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Research article

Eventual smoothness of generalized solutions to a singular chemotaxis system for urban crime in space dimension 2

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Abstract: This paper is concerned with a chemotaxis system in a two-dimensional setting as follows:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla \ln v) - \kappa uv + ru - \mu u^2 + h_1, \\ v_t = \Delta v - v + uv + h_2, \end{cases} \tag{$\star$}$$

with the parameters $\chi, \kappa, \mu > 0$ and $r \in \mathbb{R}$, and with the given functions $h_1, h_2 \geq 0$. This model was originally introduced by Short *et al* for urban crime with the particular values $\chi = 2, r = 0$ and $\mu = 0$, and the logistic source term $ru - \mu u^2$ was incorporated into ($\star$) by Heihoff to describe the fierce competition among criminals. Heihoff also proved that the initial-boundary value problem of ($\star$) possesses a global generalized solution in the two-dimensional setting. The main purpose of this paper is to show that such a generalized solution becomes bounded and smooth at least eventually. In addition, the long-time asymptotic behavior of such a solution is discussed.

Keywords: chemotaxis system; generalized solution; eventual smoothness; asymptotic behavior

1. Introduction and main results

We study a class of logarithmic chemotaxis systems with the logistic source of the following form

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla \ln v) - \kappa uv + ru - \mu u^2 + h_1, \\ v_t = \Delta v - v + uv + h_2, \end{cases} \tag{1.1}$$

with the parameters $\chi, \kappa, \mu > 0$ and $r \in \mathbb{R}$. This model was proposed by Short *et al.* to describe the propagation of criminal activities with the particular values $\chi = 2, r = 0$ and $\mu = 0$ ([1, 2]), in which $u(x, t)$ denotes the density of criminals, $v(x, t)$ represents an abstract so-called attractiveness, the given function h_1 denotes the density of additional criminals and h_2 describes the source of attractiveness. The logistic source term, i.e., $ru - \mu u^2$, is a fairly standard addition to chemotaxis models. Here, it was incorporated into the Short *et al.* model by Heihoff ([3]) to model the fierce competition among criminals for, e.g., good targets, which are limited resources. We refer to [4–10] for further developments of the Short *et al.* and to [11, 12] for a review.

Mathematical analysis on (1.1) is still at quite an early stage and there are only a few relative results. For instance, for the Short *et al.* model, i.e., $r = 0$ and $\mu = 0$, the local classical solution was obtained in [13], which is globally provided that either $n = 1$ ([14, 15]), or $n \geq 2$ and $\chi < \frac{2}{n}$ ([16, 17]), or the initial data and the given functions h_1 and h_2 are assumed to be small ([18, 19]). As to the radial renormalized solvability, the global existence was established provided that either $n = 2$ ([20]) or $n = 3$ and $\chi \in (0, \sqrt{3})$ ([21]); without requiring the symmetry hypothesis, the generalized solvability was obtained in [22] for any $\chi > 0$ and $n = 2$. In addition, when Δu in the first equation in (1.1) is replaced by $\nabla \cdot (\nabla u^m)$ with some $m > 0$, the globally weak solvability was obtained in the two-dimensional setting provided that either $m > \frac{3}{2}$ ([23]) or $m > 1$ and $\chi < \frac{\sqrt{3}}{2}$ ([24]). We would like to remark that a reduced crime model, i.e., $\tau u_t = \Delta u - \chi \nabla \cdot (u \nabla \ln v)$ and $v_t = \Delta v - v + uv$, admits an unbounded solution for appropriately large initial data, provided that $n \geq 3$, $\chi > 0$, and $\tau > 0$ is enough small ([25]). Finally, we mention there appear various studies on the variants of Short *et al.* model, see [26–29].

For the case of $r \in \mathbb{R}$ and $\mu > 0$, the corresponding initial-boundary value problem admits a generalized solution (in the sense of Definition 1.1 below) in the two-dimensional setting ([3]). To illustrate how critical the interaction between the term $-\mu u^2$ in the first equation and the growth term $+uv$ in the second equation is, the stronger logistic source, $-\mu u^{2+\alpha}$, with $\alpha > 0$ for $n = 2, 3$ ([3, 30]) or $\alpha > \frac{n}{4} - 1$ for $n \geq 4$ ([30]), was proved to be enough for the global existence of a classical solution. This also indicates that the regularity of the generalized solution structured in [3] is not enough to trigger a bootstrap argument to improve the regularity of such a solution, and thereby it is not known whether or not this generalized solution develops singularities. Therefore, motivated by [26, 31–34], the main purpose of this paper is to reveal that the global generalized solution established in [3] at least eventually becomes bounded and smooth, and approaches spatial equilibria in the large time limit.

Precisely, we will present the eventual smoothness of the global generalized solution of the initial-boundary value problem:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla \ln v) - \kappa uv + ru - \mu u^2 + h_1, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + uv + h_2, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where ν denotes the exterior normal vector to the boundary $\partial\Omega$ and the initial data (u_0, v_0) fulfills that

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) \ \text{ with } \ u_0 \geq 0 \ \text{ and } \ u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\overline{\Omega}) \ \text{ with } \ \inf\limits_{x \in \overline{\Omega}} v_0 > 0. \end{cases} \tag{1.3}$$

In order to specify the setup for our analysis, we assume throughout the sequel that

$$0 \leq h_i \in C^1(\overline{\Omega} \times [0, \infty)) \cap L^\infty(\Omega \times (0, \infty)), \quad i = 1, 2, \tag{1.4}$$

with the additional properties that

$$\inf_{t > 0} \int_\Omega h_2(x, t) dx > 0, \tag{1.5}$$

$$\int_0^\infty \int_\Omega h_1(\cdot,t)dxds < \infty, \tag{1.6}$$

$$\int_0^\infty \int_\Omega |h_2(\cdot,t) - h_{2,\infty}(\cdot)|^2 dxds < \infty \tag{1.7}$$

with some $h_{2,\infty} \in C^1(\overline{\Omega})$.

Now, we briefly review the concept of generalized solution used in [3] for the initial-boundary value problem (1.2) as follows:

Definition 1.1. *A pair of nonnegative functions* (u, v) *is called a global generalized solution to the initial-boundary value problem* (1.2) *if for any* $T > 0$,
1) it holds that for any $q < \infty$

$$\begin{cases} v \in L^\infty(0,T; L^q(\Omega)), \quad \ln v \in L^2(0,T; W^{1,2}(\Omega)), \\ u \in L^2(\Omega \times (0,T)) \cap L^\infty(0,T; L^1(\Omega)), \quad \ln(1+u) \in L^2(0,T; W^{1,2}(\Omega)), \\ uv \in L^1(\Omega \times (0,T)), \quad v^{-1} \in L^\infty(\Omega \times (0,T)); \end{cases} \tag{1.8}$$

2) it holds that

$$\int_\Omega u(\cdot,t)dx + \int_0^t \int_\Omega \left(\kappa uv + \mu u^2\right) dxds \le \int_\Omega u_0 dx + \int_0^t \int_\Omega (ru + h_1)\, dxds, \;\; a.e., \text{ in } [0,T]; \tag{1.9}$$

3) it holds that for $0 \le \varphi(x,t) \in C_0^\infty(\overline{\Omega} \times [0,T))$ *with* $\nabla\varphi \cdot \nu|_{\partial\Omega\times(0,T)} = 0$

$$\begin{aligned} &- \int_0^T \int_\Omega \ln(u+1)\varphi_t dxdt - \int_\Omega \ln(u_0+1)\,\varphi(\cdot,0)dx \\ &\ge \int_0^T \int_\Omega \ln(u+1)\Delta\varphi dxdt + \int_0^T \int_\Omega |\nabla \ln(u+1)|^2 \varphi dxdt \\ &- \chi \int_0^T \int_\Omega \frac{u}{u+1}(\nabla \ln(u+1)\cdot \nabla \ln v)\varphi dxdt + \chi \int_0^T \int_\Omega \frac{u}{u+1}\nabla \ln v \cdot \nabla\varphi dxdt \\ &- \int_0^T \int_\Omega \frac{\kappa uv}{u+1}\varphi dxdt + \int_0^T \int_\Omega \frac{ru}{u+1}\varphi dxdt - \int_0^T \int_\Omega \frac{\mu u^2}{u+1}\varphi dxdt + \int_0^T \int_\Omega h_1 \varphi dxdt; \end{aligned} \tag{1.10}$$

4) it holds that for all $\varphi \in L^\infty(0,T; L^q(\Omega)) \cap L^2\left(0,T; W^{1,2}(\Omega)\right)$ *with* $\varphi_t \in L^2(\Omega \times (0,T))$, *compact support in* $\overline{\Omega} \times [0,T)$ *and* $q < \infty$

$$\begin{aligned} &\int_0^T \int_\Omega v\varphi_t dxdt + \int_\Omega v_0 \varphi(\cdot,0)dx \\ &= \int_0^T \int_\Omega \nabla v \cdot \nabla\varphi dxdt + \int_0^T \int_\Omega v\varphi dxdt - \int_0^T \int_\Omega uv\varphi dxdt - \int_0^T \int_\Omega h_2 \varphi dxdt. \end{aligned} \tag{1.11}$$

With Definition 1.1 at hand, letting

$$\eta := \min\left\{\inf_{x\in\Omega} v_0(x)e^{-1}, \frac{1}{4\pi}e^{-1-\frac{(\operatorname{diam}\Omega)^2}{2}}\left\{\inf_{s>0}\int_\Omega h_2(\cdot,s)dx\right\}\right\}, \tag{1.12}$$

our main results read as follows.

Theorem 1.1. *Assume that* (1.3)–(1.7) *hold. Let* $\kappa, \chi, \mu > 0$, $r \in \mathbb{R}$ *and* $\Omega \subset \mathbb{R}^2$ *be a bounded convex domain with smooth boundary, and let* (u, v) *be a generalized solution of* (1.2) *in the sense of Definition 1.1. Under the additional assumption that* $r < \kappa\eta$ *with* η *determined by* (1.12)*, there exists* $t_0 > 0$*, with the properties that* $u(x, t) \geq 0$ *and* $v(x, t) > 0$ *for any* $x \in \overline{\Omega}$ *and any* $t \geq t_0$*, and*

$$u \in C^{2,1}(\overline{\Omega} \times [t_0, \infty)), \quad v \in C^{2,1}(\overline{\Omega} \times [t_0, \infty)), \tag{1.13}$$

and that (u, v) *solves the initial-boundary value problem* (1.2) *classically in* $\Omega \times (t_0, \infty)$*. Moreover,* (u, v) *fulfills that*

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t) - v_\infty(\cdot)\|_{L^\infty} \to 0, \quad \text{as } t \to \infty, \tag{1.14}$$

where v_∞ *denotes the solution of the boundary value problem*

$$\begin{cases} -\Delta v_\infty + v_\infty = h_{2,\infty}, & x \in \Omega, \\ \nabla v_\infty \cdot \nu = 0, & x \in \partial\Omega. \end{cases} \tag{1.15}$$

Technical strategy and structure of the article

The objective of this paper, motivated by [26, 31–34], is to present that the global generalized solution of the initial-boundary value problem (1.2) at least eventually becomes bounded and smooth, and approaches spatial equilibria in a large time limit. To this end, the key steps are to establish a series of uniform *a-priori* estimates, in which the starting point is to get the uniform-in-(ε, t) lower bound for v_ε, see Lemma 2.2. We would like to remark that, for the linear signal production mechanism the combinational functional of the form

$$\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}|\nabla \widehat{v_\varepsilon}|^2 + \frac{1}{e} dx$$

where $\widehat{v_\varepsilon} := v_\varepsilon - v_\infty$ and v_∞ is a classical solution to the boundary value problem (1.15), is usually adopted to get the desired *a-priori* estimates (e.g., [35]). However, thanks to the presence of the nonlinear signal production mechanism, such functional is invalid for our case. Here, our novelty of the analysis consists of tracking the time evolution of the combinational functional of the form

$$\int_\Omega b u_\varepsilon + \frac{1}{2}u_\varepsilon^2 + \frac{1}{2}|\nabla \widehat{v_\varepsilon}|^2 dx, \quad t \geq T_0$$

with some waiting time T_0 and some $b > 0$, see Lemmas 3.4 and 3.5. From this, the key L^2-bound of u_ε is obtained, and an application of the standard bootstrap techniques shows that the generalized solution established in [3] becomes bounded and smooth at least eventually.

The rest of this paper is arranged as follows. Some preliminaries are given in Section 2. *A-priori* estimates are established in Section 3. Section 4 is devoted to showing the eventual smoothness, and the last section presents the large-time behavior desired in Theorem 1.1.

2. Preliminaries

A generalized solution of the initial-boundary value problem (1.2) can be obtained by an approximation procedure ([3,22]). Accordingly, we shall consider the following approximate problem

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \chi\nabla\cdot(u_\varepsilon\nabla\ln v_\varepsilon) - \kappa u_\varepsilon v_\varepsilon + r u_\varepsilon - \mu u_\varepsilon^2 + h_1, & x\in\Omega,\ t>0,\\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + \dfrac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + h_2, & x\in\Omega,\ t>0,\\ \dfrac{\partial u_\varepsilon}{\partial\nu} = \dfrac{\partial v_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x\in\Omega. \end{cases} \tag{2.1}$$

An application of the strategy invoking the contraction mapping principle and the well-known pointwise positivity property of the Neumann heat semigroup, as in [13, 16, 36, 37], ensures the global existence of the classical solution to the approximate problems (2.1).

Lemma 2.1. *Let the assumptions* (1.3)–(1.4) *hold. For each* $\varepsilon \in (0,1)$*, there exists a unique pair* $(u_\varepsilon, v_\varepsilon)$ *of positive functions, with the properties that for any* $T > 0$ *and* $\iota > 2$

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega}\times[0,T])\cap C^{2,1}(\overline{\Omega}\times(0,T]),\\ v_\varepsilon \in C^0(0,T;W^{1,\iota}(\overline{\Omega}))\cap C^{2,1}(\overline{\Omega}\times(0,T]), \end{cases}$$

such that $(u_\varepsilon, v_\varepsilon)$ *solves the approximate problem* (2.1) *classically in* $\Omega\times[0,\infty)$.

Proof. By a slight adaptation of the proof of [3, Lemma 2.3] (see also [22]), we can easily get the desired results.

Note that thanks to the non-negativity of (u_ε, h_2) and the variation-of-constants formula for v_ε, namely,

$$v_\varepsilon(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}\left(\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + h_2\right)(\cdot,s)ds, \tag{2.2}$$

it is clear that

$$v_\varepsilon(\cdot,t) \ge e^{t(\Delta-1)}v_0 \ge e^{-t}\inf_{x\in\Omega} v_0(x), \quad t>0, \tag{2.3}$$

which is adequate for establishing the global existence of generalized solutions, see [3]. However, to get eventual smoothness of generalized solutions, the uniform-in-t lower bound for v_ε will be necessary.

Lemma 2.2. *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded convex domain with smooth boundary and* (1.3)–(1.7) *hold. Then we have*

$$v_\varepsilon(\cdot,t) \ge \eta, \quad t>0, \tag{2.4}$$

where η *is determined by* (1.12).

Proof. Thanks to (2.3), we have

$$v_\varepsilon(\cdot,t) \geq e^{-1} \inf_{x\in\Omega} v_0(x) \quad \text{for all } t \leq 1. \tag{2.5}$$

For $t > 1$, due to the convexity of Ω, the well-known pointwise positivity property of the Neumann heat semigroup ensures that

$$e^{t\Delta} f \geq \frac{1}{4\pi t} e^{-\frac{(\operatorname{diam}\Omega)^2}{4t}} \int_\Omega f dx, \quad t > 0,$$

where $f \in C^0(\overline{\Omega})$ (cf. [38, Lemma 2.3] and [39, Lemma 3.1]), which, combined with (2.2), (1.5) and the non-negativity of (u_ε, v_0), implies that

$$\begin{aligned} v_\varepsilon(\cdot,t) &\geq \int_0^{t-\frac{1}{2}} e^{(t-s)(\Delta-1)} h_2(\cdot,s) ds \\ &\geq \int_0^{t-\frac{1}{2}} e^{-(t-s)} \frac{1}{4\pi(t-s)} e^{-\frac{(\operatorname{diam}\Omega)^2}{4(t-s)}} \int_\Omega h_2(\cdot,s) dx ds \quad \text{for all } t > 1. \end{aligned}$$

It follows that for $t > 1$

$$\begin{aligned} v_\varepsilon(\cdot,t) \geq& \frac{1}{4\pi} \left\{ \inf_{s>0} \int_\Omega h_2(\cdot,s) dx \right\} \int_{\frac{1}{2}}^t e^{-s} s^{-1} e^{-\frac{(\operatorname{diam}\Omega)^2}{4s}} ds \\ \geq& \frac{1}{4\pi} \left\{ \inf_{s>0} \int_\Omega h_2(\cdot,s) dx \right\} \int_{\frac{1}{2}}^1 e^{-s} s^{-1} e^{-\frac{(\operatorname{diam}\Omega)^2}{4s}} ds. \end{aligned}$$

Based on this, we further get that

$$v_\varepsilon(\cdot,t) \geq \frac{1}{4\pi} \left\{ \inf_{s>0} \int_\Omega h_2(\cdot,s) dx \right\} \cdot e^{-1} e^{-\frac{(\operatorname{diam}\Omega)^2}{2}}.$$

This, together with (2.5), entails the desired (2.4).

Next, we are concerned with the decay in a linear differential inequality, which is an extended version of [22, Lemma 2.5].

Lemma 2.3. *Let $\varepsilon \in (0,1)$, $y_\varepsilon \in C^1([0,\infty))$ be non-negative functions satisfying*

$$y_\varepsilon(0) = m \tag{2.6}$$

with some positive constant m independent of ε. If there exist a positive constant k and a nonnegative function $g_\varepsilon(t) \in C([0,\infty)) \cap L^\infty([0,\infty))$ which satisfies

$$\lim_{t\to\infty} \int_t^{t+1} g_\varepsilon(s) ds = 0 \quad \textit{uniformly in} \quad \varepsilon, \tag{2.7}$$

$$\|g_\varepsilon\|_{L^\infty(0,\infty)} \leq \mu \quad \text{for some } \mu \text{ independent of } (\varepsilon, t), \tag{2.8}$$

such that for each $\varepsilon > 0$,

$$y'_\varepsilon(t) + k y_\varepsilon(t) \leq g_\varepsilon(t) \quad \textit{for all} \quad t > 0, \tag{2.9}$$

then

$$y_\varepsilon(t) \to 0 \quad \textit{as} \quad t \to \infty \quad \textit{uniformly in} \quad \varepsilon. \tag{2.10}$$

At the end of this section, we recall the result on the solvability of the boundary value (1.15), which directly follows from [40].

Lemma 2.4. *For given* $h_{2,\infty} \in C^1(\overline{\Omega})$*, the problem* (1.15) *possesses a unique classical solution* v_∞ *fulfilling that* $v_\infty \in C^{2+\theta}(\overline{\Omega})$ *for some* $\theta \in (0,1)$.

3. *A-priori* estimates

A straightforward consequence of Lemma 2.2 is the following L^1-decay on the component u_ε.

Lemma 3.1. *Let all assumptions in Theorem 1.1 be fulfilled. Then there exists* $C > 0$*, independent of* (ε, t)*, such that*

$$\int_\Omega v_\varepsilon(\cdot,t)dx + \int_\Omega u_\varepsilon(\cdot,t)dx + \int_0^t \int_\Omega u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)dxds + \int_0^t \int_\Omega u_\varepsilon^2(\cdot,s)dxds \le C, \quad t > 0, \tag{3.1}$$

and that

$$\int_\Omega u_\varepsilon(\cdot,t)dx \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon, \tag{3.2}$$

$$\int_t^{t+1} \int_\Omega u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s)dxds + \int_t^{t+1} \int_\Omega u_\varepsilon^2(\cdot,s)dxds \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon. \tag{3.3}$$

Proof. Invoking (2.4) and taking $c_1 \in (0,\kappa)$, we obtain

$$\frac{d}{dt}\int_\Omega u_\varepsilon dx + (\kappa - c_1)\eta \int_\Omega u_\varepsilon dx + c_1 \int_\Omega u_\varepsilon v_\varepsilon dx + \mu \int_\Omega u_\varepsilon^2 dx \le r \int_\Omega u_\varepsilon dx + \int_\Omega h_1 dx.$$

Under the assumption that $r < \kappa\eta$, we can further take c_1 sufficiently close to 0 such that

$$c_2 := (\kappa - c_1)\eta - r > 0,$$

and thereby get

$$\frac{d}{dt}\int_\Omega u_\varepsilon dx + c_2 \int_\Omega u_\varepsilon dx + c_1 \int_\Omega u_\varepsilon v_\varepsilon dx + \mu \int_\Omega u_\varepsilon^2 dx \le \int_\Omega h_1 dx. \tag{3.4}$$

We now integrate the second equation in (2.1) over Ω to obtain

$$\frac{d}{dt}\int_\Omega v_\varepsilon dx + \int_\Omega v_\varepsilon dx = \int_\Omega u_\varepsilon v_\varepsilon dx + \int_\Omega h_2 dx,$$

which, together with (3.4), ensures

$$\frac{d}{dt}\left\{\int_\Omega u_\varepsilon dx + c_1 \int_\Omega v_\varepsilon dx\right\} + c_2 \int_\Omega u_\varepsilon dx + c_1 \int_\Omega v_\varepsilon dx + \mu \int_\Omega u_\varepsilon^2 dx \le \int_\Omega h_1 dx + \int_\Omega h_2 dx.$$

Setting $y(t) := \int_\Omega u_\varepsilon dx + c_1 \int_\Omega v_\varepsilon dx$ and $c_3 := \min\{c_2, 1\}$, it follows from (1.4) that

$$y'(t) + c_3 y(t) \le c_4 := \|h_1\|_{L^\infty(\Omega\times(0,\infty))}|\Omega| + \|h_2\|_{L^\infty(\Omega\times(0,\infty))}|\Omega|.$$

A standard ODE technique shows that

$$\int_\Omega u_\varepsilon dx + c_1\int_\Omega v_\varepsilon dx \le c_5 := \max\left\{\int_\Omega u_0 dx + c_1\int_\Omega v_0 dx, \frac{c_4}{c_3}\right\}, \quad t>0. \tag{3.5}$$

On the other hand, integrating (3.4) over $[0,t]$, for any $t>0$ we infer that

$$\int_\Omega u_\varepsilon dx + c_2\int_0^t\int_\Omega u_\varepsilon dxds + c_1\int_0^t\int_\Omega u_\varepsilon v_\varepsilon dxds + \mu\int_0^t\int_\Omega u_\varepsilon^2 dxds \le \int_\Omega u_0 dx + \int_0^t\int_\Omega h_1 dx,$$

which, with the help of (1.6) and (3.5), ensures (3.1).

Moreover, thanks to (1.6) and (1.4), it follows that

$$\int_t^{t+1}\int_\Omega h_1 dxds \to 0, \quad \text{as } t\to\infty,$$

which, together with Lemma 2.3 and (3.4), entails that the decay (3.2) holds as desired. Integrating (3.4) over $[t,t+1]$, for any $t>0$ we have

$$\int_\Omega u_\varepsilon(\cdot,t+1)dx + c_1\int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon dxds + \mu\int_t^{t+1}\int_\Omega u_\varepsilon^2 dxds \le \int_\Omega u_\varepsilon(\cdot,t)dx + \int_t^{t+1}\int_\Omega h_1 dxds.$$

Recalling (1.6) and (3.2), we arrive at (3.3).

To proceed further, we track the time evolution of $\|v_\varepsilon(\cdot,t)-v_\infty(\cdot)\|_{L^2}$, where v_∞ is classical solution of (1.15). For convenience, we set $\widehat{v_\varepsilon} := v_\varepsilon - v_\infty$. Thanks to (1.15) and (2.1), for $(u_\varepsilon, v_\varepsilon)$ given in Lemma 2.2, the initial-boundary value problem

$$\begin{cases} \widehat{v_\varepsilon}_t = \Delta\widehat{v_\varepsilon} - \widehat{v_\varepsilon} + \dfrac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + h_2 - h_{2,\infty}, & x\in\Omega,\ t>0,\\ \nabla\widehat{v_\varepsilon}\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ \widehat{v_\varepsilon}(x,0) = v_0(x) - v_\infty(x), & x\in\Omega \end{cases} \tag{3.6}$$

admits a unique classical solution $\widehat{v_\varepsilon}$.

Lemma 3.2. *Let all assumptions in Theorem 1.1 be in force. Then there exists $C>0$, independent of (ε,t), such that*

$$\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2 \le C, \quad t>0 \tag{3.7}$$

and

$$\int_0^t\int_\Omega |\nabla\widehat{v_\varepsilon}|^2 dxds + \int_0^t\int_\Omega |\widehat{v_\varepsilon}|^2 dxds \le C, \quad t>0. \tag{3.8}$$

Proof. Testing the first equation of (3.6) with $\widehat{v_\varepsilon}$, yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\widehat{v_\varepsilon}|^2 dx \le -\int_\Omega |\nabla\widehat{v_\varepsilon}|^2 dx - \int_\Omega |\widehat{v_\varepsilon}|^2 dx + \int_\Omega \frac{\widehat{v_\varepsilon} u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} dx + \int_\Omega (h_2 - h_{2,\infty})\widehat{v_\varepsilon} dx, \quad t>0.$$

Using Hölder's inequality and recalling the definition of $\widehat{v_\varepsilon}$, we have

$$\begin{aligned}\int_\Omega \frac{\widehat{v_\varepsilon} u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} dx =& \int_\Omega \frac{\widehat{v_\varepsilon}^2 u_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} dx + \int_\Omega \frac{v_\infty u_\varepsilon \widehat{v_\varepsilon}}{1+\varepsilon u_\varepsilon v_\varepsilon} dx \\ \leq& \|u_\varepsilon\|_{L^2} \|\widehat{v_\varepsilon}\|_{L^4}^2 + \|v_\infty\|_{L^\infty} \|u_\varepsilon\|_{L^2} \|\widehat{v_\varepsilon}\|_{L^2}.\end{aligned}$$

An application of the Gagliardo-Nirenberg inequality and Young's inequality implies that

$$\begin{aligned}\|u_\varepsilon\|_{L^2} \|\widehat{v_\varepsilon}\|_{L^4}^2 \leq& c_1 \|u_\varepsilon\|_{L^2} (\|\widehat{v_\varepsilon}\|_{L^2} \|\nabla \widehat{v_\varepsilon}\|_{L^2} + \|\widehat{v_\varepsilon}\|_{L^2}^2) \\ \leq& \frac{1}{4} \|\nabla \widehat{v_\varepsilon}\|_{L^2}^2 + c_2 \|u_\varepsilon\|_{L^2}^2 \|\widehat{v_\varepsilon}\|_{L^2}^2 + \frac{1}{4} \|\widehat{v_\varepsilon}\|_{L^2}^2. \end{aligned} \tag{3.9}$$

In addition, we have

$$\int_\Omega (h_2 - h_{2,\infty}) \widehat{v_\varepsilon} dx \leq \frac{1}{4} \|\widehat{v_\varepsilon}\|_{L^2}^2 + \int_\Omega |h_2 - h_{2,\infty}|^2 dx,$$

$$\|v_\infty\|_{L^\infty} \|u_\varepsilon\|_{L^2} \|\widehat{v_\varepsilon}\|_{L^2} \leq \frac{1}{4} \|\widehat{v_\varepsilon}\|_{L^2}^2 + \|v_\infty\|_{L^\infty}^2 \|u_\varepsilon\|_{L^2}^2.$$

Collecting these, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt} \int_\Omega |\widehat{v_\varepsilon}|^2 dx + \frac{3}{4} \int_\Omega |\nabla \widehat{v_\varepsilon}|^2 dx + \frac{1}{4} \int_\Omega |\widehat{v_\varepsilon}|^2 dx \\ \leq& c_2 \|u_\varepsilon\|_{L^2}^2 \|\widehat{v_\varepsilon}\|_{L^2}^2 + \|v_\infty\|_{L^\infty}^2 \|u_\varepsilon\|_{L^2}^2 + \int_\Omega |h_2 - h_{2,\infty}|^2 dx, \quad t > 0. \end{aligned} \tag{3.10}$$

Setting $y(t) = \|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2$ and $a(t) = \|u_\varepsilon(\cdot,t)\|_{L^2}^2$, it follows that

$$y'(t) + \frac{1}{2} y(t) \leq 2c_2 a(t) y(t) + b(t), \quad b(t) := 2\|v_\infty\|_{L^\infty}^2 \|u_\varepsilon\|_{L^2}^2 + 2\int_\Omega |h_2 - h_{2,\infty}|^2 dx.$$

A standard ODE technique shows

$$y(t) \leq y(0) e^{2c_2 \int_0^t a(s)ds - \frac{1}{2}t} + e^{2c_2 \int_0^t a(s)ds - \frac{1}{2}t} \int_0^t b(s) e^{-2c_2 \int_0^s a(\tau)d\tau + \frac{1}{2}s} ds.$$

Note that, thanks to (3.1), there exists $c_3 > 0$, independent of (ε, t), such that $\int_0^t a(s)ds \leq c_3$. Hence, we arrive at

$$y(t) \leq c_3 y(0) e^{-\frac{1}{2}t} + c_3 e^{-\frac{1}{2}t} \int_0^t b(s) e^{\frac{1}{2}s} ds.$$

Using (1.7) and (3.1) again, there exists $c_4 > 0$, independent of (ε, t), such that

$$c_3 e^{-\frac{1}{2}t} \int_0^t b(s) e^{\frac{1}{2}s} ds \leq c_3 \int_0^t b(s) ds \leq c_4, \quad t > 0.$$

Hence, there exists $C > 0$, independent of (ε, t), such that (3.7) holds. Moreover, thanks to (3.10) we can find $c_5 > 0$, independent of (ε, t), such that

$$\frac{1}{2}\frac{d}{dt} \int_\Omega |\widehat{v_\varepsilon}|^2 dx + \frac{3}{4} \int_\Omega |\nabla \widehat{v_\varepsilon}|^2 dx + \frac{1}{4} \int_\Omega |\widehat{v_\varepsilon}|^2 dx \leq c_5 \|u_\varepsilon\|_{L^2}^2 + \int_\Omega |h_2 - h_{2,\infty}|^2 dx, \quad t > 0. \tag{3.11}$$

We now integrate this equation over $[0,t]$ to get

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\widehat{v_\varepsilon}|^2dx+\frac{3}{4}\int_0^t\int_\Omega |\nabla\widehat{v_\varepsilon}|^2dxds+\frac{1}{4}\int_0^t\int_\Omega|\widehat{v_\varepsilon}|^2dxds\\ \le&\frac{1}{2}\|v_0-v_\infty\|_{L^2}^2+c_5\int_0^t\|u_\varepsilon\|_{L^2}^2ds+\int_0^t\int_\Omega|h_2-h_{2,\infty}|^2dxds,\end{aligned}$$

which, combined with (3.1) and (1.7), gives us the desired (3.8).

We would like to remark that although we have obtained (3.11) and can infer from (3.3) and (1.7) that for any $\varepsilon\in(0,1)$

$$\int_t^{t+1}\|u_\varepsilon\|_{L^2}^2ds+\int_t^{t+1}\int_\Omega|h_2-h_{2,\infty}|^2dxds\to 0,\quad \text{as } t\to\infty, \tag{3.12}$$

we cannot directly get the desired decay on $\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}$ by Lemma 2.3 due to the absence of the bound of $\|u_\varepsilon\|_{L^\infty(t,t+1;L^2)}$. Here, compared with (3.2), we need a new method to get decay on $\widehat{v_\varepsilon}$.

Lemma 3.3. *Let all assumptions in Theorem 1.1 be in force. Then*

$$\int_\Omega|\widehat{v_\varepsilon}|^2(\cdot,t)dx\to 0\quad as\quad t\to\infty\quad uniformly\ in\quad \varepsilon, \tag{3.13}$$

$$\int_t^{t+1}\int_\Omega|\nabla\widehat{v_\varepsilon}|^2dxds\to 0\quad as\quad t\to\infty\quad uniformly\ in\quad \varepsilon. \tag{3.14}$$

Proof. In fact, integrating (3.11) over $[t,t+1]$ yields that

$$\begin{aligned}&\frac{1}{2}\int_\Omega|\widehat{v_\varepsilon}|^2(\cdot,t+1)dx-\frac{1}{2}\int_\Omega|\widehat{v_\varepsilon}|^2(\cdot,t)dx+\frac{3}{4}\int_t^{t+1}\int_\Omega|\nabla\widehat{v_\varepsilon}|^2dxds+\frac{1}{4}\int_t^{t+1}\int_\Omega|\widehat{v_\varepsilon}|^2dxds\\ \le&c_5\int_t^{t+1}\|u_\varepsilon\|_{L^2}^2ds+\int_t^{t+1}\int_\Omega|h_2-h_{2,\infty}|^2dxds.\end{aligned} \tag{3.15}$$

By setting $z(t):=\frac{1}{2}\int_t^{t+1}\int_\Omega|\widehat{v_\varepsilon}|^2dxds$, we have

$$z'(t)+\frac{1}{2}z(t)\le c_5\int_t^{t+1}\|u_\varepsilon\|_{L^2}^2ds+\int_t^{t+1}\int_\Omega|h_2-h_{2,\infty}|^2dxds.$$

We now infer from (1.4), (1.7) and (3.1) that there exists $C>0$, independent of (ε,t), such that

$$g_\varepsilon:(t)=\int_t^{t+1}\|u_\varepsilon\|_{L^2}^2ds+\int_t^{t+1}\int_\Omega|h_2-h_{2,\infty}|^2dxds\le C,\quad t>0,$$

which, combined with Lemma 2.3, ensures that

$$z(t):=\frac{1}{2}\int_t^{t+1}\int_\Omega|\widehat{v_\varepsilon}|^2dxds\to 0\quad \text{as}\quad t\to\infty\quad \text{uniformly in}\quad \varepsilon. \tag{3.16}$$

On the other hand, letting $y(t) := \int_\Omega \left\{(1+\mu^{-1}\|v_\infty\|_{L^\infty}^2)u_\varepsilon(\cdot,t) + \frac{1}{2}|\widehat{v_\varepsilon}|^2(\cdot,t)\right\} dx$ we can infer from (3.4) and (3.10) that there exist c_i, $i = 1, 2, 3$, independent of (ε, t), such that for any $t > 0$

$$y'(t) + c_1 y(t) + \frac{3}{4}\int_\Omega |\nabla\widehat{v_\varepsilon}|^2 dx + \left(\mu - c_2\|\widehat{v_\varepsilon}\|_{L^2}^2\right)\int_\Omega u_\varepsilon^2 dx \le c_3 \int_\Omega h_1 + |h_2 - h_{2,\infty}|^2 dx. \tag{3.17}$$

From (3.16), (3.2) and the assumptions (1.6) and (1.7), there must exist T_* large enough, independent of ε, such that

$$\begin{aligned}\frac{1}{2}\int_{T*}^{T_*+1} \|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2 dt + c_3\int_{T*}^{\infty}\int_\Omega h_1 + |h_2 - h_{2,\infty}|^2 dxds \le \frac{\mu}{16c_2},\\ (1+\mu^{-1}\|v_\infty\|_{L^\infty}^2)\int_\Omega u_\varepsilon(\cdot,t)dx \le \frac{\mu}{16c_2}, \quad t \ge T_*,\end{aligned}$$

by which the mean value theorem implies there exists $\hat{t}_0 \in (T_*, T_* + 1)$, depending on ε, such that

$$(1+\mu^{-1}\|v_\infty\|_{L^\infty}^2)\int_\Omega u_\varepsilon(\cdot,\hat{t}_0)dx + \frac{1}{2}\|\widehat{v_\varepsilon}(\cdot,\hat{t}_0)\|_{L^2}^2 + c_3\int_{\hat{t}_0}^{\infty}\int_\Omega h_1 + |h_2 - h_{2,\infty}|^2 dxds \le \frac{\mu}{8c_2}. \tag{3.18}$$

Invoking these, we can claim that

$$\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2 \le \frac{\mu}{2c_2}, \quad t \ge \hat{t}_0. \tag{3.19}$$

In fact, the continuity of $\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2$, combined with (3.18), ensures that

$$\widetilde{T} := \sup\left\{t \,\middle|\, \sup_{\hat{t}_0\le s\le t}\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2 \le \frac{\mu}{2c_2}\right\} > \hat{t}_0, \tag{3.20}$$

and so we only need to show that $\widetilde{T} = \infty$. If on the contrary, there must hold

$$\sup_{\hat{t}_0\le s\le \widetilde{T}}\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2 = \frac{\mu}{2c_2}. \tag{3.21}$$

However, it follows from (3.17) and (3.20) that for $t \in [\hat{t}_0, \widetilde{T}]$

$$y'(t) + c_1 y(t) + \frac{3}{4}\int_\Omega |\nabla\widehat{v_\varepsilon}|^2 dx + \frac{1}{2}\mu\int_\Omega u_\varepsilon^2 dx \le c_3\int_\Omega h_1 + |h_2 - h_{2,\infty}|^2 dx.$$

By employing the standard ODE techniques, we arrive at

$$\begin{aligned}y(t) \le& e^{-c_1(t-\hat{t}_0)}y(\hat{t}_0) + c_3 e^{-c_1 t}\int_{\hat{t}_0}^t e^{c_1 s}\int_\Omega h_1 + |h_2 - h_{2,\infty}|^2 dxds\\ \le& y(\hat{t}_0) + c_3\int_{\hat{t}_0}^t\int_\Omega h_1 + |h_2 - h_{2,\infty}|^2 dxds,\end{aligned}$$

which, with the help of (3.18), ensures

$$y(t) \le \frac{\mu}{8c_2}, \quad t \in [\hat{t}_0, \widetilde{T}].$$

Recalling the definition of $y(t)$, we have

$$\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2 \leq \frac{\mu}{4c_2}, \quad t \in [\hat{t}_0, \widetilde{T}],$$

which contradicts (3.21). Thus we have that $\widetilde{T} = \infty$, and prove (3.19) as desired.

Thanks to the validity of (3.19), it follows from (3.17) that

$$y'(t) + c_1 y(t) + \frac{3}{4}\int_\Omega |\nabla \widehat{v_\varepsilon}|^2 dx + \frac{\mu}{2}\int_\Omega u_\varepsilon^2 dx \leq c_3 \int_\Omega h_1 + |h_2 - h_{2,\infty}|^2 dx, \quad t \geq \hat{t}_0. \tag{3.22}$$

Based on the assumptions (1.4), (1.6), (1.7) and Lemma 2.3, (3.22) ensures

$$y(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{uniformly in} \quad \varepsilon, \tag{3.23}$$

which is enough for (3.13) by recalling the definition of $y(t)$.

To get (3.14), integrating (3.22) over $[t, t+1]$, yields

$$y(t+1) + \frac{3}{4}\int_t^{t+1}\int_\Omega |\nabla \widehat{v_\varepsilon}|^2 dxds + \frac{\mu}{2}\int_t^{t+1}\int_\Omega u_\varepsilon^2 dxds \leq y(t) + c_3 \int_t^{t+1}\int_\Omega h_1 + |h_2 - h_{2,\infty}|^2 dxds,$$

which, combined with (3.23), (1.6) and (1.7) again, entails that (3.14) holds as desired.

In the sequel, we will use (3.2), (3.3) and (3.14) to obtain the uniform in ε bound for the entropy functional, denoted by

$$\mathcal{E}_\varepsilon(t) := \frac{1}{2}\int_\Omega u_\varepsilon^2 + |\nabla \widehat{v_\varepsilon}|^2 dx, \quad t > 0. \tag{3.24}$$

To achieve it, we first manage to achieve the following estimate.

Lemma 3.4. *Let all assumptions in Theorem 1.1 hold. Then there exist $a_1, a_2, a_3 > 0$, such that for any $\varepsilon \in (0,1)$,*

$$\begin{aligned}
&\mathcal{E}'(t) + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 dx + \frac{\kappa\eta - r}{2}\int_\Omega u_\varepsilon^2 dx + \mu \int_\Omega u_\varepsilon^3 dx \\
&\quad + \left(\frac{1}{2} - a_1\|u_\varepsilon\|_{L^2}^2\left(\|\nabla \widehat{v_\varepsilon}\|_{L^2}^2 + 1\right)\right)\int_\Omega |\Delta \widehat{v_\varepsilon}|^2 dx + \left(1 - a_2\|u_\varepsilon\|_{L^2}^2\right)\int_\Omega |\nabla \widehat{v_\varepsilon}|^2 dx \\
&\leq a_3\left\{\|h_1\|_{L^1} + \|u_\varepsilon\|_{L^2}^2 + \|h_2 - h_{2,\infty}\|_{L^2}^2\right\}, \quad t > 0,
\end{aligned} \tag{3.25}$$

where η is given by (2.4).

Proof. Invoking integration by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2 dx &= \int_\Omega u_\varepsilon\left(\Delta u_\varepsilon - \chi\nabla\cdot(u_\varepsilon \nabla \ln v_\varepsilon) - \kappa u_\varepsilon v_\varepsilon + r u_\varepsilon - \mu u_\varepsilon^2 + h_1\right) dx \\
&= -\int_\Omega |\nabla u_\varepsilon|^2 dx + \chi\int_\Omega \nabla u_\varepsilon \cdot (u_\varepsilon \nabla \ln v_\varepsilon)\, dx - \kappa\int_\Omega u_\varepsilon^2 v_\varepsilon dx \\
&\quad + r\int_\Omega u_\varepsilon^2 dx - \mu\int_\Omega u_\varepsilon^3 dx + \int_\Omega h_1 u_\varepsilon dx
\end{aligned}$$

$$=:P_1+P_2+P_3+P_4+P_5+P_6.$$

Since $r<\kappa\eta$ (η given in (2.4)), it follows that $c_1:=\kappa\eta-r>0$, and thereby implies from (2.4) that

$$\begin{aligned}P_3+P_4&\leq-\kappa\eta\int_\Omega u_\varepsilon^2dx+r\int_\Omega u_\varepsilon^2dx\\&=-c_1\int_\Omega u_\varepsilon^2dx.\end{aligned}$$

And using Young's inequality yields

$$P_6\leq\frac{c_1}{2}\int_\Omega u_\varepsilon^2dx+c_2\int_\Omega h_1^2dx.$$

For P_2, Hölder's inequality and (2.4) imply

$$P_2\leq\chi\eta^{-1}\|\nabla u_\varepsilon\|_{L^2}\|u_\varepsilon\nabla v_\varepsilon\|_{L^2}\leq\chi\eta^{-1}\|\nabla u_\varepsilon\|_{L^2}\|u_\varepsilon\|_{L^4}\|\nabla v_\varepsilon\|_{L^4},$$

which, together with Young's inequality, entails

$$P_2\leq\frac{1}{4}\|\nabla u_\varepsilon\|_{L^2}^2+c_3\|u_\varepsilon\|_{L^4}^2\|\nabla v_\varepsilon\|_{L^4}^2.$$

Recalling the Gagliardo-Nirenberg inequality

$$\|f\|_{L^4}^2\leq c_4\left(\|f\|_{L^2}\|\nabla f\|_{L^2}+\|f\|_{L^2}^2\right),$$

we get

$$\|u_\varepsilon\|_{L^4}^2\leq c_5\left(\|u_\varepsilon\|_{L^2}\|\nabla u_\varepsilon\|_{L^2}+\|u_\varepsilon\|_{L^2}^2\right),$$

and infer from the elliptic estimates that

$$\|\nabla v_\varepsilon\|_{L^4}^2\leq c_6\|\nabla v_\varepsilon\|_{L^2}\|\nabla v_\varepsilon\|_{H^1}\leq c_7\|\nabla v_\varepsilon\|_{L^2}\|\Delta v_\varepsilon\|_{L^2}.$$

In view of these, we arrive at

$$\begin{aligned}\|u_\varepsilon\|_{L^4}^2\|\nabla v_\varepsilon\|_{L^4}^2\leq&c_8\left(\|u_\varepsilon\|_{L^2}\|\nabla u_\varepsilon\|_{L^2}+\|u_\varepsilon\|_{L^2}^2\right)\|\nabla v_\varepsilon\|_{L^2}\|\Delta v_\varepsilon\|_{L^2}\\\leq&\frac{1}{4}\|\nabla u_\varepsilon\|_{L^2}^2+c_8^2\|u_\varepsilon\|_{L^2}^2\|\nabla v_\varepsilon\|_{L^2}^2\|\Delta v_\varepsilon\|_{L^2}^2+c_8\|u_\varepsilon\|_{L^2}^2\|\nabla v_\varepsilon\|_{L^2}\|\Delta v_\varepsilon\|_{L^2}.\end{aligned}$$

Collecting these and using Young's inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2dx+\frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2dx+\frac{c_1}{2}\int_\Omega u_\varepsilon^2dx+\mu\int_\Omega u_\varepsilon^3dx\\\leq&c_2\int_\Omega h_1^2dx+c_8^2\|u_\varepsilon\|_{L^2}^2\|\nabla v_\varepsilon\|_{L^2}^2\|\Delta v_\varepsilon\|_{L^2}^2+c_8\|u_\varepsilon\|_{L^2}^2\|\nabla v_\varepsilon\|_{L^2}\|\Delta v_\varepsilon\|_{L^2}\\\leq&c_2\int_\Omega h_1^2dx+2c_8^2\|u_\varepsilon\|_{L^2}^2\|\nabla v_\varepsilon\|_{L^2}^2\|\Delta v_\varepsilon\|_{L^2}^2+c_9\|u_\varepsilon\|_{L^2}^2.\end{aligned}$$

Recalling $\widehat{v_\varepsilon} := v_\varepsilon - v_\infty$ and invoking Lemma 2.4, it follows that

$$\begin{aligned}\|\nabla v_\varepsilon\|_{L^2}^2\|\Delta v_\varepsilon\|_{L^2}^2 \leq& 4\left(\|\nabla\widehat{v_\varepsilon}\|_{L^2}^2 + \|\nabla v_\infty\|_{L^2}^2\right)\left(\|\Delta\widehat{v_\varepsilon}\|_{L^2}^2 + \|\Delta v_\infty\|_{L^2}^2\right)\\ \leq& c_{10}\left(\|\nabla\widehat{v_\varepsilon}\|_{L^2}^2\|\Delta\widehat{v_\varepsilon}\|_{L^2}^2 + \|\nabla\widehat{v_\varepsilon}\|_{L^2}^2 + \|\Delta\widehat{v_\varepsilon}\|_{L^2}^2 + 1\right).\end{aligned}$$

This leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2dx + \frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2dx + \frac{c_1}{2}\int_\Omega u_\varepsilon^2dx + \mu\int_\Omega u_\varepsilon^3dx\\ \leq& c_2\int_\Omega h_1^2dx + c_{11}\|u_\varepsilon\|_{L^2}^2\left(\|\nabla\widehat{v_\varepsilon}\|_{L^2}^2 + 1\right)\|\Delta\widehat{v_\varepsilon}\|_{L^2}^2 + c_{12}\|u_\varepsilon\|_{L^2}^2\|\nabla\widehat{v_\varepsilon}\|_{L^2}^2 + c_{13}\|u_\varepsilon\|_{L^2}^2.\end{aligned} \tag{3.26}$$

On the other hand, we can test the first equation in (3.6) with $-\Delta\widehat{v_\varepsilon}$ to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx + \int_\Omega|\Delta\widehat{v_\varepsilon}|^2dx + \int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx\\ =&\int_\Omega\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}(-\Delta\widehat{v_\varepsilon})dx + \int_\Omega(h_2 - h_{2,\infty})(-\Delta\widehat{v_\varepsilon})dx,\end{aligned}$$

which, with the help of Young's inequality, shows

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx + \frac{1}{2}\int_\Omega|\Delta\widehat{v_\varepsilon}|^2dx + \int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx \leq \int_\Omega u_\varepsilon^2v_\varepsilon^2dx + \int_\Omega|h_2 - h_{2,\infty}|^2dx.$$

Hölder's inequality, combined with the Gagliardo-Nirenberg inequality and the elliptic estimates, entails

$$\begin{aligned}\int_\Omega u_\varepsilon^2v_\varepsilon^2dx \leq&\|u_\varepsilon\|_{L^2}^2\|v_\varepsilon\|_{L^\infty}^2\\ \leq& c_{14}\|u_\varepsilon\|_{L^2}^2\left(\|v_\varepsilon\|_{L^2}\|\Delta v_\varepsilon\|_{L^2} + \|v_\varepsilon\|_{L^2}^2\right),\end{aligned}$$

which, based on (3.7), Lemma 2.4 and the fact that $\widehat{v_\varepsilon} := v_\varepsilon - v_\infty$, leads to

$$\begin{aligned}\int_\Omega u_\varepsilon^2v_\varepsilon^2dx \leq& c_{14}\|u_\varepsilon\|_{L^2}^2\left((\|\widehat{v_\varepsilon}\|_{L^2} + \|v_\infty\|_{L^2})(\|\Delta\widehat{v_\varepsilon}\|_{L^2} + \|\Delta v_\infty\|_{L^2}) + (\|\widehat{v_\varepsilon}\|_{L^2} + \|v_\infty\|_{L^2})^2\right)\\ \leq& c_{15}\|u_\varepsilon\|_{L^2}^2\left(\|\Delta\widehat{v_\varepsilon}\|_{L^2} + 1\right).\end{aligned}$$

In the light of Young's inequality, it follows that

$$\int_\Omega u_\varepsilon^2v_\varepsilon^2dx \leq c_{16}\|u_\varepsilon\|_{L^2}^2\left(\|\Delta\widehat{v_\varepsilon}\|_{L^2}^2 + 1\right).$$

Hence, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx + \frac{1}{2}\int_\Omega|\Delta\widehat{v_\varepsilon}|^2dx + \int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx \leq c_{16}\|u_\varepsilon\|_{L^2}^2\left(\|\Delta\widehat{v_\varepsilon}\|_{L^2}^2 + 1\right) + \int_\Omega|h_2 - h_{2,\infty}|^2dx,$$

which, together with (3.26), ensures that

$$\mathcal{E}'(t) + \frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2dx + \frac{c_1}{2}\int_\Omega u_\varepsilon^2dx + \mu\int_\Omega u_\varepsilon^3dx + \frac{1}{2}\int_\Omega|\Delta\widehat{v_\varepsilon}|^2dx + \int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx$$

$$\leq c_2 \int_\Omega h_1^2 dx + c_{11}\|u_\varepsilon\|_{L^2}^2 \left(\|\nabla\widehat{v_\varepsilon}\|_{L^2}^2 + 1\right)\|\Delta\widehat{v_\varepsilon}\|_{L^2}^2 + c_{12}\|u_\varepsilon\|_{L^2}^2\|\nabla\widehat{v_\varepsilon}\|_{L^2}^2 + c_{13}\|u_\varepsilon\|_{L^2}^2$$
$$+ c_{16}\|u_\varepsilon\|_{L^2}^2 \left(\|\Delta\widehat{v_\varepsilon}\|_{L^2}^2 + 1\right) + \int_\Omega |h_2 - h_{2,\infty}|^2 dx.$$

Note that due to (1.4), we have

$$\int_\Omega h_1^2 dx \leq \|h_1\|_{L^\infty(\Omega\times(0,\infty))}^2 |\Omega|.$$

Hence, collecting these and recalling the definition of c_1, we can get the validity of (3.25).

The uniform convergence properties previously established in Lemmas 3.1 and 3.3, combined with a continuation argument, are enough to show that there exists T_0 large enough such that the variable coefficient in (3.25) maintains nonnegativity whenever $t \geq T_0$, which shall eventually lead to the following crucial estimates.

Lemma 3.5. *There exist T_0 large enough and $a_4 > 0$, independent of ε, such that for any $\varepsilon \in (0, 1)$*

$$\int_\Omega u_\varepsilon^2(\cdot, t)dx + \int_\Omega \left|\nabla\widehat{v_\varepsilon}\right|^2(\cdot, t)dx \leq a_4, \quad t \geq T_0, \tag{3.27}$$

$$\int_s^t \int_\Omega |\nabla u_\varepsilon|^2 dx d\tau + \int_s^t \int_\Omega |\Delta\widehat{v_\varepsilon}|^2 dx d\tau \leq a_4, \quad t \geq s \geq T_0. \tag{3.28}$$

Proof. Combining with (3.4) and (3.25), and setting $y(t) := \frac{a_3}{\mu}\int_\Omega u_\varepsilon dx + \mathcal{E}_\varepsilon(t)$, there exist $c_1 > 0$ and $c_2 > 0$, independent of (ε, t), such that

$$y'(t) + c_1 y(t) + \left(\frac{1}{2} - a_1\|u_\varepsilon\|_{L^2}^2\left(\|\nabla\widehat{v_\varepsilon}\|_{L^2}^2 + 1\right)\right)\int_\Omega |\Delta\widehat{v_\varepsilon}|^2 dx$$
$$+ \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 dx + \left(\frac{3}{4} - a_2\|u_\varepsilon\|_{L^2}^2\right)\int_\Omega |\nabla\widehat{v_\varepsilon}|^2 dx$$
$$\leq c_2\left\{\|h_1\|_{L^1} + \|h_2 - h_{2,\infty}\|_{L^2}^2\right\}, \quad t > 0, \tag{3.29}$$

where a_1, a_2 and a_3 are given in (3.25).

According to the uniform convergence stated in (3.2), (3.3) and (3.14), and the assumptions (1.6) and (1.7), there must exist T_* large enough, independent of ε, such that

$$\frac{a_3}{\mu}\int_\Omega u_\varepsilon(\cdot, t)dx \leq \frac{A}{2}, \quad t \geq T_*,$$

and

$$\frac{1}{2}\int_{T_*}^{T_*+1} \|u_\varepsilon(\cdot, t)\|_{L^2}^2 dt + \frac{1}{2}\int_{T_*}^{T_*+1} \|\nabla\widehat{v_\varepsilon}(\cdot, t)\|_{L^2}^2 dt + c_2\int_{T_*}^\infty \left(\|h_1\|_{L^1} + \|h_2 - h_{2,\infty}\|_{L^2}^2\right) ds \leq \frac{A}{2},$$

where $A := \min\left\{\frac{1}{8a_2}, \frac{1}{8}\sqrt{\frac{1}{a_1} + 1} - \frac{1}{8}\right\}$. By using mean value theorem we can find $\hat{t}_0 \in (T_*, T_* + 1)$, depending on ε, such that

$$y(\hat{t}_0) + c_2\int_{\hat{t}_0}^\infty \left(\|h_1\|_{L^1} + \|h_2 - h_{2,\infty}\|_{L^2}^2\right) ds \leq A, \tag{3.30}$$

which, together with the definition of $y(\hat{t}_0)$, further implies that

$$a_2\|u_\varepsilon(\cdot,\hat{t}_0)\|_{L^2}^2 \le 2a_2A, \tag{3.31}$$

and

$$\|\nabla\widehat{v_\varepsilon}(\cdot,\hat{t}_0)\|_{L^2}^2 \le 2A. \tag{3.32}$$

We now claim that

$$a_2\|u_\varepsilon(\cdot,t)\|_{L^2}^2 \le 4a_2A, \quad t \ge \hat{t}_0, \tag{3.33}$$

$$\|\nabla\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2 \le 4A, \quad t \ge \hat{t}_0, \tag{3.34}$$

and thereby assert

$$a_1\|u_\varepsilon(\cdot,t)\|_{L^2}^2\left(\|\nabla\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2+1\right) \le 4a_1A(4A+1), \quad t \ge \hat{t}_0. \tag{3.35}$$

Indeed, the continuities of $\|\nabla\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2$ and $a_2\|u_\varepsilon(\cdot,t)\|_{L^2}^2$, invoking (3.31) and (3.32), show that

$$\widetilde{T} := \sup\left\{t\,\middle|\,\sup_{\hat{t}_0\le s\le t} a_2\|u_\varepsilon(\cdot,t)\|_{L^2}^2 \le 4a_2A, \quad \sup_{\hat{t}_0\le s\le t}\|\nabla\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2 \le 4A,\right\} > \hat{t}_0, \tag{3.36}$$

and so we only need to show that $\widetilde{T} = \infty$. If on the contrary, at least one of the following statements must hold

$$\sup_{\hat{t}_0\le s\le\widetilde{T}} a_2\|u_\varepsilon(\cdot,t)\|_{L^2}^2 = 4a_2A, \tag{3.37}$$

$$\sup_{\hat{t}_0\le s\le\widetilde{T}} \|\nabla\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2 = 4A, \tag{3.38}$$

which, together with the definition of A, further leads to

$$a_1\|u_\varepsilon(\cdot,t)\|_{L^2}^2\left(\|\nabla\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2+1\right) \le 4a_1A(4A+1) \le \frac{1}{4}, \quad t \in [\hat{t}_0,\widetilde{T}], \tag{3.39}$$

$$\frac{3}{4} - a_2\|u_\varepsilon(\cdot,t)\|_{L^2}^2 \ge \frac{3}{4} - 4a_2A \ge \frac{1}{4}, \quad t \in [\hat{t}_0,\widetilde{T}]. \tag{3.40}$$

However, it follows from (3.29), (3.39) and (3.40) that

$$y'(t) + c_1y(t) + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2dx + \frac{1}{4}\int_\Omega |\Delta\widehat{v_\varepsilon}|^2dx \le c_2\left\{\|h_1\|_{L^1} + \|h_2-h_{2,\infty}\|_{L^2}^2\right\}, \quad t \in [\hat{t}_0,\widetilde{T}].$$

This, by means of the standard ODE techniques, results in that for any $t \in [\hat{t}_0,\widetilde{T}]$

$$\begin{aligned} y(t) \le& e^{-c_1(t-\hat{t}_0)}y(\hat{t}_0) + e^{-c_1t}\int_{\hat{t}_0}^t e^{c_1s}c_2\left\{\|h_1\|_{L^1} + \|h_2-h_{2,\infty}\|_{L^2}^2\right\}ds \\ \le& y(\hat{t}_0) + c_2\int_{\hat{t}_0}^\infty \left\{\|h_1\|_{L^1} + \|h_2-h_{2,\infty}\|_{L^2}^2\right\}ds, \end{aligned}$$

which, combined with (3.30) and the definition of $y(t)$, implies

$$y(t) \leq A, \quad t \in [\hat{t}_0, \widetilde{T}].$$

Hence, recalling the definitions of $y(t)$ and A again, we must have

$$a_2\|u_\varepsilon(\cdot,t)\|_{L^2}^2 \leq 2a_2A \quad \text{and} \quad \|\nabla\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^2 \leq 2A, \quad t \in [\hat{t}_0, \widetilde{T}],$$

which contradicts (3.37) and (3.38). Thus we have that $\widetilde{T} = \infty$, and prove (3.33)–(3.35) as desired.

Based on the definition of A and the validity of (3.33)–(3.35), we see from (3.29) that

$$y'(t) + c_1y(t) + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2dx + \frac{1}{4}\int_\Omega |\Delta\widehat{v_\varepsilon}|^2dx \leq c_2\left\{\|h_1\|_{L^1} + \|h_2 - h_{2,\infty}\|_{L^2}^2\right\}, \quad t \geq \hat{t}_0. \tag{3.41}$$

Hence, using the standard ODE techniques again, yields that for any $t \geq \hat{t}_0$

$$y(t) \leq y(\hat{t}_0) + c_2\int_{\hat{t}_0}^t \left\{\|h_1\|_{L^1} + \|h_2 - h_{2,\infty}\|_{L^2}^2\right\} ds,$$

which, together with (3.30), ensures

$$y(t) \leq c_3, \quad t \geq \hat{t}_0. \tag{3.42}$$

This evidently entails (3.27).

Moreover, integrating (3.41) over $[s,t]$ with $\hat{t}_0 \leq s \leq t$ and using (1.6) and (1.7) again, we subsequently arrive at

$$y(t) + \frac{1}{2}\int_s^t\int_\Omega |\nabla u_\varepsilon|^2dxd\tau + \frac{1}{4}\int_s^t\int_\Omega |\Delta\widehat{v_\varepsilon}|^2dxd\tau \leq y(s) + c_4.$$

Based on (3.42), we get that $\mathcal{E}_\varepsilon(s) \leq c_3$ due to $s \geq \hat{t}_0$, and thereby obtain (3.28) as desired.

4. Eventual smoothness

In view of Lemma 3.5 and the boundedness criterion obtained in [41, 42] via the Moser iteration and the semigroup theory, we can get the eventual bound of the generalized solution.

Lemma 4.1. *Let T_0 be given in Lemma 3.5. Then there exists $a_5 > 0$, with the property that for any $q > 2$*

$$\|\nabla\widehat{v_\varepsilon}(\cdot,t)\|_{L^q} \leq a_5, \quad t \geq T_0 + 1. \tag{4.1}$$

Proof. By means of (3.6) and the properties of the Neumann heat semigroup (cf. [43, Lemma 1.3] and [44, Lemma 2.1]), for all $t > T_0$ and $q > 2$ we have

$$\begin{aligned}\|\nabla\widehat{v_\varepsilon}(\cdot,t)\|_{L^q} \leq& \|\nabla e^{(t-T_0)(\Delta-1)}\widehat{v_\varepsilon}(\cdot,T_0)\|_{L^q} + \int_{T_0}^t \left\|\nabla e^{(t-s)(\Delta-1)}\left(\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + h_2 - h_{2,\infty}\right)\right\|_{L^q} ds\\ \leq& c_1\left(1 + (t-T_0)^{-(\frac{1}{2}-\frac{1}{q})}\right)\|\nabla\widehat{v_\varepsilon}(\cdot,T_0)\|_{L^2}\end{aligned}$$

$$+ c_1 \int_{T_0}^{t} \left(1 + (t-s)^{-\frac{1}{2}-(\frac{1}{2}-\frac{1}{q})}\right) e^{-(t-s)} \left(\|u_\varepsilon v_\varepsilon\|_{L^2} + \|h_2 - h_{2,\infty}\|_{L^2}\right) ds,$$

which, combined with (1.4) and (3.27), reduces to

$$\|\nabla \widehat{v_\varepsilon}(\cdot,t)\|_{L^q} \le c_2 + c_2(t-T_0)^{-(\frac{1}{2}-\frac{1}{q})} + c_2 \int_{T_0}^{t} \left(1 + (t-s)^{-\frac{1}{2}-(\frac{1}{2}-\frac{1}{q})}\right) e^{-(t-s)} \|u_\varepsilon v_\varepsilon\|_{L^2} ds.$$

An application of Hölder's inequality, invoking (3.27) and Lemma 2.4, yields that for any $t \ge T_0$

$$\|u_\varepsilon v_\varepsilon\|_{L^2} \le \|u_\varepsilon\|_{L^2} \|v_\varepsilon\|_{L^\infty} \le c_3 \left(\|\widehat{v_\varepsilon}\|_{L^\infty} + 1\right),$$

which, with the help of the Gagliardo-Nirenberg inequality, entails

$$\|u_\varepsilon v_\varepsilon\|_{L^2} \le c_4 \left(\|\widehat{v_\varepsilon}\|_{L^2}^{1-\vartheta} \|\nabla \widehat{v_\varepsilon}\|_{L^q}^{\vartheta} + \|\widehat{v_\varepsilon}\|_{L^2} + 1\right),$$

where $\vartheta := \frac{q}{2(q-1)}$. By employing (3.7), we arrive at

$$\|u_\varepsilon v_\varepsilon\|_{L^2} \le c_5 \left(\|\nabla \widehat{v_\varepsilon}\|_{L^q}^{\vartheta} + 1\right).$$

Collecting these, it follows that for any $t > T_0$

$$\|\nabla \widehat{v_\varepsilon}(\cdot,t)\|_{L^q} \le c_6 + c_2(t-T_0)^{-(\frac{1}{2}-\frac{1}{q})} + c_6 \int_{T_0}^{t} \left(1 + (t-s)^{-\frac{1}{2}-(\frac{1}{2}-\frac{1}{q})}\right) e^{-(t-s)} \|\nabla \widehat{v_\varepsilon}\|_{L^q}^{\vartheta} ds.$$

Letting $K(T) := \sup_{t\in(T_0,T)} \|\nabla \widehat{v_\varepsilon}(\cdot,t)\|_{L^q}$ for any $T \in (T_0,\infty)$, we get

$$K(T) \le c_6 + c_2(t-T_0)^{-(\frac{1}{2}-\frac{1}{q})} + c_7 K^{\vartheta}(T),$$

which, by using Young's inequality, ensures

$$K(T) \le c_8 + c_2(t-T_0)^{-(\frac{1}{2}-\frac{1}{q})}, \quad t > T_0.$$

Hence, for any $t \ge T_0 + 1$, we arrive at (4.1).

Based on (4.1), we can obtain the time-independent bound for u_ε in $L^\infty(\Omega)$.

Lemma 4.2. *Let T_0 be given in Lemma 3.5. Then there exists $a_6 > 0$, such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty} \le a_6, \quad t \ge T_0 + 2. \tag{4.2}$$

Proof. From the constant variation formula associated with the first equation in (2.1), we get that for any $t > t_1 := T_0 + 1$

$$\begin{aligned} 0 \le u_\varepsilon(x,t) =& e^{(\Delta-1)(t-t_1)} u_\varepsilon(x,t_1) \\ &+ \int_{t_1}^{t} e^{(\Delta-1)(t-s)} \left(-\chi \nabla\cdot(u_\varepsilon \nabla \ln v_\varepsilon) - \kappa u_\varepsilon v_\varepsilon + r u_\varepsilon - \mu u_\varepsilon^2 + h_1 + u_\varepsilon\right) ds \\ \le& e^{(\Delta-1)(t-t_1)} u_\varepsilon(x,t_1) + \int_{t_1}^{t} e^{(\Delta-1)(t-s)} \left(-\chi \nabla\cdot(u_\varepsilon \nabla \ln v_\varepsilon) + r u_\varepsilon + h_1 + u_\varepsilon\right) ds, \end{aligned}$$

which, with the help of the properties of Neumann heat semigroup (cf. [43, Lemma 1.3] and [44, Lemma 2.1]), we can pick $c_1 > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty} \le c_1\left(1+(t-t_1)^{-\frac{1}{2}}\right)\|u_\varepsilon(\cdot,t_1)\|_{L^2} + c_1\int_{t_1}^{t}\left(1+(t-s)^{-\frac{1}{2}}\right)e^{-(t-s)}\|u_\varepsilon+h_1\|_{L^2}ds$$
$$+ c_1\int_{t_1}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{1}{3}}\right)e^{-(t-s)}\|u_\varepsilon\nabla\ln v_\varepsilon\|_{L^3}ds.$$

Hence, (3.27) and (1.4) show that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty} \le c_2 + c_2(t-t_1)^{-\frac{1}{2}} + c_1\int_{t_1}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{1}{3}}\right)e^{-(t-s)}\|u_\varepsilon\nabla\ln v_\varepsilon\|_{L^3}ds.$$

On the basis of Hölder's inequality and (2.4), it follows that

$$\|u_\varepsilon\nabla\ln v_\varepsilon\|_{L^3} \le \|u_\varepsilon\|_{L^4}\|\nabla v_\varepsilon\|_{L^{12}}\|v_\varepsilon^{-1}\|_{L^\infty} \le \eta^{-1}\|u_\varepsilon\|_{L^4}\|\nabla v_\varepsilon\|_{L^{12}},$$

which, together with (4.1) and the fact that $\widehat{v_\varepsilon} = v_\varepsilon - v_\infty$, entails

$$\|u_\varepsilon\nabla\ln v_\varepsilon\|_{L^3} \le c_3\|u_\varepsilon\|_{L^4}, \quad t \ge T_0+1.$$

Based on this, the interpolation inequality and (3.27) indicate that

$$\|u_\varepsilon\nabla\ln v_\varepsilon\|_{L^3} \le c_3\|u_\varepsilon\|_{L^2}^{\frac{1}{2}}\|u_\varepsilon\|_{L^\infty}^{\frac{1}{2}} \le c_4\|u_\varepsilon\|_{L^\infty}^{\frac{1}{2}}, \quad t \ge T_0+1.$$

Collecting these, we arrive at

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty} \le c_2 + c_2(t-t_1)^{-\frac{1}{2}} + c_5\int_{t_1}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{1}{3}}\right)e^{-(t-s)}\|u_\varepsilon\|_{L^\infty}^{\frac{1}{2}}ds.$$

Setting $K(T) := \sup_{t\in(t_1,T)}\|u_\varepsilon(\cdot,t)\|_{L^\infty}$ for any $T \in (t_1,\infty)$, we have

$$K(T) \le c_2 + c_2(t-t_1)^{-\frac{1}{2}} + c_6K^{\frac{1}{2}}(T).$$

Using Young's inequality yields

$$K(T) \le c_7 + c_2(t-t_1)^{-\frac{1}{2}}, \quad t > t_1,$$

which must lead to

$$K(T) \le c_8, \quad t \ge t_1+1.$$

This implies (4.2) directly.

A straightforward consequence of Lemmas 4.1 and 4.2, invoking the parabolic Schauder estimates [45], can be stated as follows.

Lemma 4.3. *There exists $a_7 > 0$, independent of ε and t, with the property that for some $\alpha \in (0,1)$*

$$\|u_\varepsilon(\cdot,s)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times[t,t+1])} + \|v_\varepsilon(\cdot,s)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times[t,t+1])} \le a_7, \quad t > T_0+2, \tag{4.3}$$

where T_0 is given in Lemma 3.5.

Proof. Based on Lemmas 4.1 and 4.2 and the Schauder estimates ([45]), a straightforward reasoning involving standard bootstrap techniques ensures that (4.5) holds as desired by recalling Lemma 2.4 and $\widehat{v_\varepsilon} = v_\varepsilon - v_\infty$.

Lemma 4.3, combined with the Arzelà-Ascoli theorem, is enough to prove that the generalized solution (u, v) established in [3] admits the desired regularity in Theorem 1.1.

Lemma 4.4. *Let (u, v) be a generalized solution stated in Definition 1.1, and T_0 be given in Lemma 3.5. Then there exists $a_8 > 0$ with the property that for any $q > 2$*

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,q}} + \|v^{-1}(\cdot,t)\|_{L^\infty} \le a_8, \quad t \ge T_0 + 2, \tag{4.4}$$
$$\|u(\cdot,s)\|_{C^{2,1}(\Omega\times[t,t+1])} + \|v(\cdot,s)\|_{C^{2,1}(\Omega\times[t,t+1])} \le a_8, \quad t > T_0 + 2. \tag{4.5}$$

Moreover, $u \ge 0$, $v > 0$ and (u, v) solves the initial-boundary value problem (1.2) *classically in* $\Omega \times (T_0 + 2, \infty)$.

Proof. Invoking Lemma 4.3, [3, Lemma 4.2] and the Arzelà-Ascoli theorem, there exists a subsequence of $\{\varepsilon_j\}_{j=1}^\infty$ (still expressed as $\{\varepsilon_j\}_{j=1}^\infty$) such that for any $t > T_0 + 2$, as $\varepsilon = \varepsilon_j \to 0$,

$$u_\varepsilon \to u \quad \text{in} \quad C^{2,1}(\Omega\times[t,t+1],$$
$$v_\varepsilon \to v \quad \text{in} \quad C^{2,1}(\Omega\times[t,t+1].$$

This ensures (4.5), and thereby (4.4) holds as desired by using Sobolev's inequality, (4.1), (2.4) and (4.2) again. Moreover, along the lines demonstrated in [46, Lemma 2.1], we can see that if $u \ge 0$ and $v > 0$ satisfying (4.5) and such that (u, v) is a generalized solution of (1.3) in the sense of Definition 1.1, then (u, v) also solves (1.3) in the classical sense in $\Omega \times (T_0 + 2, \infty)$.

5. Asymptotic behavior

Asymptotic behavior of the generalized solution featured in Theorem 1.1 is now almost immediate.

Lemma 5.1. *Let all assumptions in Theorem 1.1 be fulfilled. Then*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty} + \|v_\varepsilon(\cdot,t) - v_\infty(\cdot)\|_{L^\infty} \to 0, \quad \text{as } t \to \infty, \tag{5.1}$$

where v_∞ denotes the solution of the boundary value problem (1.15).

Proof. It directly follows from (3.13) that

$$\int_\Omega |\widehat{v_\varepsilon}|^2(\cdot,t)dx \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon. \tag{5.2}$$

Using Sobolev's inequality and (4.1) again, for some $r > 2$ there exist $c_2 > 0$ and $c_3 > 0$ such that

$$\begin{aligned}\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^\infty} &\le c_2\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^{\frac{r-2}{2(r-1)}}\|\widehat{v_\varepsilon}(\cdot,t)\|_{W^{1,r}}^{\frac{r}{2(r-1)}}\\ &\le c_3\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^2}^{\frac{r-2}{2(r-1)}}, \quad t \ge T_0 + 2,\end{aligned}$$

which, in conjunction with (5.2), entails

$$\|\widehat{v_\varepsilon}(\cdot,t)\|_{L^\infty} \to 0, \quad \text{as } t\to\infty \quad \text{uniformly in } \varepsilon. \tag{5.3}$$

To get the decay on $\|u_\varepsilon\|_{L^\infty}$, we further develop the method used in [47]. According to the variation-of-constants formula for u_ε, for $t_0 := T_0 + 2$ the known estimates for the Neumann heat semigroup ensure that for any $t > t_0$

$$\begin{aligned}
\|u_\varepsilon(\cdot,t)\|_{L^\infty} \le& \|e^{(\Delta-1)(t-t_0)}u_\varepsilon(\cdot,t_0)\|_{L^\infty} + \chi\int_{t_0}^t \left\|e^{(\Delta-1)(t-s)}\nabla\cdot(u_\varepsilon\nabla\ln v_\varepsilon)\right\|_{L^\infty} ds\\
&+\int_{t_0}^t \left\|e^{(\Delta-1)(t-s)}(ru_\varepsilon+h_1+u_\varepsilon)\right\|_{L^\infty} ds\\
\le& c_4(1+(t-t_0)^{-\frac13})e^{-\delta(t-t_0)}\|u_\varepsilon(\cdot,t_0)\|_{L^3}\\
&+c_4\int_{t_0}^t (1+(t-s)^{-(\frac13+\frac12)})e^{-\delta(t-s)}\,\|u_\varepsilon\nabla\ln v_\varepsilon\|_{L^3}\,ds\\
&+c_4\int_{t_0}^t (1+(t-s)^{-\frac13})e^{-\delta(t-s)}\,\|ru_\varepsilon+h_1+u_\varepsilon\|_{L^3}\,ds\\
=:& V_1+V_2+V_3,
\end{aligned}$$

with some $c_4 > 0$ and $\delta > 0$. As a consequence of (4.4), we can find $c_5 > 0$, independent of ε, such that

$$\begin{aligned}
V_1 \le& c_5(1+(t-t_0)^{-\frac13})e^{-\delta(t-t_0)}\\
\le& 2c_5e^{t_0}e^{-t}, \quad t\ge t_0+1,
\end{aligned}$$

which clearly implies that for fixed t_0

$$V_1 \to 0, \quad \text{as } t\to\infty \quad \text{uniformly in } \varepsilon. \tag{5.4}$$

For V_2, Hölder's inequality, combined with (2.4), (4.4) and Lemma 2.4, entails

$$\begin{aligned}
\|u_\varepsilon\nabla\ln v_\varepsilon\|_{L^3} \le& \|v_\varepsilon^{-1}\|_{L^\infty}\|u_\varepsilon\|_{L^6}\|\nabla v_\varepsilon\|_{L^6}\\
\le& \|v_\varepsilon^{-1}\|_{L^\infty}\|u_\varepsilon\|_{L^6}\left(\|\nabla\widehat{v_\varepsilon}\|_{L^6}+\|\nabla v_\infty\|_{L^6}\right)\\
\le& c_6\|u_\varepsilon\|_{L^6}, \quad t\ge t_0.
\end{aligned}$$

By further assuming that $t > 2t_0$ and letting

$$\begin{aligned}
V_{21} =& c_4c_6\int_{t_0}^{\frac t2} (1+(t-s)^{-\frac56})e^{-\delta(t-s)}\,\|u_\varepsilon\|_{L^6}\,ds,\\
V_{22} =& c_4c_6\int_{\frac t2}^{t} (1+(t-s)^{-\frac56})e^{-\delta(t-s)}\,\|u_\varepsilon\|_{L^6}\,ds,
\end{aligned}$$

it follows that

$$V_2 \le V_{21}+V_{22}, \quad t > 2t_0.$$

For V_{21}, using (4.4) again we have

$$\begin{aligned} V_{21} &= c_4c_6 \int_{\frac{t}{2}}^{(t-t_0)} (1 + s^{-\frac{5}{6}})e^{-\delta s}\, \|u_\varepsilon\|_{L^6}\, ds \\ &\le c_7 \int_{\frac{t}{2}}^{t} (1 + s^{-\frac{5}{6}})e^{-\delta s} ds. \end{aligned}$$

Due to the fact that

$$\int_0^\infty (1 + s^{-\frac{5}{6}})e^{-\delta s} ds \le c_8,$$

we infer that

$$V_{21} \to 0, \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon. \tag{5.5}$$

For V_{22}, an application of the interpolation inequality and (4.4) yields that

$$\|u_\varepsilon(\cdot, t)\|_{L^6} \le \|u_\varepsilon(\cdot, t)\|_{L^1}^{\frac{1}{6}} \|u_\varepsilon(\cdot, t)\|_{L^\infty}^{\frac{5}{6}} \le c_9 \|u_\varepsilon(\cdot, t)\|_{L^1}^{\frac{1}{6}}, \quad t > 2t_0.$$

Invoking this, we arrive at

$$\begin{aligned} V_{22} \le& c_{10} \sup_{s > \frac{t}{2}} \|u_\varepsilon(\cdot, s)\|_{L^1}^{\frac{1}{6}} \int_{\frac{t}{2}}^{t} (1 + (t - s)^{-\frac{5}{6}})e^{-\delta(t-s)} ds \\ \le& c_{11} \sup_{s > \frac{t}{2}} \|u_\varepsilon(\cdot, s)\|_{L^1}^{\frac{1}{6}}, \quad t > 2t_0, \end{aligned}$$

which, combined with (3.2), entails

$$V_{22} \to 0, \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon.$$

This, together with (5.5), implies

$$V_2 \to 0, \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon. \tag{5.6}$$

Similarity, we set

$$\begin{aligned} V_{31} &= c_4 \int_{t_0}^{\frac{t}{2}} (1 + (t - s)^{-\frac{1}{3}})e^{-\delta(t-s)}\, \|(r + 1)u_\varepsilon + h_1\|_{L^3}\, ds, \\ V_{32} &= c_4 \int_{\frac{t}{2}}^{t} (1 + (t - s)^{-\frac{1}{3}})e^{-\delta(t-s)}\, \|(r + 1)u_\varepsilon + h_1\|_{L^3}\, ds, \end{aligned}$$

and thereby get

$$V_3 \le V_{31} + V_{32}, \quad t > 2t_0.$$

Similar to (5.5), we can infer from (4.4), (1.4) and Hölder's inequality that

$$V_{31} \le c_{12} \int_{\frac{t}{2}}^{t} (1 + s^{-\frac{1}{3}})e^{-\delta s} ds, \quad t > 2t_0,$$

and hence

$$V_{31} \to 0, \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon. \tag{5.7}$$

Similar to the estimate for V_{22}, it follows from the interpolation inequality and (4.4) that

$$c_4 \int_{\frac{t}{2}}^{t} (1 + (t-s)^{-\frac{1}{3}}) e^{-\delta(t-s)} \, \|(r+1)u_\varepsilon\|_{L^3} \, ds \le c_{13} \sup_{s > \frac{t}{2}} \|u_\varepsilon(\cdot, s)\|_{L^1}^{\frac{1}{3}}, \quad t > 2t_0,$$

which, combined with (3.2), entails

$$c_4 \int_{\frac{t}{2}}^{t} (1 + (t-s)^{-\frac{1}{3}}) e^{-\delta(t-s)} \, \|(r+1)u_\varepsilon\|_{L^3} \, ds \to 0, \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon.$$

On the other hand, the interpolation inequality and (1.4) imply

$$\|h_1\|_{L^3} \le \|h_1\|_{L^1}^{\frac{1}{3}} \|h_1\|_{L^\infty}^{\frac{2}{3}} \le c_{14} \|h_1\|_{L^1}^{\frac{1}{3}}.$$

This, with the help of Hölder's inequality, ensures

$$\begin{aligned}
& c_4 \int_{\frac{t}{2}}^{t} (1 + (t-s)^{-\frac{1}{3}}) e^{-\delta(t-s)} \, \|h_1(\cdot, s)\|_{L^3} \, ds \\
\le & c_{15} \left\{ \int_{\frac{t}{2}}^{t} (1 + (t-s)^{-\frac{1}{2}}) e^{-\frac{3}{2}\delta(t-s)} ds \right\}^{\frac{2}{3}} \left\{ \int_{\frac{t}{2}}^{t} \|h_1(\cdot, s)\|_{L^3}^3 \, ds \right\}^{\frac{1}{3}} \\
\le & c_{16} \left\{ \int_{\frac{t}{2}}^{t} \|h_1(\cdot, s)\|_{L^1} ds \right\}^{\frac{1}{3}}, \quad t > 2t_0,
\end{aligned}$$

which, together with (1.6), leads to

$$c_4 \int_{\frac{t}{2}}^{t} (1 + (t-s)^{-\frac{1}{3}}) e^{-\delta(t-s)} \, \|h_1(\cdot, s)\|_{L^3} \, ds \to 0, \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon.$$

Hence, we arrive at

$$V_{32} \to 0, \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon,$$

which, in conjunction with (5.7), gives us

$$V_3 \to 0, \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon.$$

This, further combined with (5.4) and (5.6), asserts

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty} \to 0, \quad \text{as } t \to \infty \quad \text{uniformly in } \varepsilon,$$

which implies that (5.1) holds as desired by recalling (5.3) and the definition of $\widehat{v_\varepsilon}$.

Our main result on eventual smoothness and stabilization in Theorem 1.1 is in fact a by-product of our previous analysis.

Proof of Theorem 1.1. The eventual smoothness in Theorem 1.1 has been verified evidently in Lemma 4.4. For the stabilization, it readily follows from Lemma 5.1, Lemma 4.3, [3, Lemma 4.2] and the Arzelà-Ascoli theorem that (1.14) holds.

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Conflict of interest

The authors declare there is no conflicts of interest.

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