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Research article

Space-time decay rate of high-order spatial derivative of solution for 3D compressible Euler equations with damping

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Abstract: We are concerned with the space-time decay rate of high-order spatial derivatives of solutions for 3D compressible Euler equations with damping. For any integer $\ell \geq 3$, Kim (2022) showed the space-time decay rate of the $k(0 \leq k \leq \ell - 2)$th-order spatial derivative of the solution. By making full use of the structure of the system, and employing different weighted energy methods for $0 \leq k \leq \ell - 2, k = \ell - 1, k = \ell$, it is shown that the space-time decay rate of the $(\ell - 1)$th-order and ℓth-order spatial derivative of the strong solution in weighted Lebesgue space L^2_σ are $t^{-\frac{3}{4}-\frac{\ell-1}{2}+\frac{\sigma}{2}}$ and $t^{-\frac{3}{4}-\frac{\ell}{2}+\frac{\sigma}{2}}$ respectively, which are totally new as compared to that of Kim (2022) [1].

Keywords: compressible Euler equations; space-time decay rate; weighted estimate

1. Introduction and main results

In this paper, we investigate the space-time decay rate of the $(\ell - 1)$th-order and ℓth-order spatial derivatives of the strong solution for the 3D compressible Euler equations with damping, which takes the following form:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla \rho^\gamma = -\lambda \rho u, \\ (\rho, u)\,|_{t=0} = (\rho_0, u_0)\,, \end{cases} \tag{1.1}$$

where $(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+$ is the spatial coordinate and time. $\rho = \rho(x, t)$ and $u = u(x, t)$ represent the density and velocity respectively. The pressure $p = p(\rho)$ satisfies the γ-law with the adiabatic exponent $\gamma > 1$. The constant $\lambda > 0$ models the damping effect.

1.1. History of the problem

We review some closely related results to this topic as follows. For the one-dimensional Cauchy problem, [2–6] presented a series of results on convergence rates for the nonlinear diffusion waves, [7–10] considered the asymptotic behavior of weak entropy solutions in vacuum and [11] focused on the well-posedness for compressible Euler equations with physical vacuum singularity. We also refer the readers to [12, 13] for the initial-boundary value problem. For multidimensional cases, Jang and Masmoudi [14] showed the well-posedness for the compressible Euler equations with physical vacuum singularity. Wang and Yang [15] gave the global existence and the pointwise estimates of the solution with small data, Sideris et al. [16] used an equivalent reformulation of the system (1.1) to obtain the effective energy estimates. Liao et al. [17] studied the L^p convergence rate of the planar diffusion waves by using approximate Green functions and the energy method. Fang et al. [18] obtained the existence and asymptotic behavior of C^1 solutions in some Besov space by using the spectral localization method. Tan and Wang [19, 20] improved the decay result of the velocity in the L^2-norm that is $(1+t)^{-\frac{5}{4}}$ by using different methods. As for the initial boundary value problem, we refer the interested readers to [15, 21–23]. For the decay rate of the strong solution to the Cauchy problem (1.1), under the assumption that $\|(\rho_0-\bar{\rho},u_0)\|_{H^\ell}$ with the integer $\ell \geq 3$ was sufficiently small and $\|\rho_0-\bar{\rho}\|_{L^1}+\|u_0\|_{L^{\frac{3}{2}}}$ was finite, Chen and Tan [24] showed the temporal decay rate for the solution of (1.1) to be as follows:

$$\begin{aligned} \left\|\nabla^k(\rho-\bar{\rho})(t)\right\|_{L^2} &\lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}}(0\leq k\leq \ell),\\ \left\|\nabla^k u(t)\right\|_{L^2} &\lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}-\frac{1}{2}}(0\leq k\leq \ell-1),\\ \left\|\nabla^\ell u(t)\right\|_{L^2} &\lesssim (1+t)^{-\frac{3}{4}-\frac{\ell}{2}}. \end{aligned} \tag{1.2}$$

Based on (1.2), for any integer $\ell \geq 3$, Kim [1] obtained that the space-time decay rate of the solution for the system (1.1) is $O\left(t^{-\frac{3}{4}-\frac{k}{2}+\frac{\sigma}{2}}\right), 0\leq k\leq \ell-2$.

Noticing that there is no space-time decay rate of the $(\ell-1)$th-order or ℓth-order spatial derivative of solution to (1.1), the main motivation behind this paper is to significantly contribute to the resolution of this issue. More precisely, we establish the space-time decay rates of the $(\ell-1)$th-order and ℓth-order spatial derivatives of the strong solution for (1.1).

1.2. Reformulation

In this paper, as in [16, 25], we reformulate the Cauchy problem (1.1) as follows. Introduce the sound speed

$$\mu(\rho)=\sqrt{p'(\rho)},$$

and set $\bar{\mu}=\mu(\bar{\rho})$ to correspond to the sound speed at a background density $\bar{\rho}>0$. Define

$$n=\frac{2}{\gamma-1}(\mu(\rho)-\bar{\mu}).$$

Then, (1.1) can be rewritten as

$$\begin{cases} n_t+\bar{\mu}\operatorname{div}u=-u\cdot\nabla n-\nu n\operatorname{div}u,\\ u_t+\lambda u+\bar{\mu}\nabla n=-u\cdot\nabla u-\nu n\nabla n,\\ (n,u)\left|_{t=0}\right.(n_0,u_0), \end{cases} \tag{1.3}$$

where

$$\nu:=\frac{\gamma-1}{2},\quad n_0=\frac{2}{\gamma-1}\left(\nu\left(\rho_0\right)-\bar{\mu}\right).$$

1.3. Notations

C and C_i are time independent constants, which may vary in different places. L^p and H^ℓ denote the usual Lebesgue space $L^p\left(\mathbb{R}^3\right)$ and Sobolev spaces $H^\ell\left(\mathbb{R}^3\right) = W^{\ell,2}\left(\mathbb{R}^3\right)$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^\ell}$ respectively. We denote $\|(f,g)\|_X := \|f\|_X + \|g\|_X$ for simplicity. The notation $f \lesssim g$ means that $f \le Cg$. We often drop the x-dependence of the differential operators, that is $\nabla f = \nabla_x f = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)$ and ∇^k denotes any partial derivative ∂^α with the multi-index $\alpha, |\alpha| = k$. For any $\sigma \in \mathbb{R}$, denote the weighted Lebesgue space by $L^p_\sigma\left(\mathbb{R}^3\right)(2 \le p < +\infty)$, where

$$L^p_\sigma\left(\mathbb{R}^3\right) := \left\{ f(x) : \mathbb{R}^3 \to \mathbb{R}, \|f\|^p_{L^p_\sigma(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |x|^{p\sigma} |f(x)|^p dx < +\infty \right\}.$$

Then, we can define the weighted Sobolev space as follows:

$$H^s_\sigma\left(\mathbb{R}^3\right) := \left\{ f \in L^2_\sigma\left(\mathbb{R}^3\right) \mid \|f\|^2_{H^s_\sigma(\mathbb{R}^3)} := \sum_{k \le s} \left\|\nabla^k u\right\|^2_{L^2_\sigma(\mathbb{R}^3)} < +\infty \right\}.$$

1.4. Main results

We extend the work of Kim [1], which showed the space-time decay rate of the $k(0 \le k \le \ell - 2, \ell \ge 3)$th-order derivative of the strong solution for the system (1.3) as follows in Lemma 1.1. Based on (1.2) and the result of Kim [1], we can prove the space-time decay rate of the $k(0 \le k \le \ell)$th-order derivative of the strong solution. It covers the results of Kim [1]. For the convenience of the readers, we outline the space-time decay rates of the $(\ell - 1)$th-order and ℓth-order derivatives of the solution in the following Theorem 1.2.

Lemma 1.1. *(Refer to Theorem 1.2 in [1]) For any integer* $\ell \ge 3$, *the initial data* $(n_0, u_0) \in H^\ell\left(\mathbb{R}^3\right)$, *where* $\|(n_0, u_0)\|_{H^\ell}$ *is sufficiently small and* $\|n_0\|_{L^1} + \|u_0\|_{L^{\frac{3}{2}}}$ *is finite; Then, the strong solution* (n, u) *of the system* (1.3) *such that*

$$\left\|\nabla^k (n,u)(t)\right\|_{L^2_\sigma} = O\left(t^{-\frac{3}{4} - \frac{k}{2} + \frac{\sigma}{2}}\right), \tag{1.4}$$

for all $0 \le k \le \ell - 2$, $\sigma \ge 0$ *and* $t > T$, *where* T *is large enough.*

Theorem 1.2. *For any integer* $\ell \ge 3$, *the initial data* $(n_0, u_0) \in H^\ell\left(\mathbb{R}^3\right) \bigcap H^\ell_\sigma\left(\mathbb{R}^3\right)$, *where* $\|(n_0, u_0)\|_{H^\ell}$ *is sufficiently small and* $\|n_0\|_{L^1} + \|u_0\|_{L^{\frac{3}{2}}}$ *is finite; Then, the strong solution* (n, u) *of the system* (1.3) *such that*

$$\begin{aligned} \left\|\nabla^{\ell-1} (n,u)(t)\right\|_{L^2_\sigma} &= O\left(t^{-\frac{3}{4} - \frac{\ell-1}{2} + \frac{\sigma}{2}}\right), \\ \left\|\nabla^{\ell} (n,u)(t)\right\|_{L^2_\sigma} &= O\left(t^{-\frac{3}{4} - \frac{\ell}{2} + \frac{\sigma}{2}}\right), \end{aligned} \tag{1.5}$$

for all $\sigma \ge 0$ *and* $t > T$, *where* T *is large enough.*

Remark 1.3. Kim [1] did not give the $(\ell - 1)$th-order and ℓth-order spatial derivatives of the strong solution for the system (1.3) since the term $\left\|\nabla^k (n,u)(\cdot,t)\right\|_{L^\infty}$ $(k = 0, \ldots, \ell - 2)$ is involved in the following energy inequality

$$\frac{d}{dt}\hat{\mathcal{K}}(t) \le \sum_{\beta=0}^{k-1} \left(\left\|\nabla^{\beta+1} n\right\|_{L^\infty} + \left\|\nabla^{\beta+1} u\right\|_{L^\infty} \right) \hat{\mathcal{K}}(t) + \sum_{\beta=0}^{k} \hat{\mathcal{K}}(t)^{\frac{a-1}{a}} \left\|\nabla^\beta n\right\|^{\frac{2}{a}}_{L^2}$$

$$\leq Ct^{-\frac{k}{2}-\frac{3}{2}}\hat{\mathcal{K}}(t) + Ct^{-\frac{2k+3}{2a}}\hat{\mathcal{K}}(t)^{\frac{a-1}{a}}, \tag{1.6}$$

where $\hat{\mathcal{K}}(t) = \sum_{\beta=0}^{k} \left\|\nabla^{\beta}(n,u)\right\|_{L^2_\sigma}^2$ $(0 \leq k \leq \ell-2)$. Noticing that the estimate in (1.6) is the sum of the estimates of $I_j (1 \leq j \leq 5)$ given in (3.3). To prove (1.5), we need to develop some new thoughts, and our strategy can be outlined as follows. First, we make full use of the structure of (1.3) to reduce the order of the spatial derivative of the solution (see (3.6) and (3.8)). Second, we make delicate energy estimates for I_j respectively, and we employ different weighted energy methods for $0 \leq k \leq \ell-2, k = \ell-1, k = \ell$ in the process. We can refer the readers to the proofs of (3.9) and (3.17) for more details.

Now, let us outline the strategies of proving Theorem 1.2 and explain the main difficulties in the process. We use the strategy of induction, delicate weighted energy estimates and the interpolation trick to prove Theorem 1.2. According to (1.4), Theorem 1.2 holds for $k = 0$ and $k = 1$. Using the strategy of induction and delicate weighted energy estimates, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathrm{E}}(t) \leq C_0 t^{-\frac{5}{4}}\widehat{\mathrm{E}}(t) + C_1 t^{\left(-\frac{3}{4}-\frac{k}{2}\right)\frac{2}{\sigma}}\widehat{\mathrm{E}}(t)^{\frac{\sigma-1}{\sigma}} + C_3 t^{-\frac{5}{2}-k+\sigma},$$

where $\widehat{\mathrm{E}}(t) := \left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2$. Combining the interpolation trick, we can prove that Theorem 1.2 holds for $0 \leq k \leq \ell$. The main difficulty is that Lemma 2.2 does not work in weighted Lebesgue space L^2_σ. To overcome the difficulty, we fully use the structure of (1.3) to reduce the order of the spatial derivative of the solution, and we make delicate weighted energy estimates. For the sake of simplicity, we only take the trouble term $\left\langle |x|^{2\sigma}\nabla^\ell n, \nabla^\ell(n\,\mathrm{div}\,u)\right\rangle$ in (3.6) as an example. First, by fully using the structure of $(1.3)_1$ to obtain an equation of $\mathrm{div}\,u$ given in (3.4), and by substituting (3.4) into this term, one has

$$\begin{aligned}
-\left\langle |x|^{2\sigma}\nabla^\ell n, \nabla^\ell(n\,\mathrm{div}\,u)\right\rangle &= \left\langle |x|^{2\sigma}\nabla^\ell n, \nabla^\ell\left(n\frac{u\cdot\nabla n + n_t}{\lambda+\nu n}\right)\right\rangle \\
&= \left\langle |x|^{2\sigma}\nabla^\ell n, \left(\frac{n}{\lambda+\nu n}\right)u\nabla^{\ell+1}n\right\rangle + \left\langle |x|^{2\sigma}\nabla^\ell n, \left(\frac{n}{\lambda+\nu n}\right)\nabla^\ell n_t\right\rangle \\
&\quad + \left\langle |x|^{2\sigma}\nabla^\ell n, \nabla^\ell\left(\frac{n}{\lambda+\nu n}\right)u\cdot\nabla n\right\rangle + \text{good terms} \\
&:= K_1 + K_2 + K_3 + \text{good terms}.
\end{aligned} \tag{1.7}$$

Next, we will focus on the main trouble terms $K_j (j = 1,2,3)$. For the term K_1, by employing integration by parts, it holds that

$$\left\langle |x|^{2\sigma}\nabla^\ell n, u\nabla^{\ell+1}n\right\rangle = -\frac{1}{2}\left\langle \nabla(|x|^{2\sigma}u), |\nabla^\ell n|^2\right\rangle. \tag{1.8}$$

For the term K_2, we have

$$\begin{aligned}
&\left\langle |x|^{2\sigma}\nabla^\ell n, \left(\frac{n}{\lambda+\nu n}\right)\nabla^\ell n_t\right\rangle \\
&= \frac{1}{2}\frac{d}{dt}\left\langle |x|^{2\sigma}|\nabla^\ell n|^2, \left(\frac{n}{\lambda+\nu n}\right)\right\rangle - \frac{1}{2}\left\langle |x|^{2\sigma}|\nabla^\ell n|^2, \frac{d}{dt}\left(\frac{n}{\lambda+\nu n}\right)\right\rangle,
\end{aligned} \tag{1.9}$$

where $\left\|\nabla^\ell n\right\|_{L^2_\sigma}^2 - \left\langle |x|^{2\sigma}|\nabla^\ell n|^2, \left(\frac{n}{\lambda+\nu n}\right)\right\rangle$ is equivalent to $\left\|\nabla^\ell n\right\|_{L^2_\sigma}^2$, since $\|n_0\|_{H^\ell}$ is sufficiently small, and the fact that

$$\frac{d}{dt}\left(\frac{n}{\lambda+\nu n}\right) = -\frac{\lambda\left[(\bar{\mu}\operatorname{div} u + u\cdot\nabla n + \nu n \operatorname{div} u)\right]}{(\lambda+\nu n)^2}.$$

Noticing that Lemma 2.2 does not work in L^2_σ, for the term K_3, we need to employ some new ideas. The key observation here is to use Hölder's inequality skillfully to get

$$\left\||x|^{2\sigma}\nabla^\ell n\nabla^\ell\left(\frac{n}{\lambda+\nu n}\right)u\cdot\nabla n\right\|_{L^1} \lesssim \left\|\nabla^k\left(\frac{n}{\lambda+\nu n}\right)\right\|_{L^2}\left\|\nabla^k n\right\|_{L^2_\sigma}\||x|^\sigma u\|_{L^\infty}\|\nabla n\|_{L^\infty}, \tag{1.10}$$

where

$$\begin{aligned}&\||x|^\sigma u\|_{L^\infty}\\ &\lesssim \left(\left\|\nabla^2 u\right\|_{L^2_\sigma} + \|\nabla u\|_{L^2_{\sigma-1}} + \|u\|_{L^2_{\sigma-2}} + \|\nabla u\|_{L^2_\sigma} + \|u\|_{L^2_{\sigma-1}}\right) \text{(see (2.1)).}\end{aligned}$$

With (1.8)–(1.10) in hand, we can bound the trouble terms K_1,K_2 and K_3 properly.

The paper is organized as follows. In Section 2, we present some lemmas, which are used frequently throughout this paper. In Section 3, using the strategy of induction, delicate weighted energy estimates and the interpolation trick, we prove Theorem 1.2.

2. Preliminaries

Lemma 2.1. *(Gagliardo-Nirenberg inequality) Let* $0 \le i, j \le k$; *Then,*

$$\left\|\nabla^i f\right\|_{L^p} \lesssim \left\|\nabla^j f\right\|_{L^q}^{1-a}\left\|\nabla^k f\right\|_{L^r}^{a},$$

where a satisfies

$$\frac{i}{3} - \frac{1}{p} = \left(\frac{j}{3} - \frac{1}{q}\right)(1-a) + \left(\frac{k}{3} - \frac{1}{r}\right)a.$$

Especially, when $p = 3, q = r = 2, i = j = 0$ *and* $k = 1$, *combining Cauchy's inequality, we have*

$$\|f\|_{L^3} \lesssim \|f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{1}{2}} \lesssim \|f\|_{H^1};$$

when $p = \infty, q = r = 2, i = 0, j = 1$ *and* $k = 2$, *combining Cauchy's inequality, we have*

$$\|f\|_{L^\infty} \lesssim \|\nabla f\|_{L^2}^{\frac{1}{2}}\left\|\nabla^2 f\right\|_{L^2}^{\frac{1}{2}} \lesssim \|\nabla f\|_{H^1}$$

and

$$\begin{aligned}\||x|^\sigma f\|_{L^\infty} &\lesssim \left(\|\nabla(|x|^\sigma f)\|_{L^2}^{\frac{1}{2}}\left\|\nabla^2(|x|^\sigma f)\right\|_{L^2}^{\frac{1}{2}}\right)\\ &\lesssim \left(\left\|\nabla^2 f\right\|_{L^2_\sigma} + \|\nabla f\|_{L^2_{\sigma-1}} + \|f\|_{L^2_{\sigma-2}} + \|\nabla f\|_{L^2_\sigma} + \|f\|_{L^2_{\sigma-1}}\right);\end{aligned} \tag{2.1}$$

while $i = j = 0, k = 1, a = 1$ *and* $p = q = r = 2$, *combining Minkowski's inequality, we have*

$$\||x|^\sigma f\|_{L^6} \lesssim \|\nabla(|x|^\sigma f)\|_{L^2} \lesssim \left(\|\nabla f\|_{L^2_\sigma} + \|f\|_{L^2_{\sigma-1}}\right). \tag{2.2}$$

Proof. This is a special case of [26] and some simple inferences.

Lemma 2.2. *Assume that the function $f(\varrho)$ satisfies*

$$f(\varrho) \sim \varrho \text{ and } \left\|f^{(k)}(\varrho)\right\| \leq C_k \text{ for any } k \geq 1,$$

then for any integer $k \geq 0$ and $p \geq 2$, we have

$$\left\|\nabla^k f(\varrho)\right\|_{L^p} \leq C_k \left\|\nabla^k \varrho\right\|_{L^p}.$$

Proof. Refer to Lemma A.4 of [24] for $p = 2$ and the Lemma 2.2 of [27] for $p \geq 2$.

Lemma 2.3. *The vector function $f \in C_0^\infty\left(\mathbb{R}^3\right)$ and bounded scalar function g such that*

$$\left|\int_{\mathbb{R}^3} \left(\nabla |x|^{2\sigma}\right) \cdot f g \, \mathrm{d}x\right| \lesssim \|g\|_{L^2_\sigma} \|f\|_{L^2_{\sigma-1}}.$$

Proof. The left side of the above inequality can be rewritten as

$$\left|2\sigma \int_{\mathbb{R}^3} |x|^{2\sigma-2} x_j \partial_i x_j g f_i dx\right|.$$

Using Hölder's inequality, we have

$$\left|\int_{\mathbb{R}^3} \left(\nabla |x|^{2\sigma}\right) \cdot f g \, \mathrm{d}x\right| \lesssim \|g\|_{L^2_\sigma} \|f\|_{L^2_{\sigma-1}}.$$

Lemma 2.4. *(Interpolation inequality with weights) If $p, r \geqslant 1, s + n/r, \alpha + n/p, \beta + n/q > 0$ and $0 \leqslant \theta \leqslant 1$ then*

$$\|f\|_{L^r_s} \leq \|f\|^{\theta}_{L^p_\alpha} \|f\|^{1-\theta}_{L^q_\beta}$$

for $f \in C_0^\infty(\mathbb{R}^n)$ provided that

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

and

$$s = \theta\alpha + (1-\theta)\beta.$$

Especially, while $s = p = q = 2, \theta = \frac{\sigma-1}{\sigma}, s = \sigma - 1, \alpha = \sigma$ and $\beta = 0$, we have

$$\|f\|_{L^2_{\sigma-1}} \leq \|f\|^{\frac{\sigma-1}{\sigma}}_{L^2_\sigma} \|f\|^{\frac{1}{\sigma}}_{L^2}. \tag{2.3}$$

Proof. We compute

$$\begin{aligned}\int_U |x|^{sr}|f|^r dx &= \int_U |x|^{\alpha\theta r}|f|^{\theta r}|x|^{\beta(1-\theta)r}|f|^{(1-\theta)r} dx \\ &\leq \left(\int_U \left(|x|^{\alpha\theta r}|f|^{\theta r}\right)^{\frac{p}{\theta r}} dx\right)^{\frac{\theta r}{p}} \left(\int_U \left(|x|^{\beta(1-\theta)r}|f|^{(1-\theta)r}\right)^{\frac{q}{(1-\theta)r}} dx\right)^{\frac{(1-\theta)r}{q}}.\end{aligned}$$

Thus, we complete the proof of Lemma 2.4.

Lemma 2.5. *(Gronwall-type Lemma) Let $\alpha_0 > 1, \alpha_1 < 1, \alpha_2 < 1$, and $\beta_1 < 1, \beta_2 < 1$. Assume that a continuously differential function $F : [1, \infty) \to [0, \infty)$ satisfies*

$$\begin{aligned} &\frac{d}{dt}F(t) \le C_0 t^{-\alpha_0} F(t) + C_1 t^{-\alpha_1} F(t)^{\beta_1} + C_2 t^{-\alpha_2} F(t)^{\beta_2} + C_3 t^{\sigma_1 - 1}, t \ge 1 \\ &F(1) \le K_0, \end{aligned}$$

where $C_0, C_1, C_2, C_3, K_0 \ge 0$ and $\sigma_i = \frac{1-\alpha_i}{1-\beta_i} > 0$ for $i = 1, 2$. Assume that $\sigma_1 \ge \sigma_2$, then, there exists a constant C^* depending on $\alpha_0, \alpha_1, \beta_1, \alpha_2, \beta_2, K_0$ and $C_i, i = 1, 2, 3$, for all $t \ge 1$, such that $F(t) \le C^* t^{\sigma_1}$.

Proof. We can refer to Lemma 2.1 of [28].

3. The proof of Theorem 1.2

According to (1.2), for all $0 \le k \le \ell$ and $t > T$, where T is large enough, we have

$$\left\|\nabla^k (n, u)(t)\right\|_{L^2} \lesssim t^{-\frac{3}{4}-\frac{k}{2}}. \tag{3.1}$$

We will take the strategy of induction to prove Theorem 1.2 as follows. According to (1.4), Theorem 1.2 holds for $k = 0$ and $k = 1$. By the general step of induction, assume that the estimate (1.5) holds for $0 \le m \le k - 1 (2 \le k \le \ell)$, i.e.,

$$\|\nabla^m (n, u)(t)\|_{L^2_\sigma} \le O\left(t^{-\frac{3}{4}-\frac{m}{2}+\frac{\sigma}{2}}\right), \tag{3.2}$$

for $0 \le m \le k - 1$. Then, we need to verify that (3.2) holds for $m = k$. Applying ∇^k to each equation of $(1.3)_1$ and $(1.3)_2$, multiplying the $(1.3)_1$ and $(1.3)_2$ by $|x|^{2\sigma} \nabla^k n$ and $|x|^{2\sigma} \nabla^k u$ respectively, summing them up and then integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\nabla^k (n, u)\right\|^2_{L^2_\sigma} + \lambda \left\|\nabla^k u\right\|^2_{L^2_\sigma} \\ =&\bar{\mu} \int_{\mathbb{R}^3} \nabla\left(|x|^{2\sigma}\right) \cdot \nabla^k n \nabla^k u \, \mathrm{d}x - \left\langle |x|^{2\sigma} \nabla^k n, \nabla^k (u \nabla n)\right\rangle \\ &- \nu \left\langle |x|^{2\sigma} \nabla^k n, \nabla^k (n \operatorname{div} u)\right\rangle - \left\langle |x|^{2\sigma} \nabla^k u, \nabla^k (u \cdot \nabla u)\right\rangle \\ &- \nu \left\langle |x|^{2\sigma} \nabla^k u, \nabla^k (n \cdot \nabla n)\right\rangle \\ :=& \sum_{j=1}^{5} I_j. \end{aligned} \tag{3.3}$$

For $0 \le k \le \ell$, there exist the terms involving $\nabla^{k+1}(n, u)$ in (3.3). To reduce the order of $\nabla^{k+1}(n, u)$, we have to use the equations $\operatorname{div} u$ and ∇n as derived from (1.3) as follows:

$$\operatorname{div} u = -\frac{u \cdot \nabla n + n_t}{\lambda + \nu n}, \tag{3.4}$$

$$\nabla n = -\frac{u \cdot \nabla u + u_t}{\lambda + \nu n}. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2+\lambda\left\|\nabla^k u\right\|_{L^2_\sigma}^2\\
=&\bar{\mu}\int_{\mathbb{R}^3}\nabla\left(|x|^{2\sigma}\right)\cdot\nabla^k n\nabla^k u\,\mathrm{d}x-\left\langle|x|^{2\sigma}\nabla^k n,\nabla^k\left(u\nabla n\right)\right\rangle\\
&+\nu\left\langle|x|^{2\sigma}\nabla^k n,\nabla^k\left(\frac{nu\cdot\nabla n}{\lambda+\nu n}\right)\right\rangle+\nu\left\langle|x|^{2\sigma}\nabla^k n,\nabla^k\left(\frac{nn_t}{\lambda+\nu n}\right)\right\rangle-\left\langle|x|^{2\sigma}\nabla^k u,\nabla^k\left(u\cdot\nabla u\right)\right\rangle\\
&+\nu\left\langle|x|^{2\sigma}\nabla^k u,\nabla^k\left(\frac{n(u\cdot\nabla u+\lambda u)}{\lambda+\nu n}\right)\right\rangle+\nu\left\langle|x|^{2\sigma}\nabla^k u,\nabla^k\left(\frac{nu_t}{\lambda+\nu n}\right)\right\rangle\\
:=&\sum_{j=1}^{7}J_j.
\end{aligned}
\tag{3.6}
$$

Applying Lemma 2.3 and Cauchy's inequality, one has

$$
\begin{aligned}
|J_1|\lesssim&\left\|\nabla(|x|^{2\sigma})\nabla^k n\nabla^k u\right\|_{L^1}\\
\lesssim&\|\nabla^k u\|_{L^2_\sigma}\|\nabla^k n\|_{L^2_{\sigma-1}}\\
\leq&\epsilon\kappa_2\|\nabla^k u\|_{L^2_\sigma}^2+C\kappa_2(\epsilon)\|\nabla^k n\|_{L^2_{\sigma-1}}^2.
\end{aligned}
\tag{3.7}
$$

For J_2, using integration by parts, we obtain

$$
\begin{aligned}
J_2=&-\left\langle|x|^{2\sigma}\nabla^k n,\nabla^k\left(u\nabla n\right)\right\rangle\\
=&-\left\langle|x|^{2\sigma}\nabla^k n,u\nabla^{k+1}n\right\rangle-\sum_{m=1}^{k}C_k^m\left\langle|x|^{2\sigma}\nabla^k n,\nabla^m u\nabla^{k-m+1}n\right\rangle\\
=&\frac{1}{2}\left\langle\nabla(|x|^{2\sigma}u),|\nabla^k n|^2\right\rangle-\sum_{m=1}^{k}C_k^m\left\langle|x|^{2\sigma}\nabla^k n,\nabla^m u\nabla^{k-m+1}n\right\rangle.
\end{aligned}
$$

Using Minkowski's inequality, Hölder's inequality, Lemmas 2.3 and 2.1 (Gagliardo-Nirenberg inequality), (3.1) and (3.2), and Cauchy's inequality, we have

$$
\begin{aligned}
|J_2|\lesssim&\left\|\nabla(|x|^{2\sigma}u)|\nabla^k n|^2\right\|_{L^1}+\left\|\sum_{m=1}^{k}|x|^{2\sigma}\nabla^k n\nabla^m u\nabla^{k-m+1}n\right\|_{L^1}\\
\lesssim&\|u\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_{\sigma-1}}+\|\nabla u\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}^2+\sum_{m=2}^{k-2}\|\nabla^m u\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^{k-m+1}n\right\|_{L^2_\sigma}\\
&+\left\|\nabla^{k-1}u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^2 n\right\|_{L^\infty}+\left\|\nabla^k u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\|\nabla n\|_{L^\infty}\\
\lesssim&\|\nabla u\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_{\sigma-1}}+\left\|\nabla^2 u\right\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}^2+\left\|\nabla^k u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^2 n\right\|_{H^1}\\
&+\sum_{m=2}^{k-2}\left\|\nabla^{m+1}u\right\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^{k-m+1}n\right\|_{L^2_\sigma}+\left\|\nabla^{k-1}u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^3 n\right\|_{H^1}\\
\lesssim&t^{-\frac{5}{4}}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_{\sigma-1}}+t^{-\frac{7}{4}}\left(\left\|\nabla^k n\right\|_{L^2_\sigma}^2+\left\|\nabla^k u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\right)+t^{-\frac{5}{4}-\frac{k}{2}+\frac{\sigma}{2}-\frac{5}{4}}\left\|\nabla^k n\right\|_{L^2_\sigma}
\end{aligned}
$$

$$\lesssim t^{-\frac{5}{4}}\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2+t^{-\frac{5}{4}}\left\|\nabla^k n\right\|_{L^2_{\sigma-1}}^2+t^{-\frac{5}{2}-k+\sigma}. \tag{3.8}$$

For J_3, we have

$$\begin{aligned}
\frac{1}{\nu}J_3 =&\left\langle |x|^{2\sigma}\nabla^k n, \nabla^k\left(\frac{nu\cdot\nabla n}{\lambda+\nu n}\right)\right\rangle\\
=&\left\langle |x|^{2\sigma}\nabla^k n, \frac{n}{\lambda+\nu n}\nabla^k\left(u\cdot\nabla n\right)\right\rangle + k\left\langle |x|^{2\sigma}\nabla^k n, \nabla\left(\frac{n}{\lambda+\nu n}\right)\nabla^{k-1}\left(u\cdot\nabla n\right)\right\rangle\\
&+\sum_{m=2}^{k-2}C_k^m\left\langle |x|^{2\sigma}\nabla^k n, \nabla^m u\nabla^{k-m+1}n\right\rangle\\
&+\left\langle |x|^{2\sigma}\nabla^k n, \nabla^{k-1}\left(\frac{n}{\lambda+\nu n}\right)\nabla\left(u\cdot\nabla n\right)\right\rangle+\left\langle |x|^{2\sigma}\nabla^k n, \nabla^k\left(\frac{n}{\lambda+\nu n}\right)u\cdot\nabla n\right\rangle\\
:=&\sum_{j=1}^{5}J_{3,j}.
\end{aligned} \tag{3.9}$$

Using integration by parts, we obtain

$$\begin{aligned}
J_{3,1} =&\left\langle |x|^{2\sigma}\nabla^k n, \frac{n}{\lambda+\nu n}\nabla^k\left(u\cdot\nabla n\right)\right\rangle\\
=&\left\langle |x|^{2\sigma}\nabla^k n, \frac{n}{\lambda+\nu n}u\nabla^{k+1}n\right\rangle+\sum_{j=1}^{k}C_k^j\left\langle |x|^{2\sigma}\nabla^k n, \frac{n}{\lambda+\nu n}\nabla^j u\nabla^{k-j+1}n\right\rangle\\
=&-\frac{1}{2}\left\langle \nabla\left(|x|^{2\sigma}\frac{n}{\lambda+\nu n}u\right), |\nabla^k n|^2\right\rangle+\sum_{j=1}^{k}C_k^j\left\langle |x|^{2\sigma}\nabla^k n, \frac{n}{\lambda+\nu n}\nabla^j u\nabla^{k-j+1}n\right\rangle
\end{aligned}$$

Applying Minkowski's inequality, Hölder's inequality, Lemmas 2.3 and 2.1 (Gagliardo-Nirenberg inequality), (3.1), (3.2) and Cauchy's inequality, we have

$$\begin{aligned}
|J_{3,1}|\lesssim&\left\|\frac{n}{\lambda+\nu n}\right\|_{L^\infty}\left[\left\|\nabla(|x|^{2\sigma})|\nabla^k n|^2\right\|_{L^1}+\sum_{j=1}^{k}\left\||x|^{2\sigma}\nabla^k n\nabla^j u\nabla^{k-j+1}n\right\|_{L^1}\right]\\
&+\left\||x|^{2\sigma}\nabla\left(\frac{n}{\lambda+\nu n}\right)u|\nabla^k n|^2\right\|_{L^1}\\
\lesssim&\|n\|_{L^\infty}\Bigg[\|u\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_{\sigma-1}}+\|\nabla u\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}^2\\
&+\sum_{j=2}^{k-2}\left\|\nabla^j u\right\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^{k-j+1}n\right\|_{L^2_\sigma}\\
&+\left\|\nabla^{k-1}u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^2 n\right\|_{L^\infty}+\left\|\nabla^k u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\|\nabla n\|_{L^\infty}\Bigg]\\
&+\|\nabla n\|_{L^\infty}\|u\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}^2\\
\lesssim&\|\nabla n\|_{H^1}\Bigg[\|\nabla u\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_{\sigma-1}}+\left\|\nabla^2 u\right\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}^2\\
&+\sum_{j=2}^{k-2}\left\|\nabla^{j+1}u\right\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^{k-j+1}n\right\|_{L^2_\sigma}
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\nabla^{k-1}u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^3 n\right\|_{H^1}+\left\|\nabla^k u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^2 n\right\|_{H^1}\Big]\\
&+\left\|\nabla^2 n\right\|_{H^1}\|\nabla u\|_{H^1}\left\|\nabla^k n\right\|^2_{L^2_\sigma}\\
\lesssim\, & t^{-\frac{5}{2}}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_{\sigma-1}}+t^{-3}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+t^{-\frac{5}{4}-\frac{k}{2}+\frac{\sigma}{2}-\frac{5}{2}}\left\|\nabla^k n\right\|_{L^2_\sigma}\\
&+t^{-\frac{5}{4}-\frac{k}{2}+\frac{\sigma}{2}-3}\left\|\nabla^k n\right\|_{L^2_\sigma}+t^{-3}\left\|\nabla^k u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\\
\lesssim\, & t^{-\frac{5}{2}}\left\|\nabla^k (n,u)\right\|^2_{L^2_\sigma}+t^{-\frac{5}{2}}\left\|\nabla^k n\right\|^2_{L^2_{\sigma-1}}+t^{-\frac{5}{2}-k+\sigma}. \qquad (3.10)
\end{aligned}
$$

Applying Minkowski's inequality, Hölder's inequality, Lemmas 2.2 and 2.1 (Gagliardo-Nirenberg inequality), (3.1), (3.2) and Cauchy's inequality, we obtain

$$
\begin{aligned}
|J_{3,2}|\lesssim\, & \|\nabla n\|_{L^\infty}\Bigg[\|u\|_{L^\infty}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+\sum_{j=1}^{k-2}\left\|\nabla^j u\right\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^{k-1-j+1}n\right\|_{L^2_\sigma}\\
&+\left\|\nabla^{k-1}u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\|\nabla n\|_{L^\infty}\Bigg]\\
\lesssim\, & \left\|\nabla^2 n\right\|_{H^1}\Bigg[\|\nabla u\|_{H^1}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+\sum_{j=1}^{k-2}\left\|\nabla^{j+1}u\right\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^{k-1-j+1}n\right\|_{L^2_\sigma}\\
&+\left\|\nabla^{k-1}u\right\|_{L^2_\sigma}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^2 n\right\|_{H^1}\Bigg]\\
\lesssim\, & t^{-3}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+t^{-\frac{5}{4}-\frac{k}{2}+\frac{\sigma}{2}-\frac{5}{2}}\left\|\nabla^k n\right\|_{L^2_\sigma}\\
\lesssim\, & t^{-3}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+t^{-\frac{5}{2}-k+\sigma}, \qquad (3.11)\\
|J_{3,3}|\lesssim\, & \sum_{m=2}^{k-2}\|\nabla^m n\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}\sum_{j=0}^{k-m}\left\|\nabla^j u\right\|_{L^\infty}\left\|\nabla^{k-m-j+1}n\right\|_{L^2_\sigma}\\
\lesssim\, & \sum_{m=2}^{k-2}\left\|\nabla^{m+1}n\right\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}\sum_{j=0}^{k-m}\left\|\nabla^{j+1}u\right\|_{H^1}\left\|\nabla^{k-m-j+1}n\right\|_{L^2_\sigma}\\
\lesssim\, & t^{-\frac{5}{4}-\frac{k}{2}+\frac{\sigma}{2}-\frac{5}{2}}\left\|\nabla^k n\right\|_{L^2_\sigma}\\
\lesssim\, & t^{-5}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+t^{-\frac{5}{2}-k+\sigma}, \qquad (3.12)\\
|J_{3,4}|\lesssim\, & \left\|\nabla^{k-1}\left(\frac{n}{\lambda+\nu n}\right)\right\|_{L^3}\left\|\nabla^k n\right\|_{L^2_\sigma}\left(\|u\|_{L^\infty}\left\|\nabla^2 n\right\|_{L^6_\sigma}+\|\nabla u\|_{L^\infty}\|\nabla n\|_{L^6_\sigma}\right)\\
\lesssim\, & \left\|\nabla^{k-1}n\right\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}\Big[\|\nabla u\|_{H^1}\left(\left\|\nabla^3 n\right\|_{L^2_\sigma}+\left\|\nabla^2 n\right\|_{L^2_{\sigma-1}}\right)\\
&+\left\|\nabla^2 u\right\|_{H^1}\left(\left\|\nabla^2 n\right\|_{L^2_\sigma}+\|\nabla n\|_{L^2_{\sigma-1}}\right)\Big]\\
\lesssim\, & t^{-5}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+t^{-\frac{5}{2}-k+\sigma}, \qquad (3.13)\\
|J_{3,5}|\lesssim\, & \left\|\nabla^k\left(\frac{n}{\lambda+\nu n}\right)\right\|_{L^2}\left\|\nabla^k n\right\|_{L^2_\sigma}\||x|^\sigma u\|_{L^\infty}\|\nabla n\|_{L^\infty}\\
\lesssim\, & \left\|\nabla^k n\right\|_{L^2}\left\|\nabla^k n\right\|_{L^2_\sigma}\left(\left\|\nabla^2 u\right\|_{L^2_\sigma}+\|\nabla u\|_{L^2_{\sigma-1}}+\|u\|_{L^2_{\sigma-2}}+\|\nabla u\|_{L^2_\sigma}+\|u\|_{L^2_{\sigma-1}}\right)\left\|\nabla^2 n\right\|_{H^1}\\
\lesssim\, & t^{-5}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+t^{-\frac{5}{2}-k+\sigma}. \qquad (3.14)
\end{aligned}
$$

Substituting (3.10)–(3.14) into (3.9), we have

$$|J_3| \lesssim t^{-\frac{5}{2}} \left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2 + t^{-\frac{5}{2}} \left\|\nabla^k n\right\|_{L^2_{\sigma-1}}^2 + t^{-\frac{5}{2}-k+\sigma}. \tag{3.15}$$

For J_4, we have to use the equation n_t derived from $(1.3)_1$ as follows:

$$n_t = -\left(\bar{\mu}\,\mathrm{div}\,u + u\cdot\nabla n + \nu n\,\mathrm{div}\,u\right). \tag{3.16}$$

Applying (3.16), J_4 can be rewritten as

$$\begin{aligned}
\frac{1}{\nu}J_4 =& \left\langle |x|^{2\sigma}\nabla^k n, \frac{n}{\lambda+\nu n}\nabla^k n_t\right\rangle + \sum_{m=1}^{k} C_k^m \left\langle |x|^{2\sigma}\nabla^k n, \nabla^m\left(\frac{n}{\lambda+\nu n}\right)\nabla^{k-m}n_t\right\rangle \\
=&\frac{1}{2}\frac{d}{dt}\left\langle |x|^{2\sigma}|\nabla^k n|^2, \frac{n}{\lambda+\nu n}\right\rangle - \frac{1}{2}\left\langle |x|^{2\sigma}|\nabla^k n|^2, \frac{d}{dt}\left(\frac{n}{\lambda+\nu n}\right)\right\rangle \\
&+ \sum_{m=1}^{k} C_k^m \left\langle |x|^{2\sigma}\nabla^k n, \nabla^m\left(\frac{n}{\lambda+\nu n}\right)\nabla^{k-m}n_t\right\rangle \\
=&\frac{1}{2}\frac{d}{dt}\left\langle |x|^{2\sigma}|\nabla^k n|^2, \frac{n}{\lambda+\nu n}\right\rangle + \frac{\lambda}{2}\left\langle |x|^{2\sigma}|\nabla^k n|^2, \frac{(\bar{\mu}\,\mathrm{div}\,u + u\cdot\nabla n + \nu n\,\mathrm{div}\,u)}{(\lambda+\nu n)^2}\right\rangle \\
&- k\left\langle |x|^{2\sigma}\nabla^k n, \nabla\left(\frac{n}{\lambda+\nu n}\right)\nabla^{k-1}\left(\bar{\mu}\,\mathrm{div}\,u + u\cdot\nabla n + \nu n\,\mathrm{div}\,u\right)\right\rangle \\
&- \sum_{m=2}^{k-2} C_k^m \left\langle |x|^{2\sigma}\nabla^k n, \nabla^m\left(\frac{n}{\lambda+\nu n}\right)\nabla^{k-m}\left(\bar{\mu}\,\mathrm{div}\,u + u\cdot\nabla n + \nu n\,\mathrm{div}\,u\right)\right\rangle \\
&- k\left\langle |x|^{2\sigma}\nabla^k n, \nabla^{k-1}\left(\frac{n}{\lambda+\nu n}\right)\nabla\left(\bar{\mu}\,\mathrm{div}\,u + u\cdot\nabla n + \nu n\,\mathrm{div}\,u\right)\right\rangle \\
&- \left\langle |x|^{2\sigma}\nabla^k n, \nabla^k\left(\frac{n}{\lambda+\nu n}\right)\left(\bar{\mu}\,\mathrm{div}\,u + u\cdot\nabla n + \nu n\,\mathrm{div}\,u\right)\right\rangle \\
:=&\frac{1}{2}\frac{d}{dt}\left\langle |x|^{2\sigma}|\nabla^k n|^2, \frac{n}{\lambda+\nu n}\right\rangle + \frac{\lambda}{2}J_{4,1} + J_{4,2} + J_{4,3} + J_{4,4} + J_{4,5}.
\end{aligned} \tag{3.17}$$

Applying Minkowski's inequality, Hölder's inequality, Lemmas 2.2 and 2.1 (Gagliardo-Nirenberg inequality), (3.1), (3.2) and Cauchy's inequality, we obtain

$$\begin{aligned}
|J_{4,1}| \lesssim& \left\|\frac{1}{(\lambda+\nu n)^2}\right\|_{L^\infty} \left\|\nabla^k n\right\|_{L^2_\sigma}^2 \left(\|\nabla u\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla n\|_{L^\infty} + \|n\|_{L^\infty}\|\nabla u\|_{L^\infty}\right) \\
\lesssim& \|\nabla n\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}^2 \left(\left\|\nabla^2 u\right\|_{H^1} + \|\nabla u\|_{H^1}\left\|\nabla^2 n\right\|_{H^1} + \|\nabla n\|_{H^1}\left\|\nabla^2 u\right\|_{H^1}\right) \\
\lesssim& t^{-3}\left\|\nabla^k n\right\|_{L^2_\sigma}^2,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
|J_{4,2}| \lesssim& \|\nabla n\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}\Bigg(\left\|\nabla^k u\right\|_{L^2_\sigma} + \|(n,u)\|_{L^\infty}\left\|\nabla^k(n,u)\right\|_{L^2_\sigma} \\
&+ \sum_{j=1}^{k-2}\left\|\nabla^j(n,u)\right\|_{L^\infty}\left\|\nabla^{k-1-j+1}(n,u)\right\|_{L^2_\sigma} + \left\|\nabla^{k-1}(n,u)\right\|_{L^2_\sigma}\|\nabla(n,u)\|_{L^\infty}\Bigg) \\
\lesssim& \left\|\nabla^2 n\right\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}\left(\left\|\nabla^k u\right\|_{L^2_\sigma} + \|\nabla(n,u)\|_{H^1}\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{k-2}\left\|\nabla^{j+1}(n,u)\right\|_{H^1}\left\|\nabla^{k-1-j+1}(n,u)\right\|_{L^2_\sigma}+\left\|\nabla^{k-1}(n,u)\right\|_{L^2_\sigma}\left\|\nabla^2(n,u)\right\|_{H^1}\Bigg)\\
&\lesssim t^{-\frac{7}{4}}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^k u\right\|_{L^2_\sigma}+t^{-3}\left\|\nabla^k n\right\|_{L^2_\sigma}\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}+t^{-\frac{5}{4}-\frac{k}{2}+\frac{\sigma}{2}-\frac{5}{2}}\left\|\nabla^k n\right\|_{L^2_\sigma}\\
&\lesssim t^{-3}\left\|\nabla^k(n,u)\right\|^2_{L^2_\sigma}+t^{-\frac{5}{2}-k+\sigma},
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
|J_{4,3}|\lesssim&\sum_{m=2}^{k-2}\|\nabla^m n\|_{L^\infty}\left\|\nabla^k n\right\|_{L^2_\sigma}\left(\left\|\nabla^{k-m+1}u\right\|_{L^2_\sigma}+\sum_{j=0}^{k-m}\left\|\nabla^j(n,u)\right\|_{L^\infty}\left\|\nabla^{k-m-j+1}(n,u)\right\|_{L^2_\sigma}\right)\\
\lesssim&\sum_{m=2}^{k-2}\left\|\nabla^{m+1}n\right\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}\left(\left\|\nabla^{k-m+1}u\right\|_{L^2_\sigma}+\sum_{j=0}^{k-m}\left\|\nabla^{j+1}(n,u)\right\|_{H^1}\left\|\nabla^{k-m-j+1}(n,u)\right\|_{L^2_\sigma}\right)\\
\lesssim& t^{-\frac{5}{4}-\frac{k}{2}+\frac{\sigma}{2}-\frac{5}{4}}\left\|\nabla^k n\right\|_{L^2_\sigma}+t^{-\frac{5}{4}-\frac{k}{2}+\frac{\sigma}{2}-\frac{5}{2}}\left\|\nabla^k n\right\|_{L^2_\sigma}\\
\lesssim& t^{-\frac{5}{2}}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+t^{-\frac{5}{2}-k+\sigma},
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
|J_{4,4}|\lesssim&\left\|\nabla^{k-1}\left(\frac{n}{\lambda+\nu n}\right)\right\|_{L^3}\left\|\nabla^k n\right\|_{L^2_\sigma}\Big(\left\|\nabla^2 u\right\|_{L^6_\sigma}\\
&+\|(n,u)\|_{L^\infty}\left\|\nabla^2(n,u)\right\|_{L^6_\sigma}+\|\nabla(n,u)\|_{L^\infty}\|\nabla(n,u)\|_{L^6_\sigma}\Big)\\
\lesssim&\left\|\nabla^{k-1}n\right\|_{H^1}\left\|\nabla^k n\right\|_{L^2_\sigma}\Big[\left(\left\|\nabla^3 u\right\|_{L^2_\sigma}+\left\|\nabla^2 u\right\|_{L^2_{\sigma-1}}\right)+\|\nabla u\|_{H^1}\left(\left\|\nabla^3 n\right\|_{L^2_\sigma}+\left\|\nabla^2 n\right\|_{L^2_{\sigma-1}}\right)\\
&+\left\|\nabla^2 u\right\|_{H^1}\left(\left\|\nabla^2 n\right\|_{L^2_\sigma}+\|\nabla n\|_{L^2_{\sigma-1}}\right)\Big]\\
\lesssim& t^{-\frac{5}{4}-\frac{k}{2}+\frac{\sigma}{2}-\frac{5}{4}}\left\|\nabla^k n\right\|_{L^2_\sigma}+t^{-\frac{5}{4}-\frac{k}{2}+\frac{\sigma}{2}-\frac{5}{2}}\left\|\nabla^k n\right\|_{L^2_\sigma}\\
\lesssim& t^{-\frac{5}{2}}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+t^{-\frac{5}{2}-k+\sigma},
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
|J_{4,5}|\lesssim&\left\|\nabla^k\left(\frac{n}{\lambda+\nu n}\right)\right\|_{L^2}\left(\||x|^\sigma u\|_{L^\infty}+\left\|\nabla^k n\right\|_{L^2_\sigma}\||x|^\sigma(n,u)\|_{L^\infty}\|\nabla(n,u)\|_{L^\infty}\right)\\
\lesssim&\left\|\nabla^k n\right\|_{L^2}\left\|\nabla^k n\right\|_{L^2_\sigma}\Big[(\left\|\nabla^2(n,u)\right\|_{L^2_\sigma}+\|\nabla(n,u)\|_{L^2_{\sigma-1}}+\|(n,u)\|_{L^2_{\sigma-2}}\\
&+\|\nabla(n,u)\|_{L^2_\sigma}+\|(n,u)\|_{L^2_{\sigma-1}})(1+\left\|\nabla^2(n,u)\right\|_{H^1})\Big]\\
\lesssim& t^{-\frac{5}{2}}\left\|\nabla^k n\right\|^2_{L^2_\sigma}+t^{-\frac{5}{2}-k+\sigma}.
\end{aligned}
\tag{3.22}
$$

Substituting (3.18)–(3.22) into (3.17), we have

$$
|J_4|\leq\frac{\nu}{2}\frac{d}{dt}\left\langle|x|^{2\sigma}|\nabla^k n|^2,\frac{n}{\lambda+\nu n}\right\rangle+Ct^{-\frac{5}{2}}\left\|\nabla^k(n,u)\right\|^2_{L^2_\sigma}+Ct^{-\frac{5}{2}-k+\sigma}.
\tag{3.23}
$$

Using the same arguments as J_2, J_3 and J_4 for J_5, J_6 and J_7 respectively, we have

$$
\begin{aligned}
|J_5|\lesssim& t^{-\frac{5}{4}}\left\|\nabla^k(n,u)\right\|^2_{L^2_\sigma}+t^{-\frac{5}{4}}\left\|\nabla^k u\right\|^2_{L^2_{\sigma-1}}+t^{-\frac{5}{2}-k+\sigma},\\
|J_6|\lesssim& t^{-\frac{5}{2}}\left\|\nabla^k(n,u)\right\|^2_{L^2_\sigma}+t^{-\frac{5}{2}}\left\|\nabla^k u\right\|^2_{L^2_{\sigma-1}}+t^{-\frac{5}{2}-k+\sigma},\\
|J_7|\leq&\frac{\nu}{2}\frac{d}{dt}\left\langle|x|^{2\sigma}|\nabla^k u|^2,\frac{n}{\lambda+\nu n}\right\rangle+Ct^{-\frac{5}{2}}\left\|\nabla^k(n,u)\right\|^2_{L^2_\sigma}+Ct^{-\frac{5}{2}-k+\sigma}.
\end{aligned}
\tag{3.24}
$$

Substituting (3.7), (3.8), (3.15), (3.23) and (3.24) into (3.6), and noticing that ϵ is small enough, then there exists a large enough T such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2+\nu\left\langle|x|^{2\sigma}(|\nabla^k n|^2+|\nabla^k u|^2),\frac{n}{\lambda+\nu n}\right\rangle\right]+\frac{\lambda}{2}\left\|\nabla^k u\right\|_{L^2_\sigma}^2\\&\lesssim t^{-\frac{5}{4}}\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2+t^{-\frac{5}{4}}\left\|\nabla^k(n,u)\right\|_{L^2_{\sigma-1}}^2+t^{-\frac{5}{2}-k+\sigma},\end{aligned}\tag{3.25}$$

for all $t > T$. Defining

$$H(t)=\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2+\nu\left\langle|x|^{2\sigma}(|\nabla^k n|^2+|\nabla^k u|^2),\frac{n}{\lambda+\nu n}\right\rangle,$$

it is obvious that there exist two positive constants $\overline{C}$ and $\underline{C}$ such that $\underline{C}\left\|\nabla^k(p,u)\right\|_{L^2_\sigma}^2\leq H(t)\leq\overline{C}\left\|\nabla^k(p,u)\right\|_{L^2_\sigma}^2$. Thus, $H(t)$ is equivalent to $\left\|\nabla^k(p,u)\right\|_{L^2_\sigma}^2$, and (3.25) can be rewritten as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2+\frac{\lambda}{2}\left\|\nabla^k u\right\|_{L^2_\sigma}^2\\&\lesssim t^{-\frac{5}{4}}\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2+t^{-\frac{5}{4}}\left\|\nabla^k(n,u)\right\|_{L^2_{\sigma-1}}^2+t^{-\frac{5}{2}-k+\sigma}.\end{aligned}\tag{3.26}$$

Substituting (2.3) and (3.1) into (3.26), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2+\frac{\lambda}{2}\left\|\nabla^k u\right\|_{L^2_\sigma}^2\\&\lesssim t^{-\frac{5}{4}}\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2+\|\nabla^k(n,u)\|_{L^2_\sigma}^{\frac{2(\sigma-1)}{\sigma}}\|\nabla^k(n,u)\|_{L^2}^{\frac{2}{\sigma}}+t^{-\frac{5}{2}-k+\sigma}\\&\lesssim t^{-\frac{5}{4}}\|\nabla^k(n,u)\|_{L^2_\sigma}^2+t^{\left(-\frac{3}{4}-\frac{k}{2}\right)\frac{2}{\sigma}}\|\nabla^k(n,u)\|_{L^2_\sigma}^{\frac{2(\sigma-1)}{\sigma}}+t^{-\frac{5}{2}-k+\sigma}.\end{aligned}\tag{3.27}$$

Denoting $\widehat{\mathrm{E}}(t):=\left\|\nabla^k(n,u)\right\|_{L^2_\sigma}^2$, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathrm{E}}(t)\leq C_0t^{-\frac{5}{4}}\widehat{\mathrm{E}}(t)+C_1t^{\left(-\frac{3}{4}-\frac{k}{2}\right)\frac{2}{\sigma}}\widehat{\mathrm{E}}(t)^{\frac{\sigma-1}{\sigma}}+C_3t^{-\frac{5}{2}-k+\sigma}.$$

If $\sigma>\frac{3}{2}+k$, then we can apply Lemma 2.5 with $\alpha_0=\frac{5}{4}>1,\alpha_1=\left(\frac{3}{4}+\frac{k}{2}\right)\frac{2}{\sigma}<1$ and $\beta_1=\frac{\sigma-1}{\sigma}<1$, $\sigma_1=\frac{1-\alpha_1}{1-\beta_1}=-\frac{3}{2}-k+\sigma$. Thus,

$$\widehat{\mathrm{E}}(t)\leq Ct^{-\frac{3}{2}-k+\sigma},\tag{3.28}$$

for all $t > T$. The Theorem 1.2 is proved for all $\sigma>\frac{3}{2}+k$ and the conclusion for the case of $[0,\frac{3}{2}+k]$ is proved by Lemma 2.4 (Interpolation inequality with weights). More precisely, by combining (3.1) and (3.28), we have

$$\|\nabla^k(n,u)(t)\|_{L^2_{\sigma_0}}\lesssim\|\nabla^k(n,u)(t)\|_{L^2}^{1-\frac{\sigma_0}{\sigma}}\|\nabla^k(n,u)(t)\|_{L^2_\sigma}^{\frac{\sigma_0}{\sigma}}\lesssim t^{-\frac{3}{2}-k+\frac{\sigma_0}{2}},\tag{3.29}$$

for all $t > T$ and $\sigma_0\in[0,\sigma]$, where $[0,\frac{3}{2}+k]\subset[0,\sigma](\sigma>\frac{3}{2}+k)$. Thus, we have covered the proof of Theorem 1.2.

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Conflict of interest

The author declares that there is no conflicts of interest.

References

1. J. Kim, Space-time decay rate for 3D compressible Euler equations with damping, *J. Evol. Equations*, **22** (2022), 1424–3199. https://doi.org/10.1007/s00028-022-00830-6
2. L. Hsiao, T. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, *Commun. Math. Phys.*, **143** (1992), 599–605. http://projecteuclid.org/euclid.cmp/1104249084
3. L. Hsiao, T. Liu, Nonlinear diffusive phenomena of nonlinear hyperbolic systems, *Chin. Ann. Math. Ser. B*, **14** (1993), 465–480.
4. L. Hsiao, D. Serre, Global existence of solutions for the system of compressible adiabatic flow through porous media, *SIAM J. Math. Anal.*, **27** (1996), 70–77. https://doi.org/10.1137/S0036141094267078
5. K. Nishihara, W. Wang, T. Yang, L_p-convergence rate to nonlinear diffusion waves for p-system with damping, *J. Differ. Equations*, **161** (2000), 191–21. https://doi.org/10.1006/jdeq.1999.3703
6. H. Zhao, Convergence to strong nonlinear diffusion waves for solutions of p-system with damping, *J. Differ. Equations*, **174** (2001), 200–236. https://doi.org/10.1006/jdeq.2000.3936
7. Q. Chen, Z. Tan,Time decay of solutions to the compressible Euler equations with damping, *Kinet. Rel. Models*, **166** (2003), 359–376. https://doi.org/10.1007/s00205-002-0234-5
8. F. Huang, P. Marcati, R. Pan, Convergence to the Barenblatt solution for the compressible Euler equations with damping and vacuum, *Arch. Ration. Mech. Anal.*, **176** (2005), 1–24. https://doi.org/10.1007/s00205-004-0349-y
9. F. Huang, R. Pan, Asymptotic behavior of the solutions to the damped compressible Euler equations with vacuum, *J. Differ. Equations*, **220** (2006), 207–233. https://doi.org/10.1016/j.jde.2005.03.0125
10. C. Zhu, Convergence rates to nonlinear diffusion waves for weak entropy solutions to p-system with damping, *Sci. China Ser. A*, **46** (2003), 562–575. https://doi.org/10.1360/03ys9057
11. J. Jang, N. Masmoudi, Well-posedness for compressible Euler equations with physical vacuum singularity, *Commun. Pure Appl. Math.*, **62** (2009), 1327–1385. https://doi.org/10.1002/cpa.20285

12. R. Duan, M. Jiang, Y. Zhang, Boundary effect on asymptotic behavior of solutions to the p-system with time-dependent damping, *Adv. Math. Phys.*, (2020), 3060867. https://doi.org/10.1155/2020/3060867
13. L. Hsiao, R. Pan, The damped p-system with boundary effects, in *Nonlinear PDE's, Dynamics and Continuum Physics*, **255** (2000), 109–123. https://doi.org/10.1090/conm/255/03977
14. J. Jang, N. Masmoudi, Well-posedness of compressible Euler equations in a physical vacuum, *Commun. Pure Appl. Math.*, **68** (2015), 61–111. https://doi.org/10.1002/cpa.21517
15. W. Wang, T. Yang, The pointwise estimates of solutions for Euler equations with damping in multi-dimensions, *J. Differ. Equations*, **173** (2001), 410–450. https://doi.org/10.1006/jdeq.2000.3937
16. T. C. Sideris, B. Thomases, D. Wang, Long time behavior of solutions to the 3D compressible Euler equations with damping, *Commun. Partial Differ. Equations*, **28** (2003), 795–816. https://doi.org/10.1081/PDE-120020497
17. J. Liao, W. Wang, T, Yang, L^p convergence rates of planar waves for multi-dimensional Euler equations with damping, *J. Differ. Equations*, **247** (2009), 303–329. https://doi.org/10.1016/j.jde.2009.03.011
18. D. Fang, J. Xu, Existence and asymptotic behavior of C^1 solutions to the multi-dimensional compressible Euler equations with damping, *Nonlinear Anal. Theory Methods Appl.*, **70** (2009), 244–261. https://doi.org/10.1016/j.na.2007.11.049
19. Z. Tan, Y. Wang, On hyperbolic-dissipative systems of composite type, *J. Differ. Equations*, **260** (2016), 1091–1125. https://doi.org/10.1016/j.jde.2015.09.025
20. Z. Tan, Y. Wang, Global solution and large-time behavior of the $3D$ compressible Euler equations with damping, *J. Differ. Equations*, **254** (2013), 1686–1704. https://doi.org/10.1016/j.jde.2012.10.026
21. M. Jiang, Y. Zhang, Existence and asymptotic behavior of global smooth solution for p-system with nonlinear damping and fixed boundary effect, *Math. Methods Appl. Sci.*, **37** (2014), 2585–2596. https://doi.org/10.1002/mma.2998
22. R. Pan, K. Zhao, The 3D compressible Euler equations with damping in a bounded domain, *J. Differ. Equations*, **246** (2009), 581–596. https://doi.org/10.1016/j.jde.2008.06.007
23. Y. Zhang, Z. Tan, Existence and asymptotic behavior of global smooth solution for p-system with damping and boundary effect, *Nonlinear Anal. Theory Methods Appl.*, **72** (2010), 605–619. https://doi.org/10.1016/j.na.2009.10.046
24. Q. Chen, Z. Tan, Time decay of solutions to the compressible Euler equations with damping, *Kinet. Rel. Models*, **7** (2014), 605–619. https://doi.org/10.3934/krm.2014.7.605
25. Z. Tan, G. Wu, Large time behavior of solutions for compressible Euler equations with damping in $\mathbb{R}^3$, *J. Diffe. Equations*, **252** (2012), 1546–156. https://doi.org/10.1016/j.jde.2011.09.003
26. L. Nirenberg, On elliptic partial differential equations, in *Il principio di minimo e sue applicazioni alle equazioni funzionali*, **13** (1959), 115–162.
27. Z. Tan, Y. Wang, F. Xu, Large-time behavior of the full compressible Euler-Poisson system without the temperature damping, *Discrete Contin. Dyn. Syst.*, **36** (2016), 1583–1601. https://doi.org/10.3934/dcds.2016.36.1583

28. S. Weng, Space-time decay estimates for the incompressible viscous resistive MHD and Hall-MHD equations, *J. Funct. Anal.*, **270** (2016), 2168–218. https://doi.org/10.1016/j.jfa.2016.01.021

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