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Research article

Existence and multiplicity of solutions for fractional $p(x)$-Kirchhoff-type problems

Zhiwei Hao* **and Huiqin Zheng**

School of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan 411201, China

* **Correspondence:** Email: haozhiwei@hnust.edu.cn.

Abstract: In this paper, we deal with the existence and multiplicity of solutions for fractional $p(x)$-Kirchhoff-type problems as follows:

$$\begin{cases} M\Big(\int_Q \frac{1}{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\Big)(-\Delta_{p(x)})^s v(x) & \\ \quad = \lambda |v(x)|^{r(x)-2} v(x), & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^d \backslash \Omega, \end{cases}$$

where $(-\triangle_{p(x)})^s$ is the fractional $p(x)$-Laplacian. Different from the previous ones which have recently appeared, we weaken the condition of M and obtain the existence and multiplicity of solutions via the symmetric mountain pass theorem and the theory of the fractional Sobolev space with variable exponents.

Keywords: the symmetryic mountain pass theorem; Kirchhoff-type problem; fractional $p(x)$-Laplacian; fractional Sobolev space with variable exponents

1. Introduction and main results

In [1], Kirchhoff studied a stationary version of the equation

$$\rho_0 \frac{\partial^2 v}{\partial t^2} - \Big(\frac{p_1}{h_1} + \frac{E_0}{2L}\int_0^L \Big|\frac{\partial v}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 v}{\partial t^2} = 0, \tag{1.1}$$

where ρ_0, p_1, h_1, L and E_0 are constants. Such equation extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the string during the vibrations. It is worthwhile to note that the Eq (1.1) received much attention only after Lions [2] put forward an abstract framework to the Eq (1.1). After this work, various equations of Kirchhoff-type have been studied extensively. For instance, many researchers have studied the Kirchhoff-type equations involving the p-Laplacian, which can be found in [3–6], $p(x)$-Laplacian (see, for example, [7–11]) and fractional $p(x)$-Laplacian (see [12–15]).

Recently, lots of researchers have been interested in the Kirchhoff-type equations involving the p-Laplacian (see [16, 17]). In [5], Liu proved the existence of infinite solutions for the p-Kirchhoff-type problems via the fountain theorem. Since the infimum of its principal eigenvalue is zero, the p-Laplacian is not homogenous, and generally it does not have the alleged first eigenvalue. Hence, more and more attention has been given to partial differential equations with nonstandard growth conditions. Dai and Hao [18] investigated the existence and multiplicity of solutions to Kirchhoff-type problems associated with the $p(x)$-Laplacian via a direct variational approach. The $p(x)$-Laplacian has more complex nonlinear properties than the p-Laplacian, and we can refer to [11, 19] for more details about it.

In the last few years, many researchers have tended to focus on the fractional $p(x)$-Kirchhoff-type problems. Kaufmann, Rossi and Vidal [20] introduced the fractional $p(x)$-Laplacian $\triangle_{p(x)}\ v = \text{div}(|\bigtriangledown v|^{p(x)-2})$ (see, for example, [21,22]). In [23], the authors investigated the fractional $p(x)$-Laplace operator and associated fundamental properties about new fractional Sobolev spaces with variable exponents. In [14], by using dint of the variational methods, Azroul et al. investigated the existence of solutions for the Kirchhoff-type problems involving fractional $p(x)$-Laplacian as follows:

$$(P_M^s)\begin{cases} M\Big(\int_Q \frac{1}{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\Big)(-\Delta_{p(x)})^s v(x) & \\ = \lambda |v(x)|^{r(x)-2} v(x), & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^d \backslash \Omega, \end{cases}$$

where $M \in Q_1$, i.e., M satisfies the following: there exist $0 < a_1 \leq a_2$ and $\beta > 1$ such that

$$a_1 \tau^{\beta-1} \leq M(\tau) \leq a_2 \tau^{\beta-1} \quad \text{for all } \tau \in \mathbb{R}^d.$$

In [24], applying the symmetric mountain pass theorem, Azroul, Benkirane and Shimi resolved the existence solutions to the following Kirchhoff-type problems involving fractional $p(x,\cdot)$-Laplacian in $\mathbb{R}^d$:

$$\begin{cases} M\Big(\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\Big)(-\Delta_{p(x,\cdot)})^s v(x) & \\ \quad + |v|^{\bar{p}(x)-2} v = f(x,v), & \text{in } \mathbb{R}^d, \\ v \in W^{s,p(x,y)}(\mathbb{R}^d), & \end{cases}$$

where $M \in Q_2$, i.e., the continuous function $M : \mathbb{R}_0^+ := [0, +\infty) \to \mathbb{R}_0^+$ satisfies the following conditions:

(M_1): Let $\epsilon_0 > 0$ and $\alpha \in (1, (p_s^*)_-/p_+)$. Suppose that

$$kM(k) \leq \alpha \widehat{M}(k) \quad \text{for all } k \geq \epsilon_0, \tag{1.2}$$

where

$$\widehat{M}(k) = \int_0^k M(\epsilon_0) d\epsilon_0$$

and $(p_s^*)_-$, p_+ and p_- will be introduced in Section 2.

(M_2) : let $\epsilon > 0$. Suppose $l = l(\epsilon) > 0$ such that

$$M(k) \geq l \quad \text{for all} \quad k \geq \epsilon. \tag{1.3}$$

By comparison their definitions, we find that the condition of Q_2 is weaker than Q_1. Hence the spontaneous question is to show which results we can obtain if we replace $M \in Q_1$ by $M \in Q_2$ in [14].

Inspired by the fractional Sobolev spaces with variable exponents (for more details see [23]) and the papers referred to above, we aim to deal with the existence and multiplicity solutions to the fractional $p(x)$-Kirchhoff-type problem (P_M^s) which is introduced in [14], where

- $M \in Q_2$ and λ is a positive real constant;
- $Q := \mathbb{R}^{2d} \backslash (\Omega^c \times \Omega^c)$, where $\Omega \in \mathbb{R}^d$ is a Lipschitz bounded open domain and $\Omega^c = \mathbb{R}^d \setminus \Omega$, $d \geq 3$;
- the continuous functions $r : \overline{\Omega} \to (1, +\infty)$ and $p : \overline{Q} \to (1, +\infty)$ are bounded;
- for $s \in (0, 1)$, the operator $(-\Delta_{p(x)})^s$ is the following fractional $p(x)$-Laplacian

$$(-\Delta_{p(x)})^s v(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{|v(x) - v(y)|^{p(x)-2}(v(x) - v(y))}{|x - y|^{d+sp(x,y)}} dy \quad \text{for all } x \in \mathbb{R}^d,$$

where p.v. stands for Cauchy principle value for brevity.

We introduce our main conclusions and results as follows:

Theorem 1.1. *For the continuous function $r : \overline{\Omega} \to (1, +\infty)$, let*

$$1 < r_- := \inf_{x \in \overline{\Omega}} r(x) \leq r(x) < r_+ := \sup_{x \in \overline{\Omega}} r(x) < p_s^*(x). \tag{1.4}$$

Suppose that M satisfies (1.3) and $r \in C_+(\overline{\Omega})$ such that

$$1 < r(x) \leq r_+ < p_- \quad \textit{for all} \quad x \in \overline{\Omega}, \tag{1.5}$$

then problem (P_M^s) has a nontrivial weak solution, if there is $\lambda_1 > 0$ such that

$$\lambda_1 < \lambda < +\infty.$$

Theorem 1.2. *Suppose that $p \in C_+(\overline{Q})$ is symmetric with $sp_+ < d$ and $s \in (0, 1)$. Let $M \in Q_2$, $r \in C_+(\overline{Q})$ with*

$$\alpha p_+ < r_-, \tag{1.6}$$

$$r_+ < p_s^*(x) \quad \textit{for all} \quad x \in \overline{\Omega}. \tag{1.7}$$

Then problem (P_M^s) has a sequence $\{u_n\}_n$ of nontrivial solutions, if there is a constant $c_1 > 0$ such that

$$0 < \lambda < c_1.$$

Remark 1.3. *We discuss the $p(x)$-Kirchhoff-type problem (P_M^s) in two situations: if r satisfies (1.5), we apply the direct variational methods; we utilize the symmetric mountain pass theorem if r satisfies (1.6).*

The paper is organized as follows: we introduce the fractional Sobolev spaces with variable exponents and some necessary properties of variable Lebesgue spaces in Section 2. In the end, Section 3 gives the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

In this section, we present some useful properties of the fractional Sobolev spaces with variable exponents and the generalized Lebesgue spaces. We can refer to [17, 18, 20, 25, 26] and the references therein for more details.

We always suppose that Ω is a Lipschitz bounded open domain in $\mathbb{R}^d$ and $\overline{\Omega}$ is a closure of Ω. Consider a continuous function $r:\overline{\Omega}\to(1,+\infty)$ and set

$$C_+(\overline{\Omega})=\{r\in(\overline{\Omega})^c: r(x)>1 \quad \text{for all} \quad x\in\overline{\Omega}\}.$$

For a measurable function v and any $r\in C_+(\overline{\Omega})$, the modular functional $\rho_{r(x)}(v)$ is defined by

$$\rho_{r(x)}(v)=\int_\Omega |v(x)|^{r(x)}dx$$

The variable Lebesgue space is defined as

$$L^{r(x)}:=L^{r(x)}(\Omega)=\{v:\rho_{r(x)}(\lambda v)<+\infty\}$$

equipped with the norm

$$\|v\|_{L^{r(x)}}=\inf\{\lambda>0:\rho_{r(x)}(f/\lambda)\le 1\}.$$

Suppose $r\in C_+(\overline{\Omega})$ such that $\frac{1}{r(x)}+\frac{1}{r'(x)}=1$, where $r'(x)$ is the conjugate exponent of $r(x)$. Then the Hölder inequality is as follows:

Lemma 2.1 ([20]). *Let $v\in L^{r(x)}$ and $u\in L^{r'(x)}$. There exists a positive constant c such that*

$$\Big|\int_\Omega v(x)u(x)dx\Big|\le c\|v\|_{L^{r(x)}}\|u\|_{L^{r'(x)}}.$$

Proposition 2.2 ([27]). *Let $v\in L^{r(x)}$. The following properties hold:*

(i) $\|v\|_{L^{r(x)}}=1\ (resp.>1,<1)\Leftrightarrow\rho_{r(x)}(v)=1\ (resp.>1,<1)$;

(ii) $\|v\|_{L^{r(x)}}<1\Rightarrow\|v\|^{r_+}_{L^{r(x)}}\le\rho_{r(x)}(v)\le\|v\|^{r_-}_{L^{r(x)}}$;

(iii) $\|v\|_{L^{r(x)}}>1\Rightarrow\|v\|^{r_-}_{L^{r(x)}}\le\rho_{r(x)}(v)\le\|v\|^{r_+}_{L^{r(x)}}$;

(iv) $\lim_{k\to+\infty}\|v_k-v\|_{L^{r(x)}}=0\Leftrightarrow\lim_{k\to+\infty}\rho_{r(x)}(v_k-v)=0.$

We show the following proposition, which is from Theorems 1.6 and 1.10 in [26].

Proposition 2.3. *Suppose $1<r_-\le r(x)\le r_+<\infty$; then, $(L^{r(x)},\|\cdot\|_{L^{r(x)}})$ is a reflexive uniformly convex and separable Banach space.*

Let the continuous function $p:\overline{Q}\to(1,+\infty)$ be bounded. We set

$$1<p_-:=\inf_{(x,y)\in\overline{Q}}p(x,y)\le p(x,y)\le p_+:=\sup_{(x,y)\in\overline{Q}}p(x,y) \tag{2.1}$$

and p is symmetric, if $p(x, y)$ satisfies the following

$$p(x, y) = p(y, x) \quad \text{for all} \quad (x, y) \in \overline{Q}. \tag{2.2}$$

We assume that

$$\bar{p}(x) := p(x, x) \quad \text{for all} \quad x = y.$$

Throughout this paper, $s \in (0, 1)$ and the fractional Sobolev space with variable exponents is defined in [14] given by

$$S = \left\{ \begin{array}{l} v : \mathbb{R}^d \to \mathbb{R} \text{ measurable such that } v|_\Omega \in L^{\bar{p}(x)} \text{ with} \\ \int_Q \frac{|v(x) - v(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{d+sp(x,y)}} dxdy < +\infty \text{ for some } \lambda > 0 \end{array} \right\}.$$

The norm of S is as follows

$$\|v\|_S = \|v\|_{L^{\bar{p}(x)}} + [v]_S,$$

where $[v]_S$ is defined by

$$[v]_S = [v]_{s,p(x,y)}(Q) = \inf \left\{ \lambda > 0 : \int_Q \frac{|v(x) - v(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{d+sp(x,y)}} dxdy \leq 1 \right\}.$$

Also, $(S, \|\cdot\|_S)$ is a separable reflexive Banach space which is introduced in [14].

Now, we denote the linear subspace of S given by

$$S_0 = \left\{ v \in S : v = 0 \quad a.e. \text{ in } \mathbb{R}^d \backslash \Omega \right\};$$

the modular norm is as follows

$$\|v\|_{S_0} := [v]_S = \inf \left\{ \lambda > 0 : \int_Q \frac{|v(x) - v(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{d+sp(x,y)}} dxdy \leq 1 \right\}.$$

We know that $(S_0, \|\cdot\|_{S_0})$ is a separable, reflexive and uniformly convex Banach space (see Lemma 2.3 in [14]).

We denote the modular $\rho_{p(x,y)} : S_0 \to \mathbb{R}$ by

$$\rho_{p(x,y)}(v) = \int_Q \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{d+sp(x,y)}} dxdy,$$

where

$$\|v\|_{\rho_{p(x,y)}(v)} = \inf \left\{ \lambda > 0 : \rho_{p(x,y)}(v/\lambda) \leq 1 \right\} = [v]_S.$$

Similarly to Proposition 2.1, $\rho_{p(x,y)}$ has the following property:

Lemma 2.4 ([13]). *Let p satisfy (2.1) and $s \in (0, 1)$. Suppose $v \in S_0$; we can obtain*

(i) $\|v\|_{S_0} \leq 1 \Rightarrow \|v\|_{S_0}^{p_+} \leq \rho_{p(x,y)}(v) \leq \|v\|_{S_0}^{p_-}$;

(ii) $\|v\|_{S_0} \geq 1 \Rightarrow \|v\|_{S_0}^{p_-} \leq \rho_{p(x,y)}(v) \leq \|v\|_{S_0}^{p_+}$.

We will introduce a continuous compact embedding theorem as follows.

Theorem 2.5 ([13]). *Let $s \in (0,1)$ and p satisfy* (2.1) *and* (2.2) *with $sp_+ < d$. If r satisfies* (1.4)*, i.e.,*

$$1 < r_- \leq r(x) < p_s^*(x) := \frac{d\bar{p}(x)}{d - s\bar{p}(x)} \quad \text{for all} \quad x \in \overline{\Omega}.$$

Then we have

$$\|v\|_{L^{r(x)}} \leq c\|v\|_S \quad \text{for any} \quad v \in S,$$

where c is a positive constant depending on p, s, r, d and Ω. In other words, the embedding $S \hookrightarrow L^{r(x)}$ is continuous and this embedding is compact.

Remark 2.6. *(i) Theorem* 2.5 *still holds if we replace S by S_0.*

(ii) Since $1 < r_- \leq r(x) < p_s^*(x)$, then according to Theorem 2.5*, we can get that $\|\cdot\|_{S_0}$ and $\|\cdot\|_S$ are equivalent on S_0.*

We need to introduce the functional $\mathcal{L} : S_0 \to S_0^*$ defined by

$$\langle \mathcal{L}(v), \varphi \rangle = \int_Q \frac{|v(x) - v(y)|^{p(x,y)-2}(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{d+sp(x,y)}} dxdy$$

for all $\varphi \in S_0$, where S_0^* is the dual space of S_0.

Lemma 2.7 ([23]). *Suppose that p satisfies* (2.1)*,* (2.2) *and $s \in (0,1)$. Then the following results hold:*

(i) $\mathcal{L}$ is a bounded and strictly monotone operator;

(ii) $\mathcal{L}$ is a homeomorphism;

(iii) $\mathcal{L}$ is a mapping of type (S_+), i.e., $v_n \to v$ in S_0, if $v_n \rightharpoonup v$ in S_0 and $\mathcal{L}$

$$\limsup_{n \to +\infty} \langle \mathcal{L}(v_n) - \mathcal{L}(v), v_n - v \rangle \leq 0.$$

3. Proof of the main results

Definition 3.1. *We say that $v \in S_0$ is a weak solution of problem (P_M^s), if*

$$M(\sigma_{p(x,y)}(v)) \int_Q \frac{|v(x) - v(y)|^{p(x,y)-2}(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{d+sp(x,y)}} dxdy - \lambda \int_\Omega |v(x)|^{r(x)-2} v(x)\varphi(x) dx = 0$$

for all $\varphi \in S_0$, where

$$\sigma_{p(x,y)}(v) = \int_Q \frac{1}{p(x,y)} \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{d+sp(x,y)}} dxdy.$$

For the purpose of formulating the variational method of problem (P_M^s), we present the functional $I_\lambda : S_0 \to \mathbb{R}$ given by

$$\begin{aligned} I_\lambda(v) &= \widehat{M}\Big(\int_Q \frac{1}{p(x,y)} \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{d+sp(x,y)}} dxdy\Big) - \lambda \int_\Omega \frac{1}{r(x)} |v(x)|^{r(x)} dx \\ &= \widehat{M}(\sigma_{p(x,y)}(v)) - \lambda \int_\Omega \frac{1}{r(x)} |v(x)|^{r(x)} dx. \end{aligned}$$

It is not tough to demonstrate that $I_\lambda \in C^1(S_0, \mathbb{R})$ and I_λ is well defined. Moreover, for all $v, \varphi \in S_0$, the Gateaux derivative of I_λ is introduced by

$$\begin{aligned}\langle I_\lambda'(v), \varphi\rangle &= M(\sigma_{p(x,y)})(v)\int_Q \frac{|\, v(x)-v(y)\,|^{p(x,y)-2}\,(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+sp(x,y)}}dxdy \\ &- \lambda\int_\Omega |v(x)|^{r(x)-2}v(x)\varphi(x)dx = 0.\end{aligned}$$

Thus, the weak solutions of (P_M^s) correspond to the critical points of I_λ.

To prove Theorem 1.1, we need to introduce the next result:

Lemma 3.2. *For $\lambda \in \mathbb{R}$, the functional I_λ is coercive on S_0.*

Proof. We assume $\|v\|_{S_0} > 1$. By (M_2) we obtain that

$$I_\lambda(v) \geq l\sigma_{p(x,y)}(v) - \frac{\lambda}{r_-}\rho_{p(x,y)}(v).$$

According to Remark 2.6 (i), we have

$$I_\lambda(v) \geq \frac{l}{p_+}\|v\|_{S_0}^{p_-} - \frac{\lambda c^{r_-}}{r_-}\min\{\|v\|_{S_0}^{r_-},\ \|v\|_{S_0}^{r_+}\},$$

where c is a positive constant depending on $d,\ s,\ p,\ r$ and Ω. It follows from (1.5) that

$$I_\lambda(v) \to \infty, \qquad as\ \ \|v\|_{S_0} \to \infty.$$

Next, we prove Theorem 1.1.

Proof of Theorem 1.1: According to Lemma 3.2, we know that I_λ is coercive on S_0. Moreover, I_λ is weakly lower semi-continuous on S_0. Applying Theorem 1.2 in [28], we find that there exists $\bar{v}_1 \in S_0$ which is a global minimizer of I_λ; thus, the problem (P_M^s) has a weak solution.

Now, we claim that the weak solution $\bar{v}_1$ is nontrivial for all λ large enough. Indeed, let $\delta_0 > 1$ and $|\Omega_1| > 0$, where Ω_1 is an open subset of Ω. Suppose $\eta_0 \in C_0^\infty(\overline{\Omega})$ such that $\eta_0(x)$ satisfies

$$\begin{cases}\eta_0(x) = \delta_0 & \text{for all } \ x \in \overline{\Omega}_1, \\ 0 \leq \eta_0(x) \leq \delta_0 & \text{for all } \ x \in \Omega \setminus \Omega_1.\end{cases}$$

Then we have that

$$\begin{aligned}I_\lambda(\eta_0) &= \widehat{M}(\sigma_{p(x,y)}(\eta_0)) - \lambda\int_\Omega \frac{1}{r(x)}|\eta_0(x)|^{r(x)}dx \\ &\leq c_3 - \frac{\lambda}{r_+}\int_\Omega |\eta_0(x)|^{r(x)}dx \leq c_3 - \frac{\lambda}{r_+}\delta_0^{r_-}|\Omega_1|,\end{aligned}$$

where c_3 is a constant. Therefore, we get

$$I_\lambda(\eta_0) < 0 \ \text{ for all } \ \lambda \in (\lambda_1, +\infty),$$

if the nonnegative λ_1 is large enough. The proof is now complete.

We say that I_λ satisfies $(Ce)_c$-condition for any $c \in \mathbb{R}$ if every sequence $\{v_n\}$ such that

$$I_\lambda(v_n) \to c, \qquad \|I'_\lambda(v_n)\|_{S_0^*}(1 + \|v_n\|_{S_0}) \to 0$$

has a strongly convergent subsequence in S_0. In order to prove Theorem 1.2, we need the symmetric mountain pass theorem as follows.

Theorem 3.3 ([29, 30]). *For the infinite dimensional Banach space S, we define*

$$S = \bigoplus_{j=1}^{2} S_j,$$

where S_2 is finite dimensional. Suppose $I \in C^1(S, \mathbb{R})$, if I satisfies the following
(1) $I(0) = 0$, $I(-v) = I(v)$ for all $v \in S$;
(2) for all $c > 0$, I satisfies $(Ce)_c$-condition;
(3) suppose constants ρ, a are positive, and we have $I|_{\partial B_\rho \cap Z} \geq a$;
(4) for each finite dimensional subspace $\tilde{S} \subset S$, we obtain $I(v) \leq 0$ on $\tilde{S} \backslash B_r$, if $r = r(\tilde{S})$ is a positive constant.

Then I possesses an unbounded sequence of critical values.

The following result shows that the functional I_λ satisfies the geometrical condition of the mountain pass.

Lemma 3.4. *Let $c_1 > 0$ and $v \in S_0$ with $\|v\|_{S_0} = \rho > 0$. Then for each $\lambda \in (0, c_1)$, we can choose $a > 0$ such that $I_\lambda|_{\partial B_\rho \cap Z} \geq a$.*

Proof. Suppose $v \in S_0$ and $\rho \in (0, 1)$ such that $\|u\|_{S_0} = \rho$. It follows from Theorem 2.5, (M_1), (1.6) and (1.7) that

$$\begin{aligned} I_\lambda(v) &= \widehat{M}(\sigma_{p(x,y)}(v)) - \lambda \int_\Omega \frac{1}{r(x)} |v(x)|^{r(x)} dx \\ &\geq \widehat{M}(1)(\sigma_{p(x,y)}(v))^\alpha - \frac{\lambda}{r_-} \int_\Omega |v(x)|^{r(x)} dx \\ &\geq \frac{\widehat{M}(1)}{(p_+)^\alpha} (\rho_{p(x,y)}(v))^\alpha - \frac{\lambda}{r_-} \rho_{r(x)}(v) \\ &\geq \frac{\widehat{M}(1)}{(p_+)^\alpha} \|v\|_{S_0}^{\alpha p_+} - \frac{\lambda}{r_-} \|v\|_{L^{r(x)}}^{r_+} \\ &\geq \frac{\widehat{M}(1)}{(p_+)^\alpha} \|v\|_{S_0}^{\alpha p_+} - \frac{\lambda c^{r_+}}{r_-} \|v\|_{S_0}^{r_+} \\ &\geq \rho^{\alpha p_+} \left(\frac{\widehat{M}(1)}{(p_+)^\alpha} - \frac{\lambda c^{r_+}}{r_-} \rho^{r_+ - \alpha p_+} \right). \end{aligned}$$

Thus, choosing ρ even smaller, we have

$$I_\lambda(v) > 0,$$

since $\alpha p_+ < r_- < r_+$.

Lemma 3.5. *For each finite dimensional subspace $\widetilde{S} \subset S_0$, $v \in \widetilde{S}$ and $\lambda \in \mathbb{R}$, there exists $r = r(\widetilde{S}) > 0$ such that*

$$I_\lambda(v) \le 0,$$

where $\|v\|_{S_0} \ge r$.

Proof. Suppose $\phi \in C_0^\infty(\Omega)$ with $\phi > 0$. According to (M_1), one must have

$$\widehat{M}(k) \le \widehat{M}(1)k^\alpha \text{ for all } k \ge 1. \tag{3.1}$$

Therefore, by (3.1), we have

$$\begin{aligned} I_\lambda(t\phi) &= \widehat{M}(\sigma_{p(x,y)}(t\phi)) - \lambda \int_\Omega \frac{1}{r(x)}|t\phi|^{r(x)}dx \\ &\le \widehat{M}(1)t^{\alpha p_+}(\sigma_{p(x,y)}(\phi))^\alpha - \frac{\lambda}{r_+}\int_\Omega |t\phi|^{r(x)}dx \\ &\le \frac{\widehat{M}(1)}{(p_-)^\alpha}t^{\alpha p_+}(\rho_{p(x,y)}(\phi))^\alpha - t^{r_-}\frac{\lambda}{r_+}\int_\Omega |\phi|^{r(x)}dx. \end{aligned}$$

It follows from (1.6) that

$$\lim_{t\to\infty} I_\lambda(t\phi) = -\infty.$$

Thus, there exists a large $t > 1$ such that

$$I_\lambda(v) \le 0.$$

The statement holds.

Lemma 3.6. *The functional I_λ satisfies condition $(Ce)_c$ in S_0.*

Proof. For all $c \in \mathbb{R}$, suppose a sequence $\{v_n\} \subset S_0$ such that

$$I_\lambda(v_n) \xrightarrow[n\to+\infty]{} c_2,$$

$$I'_\lambda(v_n) \xrightarrow[n\to+\infty]{} 0. \tag{3.2}$$

First, we claim that $\{v_n\} \subset S_0$ is bounded. Arguing by contrary, passing eventually to a subsequence, still denote by v_n, we assume that $\lim\limits_{n\to+\infty} \|v_n\|_{S_0} = +\infty$. Hence, for all n, we can consider that $\|v_n\|_{S_0} > 1$. According to (3.2), Lemma 2.4, (M_1) and (M_2), we get that

$$
\begin{aligned}
1 &+ c_2 + \|v_n\|_{S_0} \\
&\geq I_\lambda(v_n) - \frac{1}{\alpha p_+}\langle I'_\lambda(v_n), v_n\rangle \\
&= \widehat{M}(\sigma_{p(x,y)}(v_n)) - \lambda\int_\Omega \frac{\lambda}{r(x)}|v_n(x)|^{r(x)}dx \\
&- \frac{1}{\alpha p_+}M(\sigma_{p(x,y)}(v_n))\int_Q \frac{|\, v_n(x) - v_n(y)\,|^{p(x,y)}}{|\, x-y\,|^{d+sp(x,y)}}dxdy + \frac{\lambda}{\alpha p_+}\int_\Omega |v_n(x)|^{r(x)}dx \\
&\geq \widehat{M}(\sigma_{p(x,y)}(v_n)) - \frac{1}{\alpha p_+}M(\sigma_{p(x,y)}(v_n))\int_Q \frac{|\, v_n(x) - v_n(y)\,|^{p(x,y)}}{|\, x-y\,|^{d+sp(x,y)}}dxdy \\
&- \frac{\lambda}{r_-}\int_\Omega |v_n(x)|^{r(x)}dx + \frac{\lambda}{\alpha p_+}\int_\Omega |v_n(x)|^{r(x)}dx \\
&\geq l\sigma_{p(x,y)}(v_n) - \frac{1}{p_+}\widehat{M}(1)(\sigma_{p(x,y)}(v_n))^{\alpha-1}\rho_{p(x,y)}(v_n) - (\frac{\lambda}{r_-} - \frac{\lambda}{\alpha p_+})\rho_{r(x)}(v_n) \\
&\geq \frac{l}{p_+}\rho_{p(x,y)}(v_n) - \frac{\widehat{M}(1)}{p_+(p_-)^{\alpha-1}}(\rho_{p(x,y)}(v_n))^\alpha - (\frac{\lambda}{r_-} - \frac{\lambda}{\alpha p_+})\min\{\|v_n\|^{r_+}_{L^{r(x)}}, \|v_n\|^{r_-}_{L^{r(x)}}\} \\
&\geq \frac{l}{p_+}\|v_n\|^{p_-}_{S_0} - \frac{\widehat{M}(1)}{(p_-)^\alpha}\|v_n\|^{\alpha p_+}_{S_0} - (\frac{\lambda c^{r_-}}{r_-} - \frac{\lambda c^{r_-}}{\alpha p_+})\min\{\|v_n\|^{r_+}_{S_0}, \|v_n\|^{r_-}_{S_0}\},
\end{aligned}
$$

when n is large enough. Dividing the above inequality by $\|v_n\|_{S_0}$. According to $\alpha p_+ < r_- < r_+$, we have $-(\frac{\lambda}{r_-} - \frac{\lambda}{\alpha p_+}) > 0$. By passing to the limit as $n \to +\infty$, we can get a contradiction.

Since S_0 is reflexive, then we can assume that $v_n \rightharpoonup \bar{v}$ in S_0. It follows from (3.2) that

$$
\lim_{n\to+\infty}\langle I'_\lambda(v_n), v_n - \bar{v}\rangle = 0,
$$

that is

$$
\begin{aligned}
M(\sigma_{p(x,y)})(v_n)\int_Q &\frac{|v_n(x) - v_n(y)|^{p(x,y)-2}(v_n(x) - v_n(y))((v_n(x) - v_n(y)) - (\bar{v}(x) - \bar{v}(y)))}{|x-y|^{d+sp(x,y)}} \\
&dxdy - \lambda\int_\Omega |v_n(x)|^{r(x)-2}v_n(x)(v_n(x) - \bar{v}(x))dx = 0.
\end{aligned}
\tag{3.3}
$$

Moreover, due to $r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$ and Remark 2.6 (i), we can conclude that $v_n \to \bar{v}$ in $L^{r(x)}$. Hence according to Lemma 2.1 and the proof of Theorem 3.1 in [14], we have that

$$
\lim_{n\to+\infty}\int_\Omega |v_n|^{r(x)-2}v_n(v_n - \bar{v})dx = 0. \tag{3.4}
$$

Since $\{v_n\}$ is bounded in S_0, if necessary, we can suppose that

$$
\sigma_{p(x,y)}(v_n) \xrightarrow[n\to+\infty]{} t_2 \geq 0.
$$

If $t_2 = 0$, then $\{v_n\} \to \bar{v} = 0$ in S_0 and the proof is complete. If $t_2 > 0$, according to the continuous function M, we have

$$M(\sigma_{p(x,y)}(v_n)) \xrightarrow[n\to+\infty]{} M(t_2) \geq 0.$$

Thus, for n large enough and $M \in Q_2$, we get that

$$0 < l < M(\sigma_{p(x,y)}(v_n)) < c_3. \tag{3.5}$$

Combining (3.3)–(3.5), we deduce that

$$\lim_{n\to+\infty} \int_\Omega \frac{|v_n(x) - v_n(y)|^{p(x,y)-2}(v_n(x) - v_n(y))((v_n(x) - v_n(y)) - (\bar{v}(x) - \bar{v}(y)))}{|x - y|^{d+sp(x,y)}} dxdy = 0. \tag{3.6}$$

Using (3.6), Lemma 2.7 (iii) and $v_n \rightharpoonup \bar{v}$ in S_0, we obtain that

$$\begin{cases} \lim\limits_{n\to+\infty} \langle \mathcal{L}(v_n), v_n - \bar{v}\rangle \leq 0 \\ v_n \rightharpoonup \bar{v} \ in\ S_0 \\ \mathcal{L} \text{ is a mapping of type } (S_+) \end{cases} \Rightarrow \ v_n \to \bar{v} \ in\ S_0.$$

Moreover, according to (3.2), we have

$$\lim_{n\to+\infty} I_\lambda(v_n) = I_\lambda(\bar{v}) = c_2 \ \text{ and } \ I'_\lambda(\bar{v}) = 0.$$

This completes the proof of Lemma 3.6.

Proof of Theorem 1.2: Clearly, $I_\lambda(0) = 0$ and $I_\lambda(-v) = I_\lambda(v)$. According to Theorem 3.3 and Lemmas 3.4–3.6, we deduce that Theorem 1.2 holds.

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Conflict of interest

The authors declare that there is no conflict of interest.

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