Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(6): 3362–3380.
DOI: 10.3934/era.2023170
Received: 26 January 2023
Revised: 21 March 2023
Accepted: 04 April 2023
Published: 17 April 2023

Research article

The effects of cross-diffusion and logistic source on the boundedness of solutions to a pursuit-evasion model

Chang-Jian Wang*and Zi-Han Zheng

School of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, China

* **Correspondence:** Email: cjwang0719@126.com.

Abstract: We study the following quasilinear pursuit-evasion model:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u(u+1)^{\alpha}\nabla w) + u(\lambda_1 - \mu_1 u^{r_1-1} + av), & x\in\Omega,\ t>0,\\ v_t = \Delta v + \xi\nabla\cdot(v(v+1)^{\beta}\nabla z) + v(\lambda_2 - \mu_2 v^{r_2-1} - bu), & x\in\Omega,\ t>0,\\ 0 = \Delta w - w + v, & x\in\Omega,\ t>0,\\ 0 = \Delta z - z + u, & x\in\Omega,\ t>0,\end{cases}$$

in a smooth and bounded domain $\Omega\subset\mathbb{R}^n (n\geq 1)$, where $a, b, \chi, \xi, \lambda_1, \lambda_2, \mu_1, \mu_2 > 0$, $\alpha, \beta\in\mathbb{R}$, and $r_1, r_2 > 1$. When $r_1 > \max\{1, 1+\alpha\}, r_2 > \max\{1, 1+\beta\}$, it has been proved that if $\min\{(r_1-1)(r_2-\beta-1), (r_1-\alpha-1)(r_2-\beta-1)\} > \frac{(n-2)_+}{n}$, then for some suitable nonnegative initial data u_0 and v_0, the system admits a unique globally classical solution which is bounded in $\Omega\times(0,\infty)$.

Keywords: boundedness criteria; pursuit-evasion model; cross-diffusion; logistic source

1. Introduction

This paper is concerned with the quasilinear pursuit-evasion model:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u(u+1)^{\alpha}\nabla w) + u(\lambda_1 - \mu_1 u^{r_1-1} + av), & x\in\Omega,\ t>0,\\ v_t = \Delta v + \xi\nabla\cdot(v(v+1)^{\beta}\nabla z) + v(\lambda_2 - \mu_2 v^{r_2-1} - bu), & x\in\Omega,\ t>0,\\ 0 = \Delta w - w + v, & x\in\Omega,\ t>0,\\ 0 = \Delta z - z + u, & x\in\Omega,\ t>0,\end{cases} \tag{1.1}$$

with homogeneous Neumann boundary condition $\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = \frac{\partial z}{\partial\nu} = 0$ and initial data $u(x,0) = u_0(x), v(x,0) = v_0(x)$, where $\Omega\subset\mathbb{R}^n (n\geq 1)$ is a bounded and smooth domain, ν denotes the outward unit normal vector on $\partial\Omega$, and the parameters satisfy $a, b, \chi, \xi, \lambda_1, \lambda_2, \mu_1, \mu_2 > 0$, $\alpha, \beta \in \mathbb{R}$ and $r_1, r_2 > 1$. The initial data u_0 and v_0 are assumed to fulfill

$$u_0, v_0 \in C^0(\overline{\Omega}) \ \text{ with } u_0, v_0 \geq 0 \ \text{ in } \ \overline{\Omega}. \tag{1.2}$$

Here, u and v denote the population densities of the predators and prey, respectively. w and z represent the chemical substances released by prey and predators, respectively. In the current work, we shall reveal the effects of cross-diffusion and logistic source on the boundedness of solutions to model (1.1). Before stating our main results, let us recall some existing results on chemotaxis and predator-prey models.

The classical mathematical model for chemotaxis was proposed by Keller and Segel [1] to describe the aggregation of cellular slime molds, as follows:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \end{cases} \tag{1.3}$$

where $u(x,t)$ represents the density of cells, and $v(x,t)$ denotes the concentration of the chemical signal produced by cells. Recently, the system (1.3) has been studied extensively, and a lot of valuable theoretical results have been obtained by scholars [2–4]. Among them, one of the main issues related to (1.3) is to study whether there is a globally in-time bounded solution or when blow-up occurs. For instance, when $\tau = 1$, it has been proved that the classical solution of system (1.3) is globally bounded for $n = 1$ [5]. For $n = 2$, there exist initial data (u_0, u_0) such that the integral $\int_\Omega u_0 dx = \frac{4\pi}{\chi}$ [6–9] is the critical mass between boundedness and blow-up (i.e., when $\int_\Omega u_0 dx < \frac{4\pi}{\chi}$, the classical solution is globally bounded in time, and when $\int_\Omega u_0 dx > \frac{4\pi}{\chi}$ the solution blows up). In higher dimension setting $n \geq 3$, for any prescribed total mass $m = \int_\Omega u_0 > 0$, the system (1.3) possesses finite time blow-up solutions or unbounded solutions [10, 11]. When the second equation in system (1.3) is replaced by $v_t = \Delta v - v + g(u)$, with $g(u) \in C^1([0, +\infty))$ and $0 \leq g(u) \leq K u^\alpha$ for some constants $K, \alpha > 0$, Liu and Tao [12] obtained that the system (1.3) has a globally bounded classical solution if $0 < \alpha < \frac{2}{n}$. In addition, if Ω is a ball, and the initial data satisfy some suitable conditions, with the second equation degenerating into an elliptic equation, Winkler [13] obtained the critical exponent for blow-up and boundedness.

In order to investigate the proliferation and death of cell population, some interesting dynamical properties of solutions to the following chemotaxis-growth model have been established:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega, \ t > 0, \\ \tau v_t = \Delta v - v + g(u), & x \in \Omega, \ t > 0. \end{cases} \tag{1.4}$$

Here, it is worth mentioning that logistic-type growth restrictions somewhat benefit the global boundedness of solutions to system (1.4). For instance, in the case $\tau = 0$, when $f(u) \leq u(a - bu)$ and $g(u) = u$ with $a, b > 0$, Tello and Winkler [14] proved that the classical solution of system (1.4) is globally bounded whenever $\frac{n-2}{n}\chi < b$. When considering the more general forms of $f(u)$ and $g(u)$ with $f(u) \leq u(a - bu^s)$ and $g(u) = u^k$ for $k, s > 0$, Wang and Xiang [15] showed that the classical solutions of system (1.4) are globally bounded if either $s > k$ or $s = k$ with $\frac{kn-2}{kn}\chi < b$. As for $f(u) = au - bu^s$ and $g(u) = u$ with $s > 1, a \geq 0, b > 0$, Winkler [16] introduced a concept of very weak solutions and established the conditions of global existence and boundedness for such solutions. In the case $\tau = 1$, when $g(u) = u$ and $-c_0(u + u^s) \le f(u) \le a - bu^s$ with some $s > 1, b, c_0 > 0$ and $a \ge 0$, by an appropriate definition of very weak solutions, Viglialoro [17] constructed such global solutions under the assumptions that $n \ge 2$ and $s > 1 - \frac{2}{n}$. In [18], a relaxation of these hypotheses could be achieved so as to ensure solvability even for any $s > \frac{2n+4}{n+4}$ with $n \ge 2$.

Based on the so-called volume-filling effect proposed by Hillen and Painter [19], the self-diffusion functions and chemotactic sensitivity functions may have nonlinear forms of the cell density. Such mechanism can be described as the following system:

$$\begin{cases} u_t = \nabla \cdot (D(u)u) - \nabla \cdot (S(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.5}$$

where $D(u) > 0$ describes the strength of diffusion, and $S(u) > 0$ denotes the strength of chemoattractant. For the case $\tau = 1$, when $\Omega \subset \mathbb{R}^n (n \ge 2)$ is a ball, and $f(u) = 0$, the existence of the blow-up solution has been studied by Winkler [20], and it depends on the value of $\frac{S(u)}{D(u)}$. Namely, if $\frac{S(u)}{D(u)} \ge cu^\alpha$ with $\alpha > \frac{2}{n}$ and some constant $c > 0$ for all $u > 1$, then for any $M > 0$ there exist solutions that blow up in either finite or infinite time with mass $\int_\Omega u_0 = M$. Later on, Tao and Winkler [21] showed that this blow-up result is optimal, i.e., if $\frac{S(u)}{D(u)} \le cu^\alpha$ with $\alpha < \frac{2}{n}, n \ge 1$ and some constant $c > 0$ for all $u > 1$, then the system (1.5) possesses globally bounded classical solutions. For the case $\tau = 0$, if the second equation in (1.5) is replaced by $0 = \Delta v - \mu(t) + u$ with $\mu(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$, Lin-Mu-Zhong [22] obtained a finite time blow-up result in higher dimensions with $n \ge 5$. For more results on global boundedness or blow-up of solutions related to system (1.5), readers can refer to [23–30] for more details.

In a realistic environment, the relationships among biological species may be more complicated, and the predator-prey mechanism sometimes should be considered. The general model can be written as:

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v) + f(u,v), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v + \xi\nabla \cdot (v\nabla u) + g(u,v), & x \in \Omega,\ t > 0, \end{cases} \tag{1.6}$$

where u denotes the density of predator population, v represents the density of prey population, f and g are functional response functions describing the interaction between two species, and $\chi, \xi > 0$ are constants denoting the strengths of attraction and repulsion, respectively. When $f(u,v) = g(u,v) = 0$, for one-dimensional setting, Tao and Winkler [31] obtained the global existence of weak solutions. Subsequently, when $f(u,v) = u(\lambda_1 - u + a_1 v)$ and $g(u,v) = v(\lambda_2 - v - a_2 u)$ with $a_1, a_2, \lambda_1, \lambda_2 > 0$, the existence theory and qualitative analysis were established by Tao and Winkler [32]. Inter alia, the predator-prey models with prey-taxis or predator-taxis have also been widely studied. From a numerical point of view, the authors [33] showed that initial conditions and the form of functional response functions play important roles in the pattern formation for a predator-prey model with prey-taxis and diffusion. Tello and Wrzosek [34] studied an indirect prey-taxis predator-prey model and proved the global existence of solutions in any space dimension. In [35], the stability of globally classical solutions for a prey-predator model with indirect predator-taxis was considered. In addition, some other relevant results can be found in [36, 37].

Referring to the subsystems (1.3), (1.4) and (1.6) mentioned above, when taking two taxis mechanisms into account, we arrive at the following indirect pursuit-evasion model:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla w) + f(u,v), & x\in\Omega,\ t>0,\\ v_t = \Delta v + \xi\nabla\cdot(v\nabla z) + g(u,v), & x\in\Omega,\ t>0,\\ \tau_1 w_t = \Delta w - w + v, & x\in\Omega,\ t>0,\\ \tau_2 z_t = \Delta z - z + u, & x\in\Omega,\ t>0. \end{cases} \tag{1.7}$$

Recently, the researches on such model mainly focus on the global existence and long time dynamic behavior of corresponding classical (or weak) solutions. For instance, the existence and uniqueness of non-negative bounded weak solutions to the indirect pursuit-evasion model have been studied for $n = 2$ in [38,39]. In the case $\tau_1 = \tau_2 = 0$, when $f(u,v) = u(\lambda - u + av)$ and $g(u,v) = v(\mu - v - bu)$ with $a, b, \mu, \lambda > 0$, Li-Tao-Winkler [40] proved that the system admits globally bounded smooth solutions, moreover, they also derived the qualitative properties of classical solutions. When $f(u,v) = u(\lambda_1 - \mu_1 u^{r_1-1} + av)$ and $g(u,v) = v(\lambda_2 - \mu_2 v^{r_2-1} - bu)$ with $a, b, \mu_1, \mu_2, \lambda_1, \lambda_2 > 0$ and $r_1, r_2 > 1$, it was showed that the boundedness conditions of solutions depend on r_1, r_2 and dimension n with $(r_1 - 1)(r_2 - 1) > \frac{(n-2)_+}{n}$ for $n \geq 1$ by Zheng and Zhang [41]. In the case $\tau_1 = 1, \tau_2 = 0$, Liu and Liu [42] studied the global existence and boundedness of classical solutions by estimating L^p-norm of u and v, and they also showed the large time behavior and convergence rate of solutions. In the case $\tau_1 = \tau_2 = 1$, when $f(u,v) = u(\lambda - u + av)$ and $g(u,v) = v(\mu - v - bu)$ with $a, b, \mu, \lambda > 0$, the conditions for global existence of solutions were established by Qi and Ke [43], and under some exact smallness conditions on χ and ξ, the convergence with respect to $L^\infty(\Omega)-$norm of solutions was also derived. In Lebesgue spaces, by using the de Giorgi method and estimates, Amorim and Telch [44] obtained the conditions for global well-posedness and boundedness of solutions to an indirect pursuit-evasion system. Later on, the similar problem to a quasilinear parabolic predator-prey system with pursuit-evasion was considered in [45]. Telch [46] generalized the results of [44] without considering any assumption under the asymptotic behaviour of pheromone production of the predator as therein.

Motivated by the work mentioned above, we are interested in an indirect pursuit-evasion model with cross-diffusion and generalized logistic source. The purpose of this paper is to detect the possible effects resulting from the cross-diffusion and logistic source on the boundedness of solutions to system (1.1). We state our main results to system (1.1) as follows.

Theorem 1.1. *Assume that* $\Omega \subset \mathbb{R}^n (n \geq 1)$ *is a bounded and smooth domain. Let* $\lambda_1, \lambda_2, \mu_1, \mu_2, \chi, \xi, a, b > 0$, $\alpha, \beta \in \mathbb{R}$ *and* $r_1, r_2 > 1$. *When* $r_1 > \max\{1, 1+\alpha\}$ *and* $r_2 > \max\{1, 1+\beta\}$, *if* $\min\{(r_1-1)(r_2-\beta-1), (r_1-\alpha-1)(r_2-\beta-1)\} > \frac{(n-2)_+}{n}$ *then for any nonnegative initial data* $u_0(x)$ *and* $v_0(x)$ *satisfying* (1.2)*, the system* (1.1) *admits a unique globally classical solution, which is bounded in the sense that there exists* $C > 0$ *satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \tag{1.8}$$

for all $t > 0$.

The main difficulty of this paper comes from the cross-diffusion and generalized logistic source. We shall use the Gagliardo-Nirenberg inequality to deal with the term $\int_\Omega z^{\frac{p+r_2-1}{r_2-\beta-1}}$ generated by the cross-diffusion and logistic source in proving $L^p(\Omega)-$boundedness of u and v in Lemma 3.3. Moreover, we need to modify the Moser-type iteration developed in [21] to obtain the $L^\infty(\Omega)$ boundedness of u and v.

Remark 1.2. *The nonlinear cross-diffusion term* $-\chi\nabla\cdot(u(u+1)^\alpha\nabla w)$ *means that the predators move toward the higher concentration of the chemical substances produced by prey and the other term* $\xi\nabla\cdot(v(v+1)^\beta\nabla z)$ *describes that prey moves away from the higher concentration of the chemical substances secreted by predators. This tendency movement of predators and prey can prevent the population from overcrowding and reach limited saturation. When the predator moves towards the high-concentration chemicals produced by the prey or the prey is far away from the high-concentration chemicals secreted by the predator, the predator and prey will affect the intraspecific struggle due to their respective aggregation, thus affecting the dynamics.*

Remark 1.3. *The boundedness result established in Theorem 1.1 is more generalized than the previous ones. Namely, when* $\alpha=\beta=0$, *the boundedness criteria is consistent with the one developed in [41]. Moreover, compared to the boundedness result obtained by Li-Tao-Winkler [40] with* $\alpha=\beta=0$, $r_1=r_2=2$ *and* $n\le 3$, *this paper improves the dimension of the classical solutions with* $n\ge 1$.

Remark 1.4. *Telch [45] studied a chemotaxis quasilinear parabolic predator-prey system with pursuit-evasion dynamics. For some suitable initial data, he obtained the boundedness conditions of globally classical solutions by using maximal Sobolev regularity, which only depend on the parameters of the system. Although the system* (1.1) *can be considered as a special case of [45], the boundedness conditions obtained in this paper are based on the method of Moser-type iteration, which depend not only on the parameters of the system but also on the dimensions of the space.*

The rest of this paper is carried out as follows. In Section 2, we state a local existence lemma to model (1.1) and introduce a Gagliardo-Nirenberg inequality, which are crucial in the proof of Theorem 1.1. In Section 3, we establish the $L^p(\Omega)$ estimates to u and v, and then modify the Moser-type iteration developed in [21] to prove the main result.

2. Preliminaries

In this section, we first state a lemma on the local existence and uniqueness of the classical solutions for the system (1.1). The proof is quite standard relying on Schauder fixed theorem, we refer readers to [47, 48] for more details.

Lemma 2.1. *Assume that* $\Omega\subset\mathbb{R}^n (n\ge 1)$ *is a bounded and smooth domain. For any nonnegative initial data* u_0 *and* v_0 *satisfying* (1.2), *there exists* $T_{\max}\in(0,+\infty]$ *such that the system* (1.1) *possesses a unique nonnegative classical solution* (u,v,w,z) *fulfilling*

$$\begin{cases} u\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})),\\ v\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})),\\ w\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,0}(\overline{\Omega}\times(0,T_{\max})),\\ z\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,0}(\overline{\Omega}\times(0,T_{\max})). \end{cases} \tag{2.1}$$

Furthermore, if $T_{\max}<\infty$, *then* $\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}\to\infty$ *as* $t\nearrow T_{\max}$.

Nextly, we introduce a Gagliardo-Nirenberg inequality and the proof can be found in [49].

Lemma 2.2. *Assume that $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded and smooth domain. Let $1 \leq p, q, r, s \leq \infty$ and $j, m \in \mathbb{N}$ with $j \in [0, m)$. For any $\alpha_0 \in [\frac{j}{m}, 1]$, there exist $C_1, C_2 > 0$ depending only on $n, m, j, q, r, s, \alpha_0$, such that the derivatives $D^j w$ satisfy the following inequality*

$$\|D^j w\|_{L^p(\Omega)} \leq C_1 \|D^m w\|^{\alpha_0}_{L^r(\Omega)} \|w\|^{1-\alpha_0}_{L^q(\Omega)} + C_2 \|w\|_{L^s(\Omega)}, \tag{2.2}$$

where $\frac{1}{p} = \frac{j}{n} + (\frac{1}{r} - \frac{m}{n})\alpha_0 + \frac{1-\alpha_0}{q}$, for any $w \in L^q(\Omega)$ with $D^m w \in L^r(\Omega)$ and $w \in L^s(\Omega)$.

3. The proof of the main result

In this section, we will obtain some priori estimates of solutions by using the Sobolev embedding theorem and the L^p- estimate of the parabolic and elliptic equations.

Lemma 3.1. *Assume that the conditions in Lemma 2.1 hold and the parameters $r_1, r_2 > 1$, then there exists $C > 0$ such that*

$$\int_\Omega u + \int_\Omega v \leq C \ \textit{for all } t \in (0, T_{\max}). \tag{3.1}$$

Proof. Integrating the first and second equations of system (1.1) over Ω, we deduce that

$$\frac{d}{dt}\int_\Omega u + \int_\Omega u = \int_\Omega u(\lambda_1 - \mu_1 u^{r_1 - 1} + av) + \int_\Omega u \tag{3.2}$$

and

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v = \int_\Omega v(\lambda_2 - \mu_2 v^{r_2 - 1} - bu) + \int_\Omega v \tag{3.3}$$

for all $t \in (0, T_{\max})$. Integrating the sum of b times (3.2) and a times (3.3) and employing Young's inequality, we can derive that there exists $c_1 > 0$ such that

$$\begin{aligned} b\frac{d}{dt}\int_\Omega u + b\int_\Omega u + a\frac{d}{dt}\int_\Omega v + a\int_\Omega v &= b(\lambda_1 + 1)\int_\Omega u - b\mu_1 \int_\Omega u^{r_1} + a(\lambda_2 + 1)\int_\Omega v - a\mu_2 \int_\Omega v^{r_2} \\ &\leq -\frac{b\mu_1}{2}\int_\Omega u^{r_1} - \frac{a\mu_2}{2}\int_\Omega v^{r_2} + c_1 \leq c_1 \end{aligned} \tag{3.4}$$

for all $t \in (0, T_{\max})$. Therefore, with an application of ODE comparison, there exists $c_2 > 0$ such that

$$\int_\Omega u + \int_\Omega v \leq c_2 \ \text{ for all } t \in (0, T_{\max}). \tag{3.5}$$

Based on Lemma 3.1, the following lemma enables us to get the better properties of z and w than L^1-boundedness.

Lemma 3.2. *Assume that the conditions in Lemma 2.1 hold. Then there exists $C > 0$ such that w and z possess the following properties*

$$\int_\Omega z^{l_0} + \int_\Omega w^{l_0} \le C \ \textit{for all } t \in (0, T_{\max}), \tag{3.6}$$

with $l_0 \in \left[1, \frac{n}{(n-2)_+}\right)$.

Proof. Combining Lemma 3.1 with the third and fourth equations of (1.1), there exist $c_3, c_4 > 0$ such that

$$\int_\Omega z(x,t) = \int_\Omega u(x,t) \le c_3 \tag{3.7}$$

and

$$\int_\Omega w(x,t) = \int_\Omega v(x,t) \le c_4 \tag{3.8}$$

for all $t \in (0, T_{\max})$. Using the classical result by Berzis and Strauss [50] and the Minkowski inequality, we find that there exist $c_5, c_6, c_7 > 0$ and $\tilde{c}_5, \tilde{c}_6, \tilde{c}_7 > 0$ such that

$$\|z(\cdot,t)\|_{w^{1,l}(\Omega)} \le c_5\|\Delta z(\cdot,t) - z(\cdot,t)\|_{L^1(\Omega)} \le c_6\|u(\cdot,t)\|_{L^1(\Omega)} \le c_7 \tag{3.9}$$

and

$$\|w(\cdot,t)\|_{w^{1,l}(\Omega)} \le \tilde{c}_5\|\Delta w(\cdot,t) - w(\cdot,t)\|_{L^1(\Omega)} \le \tilde{c}_6\|v(\cdot,t)\|_{L^1(\Omega)} \le \tilde{c}_7 \tag{3.10}$$

for all $t \in (0, T_{\max})$ and $l \in [1, \frac{n}{(n-1)_+})$. Thus, by the Sobolev embedding theorem, one can see that

$$\|z(\cdot,t)\|_{L^{l_0}(\Omega)} \le c_8 \text{ and } \|w(\cdot,t)\|_{L^{l_0}(\Omega)} \le c_8 \text{ for all } t \in (0, T_{\max}) \text{ and } l_0 \in [1, \frac{n}{(n-2)_+}),$$

with some $c_8 > 0$. The proof of Lemma 3.2 is completed.

To obtain the global boundedness of the classical solutions to system (1.1), the following L^p-estimates of components u and v are crucial.

Lemma 3.3. *Assume that the conditions in Theorem 1.1 hold. For any $p > \max\{1, 1-\alpha, 1-\beta\}$, we can find $C > 0$ such that*

$$\int_\Omega v^p + \int_\Omega u^p \le C \ \textit{for all } t \in (0, T_{\max}). \tag{3.11}$$

Proof. Multiplying the second equation in system (1.1) by $(v+1)^{p-1}$ and integrating by parts over Ω, we can obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + (p-1)\int_\Omega (v+1)^{p-2}|\nabla v|^2 + \int_\Omega (v+1)^p \\
&= -(p-1)\xi\int_\Omega (v+1)^{p+\beta-1}\nabla v\cdot\nabla z + (p-1)\xi\int_\Omega (v+1)^{p+\beta-2}\nabla v\cdot\nabla z + (\lambda_2+1)\int_\Omega v(v+1)^{p-1}
\end{aligned}$$

$$
\begin{aligned}
&-\mu_2\int_\Omega v^{r_2}(v+1)^{p-1}-b\int_\Omega uv(v+1)^{p-1}\\
\le&\frac{(p-1)\xi}{p+\beta}\int_\Omega(v+1)^{p+\beta}\Delta z-\frac{(p-1)\xi}{p+\beta-1}\int_\Omega(v+1)^{p+\beta-1}\Delta z+(\lambda_2+1)\int_\Omega(v+1)^p-\mu_2\int_\Omega v^{p+r_2-1}\\
\le&\frac{(p-1)\xi}{p+\beta}\int_\Omega(v+1)^{p+\beta}(z-u)+\frac{(p-1)\xi}{p+\beta-1}\int_\Omega(v+1)^{p+\beta-1}u+(\lambda_2+1)\int_\Omega(v+1)^p-\mu_2\int_\Omega v^{p+r_2-1}
\end{aligned}
\tag{3.12}
$$

for all $t\in(0,T_{\max})$, where the equation $\Delta z=z-u$ has been used here. In view of Lemmas 3.1 and 3.2, using Young's inequality and basic inequality $(x+y)^\vartheta\le 2^\vartheta(x^\vartheta+y^\vartheta)$ with $x,y>0$ and $\vartheta>1$, we can get

$$
\begin{aligned}
\frac{(p-1)\xi}{p+\beta-1}\int_\Omega(v+1)^{p+\beta-1}u&\le\frac{(p-1)\xi}{p+\beta}\int_\Omega(v+1)^{p+\beta}u+c_9\int u\\
&\le\frac{(p-1)\xi}{p+\beta}\int_\Omega(v+1)^{p+\beta}u+c_{10}
\end{aligned}
\tag{3.13}
$$

as well as

$$
\begin{aligned}
\frac{(p-1)\xi}{p+\beta}\int_\Omega(v+1)^{p+\beta}z&\le\frac{2^{p+\beta}(p-1)\xi}{p+\beta}\int_\Omega v^{p+\beta}z+\frac{2^{p+\beta}(p-1)\xi}{p+\beta}\int_\Omega z\\
&\le\frac{\mu_2}{2}\int_\Omega v^{p+r_2-1}+c_{11}\int_\Omega z^{\frac{p+r_2-1}{r_2-\beta-1}}+c_{12}
\end{aligned}
\tag{3.14}
$$

for all $t\in(0,T_{\max})$ with some constants $c_9,c_{10},c_{11},c_{12}>0$. Therefore, combining (3.12), (3.24) and (3.14), one may obtain

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega(v+1)^p+(p-1)\int_\Omega(v+1)^{p-2}|\nabla v|^2+\int_\Omega(v+1)^p\\
\le&-\frac{\mu_2}{2}\int_\Omega v^{p+r_2-1}+c_{11}\int_\Omega z^{\frac{p+r_2-1}{r_2-\beta-1}}+(\lambda_2+1)\int_\Omega(v+1)^p+c_{12}\ \text{ for all } t\in(0,T_{\max}).
\end{aligned}
\tag{3.15}
$$

Next, multiplying the first equation in (1.1) by $(u+1)^{p-1}$, integrating by parts over Ω and applying the identity $\Delta w=w-v$, we get

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega(u+1)^p+(p-1)\int_\Omega(u+1)^{p-2}|\nabla u|^2+\int_\Omega(u+1)^p\\
\le&\frac{(p-1)\chi}{p+\alpha}\int_\Omega\nabla(u+1)^{p+\alpha}\cdot\nabla w-\frac{(p-1)\chi}{p+\alpha-1}\int_\Omega\nabla(u+1)^{p+\alpha-1}\cdot\nabla w+(\lambda_1+1)\int_\Omega(u+1)^p\\
&-\mu_1\int_\Omega u^{p+r_1-1}+a\int_\Omega(u+1)^pv\\
\le&-\frac{(p-1)\chi}{p+\alpha}\int_\Omega(u+1)^{p+\alpha}\Delta w+\frac{(p-1)\chi}{p+\alpha-1}\int_\Omega(u+1)^{p+\alpha-1}\Delta w+(\lambda_1+1)\int_\Omega(u+1)^p\\
&-\mu_1\int_\Omega u^{p+r_1-1}+a\int_\Omega(u+1)^pv\\
\le&\frac{(p-1)\chi}{p+\alpha}\int_\Omega(u+1)^{p+\alpha}(v-w)+\frac{(p-1)\chi}{p+\alpha-1}\int_\Omega(u+1)^{p+\alpha-1}w+(\lambda_1+1)\int_\Omega(u+1)^p
\end{aligned}
$$

$$-\mu_1\int_\Omega u^{p+r_1-1}+a\int_\Omega (u+1)^p v \ \text{ for all } t\in(0,T_{\max}). \tag{3.16}$$

By Young's inequality , we can deduce from Lemma 3.2

$$\begin{aligned}\frac{(p-1)\chi}{p+\alpha-1}\int_\Omega (u+1)^{p+\alpha-1}w &\le \frac{(p-1)\chi}{p+\alpha}\int_\Omega (u+1)^{p+\alpha}w+c_{13}\int w\\ &\le \frac{(p-1)\chi}{p+\alpha}\int_\Omega (u+1)^{p+\alpha}w+c_{14}\end{aligned} \tag{3.17}$$

with some $c_{13}>0, c_{14}>0$. Using Young's inequality and basic inequality $(x+y)^\vartheta\le 2^\vartheta(x^\vartheta+y^\vartheta)$ with $x,y>0$ and $\vartheta>1$ once more, for $p>1-\alpha$ and $r_1>\max\{1,\alpha+1\}$, we can get from Lemma 3.1

$$\begin{aligned}\frac{(p-1)\chi}{p+\alpha}\int_\Omega (u+1)^{p+\alpha}v \le&\frac{(p-1)\chi}{p+\alpha}2^{p+\alpha}\int_\Omega u^{p+\alpha}v+\frac{(p-1)\chi}{p+\alpha}2^{p+\alpha}\int_\Omega v\\ \le&\frac{\mu_1}{4}\int_\Omega u^{p+r_1-1}+c_{14}\int_\Omega v^{\frac{p+r_1-1}{r_1-\alpha-1}}+c_{15}\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}a\int_\Omega (u+1)^p v\le&2^p a\int_\Omega u^p v+2^p a\int_\Omega v\\ \le&\frac{\mu_1}{4}\int_\Omega u^{p+r_1-1}+c_{16}\int_\Omega v^{\frac{p+r_1-1}{r_1-1}}+c_{17}\end{aligned} \tag{3.19}$$

for all $t\in(0,T_{\max})$ with certain $c_{14},c_{15},c_{16},c_{17}>0$. Substituting (3.17)–(3.19) into (3.16), we get there exists $c_{18}>0$ such that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p+(p-1)\int_\Omega (u+1)^{p-2}|\nabla u|^2+\int_\Omega (u+1)^p\\ \le&(\lambda_1+1)\int_\Omega (u+1)^p-\frac{\mu_1}{2}\int_\Omega u^{p+r_1-1}+c_{14}\int_\Omega v^{\frac{p+r_1-1}{r_1-\alpha-1}}+c_{16}\int_\Omega v^{\frac{p+r_1-1}{r_1-1}}+c_{18}.\end{aligned} \tag{3.20}$$

Since

$$r_1>\max\{1,1+\alpha\}, r_2>\max\{1,1+\beta\}$$

and

$$\min\{(r_1-1)(r_2-\beta-1),(r_1-\alpha-1)(r_2-\beta-1)\}>\frac{(n-2)_+}{n}$$

for any $p>\max\{1,1-\alpha,1-\beta\}$, so that we can find $l_0\in\left[1,\frac{n}{(n-2)_+}\right)$ which is sufficiently close to $\frac{n}{(n-2)_+}$ such that

$$\frac{p+r_2-1}{r_2-\beta-1}<(1+\frac{2l_0}{n})\cdot(p+r_1-1)<\frac{n}{(n-2)_+}\cdot(p+r_1-1) \tag{3.21}$$

and

$$\max\left\{\frac{p+r_1-1}{r_1-1},\frac{p+r_1-1}{r_1-\alpha-1}\right\}<p+r_2-1. \tag{3.22}$$

Using the Gagliardo-Nirenberg inequality in Lemma 2.2 and the L^p−theory of elliptic equation, we can find $c_{19}, c_{20}, c_{21} > 0$ such that

$$\begin{aligned}
c_{11}\int_\Omega z^{\frac{p+r_2-1}{r_2-\beta-1}} &= c_{11}\|z\|^{\frac{p+r_2-1}{r_2-\beta-1}}_{L^{\frac{p+r_2-1}{r_2-\beta-1}}(\Omega)}\\
&\le c_{19}\|\Delta z\|^{\frac{p+r_2-1}{r_2-\beta-1}k_1}_{L^{p+r_1-1}(\Omega)}\|z\|^{\frac{p+r_2-1}{r_2-\beta-1}\cdot(1-k_1)}_{L^{l_0}(\Omega)} + c_{19}\|z\|^{\frac{p+r_2-1}{r_2-\beta-1}}_{L^{l_0}(\Omega)}\\
&\le c_{20}\|\Delta z\|^{\frac{p+r_2-1}{r_2-\beta-1}k_1}_{L^{p+r_1-1}(\Omega)} + c_{20}\\
&\le c_{21}\|u\|^{\frac{p+r_2-1}{r_2-\beta-1}k_1}_{L^{p+r_1-1}(\Omega)} + c_{21},
\end{aligned}\tag{3.23}$$

where $k_1 = \frac{\frac{r_2-\beta-1}{p+r_2-1}-\frac{1}{l_0}}{\frac{1}{p+r_1-1}-\frac{2}{n}-\frac{1}{l_0}} = \frac{1-\frac{(r_2-\beta-1)l_0}{p+r_2-1}}{1-\left(\frac{1}{p+r_1-1}-\frac{2}{n}\right)l_0} \in (0,1)$ with $l_0 \in \left(1, \frac{n}{(n-2)_+}\right)$. In fact, since $r_1 > \max\{1, 1+\alpha\}, r_2 > \max\{1, 1+\beta\}$, thus $\frac{r_2-\beta-1}{p+r_2-1} > \frac{1}{p+r_1-1} - \frac{2}{n}$ for any $p > \max\{1, 1-\alpha, 1-\beta\}$. From (3.21) we can get

$$\frac{p+r_2-1}{r_2-\beta-1}k_1 = \frac{p+r_2-1}{r_2-\beta-1}\cdot\frac{1-\frac{(r_2-\beta-1)l_0}{p+r_2-1}}{1-\left(\frac{1}{p+r_1-1}-\frac{2}{n}\right)l_0} < p+r_1-1. \tag{3.24}$$

Therefore, there exists $c_{22} > 0$ such that

$$\begin{aligned}
c_{11}\int_\Omega z^{\frac{p+r_2-1}{r_2-\beta-1}} &\le c_{21}\|u\|^{\frac{p+r_2-1}{r_2-\beta-1}k_1}_{L^{p+r_1-1}(\Omega)} + c_{21}\\
&\le \frac{\mu_1}{4}\int_\Omega u^{p+r_1-1} + c_{22}.
\end{aligned}\tag{3.25}$$

Collecting (3.15) and (3.25), we can derive

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + (p-1)\int_\Omega (v+1)^{p-2}|\nabla v|^2 + \int_\Omega (v+1)^p\\
\le&(\lambda_2+1)\int_\Omega (v+1)^p - \frac{\mu_2}{2}\int_\Omega v^{p+r_2-1} + \frac{\mu_1}{4}\int_\Omega u^{p+r_1-1} + c_{24}
\end{aligned}\tag{3.26}$$

for all $t \in (0, T_{\max})$ with some $c_{23} > 0$. With an application of Young's inequality, from (3.22), we can find $c_{24} > 0$ such that

$$c_{14}\int_\Omega v^{\frac{p+r_1-1}{r_1-\alpha-1}} + c_{16}\int_\Omega v^{\frac{p+r_1-1}{r_1-1}} \le \frac{\mu_2}{4}\int_\Omega v^{p+r_2-1} + c_{24} \tag{3.27}$$

for all $t \in (0, T_{\max})$. Combining (3.20), (3.26) with (3.27) and using Young's inequality, we can obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p + (p-1)\int_\Omega (u+1)^{p-2}|\nabla u|^2 + \int_\Omega (u+1)^p + \frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p\\
&+(p-1)\int_\Omega (v+1)^{p-2}|\nabla v|^2 + \int_\Omega (v+1)^p\\
\le&(\lambda_1+1)\int_\Omega (u+1)^p - \frac{\mu_1}{4}\int_\Omega u^{p+r_1-1} + (\lambda_2+1)\int_\Omega (v+1)^p - \frac{\mu_2}{4}\int_\Omega v^{p+r_2-1} + c_{25}
\end{aligned}\tag{3.28}$$

for all $t \in (0, T_{\max})$, with some $c_{25} > 0$. Using the fact that $r_1 > \max\{1, 1+\alpha\}$ and $r_2 > \max\{1, 1+\beta\}$, from Young's inequality and basic inequality $(x+y)^\vartheta \le 2^\vartheta(x^\vartheta + y^\vartheta)$ with $x, y > 0$ and $\vartheta > 1$, we can find $c_{26}, c_{27} > 0$ such that

$$(\lambda_1 + 1)\int_\Omega (u+1)^p \le \frac{\mu_1}{4}\int_\Omega u^{p+r_1-1} + c_{26} \tag{3.29}$$

and

$$(\lambda_2 + 1)\int_\Omega (v+1)^p \le \frac{\mu_2}{4}\int_\Omega v^{p+r_2-1} + c_{27} \tag{3.30}$$

for all $t \in (0, T_{\max})$. Thus

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p + (p-1)\int_\Omega (u+1)^{p-2}|\nabla u|^2 + \int_\Omega (u+1)^p + \frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p\\&+(p-1)\int_\Omega (v+1)^{p-2}|\nabla v|^2 + \int_\Omega (v+1)^p \le c_{25} + c_{26} + c_{27}\end{aligned} \tag{3.31}$$

for all $t \in (0, T_{\max})$. Therefore, the desired results can be deduced by Gronwall's inequality. This completes the proof of Lemma 3.3.

Now we are in a position to prove Theorem 1.1. Although the method is based on the literature [21], we need to modify some steps therein. For the convenience of readers, we only give detailed proofs for the modifications.

The proof of Theorem 1.1 From Lemma 3.3, there exist $p_0 > \max\{1, 1-\alpha, 1-\beta\}$ and $c_{28} > 0$ such that

$$\int_\Omega u^{p_0} + \int_\Omega v^{p_0} \le c_{28} \ \text{ for all } t \in (0, T_{\max}). \tag{3.32}$$

By the elliptic L^p-estimates applied to the third and fourth equations in system (1.1), there exists $c_{29} > 0$ such that

$$\sup_{t\in(0,T_{\max})}\left[\|w(\cdot,t)\|_{W^{2,p_0}(\Omega)} + \|z(\cdot,t)\|_{W^{2,p_0}(\Omega)}\right] \le c_{29}. \tag{3.33}$$

Thus the Sobolev embedding theorem enables us to obtain

$$\sup_{t\in(0,T_{\max})}\left[\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)}\right] \le c_{30}, \tag{3.34}$$

with some $c_{30} > 0$. Repeating the computations in (3.12) and using (3.34), we can obtain

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + (p-1)\int_\Omega (v+1)^{p-2}|\nabla v|^2 + \int_\Omega (v+1)^p\\&\le c_{31}(p-1)\int_\Omega (v+1)^{p+\beta-1}|\nabla v| + (\lambda_2+1)\int_\Omega (v+1)^p\\&\le c_{31}(p-1)\int_\Omega (v+1)^{\frac{p-2}{2}}|\nabla v|(v+1)^{p+\beta-1-\frac{p-2}{2}} + (\lambda_2+1)\int_\Omega (v+1)^p\end{aligned} \tag{3.35}$$

for all $t \in (0, T_{\max})$ with $c_{31} > 0$. Using Young's inequality, we can get

$$\begin{aligned} & c_{31}(p-1)\int_\Omega (v+1)^{\frac{p-2}{2}}|\nabla v|(v+1)^{p+\beta-1-\frac{p-2}{2}} \\ \le & \frac{p-1}{2}\int_\Omega (v+1)^{p-2}|\nabla v|^2 + \frac{c_{31}^2(p-1)}{2}\int_\Omega (v+1)^{p+2\beta} \text{ for all } t\in(0,T_{\max}). \end{aligned} \tag{3.36}$$

From (3.35) and (3.36), it is easy to see

$$\begin{aligned} & \frac{d}{dt}\int_\Omega (v+1)^p + \frac{p(p-1)}{2}\int_\Omega (v+1)^{p-2}|\nabla v|^2 + p\int_\Omega (v+1)^p \\ \le & \frac{c_{31}^2 p(p-1)}{2}\int_\Omega (v+1)^{p+2\beta} + (\lambda_2+1)p\int_\Omega (v+1)^p \text{ for all } t\in(0,T_{\max}). \end{aligned} \tag{3.37}$$

We define $q > n+2$ such that

$$p_0 > 2\beta\left[\frac{1}{1-\frac{nq}{(n+2)(q-2)}} - 1\right]. \tag{3.38}$$

In the following, the proofs are divided into two cases with $\beta \ge 0$ and $\beta < 0$. Here, we only deal with the case $\beta \ge 0$. And for the other case, we can employ the similar process. Let $r \in \left(2, \frac{2(n+2)}{n}\right)$ sufficiently close to $\frac{2(n+2)}{n}$. Define $\theta(r) = \frac{r}{2}\cdot\frac{p_0}{p_0+2\beta}$, it is not difficult to get

$$\theta(r) \ge \frac{q}{q-2}. \tag{3.39}$$

In fact, the inequality (3.39) can be ensured by the following direct computation

$$\begin{aligned} \theta\left(\frac{2(n+2)}{n}\right) &= \frac{n+2}{n}\cdot\left(1-\frac{2\beta}{p_0+2\beta}\right) \\ &> \frac{n+2}{n}\cdot\left\{1-\frac{2\beta}{2\beta\left[\frac{1}{1-\frac{nq}{(n+2)(q-2)}}-1\right]+2\beta}\right\} \\ &= \frac{n+2}{n}\cdot\frac{nq}{(n+2)(q-2)} \\ &= \frac{q}{q-2}. \end{aligned} \tag{3.40}$$

We choose $s \in (0,2)$ sufficiently close to 2 such that

$$r < \frac{2(n+s)}{n} \tag{3.41}$$

and

$$\frac{\frac{nr}{s}-n}{\frac{2q}{q-2}\cdot\left(1-\frac{n}{2}+\frac{n}{s}\right)} < 1. \tag{3.42}$$

In fact, the inequality (3.42) can be achieved due to the fact that the expression $\frac{\frac{nr}{2}-n}{\frac{2q}{q-2}}$ satisfies

$$\frac{\frac{nr}{2}-n}{\frac{2q}{q-2}} < \frac{\frac{n}{2}\cdot\frac{2(n+2)}{n}-n}{\frac{2q}{q-2}} = 1-\frac{2}{q} < 1 \text{ as } s \to 2. \tag{3.43}$$

We continue to define

$$p_k := \frac{2}{s}p_{k-1}, \quad k \geq 1, \tag{3.44}$$

and note that $\{p_k\}_{k\in\mathbb{N}}$ is increasing and

$$m_1 \cdot \left(\frac{2}{s}\right)^k \leq p_k \leq m_2 \cdot \left(\frac{2}{s}\right)^k \tag{3.45}$$

for all $k \in \mathbb{N}$, with some $m_1, m_2 > 0$. Denoting

$$\theta_k := \frac{r}{2}\cdot\frac{p_k}{p_k+2\beta}, \ k \in \mathbb{N}, \tag{3.46}$$

it is not difficult to see that $\{\theta_k\}_{k\in\mathbb{N}}$ is increasing with $\theta_k \geq \theta_0 = \theta(r) \geq \frac{q}{q-2}$. In the following, we will deduce a recursive inequality for

$$M_k := \sup_{t\in(0,T_{\max})}\int_\Omega (v+1)^{p_k}, \ k \in \mathbb{N}. \tag{3.47}$$

To this end, we can rewrite (3.37) as

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (v+1)^{p_k} + \frac{p_k(p_k-1)}{2}\int_\Omega (v+1)^{p_k-2}|\nabla v|^2 + p_k\int_\Omega (v+1)^{p_k} \\
\leq&\frac{c_{31}^2 p_k(p_k-1)}{2}\int_\Omega (v+1)^{p_k+2\beta} + (\lambda_2+1)p_k\int_\Omega (v+1)^{p_k} \text{ for all } t \in (0,T_{\max}).
\end{aligned} \tag{3.48}$$

Employing Hölder's inequality, we get

$$\begin{aligned}
\frac{c_{31}^2}{2}\int_\Omega (v+1)^{p_k+2\beta} &\leq \frac{c_{31}^2}{2}\left[\int_\Omega (v+1)^{(p_k+2\beta)\theta_k}\right]^{\frac{1}{\theta_k}}\cdot\left[\int_\Omega 1^{\frac{\theta_k}{\theta_k-1}}\right]^{\frac{\theta_k-1}{\theta_k}} \\
&\leq c_{32}\left[\int_\Omega (v+1)^{(p_k+2\beta)\theta_k}\right]^{\frac{1}{\theta_k}} \\
&= c_{32}\|(v+1)^{\frac{p_k}{2}}\|^{\frac{r}{\theta_k}}_{L^{\frac{2(p_k+2\beta)\theta_k}{p_k}}(\Omega)} \\
&= c_{32}\|(v+1)^{\frac{p_k}{2}}\|^{\frac{r}{\theta_k}}_{L^r(\Omega)}
\end{aligned} \tag{3.49}$$

for all $t \in (0,T_{\max})$ with some $c_{32} > 0$. In view of the Gagliardo-Nirenberg inequality, there exists $c_{33} > 0$ independent of k such that

$$c_{32}\|(v+1)^{\frac{p_k}{2}}\|^{\frac{r}{\theta_k}}_{L^r(\Omega)} \leq c_{33}\|\nabla(v+1)^{\frac{p_k}{2}}\|^{\frac{ra_0}{\theta_k}}_{L^2(\Omega)}\|(v+1)^{\frac{p_k}{2}}\|^{\frac{r}{\theta_k}(1-a_0)}_{L^s(\Omega)} + c_{33}\|(v+1)^{\frac{p_k}{2}}\|^{\frac{r}{\theta_k}}_{L^s(\Omega)}. \tag{3.50}$$

Due to $p_{k-1} = \frac{p_k}{2} s$ in (3.44), we can get from (3.47)

$$c_{32} \|(v+1)^{\frac{p_k}{2}}\|_{L^r(\Omega)}^{\frac{r}{\theta_k}} \leq c_{33} M_{k-1}^{\frac{(1-a_0)r}{\theta_k s}} \cdot \left[\int_\Omega |\nabla (v+1)^{\frac{p_k}{2}}|^2 \right]^{\frac{ra_0}{2\theta_k}} + c_{33} M_{k-1}^{\frac{r}{\theta_k s}} \tag{3.51}$$

for all $t \in (0, T_{\max})$, with

$$a_0 = \frac{\frac{n}{s} - \frac{n}{r}}{1 - \frac{n}{2} + \frac{n}{s}} \in (0,1). \tag{3.52}$$

Using Young's inequality on (3.51), we deduce from (3.48)

$$\begin{aligned} &\frac{d}{dt} \int_\Omega (v+1)^{p_k} + \frac{p_k(p_k-1)}{4} \int_\Omega (v+1)^{p_k-2} |\nabla v|^2 + p_k \int_\Omega (v+1)^{p_k} \\ &\leq c_{34} \left[p_k^2 M_{k-1}^{\frac{r(1-a_0)}{\theta_k s}} \right]^{\frac{2\theta_k}{2\theta_k - ra_0}} + c_{34} p_k^2 M_{k-1}^{\frac{r}{\theta_k s}} + (\lambda_2+1) p_k \int_\Omega (v+1)^{p_k} \text{ for all } t \in (0, T_{\max}), \end{aligned} \tag{3.53}$$

with some $c_{34} > 0$, where we have used the inequality (3.42), which ensures that

$$\frac{ra_0}{2\theta_k} \leq \frac{ra_0}{2\theta_0} \leq \frac{ra_0}{\frac{2q}{q-2}} = \frac{\frac{nr}{s} - n}{\frac{2q}{q-2} \cdot (1 - \frac{n}{2} + \frac{n}{s})} < 1 \text{ for all } k \in \mathbb{N}. \tag{3.54}$$

Using Young's inequality and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} (\lambda_2+1) p_k \int_\Omega (v+1)^{p_k} &\leq c_{35} p_k \left[\|\nabla (v+1)^{\frac{p_k}{2}}\|_{L^2(\Omega)}^{2a_1} \|(v+1)^{\frac{p_k}{2}}\|_{L^s(\Omega)}^{2(1-a_1)} + \|(v+1)^{\frac{p_k}{2}}\|_{L^s(\Omega)}^2 \right] \\ &= c_{35} p_k M_{k-1}^{\frac{2(1-a_1)}{s}} \cdot \left[\int_\Omega |\nabla (v+1)^{\frac{p_k}{2}}|^2 \right]^{a_1} + c_{35} p_k M_{k-1}^{\frac{2}{s}} \\ &\leq \varepsilon \int_\Omega |\nabla (v+1)^{\frac{p_k}{2}}|^2 + \left[\varepsilon^{-\frac{a_1}{1-a_1}} (c_{35} p_k)^{\frac{1}{1-a_1}} + c_{35} p_k \right] M_{k-1}^{\frac{2}{s}} \end{aligned} \tag{3.55}$$

for all $t \in (0, T_{\max})$ and any $\varepsilon > 0$, with

$$a_1 = \frac{\frac{n}{s} - \frac{n}{2}}{1 - \frac{n}{2} + \frac{n}{s}} \in (0,1). \tag{3.56}$$

Recalling (3.53), we can get

$$\begin{aligned} \frac{d}{dt} \int_\Omega (v+1)^{p_k} + \int_\Omega (v+1)^{p_k} \leq & c_{34} \left[p_k^2 M_{k-1}^{\frac{r(1-a_0)}{\theta_k s}} \right]^{\frac{2\theta_k}{2\theta_k - ra_0}} + c_{34} p_k^2 M_{k-1}^{\frac{r}{\theta_k s}} \\ &+ \left[\varepsilon^{-\frac{a_1}{1-a_1}} (c_{35} p_k)^{\frac{1}{1-a_1}} + c_{35} p_k \right] M_{k-1}^{\frac{2}{s}}. \end{aligned} \tag{3.57}$$

To simplify this, it is easy to see

$$\frac{2r(1-a_0)}{s(2\theta_k - ra_0)} \geq \max \left\{ \frac{r}{\theta_k s}, \frac{2}{s} \right\} \text{ for all } k \geq 1. \tag{3.58}$$

In fact, (3.46) implies that $\theta_k \leq \frac{r}{2}$, which guarantees (3.58). Furthermore, it is clear that

$$\frac{1}{1-a_1} < 2 < \frac{4\theta_k}{2\theta_k - ra_0} \leq \frac{4\theta_0}{2\theta_0 - ra_0} \tag{3.59}$$

for all $k \geq 1$ and $s \in (0,2)$ with s sufficiently closing to 2. Therefore, we conclude from (3.57) and (3.45)

$$\frac{d}{dt}\int_\Omega (v+1)^{p_k} + \int_\Omega (v+1)^{p_k} \leq c_{36}\tilde{c}^k M_{k-1}^{\frac{2r(1-a_0)}{s(2\theta_k - ra_0)}} \tag{3.60}$$

for all $t \in (0, T_{\max})$ and $k \geq 1$, with some $c_{36} > 0$ and $\tilde{c} = \left(\frac{2}{s}\right)^{\frac{4\theta_0}{2\theta_0 - ra_0}} > 1$. With an application of ODE comparison, we can deduce the following recursive inequality

$$M_k \leq \max\left\{\int_\Omega (v_0+1)^{p_k}, c_{36}\tilde{c}^k M_{k-1}^{\frac{2r(1-a_0)}{s(2\theta_k - ra_0)}}\right\} \text{ for all } k \geq 1. \tag{3.61}$$

Using the similar iterative process established in [21], we can find a constant $c_{37} > 0$ such that

$$\|v\|_{L^\infty(\Omega)} \leq \|(v+1)\|_{L^\infty(\Omega)} \leq c_{37} \text{ for all } t \in (0, T_{\max}). \tag{3.62}$$

From the $L^\infty-$boundedness of v, there exists $\lambda_3 > 0$ such that

$$u(\lambda_1 - \mu_1 u^{r_1 - 1} + av) \leq u(\lambda_3 - \mu_1 u^{r_1 - 1}). \tag{3.63}$$

By the same method as in the proof of $L^\infty-$boundedness of v, we claim that there exists $c_{38} > 0$

$$\|u\|_{L^\infty(\Omega)} \leq c_{38} \text{ for all } t \in (0, T_{\max}). \tag{3.64}$$

Thus we complete the proof of Theorem 1.1.

4. Conclusions and outlook

In this paper, we have studied a general Lotka-Volterra partial differential equations with indirect pursuit-evasion dynamics and cross-diffusion mechanisms. It has been proved that the boundedness of the solutions mainly depends on the intensity of cross-diffusion and intraspecific competition of the populations, as well as the dimensions of the space. Compared to previous results, the novelty of this paper is that our boundedness conditions are more generalized, which will be more consistent with the real biological environment.

From a purely mathematical perspective, the boundedness result developed in this paper may not directly reflect some biological implications. However, such conclusion is essential in studying the persistence and long time stability of populations. In a sense, these results may indicate that the system does not lead to non-biological phenomena, such as blow-up in finite time.

Some other interesting problems related to system (1.1) are also worth exploring further, such as the qualitative analysis of system (1.1), global existence and boundedness of classical solution to fully parabolic version of system (1.1) and so on. We will consider these issues in future work.

Acknowledgments

We would like to thank the anonymous referees for many useful comments and suggestions that greatly improve the work. This work was partially supported by NSFC Grant 11901500, Scientific and Technological Key Projects of Henan Province NO. 232102310227, NO. 222102320425 and Nanhu Scholars Program for Young Scholars of XYNU NO. 2020017.

Conflict of interest

The authors declare that there is no conflicts of interest regarding the publication of this paper.

References

1. E. Keller, L. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, **26** (1970), 399–415. https://doi.org/10.1016/0022-5193(70)90092-5
2. T. Ciéslak, M. Winkler, Finite-time blow-up in a quasilinear system of chemotaxis, *Nonlinearity*, **21** (2008), 1057–1076. https://doi.org/10.1088/0951-7715/21/5/009
3. D. Horstmann, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences I, *Jahresber. DMV*, **105** (2003), 103–165.
4. R. Kowalczyk, Preventing blow-up in a chemotaxis model, *J. Math. Anal. Appl.*, **305** (2005), 566–588. https://doi.org/10.1016/j.jmaa.2004.12.009
5. K. Osaki, A. Yagi, Finite dimensional attractor for one-dimensional Keller-Segel equations, *Funkcial. Ekvac.*, **44** (2001), 441–469.
6. H. Gajewski, K. Zacharias, Global behavior of a reaction-diffusion system modelling chemotaxis, *Math. Nachr.*, **195** (1998), 77–114. https://doi.org/10.1002/mana.19981950106
7. D. Horstmann, G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, *Eur. J. Appl. Math.*, **12** (2001), 159–177. https://doi.org/10.1017/S0956792501004363
8. T. Nagai, T. Senba, K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, *Funkcial. Ekvac.*, **40** (1997), 411–433.
9. T. Senba, T. Suzuki, Parabolic system of chemotaxis: Blowup in a finite and the infinite time, *Methods Appl. Anal.*, **8** (2001), 349–367. https://doi.org/10.4310/MAA.2001.v8.n2.a9
10. M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *J. Differ. Equations*, **248** (2010), 2889–2905. https://doi.org/10.1016/j.jde.2010.02.008
11. M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.*, **100** (2013), 748–767. https://doi.org/10.1016/j.matpur.2013.01.020
12. D. Liu, Y. Tao, Boundedness in a chemotaxis system with nonlinear signal production, *Appl. Math. J. Chin. Univ. Ser. B*, **31** (2016), 379–388. https://doi.org/10.1007/s11766-016-3386-z
13. M. Winkler, A critical blow-up exponent in a chemotaxis system with nonlinear signal production, *Nonlinearity*, **31** (2018), 2031–2056. https://doi.org/10.1088/1361-6544/aaaa0e
14. J. I. Tello, M. Winkler, Chemotaxis system with logistic source, *Comm. Partial Differ. Equations*, **32** (2007), 849–877. https://doi.org/10.1080/03605300701319003

15. Z. Wang, T. Xiang, A class of chemotaxis systems with growth source and nonlinear secretion, preprient, arXiv: 1510.07204.
16. M. Winkler, Chemotaxis with logistic source: very weak global solutions and boundedness properties, *J. Math. Anal. Appl.*, **348** (2008), 708–729. https://doi.org/10.1016/j.jmaa.2008.07.071
17. G. Viglialoro, Very weak global solutions to a parabolic-parabolic chemotaxis-system with logistic source, *J. Math. Anal. Appl.*, **439** (2016), 197–212. https://doi.org/10.1016/j.jmaa.2016.02.069
18. M. Winkler, The role of superlinear damping in the construction of solutions to drift-diffusion problems with initial data in L^1, *Adv. Nonlinear Anal.*, **9** (2020), 526–566. https://doi.org/10.1515/anona-2020-0013
19. K. Painter, T. Hillen, Volume-filling and quorum-sensing in models for chemosensitive movement, *Can. Appl. Math. Q.*, **10** (2002), 501–543.
20. M. Winkler, Does a 'volume-filling effect' always prevent chemotactic collapse? *Math, Methods Appl. Sci.*, **33** (2010), 12–24. https://doi.org/10.1002/mma.1146
21. Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, *J. Differ. Equations*, **252** (2012), 692–715. https://doi.org/10.1016/j.jde.2011.08.019
22. K. Lin, C. Mu, H. Zhong, A blow-up result for a quasilinear chemotaxis system with logistic source in higher dimensions, *J. Math. Anal. Appl.*, **464** (2018), 435–455. https://doi.org/10.1016/j.jmaa.2018.04.015
23. X. Cao, Y. Tao, Boundedness and stabilization enforced by mild saturation of taxis in a producer-scrounger model, *Nonlinear Anal. Real World Appl.*, **57** (2021), 103189. https://doi.org/10.1016/j.nonrwa.2020.103189
24. X. Li, Z. Xiang, Boundedness in quasilinear Keller-Segel equations with nonlinear sensitivity and logistic source, *Discrete Contin. Dyn. Syst.*, **35** (2015), 3503–3531. https://doi.org/10.3934/dcds.2015.35.3503
25. L. Wang, C. Mu, P. Zheng, On a quasilinear parabolic-elliptic chemotaxis system with logistic source, *J. Differ. Equations*, **256** (2014), 1847–1872. https://doi.org/10.1016/j.jde.2013.12.007
26. L. Wang, Y. Li, C. Mu, Boundedness in a parabolic-parabolic quasilinear chemotaxis system with logistic source, *Discrete Contin. Dyn. Syst.*, **34** (2014), 789–802. https://doi.org/10.3934/dcds.2014.34.789
27. Q. Zhang, Y. Li, Boundedness in a quasilinear fully parabolic Keller-Segel system with logistic source, *Z. Angew. Math. Phys.*, **66** (2015), 2473–2484. https://doi.org/10.1007/s00033-015-0532-z
28. C. Wang, L. Zhao, X. Zhu, A blow-up result for attraction-repulsion system with nonlinear signal production and generalized logistic source, *J. Math. Anal. Appl.*, **518** (2023), 126679. https://doi.org/10.1016/j.jmaa.2022.126679
29. C. Wang, Y. Yang, Boundedness criteria for the quasilinear attraction-repulsion chemotaxis system with nonlinear signal production and logistic source, *Electron. Res. Arch.*, **31** (2023), 299–318. https://doi.org/10.3934/era.2023015

30. C. Wang, Y. Zhu, X. Zhu, Long time behavior of the solution to a chemotaxis system with nonlinear indirect signal production and logistic source, *Electron. J. Qual. Theory Differ. Equations*, **11** (2023), 1–21. https:// doi.org/10.14232/ejqtde.2023.1.11
31. Y. Tao, M. Winkler, A fully cross-diffusive two-component evolution system: Existence and qualitative analysis via entropy-consistent thin-film-type approximation, *J. Funct. Anal.*, **281** (2021), 109069. https://doi.org/10.1016/j.jfa.2021.109069
32. Y. Tao, M. Winkler, Existence theory and qualitative analysis for a fully cross-diffusive predator-prey system, *SIAM J. Math. Anal.*, **54** (2022), 4806–4864. https://doi.org/10.1137/21M1449841
33. A. Chakraborty, M. Singh, D. Lucy, P. Ridland, Predator-prey model with prey-taxis and diffusion, *Math. Comput. Modell.*, **46** (2007), 482–498. https://doi.org/10.1016/j.mcm.2006.10.010
34. J. I. Tello, D. Wrzosek, Predator-prey model with diffusion and indirect prey-taxis, *Math. Models Methods Appl. Sci.*, **26** (2016), 2129–2162. https://doi.org/10.1142/s0218202516400108
35. I. Ahn, C. Yoon, Global solvability of prey-predator models with indirect predator-taxis, *Z. Angew. Math. Phys.*, 72 (2021), 1–20. https://doi.org/10.1007/s00033-020-01461-y
36. H. Jin, Z. Wang, Global dynamics and spatio-temporal patterns of predator-prey systems with density-dependent motion, *Eur. J. Appl. Math.*, **32** (2021), 652–682. https://doi.org/10.1017/s0956792520000248
37. G. Ren, Y. Shi, Global boundedness and stability of solutions for prey-taxis model with handling and searching predators, *Nonlinear Anal. Real World Appl.*, **60** (2021), 103306. https://doi.org/10.1016/j.nonrwa.2021.103306
38. T. Goudon, L. Urrutia, Analysis of kinetic and macroscopic models of pursuit-evasion dynamics, *Commun. Math. Sci.*, **14** (2016), 2253–2286. https://doi.org/10.4310/CMS.2016.v14.n8.a7
39. P. Amorim, B. Telch, L. M. Villada, A reaction-diffusion predator-prey model with pursuit, evasion, and nonlocal sensing, *Math. Biosci. Eng.*, **16** (2019), 5114–5145. https://doi.org/10.3934/mbe.2019257
40. G. Li, Y. Tao, M. Winkler, Large time behavior in a predator-prey system with indirect pursuit-evasion interaction, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 4383. https://doi.org/10.3934/dcdsb.2020102
41. J. Zheng, P. Zhang, Blow-up prevention by logistic source an N-dimensional parabolic-elliptic predator-prey system with indirect pursuit-evasion interaction, *J. Math. Anal. Appl.*, **519**(2023), 126741. https://doi.org/10.1016/j.jmaa.2022.126741
42. C. Liu, B. Liu, Boundedness and asymptotic behavior in a predator-prey model with indirect pursuit-evasion interaction, *Discrete Contin. Dyn. Syst. Ser. B*,**27** (2022), 4855–4874. https://doi.org/10.3934/dcdsb.2021255
43. D. Qi, Y. Ke, Large time behavior in a predator-prey system with indirect pursuit-evasion interaction, *Discrete Contin. Dyn. Syst. Ser. B*, **27** (2022), 4531–4549. https://doi.org/10.3934/dcdsb.2021240
44. P. Amorim, B. Telch, A chemotaxis predator-prey model with indirect pursuit-evasion dynamics and parabolic signal, *J. Math. Anal. Appl.*, **500** (2021), 125128. https://doi.org/10.1016/j.jmaa.2021.125128

45. B. Telch, Global boundedness in a chemotaxis quasilinear parabolic predator-prey system with pursuit-evasion, *Nonlinear Anal. Real World Appl.*, **59** (2021), 103269. https://doi.org/10.1016/j.nonrwa.2020.103269

46. B. Telch, A parabolic-quasilinear predator-prey model under pursuit-evasion dynamics, *J. Math. Anal. Appl.*, **514** (2022), 126276. https://doi.org/10.1016/j.jmaa.2022.126276

47. Y. Wang, A quasilinear attraction-repulsion chemotaxis system of parabolic-elliptic type with logistic source, *J. Math. Anal. Appl.*, **441** (2016), 259–292. https://doi.org/10.1016/j.jmaa.2016.03.061

48. M. Winkler, K. C. Djie, Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect, *Nonlinear Anal.*, **72** (2010), 1044–1064. https://doi.org/10.1016/j.na.2009.07.045

49. L. Nirenberg, On elliptic partial differential equations, *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser.*, **13** (1959), 115–162.

50. H. Brézis, W. A. Strauss, Semi-linear second-order elliptic equations in L^1, *J. Math. Soc. Jpn.*, **25** (1973), 565–590. https://doi.org/10.2969/jmsj/02540565

AIMS Press