} \mathcal{H}^T x \varrho(x) dx = \int_{\mathbb{R}_+^n} \mathcal{H}^T x \nu(dx). \quad (4.4)$$

Combining (4.2)–(4.4) with the conclusion of Lemma 3, then

$$\begin{aligned} S(\mathcal{H}) &= \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathcal{H}^T x(s) ds \\ &= h_1 \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_1(s) ds + h_2 \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_2(s) ds + \dots + h_n \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_n(s) ds \\ &= h_1 \frac{D_n^{(1)}}{H_n} + h_2 \frac{D_n^{(2)}}{H_n} + \dots + h_n \frac{D_n^{(n)}}{H_n}. \end{aligned}$$

Moreover, $S(\mathcal{H}) = \mathcal{H}^T (A_n)^{-1} (B - \mathcal{H})$ is derived by careful calculation. Computing the gradient of $S(\mathcal{H})$, then

$$\frac{\partial S(\mathcal{H})}{\partial \mathcal{H}} = \frac{\partial \mathcal{H}^T}{\partial \mathcal{H}} (A_n)^{-1} (B - \mathcal{H}) + \frac{\partial (B - \mathcal{H})^T}{\partial \mathcal{H}} ((A_n)^{-1})^T \mathcal{H}.$$

Since $\frac{\partial \mathcal{H}^T}{\partial \mathcal{H}} = E$, then

$$\frac{\partial S(\mathcal{H})}{\partial \mathcal{H}} = (A_n)^{-1} (B - \mathcal{H}) - ((A_n)^{-1})^T \mathcal{H} = (A_n)^{-1} B - ((A_n)^{-1} + ((A_n)^{-1})^T) \mathcal{H}.$$

By solving $\frac{\partial S(\mathcal{H})}{\partial \mathcal{H}} = 0$, the following critical value is gained

$$\mathcal{H} = ((A_n)^{-1} + ((A_n)^{-1})^T)^{-1} (A_n)^{-1} B.$$

That is,

$$\mathcal{H} = [(A_n)^{-1} ((A_n) ((A_n)^{-1})^T + E)]^{-1} (A_n)^{-1} B = ((A_n) ((A_n)^{-1})^T + E)^{-1} B = \mathcal{H}^*.$$

Calculating the Hessian matrix of $S(\mathcal{H})$, we obtain

$$\frac{\partial}{\partial \mathcal{H}} \left(\frac{\partial S(\mathcal{H})}{\partial \mathcal{H}} \right) = -\frac{\partial((A_n)^{-1} + ((A_n)^T)^T \mathcal{H}}{\partial \mathcal{H}} = -((A_n)^{-1} + ((A_n)^{-1})^T). \quad (4.5)$$

Since $(A_n)^{-1} + ((A_n)^{-1})^T$ is positive semi-definit, from the existence principle of extremum value of multi-variable function we obtain that $S(\mathcal{H})$ has a global maximum value at $\mathcal{H} = \mathcal{H}^*$. Obviously, \mathcal{H}^* is also sole. Therefore, if $\mathcal{H}^* \in \mathcal{Y}$, that is $h_i^* \geq 0$ ($i = 1, 2, \dots, n$) and $D_n^{(n)}|_{h_i=h_i^*, i=1,2,\dots,n} > 0$, then we finally obtain that \mathcal{H}^* is an OHS, and MESY shown in (20).

Next, we prove the conclusion (\mathcal{A}_2) . We first assume that (\mathcal{B}_1) or (\mathcal{B}_2) stands. Suppose that $\tilde{\Gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n)$ is the OHS for model (1.2), thus $\Gamma \in \mathcal{Y}$, that is

$$D_n^{(n)}|_{h_i=\gamma_i, i=1,2,\dots,n} > 0, \quad \gamma_i \geq 0, \quad i = 1, 2, \dots, n. \quad (4.6)$$

On the other hand, if $\Gamma \in \mathcal{Y}$ is the OHS, then Γ is a unique solution of equation $\frac{\partial S(\mathcal{H})}{\partial \mathcal{H}} = 0$. Therefore, we have $(h_1^*, h_2^*, \dots, h_n^*) = (\gamma_1, \gamma_2, \dots, \gamma_n)$. Thus, condition (4.6) becomes into

$$D_n^{(n)}|_{h_i=h_i^*, i=1,2,\dots,n} > 0, \quad h_i^* \geq 0, \quad i = 1, 2, \dots, n.$$

However, this is impossible.

Finally, consider (\mathcal{B}_3) . It can be assumed that (\mathcal{B}_1) and (\mathcal{B}_2) fail to stand in this case. Therefore, $h_i^* \geq 0$ ($i = 1, 2, \dots, n$) and $D_n^{(n)}|_{h_i=h_i^*, i=1,2,\dots,n} > 0$. Thus, \mathcal{Y} is not empty, i.e., (4.2)–(4.4) hold. Denote $(A_n)^{-1} + ((A_n)^{-1})^T = (h_{ij})_{n \times n}$. Then, by calculating we have

$$h_{11} = \frac{2(a_{22}a_{33} \cdots a_{nn} + \cdots + a_{22}a_{34} \cdots a_{nn-1} + a_{23}a_{32} \cdots a_{nn})}{H_n}$$

Obviously, $h_{11} > 0$, which means that $(A_n)^{-1} + ((A_n)^{-1})^T$ is not negative semi-definite. It can be further derived from the conditions in (\mathcal{B}_3) that $(A_n)^{-1} + ((A_n)^{-1})^T$ is indefinite. Hence, there is not the OHS if \mathcal{B}_3 holds. This completes the proof. \square

Remark 1. In Theorem 3 we see that the conditions (1) and (2) are required. From the following argumentations we will find that when $n = 3$ and $n = 4$ condition (2) can bring out condition (1). Therefore, we propose the following proposition. However, when $n > 4$ whether condition (2) can bring out condition (1) still is an open problem. Thus, the following conjecture can be proposed.

Proposition 1. When $n = 3$ and $n = 4$, if condition (2) holds then condition (1) also holds.

Proof. Let $k = 3$, then condition (2) becomes to

$$w_1 a_{11} - w_2 a_{21} > 0, \quad w_2 a_{22} - w_1 a_{12} - w_3 a_{32} > 0, \quad w_3 a_{33} - w_2 a_{23} > 0. \quad (4.7)$$

From (2.2), by calculating we have

$$N_1 = -\frac{a_{12}a_{23}}{a_{11}a_{22} + a_{12}a_{21}}, \quad N_2 = -\frac{a_{12}a_{21}a_{23}}{a_{22}(a_{11}a_{22} + a_{12}a_{21})},$$

and

$$1 + \frac{a_{32}}{a_{33}} N_2 = \frac{a_{22}a_{33}(a_{11}a_{22} + a_{12}a_{21}) - a_{12}a_{21}a_{23}a_{32}}{a_{22}a_{33}(a_{11}a_{22} + a_{12}a_{21})}.$$

From (4.7) we derive

$$a_{22} \geq \frac{1}{w_2}(w_1 a_{12}) + w_3 a_{32}, \quad a_{33} \geq \frac{w_2 a_{23}}{w_3}.$$

Therefore, we further have

$$\begin{aligned} & a_{22} a_{33} (a_{11} a_{22} + a_{12} a_{21}) - a_{12} a_{21} a_{23} a_{32} \\ & > \left[\frac{1}{w_2} (w_1 a_{12} + w_3 a_{32}) \frac{1}{w_3} w_2 a_{23} (a_{11} a_{22} + a_{12} a_{21}) \right] - a_{12} a_{21} a_{23} a_{32} \\ & = \left(\frac{w_1 a_{12}}{w_2 w_3} + \frac{a_{32}}{w_2} \right) w_2 a_{23} (a_{11} a_{22} + a_{12} a_{21}) - a_{12} a_{21} a_{23} a_{32} \\ & = \left(\frac{w_1 a_{12} a_{23}}{w_3} + a_{23} a_{32} \right) (a_{11} a_{22} + a_{12} a_{21}) - a_{12} a_{21} a_{23} a_{32} \\ & = \frac{w_1 a_{12} a_{23}}{w_3} (a_{11} a_{22} + a_{12} a_{21}) + a_{11} a_{22} a_{23} a_{32} > 0. \end{aligned}$$

This shows that $1 + \frac{a_{32}}{a_{33}} N_2 > 0$.

Let $k = 4$, then condition (2) becomes to

$$\begin{aligned} w_1 a_{11} - w_2 a_{21} &> 0, \quad w_2 a_{22} - w_1 a_{12} - w_3 a_{32} > 0, \\ w_3 a_{33} - w_2 a_{23} - w_4 a_{43} &> 0, \quad w_4 a_{44} - w_3 a_{34} > 0. \end{aligned} \quad (4.8)$$

From (2.2), by calculating, we have

$$\begin{aligned} N_1 &= -\frac{a_{12} a_{23} a_{44}}{a_{43} H_2}, \quad N_2 = -\frac{(a_{12} a_{21} a_{23} a_{44} H_3 + a_{23} a_{34} a_{43} H_2^2)}{a_{22} a_{43} H_2 H_3}, \\ N_3 &= -\frac{a_{32} (a_{12} a_{21} a_{23} a_{44} H_3 + a_{23} a_{34} a_{43} H_2^2)}{a_{33} a_{22} a_{43} H_2 H_3}, \end{aligned}$$

and

$$\begin{aligned} 1 + \frac{a_{43}}{a_{44}} N_3 &= 1 - \frac{(a_{12} a_{21} a_{23} a_{32} a_{44} H_3 + a_{23} a_{32} a_{34} a_{43} H_2^2)}{a_{22} a_{33} a_{44} H_2 H_3} \\ &= \frac{a_{22} a_{33} a_{44} H_2 H_3 - a_{12} a_{21} a_{23} a_{32} a_{44} H_3 - a_{23} a_{32} a_{34} a_{43} H_2^2}{a_{22} a_{33} a_{44} H_2 H_3}. \end{aligned}$$

From (4.8) we obtain

$$a_{11} > \frac{w_2}{w_1} a_{21}, \quad a_{ii} > \frac{1}{w_i} (w_{i-1} a_{i-1i} + w_{i+1} a_{i+1i}), \quad i = 2, 3, \quad a_{44} > \frac{w_3}{w_4} a_{34}.$$

Therefore,

$$\begin{aligned} & (a_{22} a_{33} H_2 - a_{12} a_{21} a_{23} a_{32}) a_{44} H_3 \\ & > \left(\frac{1}{w_2} [w_1 a_{12} + w_3 a_{32}] \frac{1}{w_3} [w_2 a_{23} + w_4 a_{43}] [a_{11} a_{22} + a_{12} a_{21}] - a_{12} a_{21} a_{23} a_{32} \right) a_{44} H_3 \\ & \geq \left(\frac{1}{w_2} w_1 a_{12} \frac{1}{w_3} [w_2 a_{23} + w_4 a_{43}] + \frac{w_4}{w_2} a_{32} a_{43} \right) (a_{11} a_{22} + a_{12} a_{21}) a_{44} H_3 \\ & \geq \frac{w_4}{w_2} a_{32} a_{43} (a_{11} a_{22} + a_{12} a_{21}) \frac{w_3}{w_4} a_{34} H_3. \end{aligned}$$

Since

$$H_3 > (a_{11}a_{22} + a_{12}a_{21})a_{33} > (a_{11}a_{22} + a_{12}a_{21})\frac{w_2}{w_3}a_{23},$$

we further obtain

$$\begin{aligned} & (a_{22}a_{33}H_2 - a_{12}a_{21}a_{23}a_{32})a_{44}H_3 \\ & > \frac{w_4}{w_2}a_{32}a_{43}(a_{11}a_{22} + a_{12}a_{21})\frac{w_3}{w_4}a_{34}(a_{11}a_{22} + a_{12}a_{21})\frac{w_2}{w_3}a_{23} \\ & = a_{23}a_{32}a_{34}a_{43}H_2^2. \end{aligned}$$

This shows that $1 + \frac{a_{43}}{a_{44}}N_3 > 0$. □

Conjecture 1. *When $n > 4$, if condition (2) holds then condition (1) also holds.*

5. Numerical examples

To support our analysis results, we will provide the numerical examples for model (1.2). The numerical simulations can be performed qualitatively, and parameters are selected in an epidemiologically feasible way.

Example 1. *In model (1.2) we take $n = 4$, $r_1 = 1.5$, $r_2 = -0.6$, $r_3 = -0.08$, $r_4 = -0.01$, $\gamma_1 = 0.1$, $\gamma_2 = 0.15$, $\gamma_3 = 0.15$, $\gamma_4 = 0.1$, $a_{11} = 1.6$, $a_{12} = 0.2$, $a_{22} = 2.2$, $a_{21} = 2.5$, $a_{23} = 0.1$, $a_{32} = 2.5$, $a_{33} = 2$, $a_{34} = 0.3$, $a_{43} = 1.3$, $a_{44} = 2$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\sigma_3 = 0.1$ and $\sigma_4 = 0.1$*

Choose $w_1 = 2.5$, $w_2 = 1.3$, $w_3 = 0.8$ and $w_4 = 0.5$, by calculating we have $w_1a_{11} - w_2a_{21} = 0.7500 > 0$, $w_2a_{22} - w_1a_{12} - w_3a_{32} = 0.3600 > 0$, $w_3a_{33} - w_2a_{23} - w_4a_{43} = 0.8200 > 0$ and $m_4a_{44} - m_3a_{34} = 0.7600 > 0$. Proposition 1 shows that $1 + \frac{a_{43}}{a_{44}}N_3 > 0$. Thus, all four species in model (1.2) are persistent in mean which is shown in Figure 1 below. Furthermore, $h_1^* = 0.3250$, $h_2^* = 0.3914$, $h_3^* = 0.3264$, $h_4^* = 0.1281$ and $D_4^{(4)}|_{h_i=h_i^*, i=1,2,3,4} = 0.5201 > 0$ are obtained by calculating, all leading principal minors of $A_4 + (A_4^{-1})^T$ are positive, which shows that the $A_4 + (A_4^{-1})^T$ is positive semi-definite. Thus, the condition of conclusion (\mathcal{A}_1) is satisfied, then the OHS: $H^* = (0.3250, 0.3914, 0.3264, 0.1281)^T$ and MESY = 0.4321 are gained, as Figure 1 shows below.

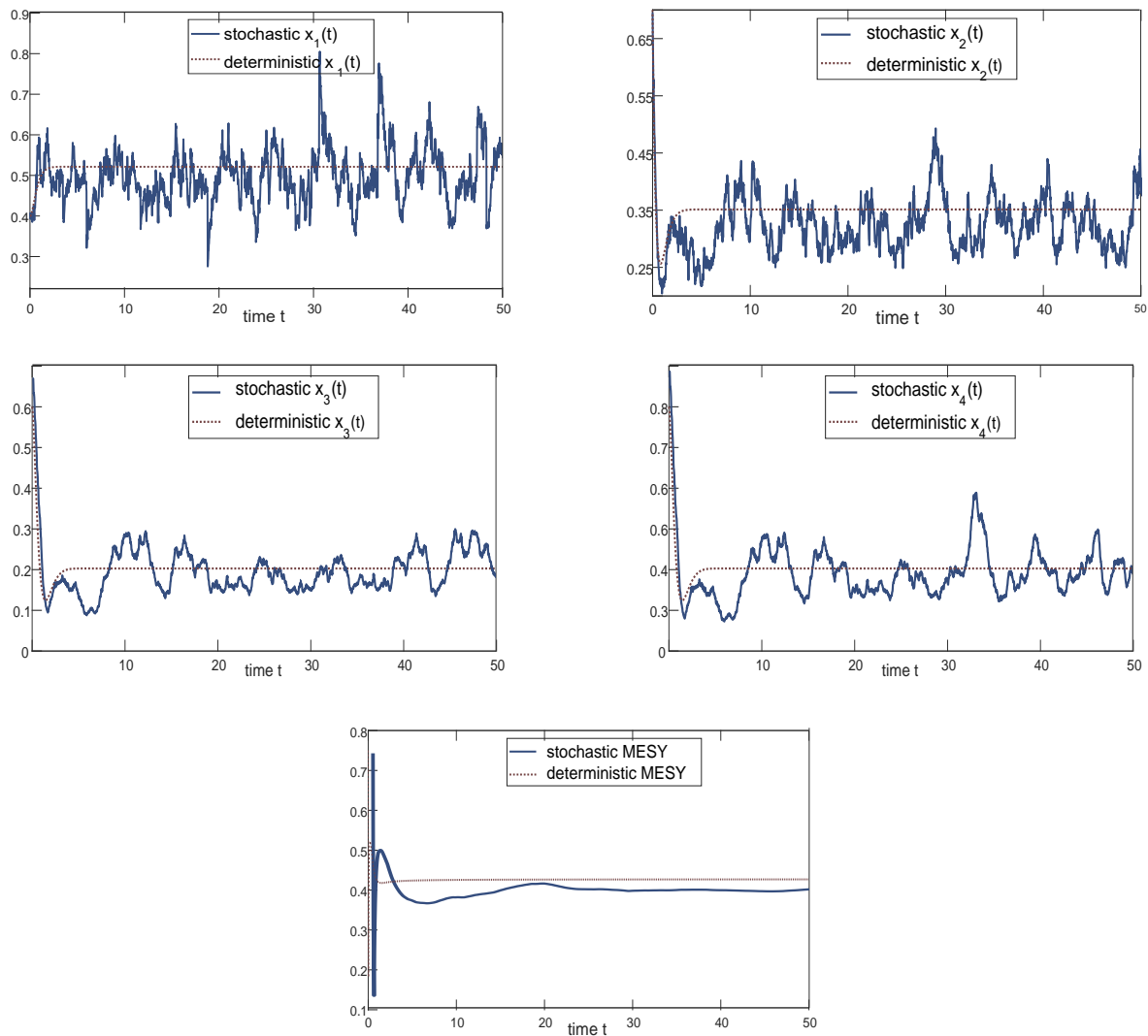


Figure 1. The sample path for the solution and MESY illustrate that species $x_i(t)$ ($i = 1, 2, 3, 4$) is persistent in the mean and MESY exists. The brown line represent the solution of the corresponding undisturbed system and the blue lines represent the solution of model (1.2).

Example 2. In model (1.2) we take $n = 4$, $r_1 = 1.5$, $r_2 = -0.5$, $r_3 = -0.03$, $r_4 = -0.01$; $\gamma_1 = 0.15$, $\gamma_2 = 0.2$, $\gamma_3 = 0.1$ and $\gamma_4 = 0.05$. The rest parameters in model (1.2) are shown in the following three cases.

Case 2.1. $a_{11} = 1.6$, $a_{12} = 0.2$, $a_{22} = 2$, $a_{21} = 2.5$, $a_{23} = 1$, $a_{32} = 2.5$, $a_{33} = 2.1$, $a_{34} = 0.2$, $a_{43} = 0.1$, $a_{44} = 2$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\sigma_3 = 0.2$ and $\sigma_4 = 0.1$.

Case 2.2. $a_{11} = 1.8$, $a_{12} = 0.2$, $a_{22} = 2$, $a_{21} = 1.5$, $a_{23} = 0.2$, $a_{32} = 1.5$, $a_{33} = 2.1$, $a_{34} = 0.8$, $a_{43} = 1.1$, $a_{44} = 2.5$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.3$ and $\sigma_4 = 0.1$.

Case 2.3. $a_{11} = 0.8$, $a_{12} = 0.2$, $a_{22} = 2$, $a_{21} = 1$, $a_{23} = 0.15$, $a_{32} = 1.5$, $a_{33} = 2$, $a_{34} = 0.1$, $a_{43} = 1$, $a_{44} = 0.08$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\sigma_3 = 0.3$ and $\sigma_4 = 0.1$.

For Case 2.1, choose $w_1 = 2.5$, $w_2 = 1.3$, $w_3 = 0.8$ and $w_4 = 1.4$, then we have $w_1 a_{11} - w_2 a_{21} =$

$0.75 > 0$, $w_2a_{22} - w_1a_{12} - w_3a_{32} = 0.10 > 0$, $w_3a_{33} - w_2a_{23} - w_4a_{43} = 0.24 > 0$ $\dot{m}_4a_{44} - m_3a_{34} = 2.64 > 0$. Furthermore, $h_1^* = 0.3621$, $h_2^* = 0.4773$, $h_3^* = 0.4665$, $h_4^* = 0.0051$ are gained by calculating. Meanwhile, $D_4^{(4)}|_{h_i=h_i^*, i=1,2,3,4} = -0.1475 < 0$. Thus the condition (\mathcal{B}_1) of (\mathcal{A}_2) is satisfied. Therefore, the OHS do not exist. The numerical simulations are given in Figure 2.

For Case 2.2, choose $w_1 = 2.2$, $w_2 = 1.1$, $w_3 = 1.1$ and $w_4 = 1.4$, then we have $w_1a_{11} - w_2a_{21} = 2.31 > 0$, $w_2a_{22} - w_1a_{12} - w_3a_{32} = 0.11 > 0$, $w_3a_{33} - w_2a_{23} - w_4a_{43} = 0.55 > 0$ $\dot{m}_4a_{44} - m_3a_{34} = 2.62 > 0$. Furthermore, $h_1^* = 0.9445 > 0$, $h_2^* = 0.2174 > 0$, $h_3^* = 0.0039 > 0$ and $h_4^* = -0.0124 < 0$ are obtained by calculating. Thus the condition (\mathcal{B}_2) of conclusion (\mathcal{A}_2) is satisfied. Therefore, the OHS do not exist. The numerical simulations are given in Figure 3.

For Case 2.3, choose $w_1 = 2.8$, $w_2 = 2$, $w_3 = 2$ and $w_4 = 2.8$, then we have $w_1a_{11} - w_2a_{21} = 0.24 > 0$, $w_2a_{22} - w_1a_{12} - w_3a_{32} = 0.44 > 0$, $w_3a_{33} - w_2a_{23} - w_4a_{43} = 0.90 > 0$ $\dot{m}_4a_{44} - m_3a_{34} = 0.024 > 0$. Furthermore, by calculating we obtain that the fourth leading principle minor of $A_4 + (A_4^{-1})^T$ is equal to $-6.2127 < 0$, which means that $A_4 + (A_4^{-1})^T$ is negative semi-definite. Thus the condition (\mathcal{B}_3) is satisfied. Therefore, the optimal harvesting strategy do not exist. The numerical simulations are given in Figure 4.

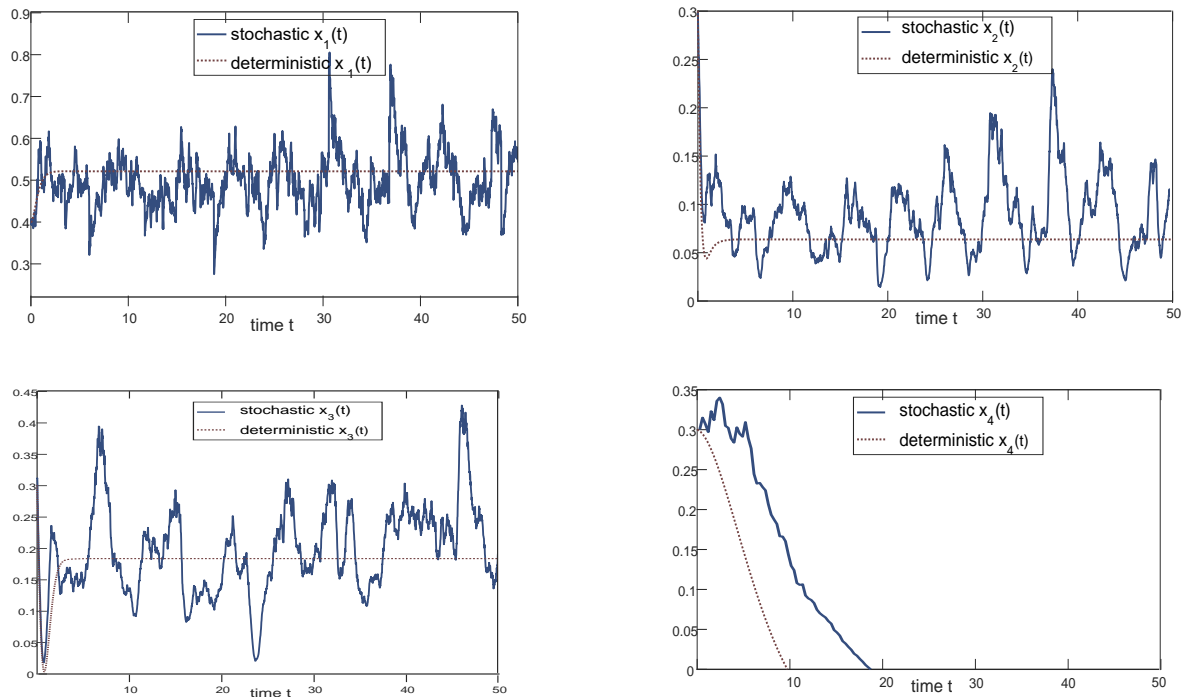


Figure 2. The sample path for the solution shows that species $x_i(t)$ ($i = 1, 2, 3$) is persistent in the mean and x_4 goes to extinction.

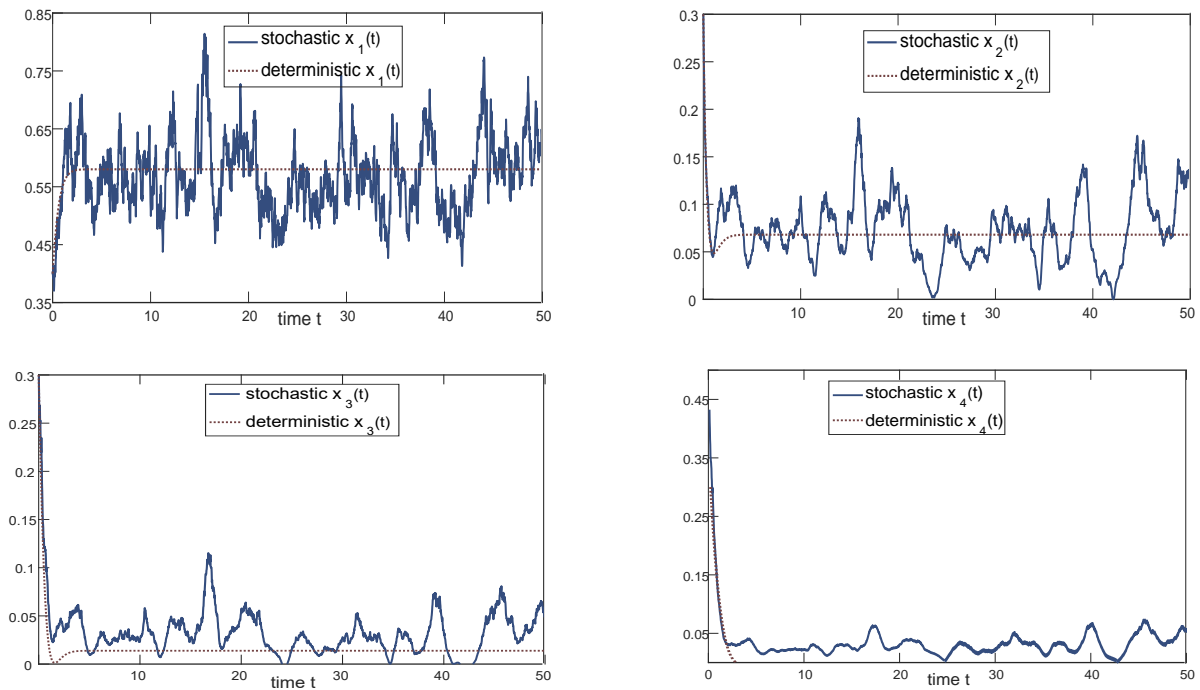


Figure 3. The sample path for the solution shows that species $x_i(t)$ ($i = 1, 2, 3$) is persistent in the mean and x_4 goes to extinction.

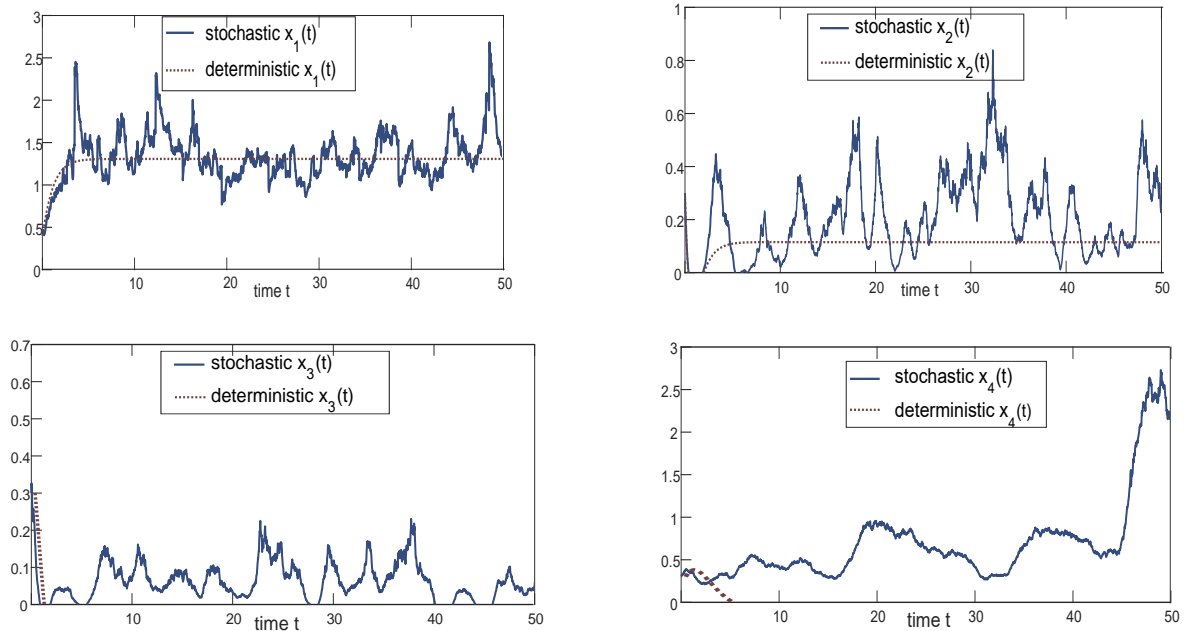


Figure 4. The sample path for the solution shows that species $x_i(t)$ ($i = 1, 2, 3$) is persistent in the mean and x_4 goes to extinction.

6. Conclusions

In this paper, based on the effect of harvesting and Lévy noises, a stochastic marine n -species system is investigated. On the basis of the research results in [23], we first establish the criterion for the stability in distribution of model (1.2). Then, the criteria for existence and nonexistence of OHS and the corresponding MESY are derived. This shows that the environmental noises affect the OHS and MESY.

Compared with the existing results, our first contribution is to investigate the stochastic stability and optimal harvesting problems for a stochastic marine n -species food chain model with Lévy noises and white noises. Second, the criteria on the stability in distribution and optimal harvesting strategy for stochastic n -species food chain model are established for the first time. Third, the noises intensity can easily affect the dynamics and MESY of marine populations, and further lead to the imbalances of marine ecology. Finally, we find that for better manage and protect marine resources, the establishment of an optimal harvesting strategy should fully consider the impact of noise intensity.

There are still some remained problems that need to be followed up. For example, in Conjecture 1, we assume if $w_1a_{11} - w_2a_{21} > 0$, $w_ia_{ii} - w_{i-1}a_{i-1i} - w_{i+1}a_{i+1i} > 0$ ($i = 2, 3, \dots, n-1$), $w_na_{nn} - w_{n-1}a_{n-1n} > 0$, then $1 + \frac{a_{nn}-1}{a_{nn}}N_{n-1} > 0$, how to prove is an open problem. Second, marine population are easily affected by climate change, consider marine population model with regime switching is necessary. Finally, more complex and realistic stochastic system with nonlinear functional response, delays and Allee effect need to be investigated. These issues may be investigated in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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