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*Research article*

# Random periodic sequence of globally mean-square exponentially stable discrete-time stochastic genetic regulatory networks with discrete spatial diffusions

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**Abstract:** This paper regards the dual effects of discrete-space and discrete-time in stochastic genetic regulatory networks via exponential Euler difference and central finite difference. Firstly, the global exponential stability of such discrete networks is investigated by using discrete constant variation formulation. In particular, the optimal exponential convergence rate is explored by solving a nonlinear optimization problem under nonlinear constraints, and an implementable computer algorithm for computing the optimal exponential convergence rate is given. Secondly, random periodic sequence for such discrete networks is investigated based on the theory of semi-flow and metric dynamical systems. The researching findings show that the spatial diffusions with nonnegative intensive coefficients have no influence on global mean square boundedness and stability, random periodicity of the networks. This paper is pioneering in considering discrete spatial diffusions, which provides a research basis for future research on genetic regulatory networks.

**Keywords:** genetic regulatory; spatial diffusion; stochastic; random periodicity; optimal convergence rate

## 1. Introduction

Genetic regulatory networks (GRNs) refer to the network containing complicated molecular interactions involving messenger ribonucleic acid (mRNAs) and protein molecules that represent the control of gene performance within an organization. Since gene performance regulates cellular functions, molecular/cellular biologists have to learn about and forecast the motions of GRNs. In addition, precise forecasting of gene control procedures can accelerate biotechnology programs. Consequently, mathematical modeling of GRNs and their behaviors are the major study trends in systems biology. Recently, GRNs have been increasingly gained the awareness of researchers because of their easy-to-understand features. These are highly useful and effective methods for describing complex and dynamical transactional relationships. Hence, GRNs are not only the foundation for the investigation of a variety of processes in biological bodies, but also have promising potential applications to systematic biology. Over the recent twenty years, a number of GRNs were already published in the literatures [1–8]. Remarkably, global exponential stability and (almost) periodicity are important and essential dynamical behaviours in GRNs, which have been extensively studied by numerous scholars (see [5, 7, 9–11]) in recent twenty years. In particular, in the case of stochastic models, the concept of random periodicity was introduced in lietrature [12] based on the theories of semi-flow and metric dynamical system, and the existence of random periodicity to several continuous time stochastic models had been discussed in literatures [12–15]. Yet, the problems of global mean-square exponential stability and random periodicity of stochastic GRNs have not been deeply addressed.

Stochastic perturbations not only separate the models from deterministic cases, but also can bring about substantial modifications in dynamic actions of GRNs, see reports [16, 17]. In general, the behaviors of stochastic systems are highly reliant on time and spatial dependence. Consequently, reaction diffusion is necessary to be taken into account and this induces the investigations of stochastic reaction diffusion systems. Besides, in biologically based network systems, the concentration of constituents is not uniform, resulting in diffusion of cytoplasm from higher to lower concentrations. As it was indicated in report [18], the concentration of cellular modules in any region has usually been related to spatial heterogeneity, i.e., mRNA and protein concentrations change according to time and space. Consequently, it is significant to consider GRNs involving the effect of spatial diffusions and more attentions (see [19–23]) had been paid to GRNs with spatial diffusions in recent years. Such as, the authors in paper [19] considered the problem of the stability of GRNs involved with spatial diffusions. With the help of the theory of stable differential equations, a novel generation requirement was created to ensure global exponential stability of reaction diffusions GRNs. Sun et al. [20] discussed $H_\infty$ state estimation to GRNs with spatial diffusions and stochastic gain fluctuations, in which the $H_\infty$ performance index is induced to evaluate the ability of the system to resist disturbances and a fuzzy model-based approach is employed to evaluate the immunity of the system. Song et al. [21] investigated delayed GRNs with spatial diffusions. By using Lyapunov functional, Wirtinger's and Halanay's inequalities, etc, a novel standard to the networks is developed to guarantee that the estimation error converges to zero. For more researches about GRNs with spatial diffusions, please refer to literatures [22, 23].

For both computational simulation and analysis, engineers frequently discretize time continuous models to evaluate their structural behaviors. The received and operated signals in digital networks are predicated on discrete time rather than continuous time. Accordingly, discrete-time GRNs have been discussed by numerous scholars, e.g., [24–27]. It is remarkable that 1) the difference methods used in papers [24–27] were Euler difference; 2) the majority of reported findings on GRNs only concerned time discrete GRNs [24–27], while the corresponding results about space discrete GRNs have not been appreciated adequately in the existing researches, probably due to the partial invalidity of the traditional methods in space-time continuous networks and the difficulty of computing the difference of Lyapunov-Krasovskii functional in discrete space and time networks. These situations trigger this discussion to discuss the space-time discrete GRNs by using exponential Euler difference for time variable and central finite difference for space variable. Notably, the exponential Euler difference is a more effective method than Euler difference (see references [28–33]).

Introduced by the above motivations, the main purpose of this article is to newly formulate a discrete-space and discrete-time stochastic GRNs by employing the methods of exponential Euler difference and finite difference. Subsequently, the constant variation formula, global existence, mean square finiteness and boundedness to the discrete-space and discrete-time GRNs are addressed. Additionally, based on the discrete-space and discrete-time constant variation formula, global mean square exponential stability and the optimal exponential convergence rate are discussed by solving nonlinear constrained optimization problem. At last, random periodicity of discrete-space and discrete-time stochastic GRNs is explored. In the following, the leading researching content and novelties are summarized as follows. 1) Discrete-space and discrete-time stochastic GRNs are newly established to expand the discrete-time GRNs [24–27]. 2) Global boundedness and exponential stability in mean square sense are discussed based on discrete-space and discrete-time constant variation formula. 3) The optimal exponential convergence rate can be gained by solving a constrained optimization problem. 4) Random periodicity of discrete-space and discrete-time stochastic GRNs is explored by the theories of semi-flow and metric dynamical systems. 5) This discussion starts the studies of global exponential stability and random periodicity of discrete-space and discrete-time stochastic GRNs.

**Plan of this paper:** In Section 2, discrete-time stochastic GRNs with discrete spatial diffusions are formulated and some important lemmas are introduced. Section 3 discusses global existence, mean square finiteness and boundedness to GRNs based on discrete constant variation formula. In Section 4, global exponential stability is addressed and the optimal exponential convergence rate can be solved by a constrained optimization problem. In addition, random periodicity of discrete space and time stochastic GRNs is displayed in Section 5. A numerical example of realizing global exponential stability with optimal convergence rate and random periodicity for discrete stochastic GRNs is given in Section 6. Section 7 states the conclusions and perspectives of this article.

**Symbols:** $\mathbb{R}^m$ denotes the space of $m$-dimensional real vectors; $\mathbb{Z}$ denotes the set of integers; $\mathbb{Z}_0 = \{0, 1, 2, \ldots\}$; $\mathbb{N} = \mathbb{Z}_0 \setminus \{0\}$; $I_J = I \cap J$, $\forall I, J \subseteq \mathbb{R}$. Let

$$\xi_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^n, \quad \xi_2 = (0, 1, \ldots, 0)^T \in \mathbb{R}^n, \quad \ldots \quad \xi_n = (0, 0, \ldots, 1)^T \in \mathbb{R}^n.$$

Define $\partial\mho_\varsigma := \bar{\mho}_\varsigma \backslash \mho_\varsigma$, where

$$\bar{\mho}_\varsigma = \left\{\varsigma = (\varsigma_1, \ldots, \varsigma_n)^T \in \mathbb{R}^n : \varsigma_q = 0, 1, \ldots, N_q, N_q \in \mathbb{N}, q = 1, 2, \ldots, n\right\},$$

$$\mho_\varsigma = \left\{\varsigma = (\varsigma_1, \ldots, \varsigma_n)^T \in \mathbb{R}^n : \varsigma_q = 1, 2, \ldots, N_q - 1, N_q \in \mathbb{N}, q = 1, 2, \ldots, n\right\}.$$

For any function $u : \bar{\mho}_\varsigma \times \mathbb{Z} \to \mathbb{R}^m$ with $u := u_k^{\langle\varsigma\rangle} = (u_{1,k}^{\langle\varsigma\rangle}, \ldots, u_{m,k}^{\langle\varsigma\rangle})^T$, denote

$$\Delta u_k^{\langle\cdot\rangle} = u_{k+1}^{\langle\cdot\rangle} - u_k^{\langle\cdot\rangle}, \quad \Delta_{\hbar_q}^2 u_k^{\langle\varsigma\rangle} = \frac{u_k^{\langle\varsigma+\xi_q\rangle} - 2u_k^{\langle\varsigma\rangle} + u_k^{\langle\varsigma-\xi_q\rangle}}{\hbar^2}, \quad \tilde{\Delta}_{\hbar_q}^2 u_k^{\langle\varsigma\rangle} = \left(\Delta_{\hbar_q}^2 + \frac{2}{\hbar^2}\right) u_k^{\langle\varsigma\rangle},$$

where $(\varsigma, k) \in \bar{\mho}_\varsigma \times \mathbb{Z}$, $\hbar > 0$, $q = 1, 2, \ldots, n$. Hereon, $\Delta$ and $\Delta_{\hbar_\cdot}^2$ denote the first order difference and second order central finite difference with respect to time and space variables (i.e., $k$ and $\varsigma$), respectively.

## 2. Space-time discrete stochastic GRNs

This section firstly gives discrete-time stochastic GRNs with discrete spatial diffusions, which can be regarded as a full dicretization scheme of continuous time stochastic GRNs with reaction diffusions.

Next, based on the theory of difference equations, constant variation formula for such discrete networks is addressed. In the end, some important inequalities such as Minkowski inequality in Lemma 2.2, etc. are recalled.

This article considers the following discrete space and time stochastic GRNs in the shape of

$$\begin{cases} \mathbf{m}_{i,k+1}^{\langle\varsigma\rangle} &= e^{-a_{i,k}h}\mathbf{m}_{i,k}^{\langle\varsigma\rangle} + \dfrac{1-e^{-a_{i,k}h}}{a_{i,k}}\Big[\displaystyle\sum_{q=1}^{n}\mu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{m}_{i,k}^{\langle\varsigma\rangle} \\ &\quad + \displaystyle\sum_{j=1}^{m} b_{ij,k}f_j(\mathbf{p}_{j,k}^{\langle\varsigma\rangle}) + \sum_{j=1}^{m}\gamma_{ij,k}\sigma_j(\mathbf{p}_{j,k}^{\langle\varsigma\rangle})w_{1j,k} + I_{i,k}\Big], \\ \mathbf{p}_{i,k+1}^{\langle\varsigma\rangle} &= e^{-c_{i,k}h}\mathbf{p}_{i,k}^{\langle\varsigma\rangle} + \dfrac{1-e^{-c_{i,k}h}}{c_{i,k}}\Big[\displaystyle\sum_{q=1}^{n}\nu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{p}_{i,k}^{\langle\varsigma\rangle} \\ &\quad + \displaystyle\sum_{j=1}^{m}\varpi_{ij,k}\eta_j(\mathbf{m}_{j,k}^{\langle\varsigma\rangle})w_{2j,k} + d_{i,k}\mathbf{m}_{i,k}^{\langle\varsigma\rangle}\Big], \quad \forall(\varsigma,k)\in\mho_\varsigma\times\mathbb{Z}, \end{cases} \tag{2.1}$$

where $\mathbf{m}_i$ and $\mathbf{p}_i$ represent the concentrations of the $i$th mRNA and $i$th protein, respectively, $i = 1,2,\ldots,m$; $\hbar$ and $h$ of less than 1 denote the space and time steps' length in order;

$$a_{i,\cdot} := a_{i,\cdot}^* + 2\sum_{q=1}^{n}\frac{\mu_{iq}}{\hbar^2}, \quad c_{i,\cdot} := c_{i,\cdot}^* + 2\sum_{q=1}^{n}\frac{\nu_{iq}}{\hbar^2};$$

$a_i^* > 0$ and $c_i^* > 0$ are the decay rates of the $i$th mRNA and $i$th protein, respectively; $d_i > 0$ is the translation rate; $\mu_{iq}$ and $\nu_{iq}$ stand for the transmission diffusion matrixes; $I_i = \sum_{j\in\mathbf{I}_i} w_{ij}$, $w_{ij} \geq 0$ is bounded and $\mathbf{I}_i$ is the set of all the $j$ which is a repressor of gene $i$; $B = (b_{ij}) \in \mathbb{R}^{m\times m}$ with

$$b_{ij} = \begin{cases} w_{ij} & \text{if transcription factor } j \text{ is an activator of gene } i, \\ 0 & \text{if there is no link from node } j \text{ to } i, \\ -w_{ij} & \text{if transcription factor } j \text{ is a repressor of gene } i; \end{cases}$$

$\gamma_{ij}$ and $\varpi_{ij}$ denote noise intensities, $i,j = 1,2,\ldots,m$; $f_j, \sigma_j, \eta_j$ take the Hill function, i.e.,

$$f_j(x) = \frac{\left(\dfrac{x}{\alpha_{1j}}\right)^{H_{1j}}}{1+\left(\dfrac{x}{\alpha_{1j}}\right)^{H_{1j}}}, \quad \sigma_j(x) = \frac{\left(\dfrac{x}{\alpha_{2j}}\right)^{H_{2j}}}{1+\left(\dfrac{x}{\alpha_{2j}}\right)^{H_{2j}}}, \quad \eta_j(x) = \frac{\left(\dfrac{x}{\alpha_{3j}}\right)^{H_{3j}}}{1+\left(\dfrac{x}{\alpha_{3j}}\right)^{H_{3j}}}, \quad \forall x\in\mathbb{R},$$

$H_{pj}$ is the Hill coefficient and $\alpha_{pj}$ is a positive constant, $p = 1,2,3$, $j = 1,2,\ldots,m$;

$$w_{1j,k} = \frac{1}{h}\left[\mathbb{B}_{1j}(kh+h) - \mathbb{B}_{1j}(kh)\right], \quad w_{2j,k} = \frac{1}{h}\left[\mathbb{B}_{2j}(kh+h) - \mathbb{B}_{2j}(kh)\right],$$

$j = 1,2,\ldots,m$; $\mathbb{B}_{11},\ldots,\mathbb{B}_{1m},\mathbb{B}_{21},\ldots,\mathbb{B}_{2m}$ are scalar mutually independent two sides standard Brown motions on complete probability space $(\Omega,\mathcal{F},\mathscr{F}_\cdot,\mathbf{P})$ with filtration

$$\mathscr{F}_k = \sigma\left\{(w_{11,s},\ldots,w_{1m,s},w_{21,s},\ldots,w_{2m,s}) : s\in(-\infty,k)_{\mathbb{Z}}\right\}, \quad \forall k\in\mathbb{Z}.$$

The Dirichlet boundary conditions of GRNs (2.1) are described as

$$\mathbf{m}_{i,k}^{\langle\varsigma\rangle}\Big|_{\varsigma\in\partial\mho_\varsigma} = 0 = \mathbf{p}_{i,k}^{\langle\varsigma\rangle}\Big|_{\varsigma\in\partial\mho_\varsigma}, \quad \forall k\in\mathbb{Z}.$$

Hereon, $\mho_\varsigma$ can be regarded as discrete form of rectangle area $\mho$ in $\mathbb{R}^m$, which is described by

$$\mho = \Big\{x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n : 0 < x_q < L_q := \hbar N_q, N_q \in \mathbb{N}, q = 1, 2, \ldots, n\Big\}.$$

Let $\mathbf{m}_{i,k}^{\langle\varsigma\rangle} = M_i(\varsigma\hbar, kh)$ and $\mathbf{p}_{i,k}^{\langle\varsigma\rangle} = P_i(\varsigma\hbar, kh)$ for $(\varsigma, k) \in \mho_\varsigma \times \mathbb{Z}$, $i = 1, 2, \ldots, m$. Then GRNs (2.1) is full discretization scheme to the following stochastic GRNs with reaction diffusions

$$\begin{cases} \dfrac{\partial}{\partial t}M_i(x,t) = \displaystyle\sum_{q=1}^{n}\frac{\partial}{\partial x_q}\Big[\mu_{iq}\frac{\partial M_i(x,t)}{\partial x_q}\Big] - a_i^*(t)M_i(x,t) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{m} b_{ij}(t)f_j(P_j(x,t)) + \sum_{j=1}^{m}\gamma_{ij}(t)\sigma_j(P_j(x,t))\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{B}_{1j}(t) + I_i(t), \\ \dfrac{\partial}{\partial t}P_i(x,t) = \displaystyle\sum_{q=1}^{n}\frac{\partial}{\partial x_q}\Big[\nu_{iq}\frac{\partial P_i(x,t)}{\partial x_q}\Big] - c_i^*(t)P_i(x,t) \\ \qquad\qquad + d_i(t)M_i(x,t) + \displaystyle\sum_{j=1}^{m}\varpi_{ij}(t)\eta_j(M_j(x,t))\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{B}_{2j}(t), \quad \forall (x,t)\in\mho\times\mathbb{R}, \end{cases} \tag{2.2}$$

where $x = (x_1, \ldots, x_n)^T \in \mho \subseteq \mathbb{R}^n$ refers to space variable, $i = 1, 2, \ldots, m$. Hereby, the discrete techniques in GRNs (2.1) are exponential Euler difference (EED in short) for time variable and finite difference method (FDM in short) for space variable. More information about EED and FDM, please refer to papers [28–33] and [34–36], respectively.

According to the technique of FDM in papers [34–36], it follows from GRNs (2.1) that

$$\begin{cases} \dfrac{\partial}{\partial t}M_i(\varsigma\hbar,t) = \displaystyle\sum_{q=1}^{n}\mu_{iq}\tilde{\Delta}^2_{\hbar_q}M_i(\varsigma\hbar,t) - \Big[a_i^*(t) + 2\sum_{q=1}^{n}\frac{\mu_{iq}}{\hbar^2}\Big]M_i(\varsigma\hbar,t) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{m} b_{ij}(t)f_j(P_j(\varsigma\hbar,t)) + \sum_{j=1}^{m}\gamma_{ij}(t)\sigma_j(P_j(\varsigma\hbar,t))\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{B}_{1j}(t) + I_i(t), \\ \dfrac{\partial}{\partial t}P_i(\varsigma\hbar,t) = \displaystyle\sum_{q=1}^{n}\nu_{iq}\tilde{\Delta}^2_{\hbar_q}P_i(\varsigma\hbar,t) - \Big[c_i^*(t) + 2\sum_{q=1}^{n}\frac{\nu_{iq}}{\hbar^2}\Big]P_i(\varsigma\hbar,t) \\ \qquad\qquad + d_i(t)M_i(\varsigma\hbar,t) + \displaystyle\sum_{j=1}^{m}\varpi_{ij}(t)\eta_j(M_j(\varsigma\hbar,t))\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{B}_{2j}(t), \quad \forall (\varsigma,t)\in\mho_\varsigma\times\mathbb{R}, \end{cases} \tag{2.3}$$

where

$$\tilde{\Delta}^2_{\hbar_q}M_i(\varsigma\hbar,\cdot) := \frac{1}{\hbar^2}[M_i(\varsigma\hbar + \xi_q\hbar,\cdot) + M_i(\varsigma\hbar - \xi_q\hbar,\cdot)]$$

and

$$\tilde{\Delta}^2_{\hbar_q}P_i(\varsigma\hbar,\cdot) := \frac{1}{\hbar^2}[P_i(\varsigma\hbar + \xi_q\hbar,\cdot) + P_i(\varsigma\hbar - \xi_q\hbar,\cdot)]$$

for $q = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$.

Define $\mathbf{m}_{i,k}^{\langle\varsigma\rangle} = M_i(\varsigma\hbar, kh)$, $\mathbf{p}_{i,k}^{\langle\varsigma\rangle} = P_i(\varsigma\hbar, kh)$, $a_{i,k} := a_i(kh)$, $b_{i,k} := b_i(kh)$, $c_{i,k} := c_i(kh)$, $d_{i,k} := d_i(kh)$, $\gamma_{ij,k} := \gamma_{ij}(kh)$, $\varpi_{ij,k} := \varpi_{ij}(kh)$ and $I_{i,k} := I_i(kh)$ for $(\varsigma, k) \in \mho_\varsigma \times \mathbb{Z}$, $i, j = 1, 2, \ldots, m$. By applying EED in literatures [28–33] into (2.3), it yields GRNs (2.1).

**Remark 2.1.** *Recently, in literatures [28–33], EED has been widely employed to study discrete-time systems arisen from many fields. Initially, the authors [29] discussed the long time behaviours of time discrete neural networks by utilizing EED. Next, EED in papers [30–32] had been used to study the time discrete stochastic models. In the wake of high level of attention on fractional calculus, papers [28, 33] established the frame of EED to multi-delay Caputo-Fabrizio fractional-order differential equations and BAM neural networks. From the viewpoints of both theories and numerical examples in literatures [28–33], it is a powerful demonstration that EED is a more precise portrayal of time-continuous systems than Euler difference. To date, nevertheless, the concerns of discrete-time GRNs with discrete spatial diffusions have been addressed by very few scholars. Thus, GRNs (2.1) has a excellent research value.*

**Remark 2.2.** *As is well-known, FDM is an important way to solve partial differential equations in the area of numerical computations. Numerous literatures paid their attentions on the researches of space discrete models arisen from many fields of science and engineering and these models were called the lattice models (see [34–36]). Up to now, it exists several reports focusing on GRNs with reaction diffusions [19–23]. To the present knowledge of the authors, there are almost no papers dealing with the study of discrete space-time GRNs. As a result, the work at hand is expected to address such a void.*

In line with the theory of difference equations, a discrete-space and discrete-time constant variation formula to GRNs (2.1) will be established as follows.

**Lemma 2.1.** *GRNs (2.1) can be given expression to*

$$\begin{cases} \mathbf{m}_{i,k}^{\langle\varsigma\rangle} = \prod_{s=k_0}^{k-1} e^{-a_{i,s}h}\mathbf{m}_{i,k_0}^{\langle\varsigma\rangle} + \sum_{v=k_0}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_{i,s}h}(1-e^{-a_{i,v}h})}{a_{i,v}} \\ \qquad \times\left[\sum_{q=1}^{n}\mu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{m}_{i,v}^{\langle\varsigma\rangle} + \sum_{j=1}^{m} b_{ij,v}f_j(\mathbf{p}_{j,v}^{\langle\varsigma\rangle}) + \sum_{j=1}^{m}\gamma_{ij,v}\sigma_j(\mathbf{p}_{j,v}^{\langle\varsigma\rangle})w_{1j,v} + I_{i,v}\right], \\ \mathbf{p}_{i,k}^{\langle\varsigma\rangle} = \prod_{s=k_0}^{k-1} e^{-c_{i,s}h}\mathbf{p}_{i,k_0}^{\langle\varsigma\rangle} + \sum_{v=k_0}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-c_{i,s}h}(1-e^{-c_{i,v}h})}{c_{i,v}} \\ \qquad \times\left[\sum_{q=1}^{n}\nu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{p}_{i,v}^{\langle\varsigma\rangle} + d_{i,v}\mathbf{m}_{i,v}^{\langle\varsigma\rangle} + \sum_{j=1}^{m}\varpi_{ij,v}\eta_j(\mathbf{m}_{j,v}^{\langle\varsigma\rangle})w_{2j,v}\right], \end{cases} \tag{2.4}$$

*where $(\varsigma, k) \in \mho_\varsigma \times [k_0, \infty)_{\mathbb{Z}}$ with some initial point $k_0 \in \mathbb{Z}$, $i = 1, 2, \ldots, m$. Besides, it holds*

$$\mathbf{m}_{i,k}^{\langle\varsigma\rangle}\Big|_{\varsigma\in\partial\mho_\varsigma} = 0 = \mathbf{p}_{i,k}^{\langle\varsigma\rangle}\Big|_{\varsigma\in\partial\mho_\varsigma}, \quad \forall k \in [k_0, \infty)_{\mathbb{Z}}, i = 1, 2, \ldots, m.$$

**Remark 2.3.** *Let $\mu_{iq} = \nu_{iq} = 0$ for $i = 1, 2, \ldots, m, q = 1, 2, \ldots, n$, and get rid of the space variable in GRNs (2.1), then Eq (2.4) is turned into*

$$\begin{cases} \mathbf{m}_{i,k} = \prod_{s=k_0}^{k-1} e^{-a_{i,s}h}\mathbf{m}_{i,k_0} + \sum_{v=k_0}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_{i,s}h}(1-e^{-a_{i,v}h})}{a_{i,v}}\left[\sum_{j=1}^{m} b_{ij,v}f_j(\mathbf{p}_{j,v}) + \sum_{j=1}^{m}\gamma_{ij,v}\sigma_j(\mathbf{p}_{j,v})w_{1j,v} + I_{i,v}\right], \\ \mathbf{p}_{i,k} = \prod_{s=k_0}^{k-1} e^{-c_{i,s}h}\mathbf{p}_{i,k_0} + \sum_{v=k_0}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-c_{i,s}h}(1-e^{-c_{i,v}h})}{c_{i,v}}\left[d_{i,v}\mathbf{m}_{i,v} + \sum_{j=1}^{m}\varpi_{ij,v}\eta_j(\mathbf{m}_{j,v})w_{2j,v}\right], \quad \forall k \in [k_0, \infty)_{\mathbb{Z}}, \end{cases} \tag{2.5}$$

*for all $i, j = 1, 2, \ldots, m$. Based on Eq (2.5), we can study the existence of various solutions for discrete-time GRNs, e.g., almost periodicity [10], almost automorphism [28], etc.*

Let $L^2(\Omega, \mathbb{R}^m)$ denote the family of all square integrable $\mathbb{R}^m$-valued random variables with the norm

$$\|u\|_2 = \max_{1\le i\le m}\left[\mathbf{E}\left|u_i\right|^2\right]^{1/2}, \quad \forall u = (u_1, \ldots, u_m)^T \in L^2(\Omega, \mathbb{R}^m),$$

in which $\mathbf{E}$ denotes the expectation operator with respect to probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathbb{B}(\bar{\mho}_\varsigma \times \mathbb{Z}, L^2(\Omega, \mathbb{R}^{2m}))$ denote the whole bounded mappings from $\bar{\mho}_\varsigma \times \mathbb{Z}$ to $L^2(\Omega, \mathbb{R}^{2m})$ and define

$$\mathbb{X} = \left\{u \in \mathbb{B}(\bar{\mho}_\varsigma \times \mathbb{Z}, L^2(\Omega, \mathbb{R}^{2m}))\middle| u_k^{\langle\varsigma\rangle} = 0, \forall(\varsigma, k) \in \partial\mho_\varsigma \times \mathbb{Z}\right\}$$

endowed with the norm

$$\|u\|_{\mathbb{X}} = \sup_{(\varsigma,k)\in\mho_\varsigma\times\mathbb{Z}} \left\|u_k^{\langle\varsigma\rangle}\right\|_2 = \sup_{(\varsigma,k)\in\mho_\varsigma\times\mathbb{Z}} \max_{1\le i\le 2m}\left[\mathbf{E}\left|u_{i,k}^{\langle\varsigma\rangle}\right|^2\right]^{1/2}, \quad \forall u = (u_1, \ldots, u_{2m})^T \in \mathbb{X}.$$

**Definition 2.1.** *A discrete-space and discrete-time stochastic process* $\mathbf{w} = (\mathbf{m}_1, \ldots, \mathbf{m}_m, \mathbf{p}_1, \ldots, \mathbf{p}_m) \in \mathbb{X}$ *is said to be the solution of GRNs (2.1) on* $[k_0, \infty)_{\mathbb{Z}}$ *if it is* $\mathscr{F}_k$*-adaptive and meets constant variation Eq (2.4),* $\forall k \in [k_0, \infty)_{\mathbb{Z}}$.

**Lemma 2.2.** ( [37]) (Minkowski inequality) *If* $X, Y \in L^2(\Omega, \mathbb{R})$, *then*

$$\left(\mathbf{E}|X + Y|^2\right)^{\frac{1}{2}} \le \left(\mathbf{E}|X|^2\right)^{\frac{1}{2}} + \left(\mathbf{E}|Y|^2\right)^{\frac{1}{2}}.$$

**Lemma 2.3.** $\mathbf{E}|w_{j,k}|^2 = \dfrac{1}{h}$ *for* $k \in \mathbb{Z}$, $j = 1, 2, \ldots, n$.

**Proof.** By the definition of $w_{j,k}$ and Itô isometric property, it derives

$$\mathbf{E}|w_{j,k}|^2 = \frac{1}{h^2}\mathbf{E}\Big(\int_{kh}^{kh+h} \mathrm{d}\mathbf{w}_j(s)\Big)^2 = \frac{1}{h^2}\mathbf{E}\int_{kh}^{kh+h} \mathrm{d}s = \frac{1}{h}, \quad \forall k \in \mathbb{Z}, j = 1, 2, \ldots, n.$$

This completes the proof.

## 3. Global existence and mean square boundedness

This section mainly focuses on global existence, global finiteness and global mean square boundedness to GRNs (2.1). In the first place, via the constant variation formula in Lemma 2.1 and the theory of stochastic calculus, global existence and finiteness of the solutions to GRNs (2.1) have been studied. Additionally, global mean square boundedness to GRNs (2.1) is addressed with the helps of constant variation Eq (2.4) and Minkowski inequality in Lemma 2.2, etc.

Set

$$\underline{a}_i^* := \inf_{k\in\mathbb{Z}} |a_{i,k}^*|, \quad a_\diamond := \min_{1\le i\le m} \inf_{k\in\mathbb{Z}} a_{i,k}, \quad \underline{a}_i := \inf_{k\in\mathbb{Z}} |a_{i,k}|, \quad \bar{a}_i := \sup_{k\in\mathbb{Z}} |a_{i,k}|,$$

$$\underline{c}_i^* := \inf_{k\in\mathbb{Z}} |c_{i,k}^*|, \quad c_\diamond := \min_{1\le i\le m} \inf_{k\in\mathbb{Z}} c_{i,k}, \quad \underline{c}_i := \inf_{k\in\mathbb{Z}} |c_{i,k}|, \quad \bar{c}_i := \sup_{k\in\mathbb{Z}} |c_{i,k}|,$$

$$\bar{d}_i := \sup_{k\in\mathbb{Z}} |d_{i,k}|, \quad \bar{I}_i := \sup_{k\in\mathbb{Z}} |I_{i,k}|, \quad \bar{b}_{ij} := \sup_{k\in\mathbb{Z}} |b_{ij,k}|,$$

$$\bar{\gamma}_{ij} := \sup_{k\in\mathbb{Z}} |\gamma_{ij,k}|, \quad \bar{\varpi}_{ij} := \sup_{k\in\mathbb{Z}} |\varpi_{ij,k}|,$$

where $i, j = 1, 2, \ldots, m$.

The initial values of GRNs (2.1) are described by

$$\mathbf{m}_{i,k_0}^{\langle\varsigma\rangle} = \delta_i^{\langle\varsigma\rangle}, \quad \mathbf{p}_{i,k_0}^{\langle\varsigma\rangle} = \rho_i^{\langle\varsigma\rangle}, \qquad \forall\varsigma \in \bar{\mho}_\varsigma, i = 1, 2, \ldots, m, \tag{3.1}$$

where $\delta.$ and $\rho.$ are $\mathscr{F}_{k_0}$-adaptive and $\mathscr{F}_{k_0+1}$-adaptive, respectively.

Assume that $\mathbf{w} = (\mathbf{m}, \mathbf{p})^T$ with $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_m)^T$ and $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_m)^T$ is a solution of GRNs (2.1) with initial values as those in (3.1). Define $\delta = (\delta_1, \ldots, \delta_m)^T$ and $\rho = (\rho_1, \ldots, \rho_m)^T$.

**Theorem 3.1.** *Suppose that $\|\delta\|_{\mathbb{X}} < \infty$, $\|\rho\|_{\mathbb{X}} < \infty$ and the assumptions below hold.*

$(E_1)$ *$a_{i,k}$, $c_{i,k}$, $\mu_{iq}$, $\nu_{iq}$, $b_{ij,k}$, $d_{i,k}$, $\gamma_{ij,k}$, $\varpi_{ij,k}$ and $I_{i,k}$ are bounded constants or sequences, $k \in \mathbb{Z}$, $q = 1, 2, \ldots, n$, $i, j = 1, 2, \ldots, m$. Further, $\min_{1\le i\le m}\{\underline{a}, \underline{c}\} > 0$.*

$(E_2)$ *$\max_{j=1,2,\ldots,m}\{|f_j(s)|, |\sigma_j(s)|, |\eta_j(s)|\} \le \varsigma_1|s| + \varsigma_2$, where $\varsigma_1, \varsigma_2$ are two positive constants.*

*Then GRNs (2.1) with initial values (3.1) possesses a solution $\mathbf{w} = (\mathbf{m}, \mathbf{p})$ with $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_m)^T$ and $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_m)^T$ on $[k_0, +\infty)_{\mathbb{Z}}$.*

**Proof.** According to initial values in (3.1), it fulfills that

- **(a)** **(Existence)** $\mathbf{m}_k^{\langle\varsigma\rangle}$ and $\mathbf{p}_k^{\langle\varsigma\rangle}$ exist for $(\varsigma, k) \in \bar{\mho}_\varsigma \times [k_0, k_0 + 1]_{\mathbb{Z}}$.
- **(b)** **(Adaptability)** $\mathbf{m}_k^{\langle\varsigma\rangle}$ and $\mathbf{p}_k^{\langle\varsigma\rangle}$ are $\mathscr{F}_k$-adaptive for $(\varsigma, k) \in \bar{\mho}_\varsigma \times [k_0, k_0 + 1]_{\mathbb{Z}}$.
- **(c)** **(Mean square finiteness)** $\left\|\mathbf{m}_k^{\langle\varsigma\rangle}\right\|_2 < \infty$ and $\left\|\mathbf{p}_k^{\langle\varsigma\rangle}\right\|_2 < \infty$ for $(\varsigma, k) \in \bar{\mho}_\varsigma \times [k_0, k_0 + 1]_{\mathbb{Z}}$.

Next, a method of mathematical induction will be employed to prove $\mathbf{m}_k^{\langle\cdot\rangle}$ and $\mathbf{p}_k^{\langle\cdot\rangle}$ satisfying **(a)**–**(c)** for $k \in [k_0, K]_{\mathbb{Z}}$, where $K \ge k_0 + 2$ is an arbitrary integer. Assume that $\mathbf{m}_{k'}^{\langle\cdot\rangle}$ and $\mathbf{p}_{k'}^{\langle\cdot\rangle}$ meet **(a)**–**(c)** for $k' \in [k_0 + 1, K]_{\mathbb{Z}}$.

In accordance to constant variation Eq (2.4), $\mathbf{m}_{i,k'+1}^{\langle\varsigma\rangle}$ and $\mathbf{m}_{i,k'+1}^{\langle\varsigma\rangle}$ are $\mathscr{F}_{k'+1}$-adaptive and exist for $\varsigma \in \bar{\mho}_\varsigma$ by the adaptability and existences of $\mathbf{m}_{i,s}^{\langle\varsigma\rangle}$ and $\mathbf{p}_{i,s}^{\langle\varsigma\rangle}$ in $(\varsigma, s) \in \bar{\mho}_\varsigma \times [k_0, k']_{\mathbb{Z}}$, $i = 1, 2, \ldots, m$. Further, by Minkowski inequality in Lemmas (2.2) and (2.3), it derives

$$\begin{aligned}
\left\|\mathbf{m}_{k'+1}^{\langle\varsigma\rangle}\right\|_2 &= \max_{1\le i\le m}\left[\mathbf{E}\left|\mathbf{m}_{i,k'+1}^{\langle\varsigma\rangle}\right|^2\right]^{\frac{1}{2}} \\
&= \max_{1\le i\le m}\Bigg\{\mathbf{E}\Bigg|\prod_{s=k_0}^{k'} e^{-a_{i,s}h}\mathbf{m}_{i,k_0}^{\langle\varsigma\rangle} + \sum_{v=k_0}^{k'}\prod_{s=v+1}^{k'}\frac{e^{-a_{i,s}h}(1 - e^{-a_{i,v}h})}{a_{i,v}} \\
&\quad \times\Bigg[\sum_{q=1}^{n}\mu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{m}_{i,v}^{\langle\varsigma\rangle} + \sum_{j=1}^{m} b_{ij,v}f_j(\mathbf{p}_{j,v}^{\langle\varsigma\rangle}) + \sum_{j=1}^{m}\gamma_{ij,v}\sigma_j(\mathbf{p}_{j,v}^{\langle\varsigma\rangle})w_{1j,v} + I_{i,v}\Bigg]\Bigg|^2\Bigg\}^{\frac{1}{2}} \\
&\le \max_{1\le i\le m} e^{\bar{a}_i h(k'-k_0+1)}\|\delta\|_{\mathbb{X}} + \max_{1\le i\le m}\frac{1 - e^{-\underline{a}_i h}}{\underline{a}_i(1 - e^{\bar{a}_i h})}\Bigg[\sum_{q=1}^{n}|\mu_{iq}|\max_{(\varsigma,s)\in\bar{\mho}_\varsigma\times[k_0,k']_{\mathbb{Z}}}\left\|\Delta_{\hbar_q}^2\mathbf{m}_s^{\langle\varsigma\rangle}\right\|_2 \\
&\quad + \sum_{j=1}^{m}\bar{b}_{ij}\varsigma_1\max_{(\varsigma,s)\in\bar{\mho}_\varsigma\times[k_0,k']_{\mathbb{Z}}}\|\mathbf{p}_s^{\langle\varsigma\rangle}\|_2 + h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\gamma}_{ij}\varsigma_1\max_{(\varsigma,s)\in\bar{\mho}_\varsigma\times[k_0,k']_{\mathbb{Z}}}\|\mathbf{p}_s^{\langle\varsigma\rangle}\|_2
\end{aligned}$$

$$+\sum_{j=1}^{m}\bar{b}_{ij}\varsigma_2+h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\gamma}_{ij}\varsigma_2+\bar{I}\Big],\quad \forall\varsigma\in\mho_{\varsigma}.$$

Similarly, it deduces

$$\begin{aligned}
\left\|\mathbf{p}_{k'+1}^{\langle\varsigma\rangle}\right\|_2 &= \max_{1\le i\le m}\left[\mathbf{E}\left|\mathbf{p}_{i,k'+1}^{\langle\varsigma\rangle}\right|^2\right]^{\frac{1}{2}}\\
&= \max_{1\le i\le m}\Bigg\{\mathbf{E}\Bigg|\prod_{s=k_0}^{k-1}e^{-c_{i,s}h}\mathbf{p}_{i,k_0}^{\langle\varsigma\rangle}+\sum_{v=k_0}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-c_{i,s}h}(1-e^{-c_{i,v}h})}{c_{i,v}}\\
&\quad\times\Big[\sum_{q=1}^{n}\nu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{p}_{i,v}^{\langle\varsigma\rangle}+d_{i,v}\mathbf{m}_{i,v}^{\langle\varsigma\rangle}+\sum_{j=1}^{m}\varpi_{ij,v}\eta_j(\mathbf{m}_{j,v}^{\langle\varsigma\rangle})w_{2j,v}\Big]\Bigg|^2\Bigg\}^{\frac{1}{2}}\\
&\le \max_{1\le i\le m}e^{\bar{c}_ih(k'-k_0+1)}\|\rho\|_{\mathbb{X}}+\max_{1\le i\le m}\frac{1-e^{-\underline{c}_ih}}{\underline{c}_i(1-e^{\bar{c}_ih})}\Big[\sum_{q=1}^{n}|\nu_{ip}|\max_{(\varsigma,s)\in\bar{\mho}_{\varsigma}\times[k_0,k']_{\mathbb{Z}}}\left\|\Delta_{\hbar_q}^2\mathbf{p}_s^{\langle\varsigma\rangle}\right\|_2\\
&\quad+\bar{d}_i\max_{(\varsigma,s)\in\bar{\mho}_{\varsigma}\times[k_0,k']_{\mathbb{Z}}}\|\mathbf{m}_s^{\langle\varsigma\rangle}\|_2+h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\varpi}_{ij}\max_{(\varsigma,s)\in\bar{\mho}_{\varsigma}\times[k_0,k']_{\mathbb{Z}}}\|\mathbf{m}_s^{\langle\varsigma\rangle}\|_2+h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\varpi}_{ij}\varsigma_2\Big],
\end{aligned}$$

where $\varsigma\in\mho_{\varsigma}$.

By the mean square finiteness of $\mathbf{u}_{\cdot,s}^{\langle\varsigma\rangle}$ and $\mathbf{p}_{\cdot,s}^{\langle\varsigma\rangle}$ in $(\varsigma,s)\in\bar{\mho}_{\varsigma}\times[k_0,k']_{\mathbb{Z}}$, it concludes

$$\left\|\tilde{\Delta}_{\hbar_q}^2\mathbf{m}_s^{\langle\varsigma\rangle}\right\|_2<\infty,\quad \left\|\mathbf{m}_{k'+1}^{\langle\varsigma\rangle}\right\|_2<\infty,\quad \left\|\tilde{\Delta}_{\hbar_q}^2\mathbf{p}_s^{\langle\varsigma\rangle}\right\|_2<\infty,\quad \left\|\mathbf{p}_{k'+1}^{\langle\varsigma\rangle}\right\|_2<\infty$$

for all $(\varsigma,s)\in\bar{\mho}_{\varsigma}\times[k_0,k']_{\mathbb{Z}}$, $q=1,2,\ldots,n$. Consequently, $\mathbf{m}_{k'+1}^{\langle\cdot\rangle}$ and $\mathbf{p}_{k'+1}^{\langle\cdot\rangle}$ satisfy mean square finiteness in **(c)**. By the arbitrariness of $K$, $\mathbf{m}_k^{\langle\cdot\rangle}$ and $\mathbf{p}_k^{\langle\cdot\rangle}$ meet **(a)**–**(c)** for $k\in[k_0,\infty)_{\mathbb{Z}}$, which induces global existence of the solution of GRNs (2.1) with initial values (3.1). This completes the proof.

Theorem 3.1 only gives global finiteness in mean square sense, the application value is relatively small. In the following, global boundedness in mean square sense of GRNs (2.1) with initial values (3.1), which has great potential for application in practical processes, will be addressed on the basis of constant variation Eq (2.4) and Minkowski inequality.

**Theorem 3.2.** *Supposing that the following conditions are valid.*

$(G_1)$ $f_j(0)=\sigma_j(0)=\eta_j(0)=0$ *and it exists positive numbers* $L_j^f$, $L_j^\sigma$ *and* $L_j^\eta$ *such that*

$$|f_j(u)-f_j(v)|\le L_j^f|u-v|,\quad |\sigma_j(u)-\sigma_j(v)|\le L_j^\sigma|u-v|,\quad |\eta_j(u)-\eta_j(v)|\le L_j^\eta|u-v|$$

*for any* $u,v\in\mathbb{R}$, $j=1,2,\ldots,m$.

$(G_2)$ $\min\{a_\diamond,c_\diamond\}>0$ *and*

$$\lambda_1:=\max_{1\le i\le m}\left\{\frac{1}{\underline{a}_i}\Big[\sum_{q=1}^{n}\frac{2|\mu_{iq}|}{\hbar^2}+\sum_{j=1}^{m}\bar{b}_{ij}L_j^f+h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\gamma}_{ij}L_j^\sigma\Big],\frac{1}{\underline{c}_i}\Big[\sum_{q=1}^{n}\frac{2|\nu_{iq}|}{\hbar^2}+\bar{d}_i+h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\varpi}_{ij}L_j^\eta\Big]\right\}<1.$$

*Then the solution of GRNs (2.1) is global mean square boundedness, i.e.,*

$$\max_{(\varsigma,k)\in\mho_\varsigma\times[k_0,\infty)_{\mathbb{Z}}}\left\{\left\|\mathbf{m}_k^{\langle\varsigma\rangle}\right\|_2,\left\|\mathbf{p}_k^{\langle\varsigma\rangle}\right\|_2,\right\}\leq\frac{\lambda_0}{1-\lambda_1},$$

*where* $\lambda_0:=\max\limits_{1\leq i\leq m}\left\{\|\delta\|_{\mathbb{X}}+\dfrac{1}{\underline{a}_i}\bar{I}_i,\|\rho\|_{\mathbb{X}}\right\}$.

**Proof.** For any constant $K\in[k_0,\infty)_{\mathbb{Z}}$, define

$$\mathcal{M}_K:=\max_{(\varsigma,k)\in\mho_\varsigma\times[k_0,K]_{\mathbb{Z}}}\left\|\mathbf{m}_k^{\langle\varsigma\rangle}\right\|_2,\quad\mathcal{P}_K:=\max_{(\varsigma,k)\in\mho_\varsigma\times[k_0,K]_{\mathbb{Z}}}\left\|\mathbf{p}_k^{\langle\varsigma\rangle}\right\|_2,\quad\mathcal{Z}_K:=\max_{(\varsigma,k)\in\mho_\varsigma\times[k_0,K]_{\mathbb{Z}}}\left\|\mathbf{z}_k^{\langle\varsigma\rangle}\right\|_2.$$

In view of Theorem 3.1, $\mathcal{M}_K,\mathcal{P}_K$ and $\mathcal{Z}_K$ are finite. With the helps of Eq (2.4), Minkowski inequality in Lemmas (2.2) and (2.3), it computes

$$\begin{aligned}\left\|\mathbf{m}_k^{\langle\varsigma\rangle}\right\|_2&=\max_{1\leq i\leq m}\left[\mathbf{E}\left|\mathbf{m}_{i,k}^{\langle\varsigma\rangle}\right|^2\right]^{\frac{1}{2}}\\&=\max_{1\leq i\leq m}\Bigg\{\mathbf{E}\Bigg|\prod_{s=k_0}^{k-1}e^{-a_{i,s}h}\mathbf{m}_{i,k_0}^{\langle\varsigma\rangle}+\sum_{v=k_0}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_{i,s}h}(1-e^{-a_{i,v}h})}{a_{i,v}}\\&\quad\times\Big[\sum_{q=1}^{n}\mu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{m}_{i,v}^{\langle\varsigma\rangle}+\sum_{j=1}^{m}b_{ij,v}f_j(\mathbf{p}_{j,v}^{\langle\varsigma\rangle})+\sum_{j=1}^{m}\gamma_{ij,v}\sigma_j(\mathbf{p}_{j,v}^{\langle\varsigma\rangle})w_{1j,v}+I_{i,v}\Big]\Bigg|^2\Bigg\}^{\frac{1}{2}}\\&\leq\|\delta\|_{\mathbb{X}}+\max_{1\leq i\leq m}\frac{1-e^{-\underline{a}_ih}}{\underline{a}_i}\sum_{v=k_0}^{k-1}e^{-\underline{a}_ih(k-v-1)}\Bigg\{\mathbf{E}\Big[\sum_{q=1}^{n}|\mu_{iq}|\left|\Delta_{\hbar_q}^2\mathbf{m}_{i,v}^{\langle\varsigma\rangle}\right|\\&\quad+\sum_{j=1}^{m}\bar{b}_{ij}L_j^f\left|\mathbf{p}_{j,v}^{\langle\varsigma\rangle}\right|+\sum_{j=1}^{m}\bar{\gamma}_{ij}L_j^\sigma\left|\mathbf{p}_{j,v}^{\langle\varsigma\rangle}\right|\left|w_{1j,v}\right|+\bar{I}_i\Big]^2\Bigg\}^{\frac{1}{2}}\\&\leq\|\delta\|_{\mathbb{X}}+\max_{1\leq i\leq m}\frac{1}{\underline{a}_i}\Big[\sum_{q=1}^{n}\frac{2|\mu_{iq}|}{\hbar^2}\mathcal{M}_K+\sum_{j=1}^{m}\bar{b}_{ij}L_j^f\mathcal{P}_K+h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\gamma}_{ij}L_j^\sigma\mathcal{P}_K+\bar{I}_i\Big]\\&\leq\|\delta\|_{\mathbb{X}}+\max_{1\leq i\leq m}\frac{1}{\underline{a}_i}\bar{I}_i+\max_{1\leq i\leq m}\frac{1}{\underline{a}_i}\Big[\sum_{q=1}^{n}\frac{2|\mu_{iq}|}{\hbar^2}+\sum_{j=1}^{m}\bar{b}_{ij}L_j^f+h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\gamma}_{ij}L_j^\sigma\Big]\mathcal{Z}_K\\&\leq\lambda_0+\lambda_1\mathcal{Z}_K,\quad\forall(\varsigma,k)\in\mho_\varsigma\times[k_0,K]_{\mathbb{Z}}.\end{aligned}\tag{3.2}$$

Similarly,

$$\begin{aligned}\left\|\mathbf{p}_k^{\langle\varsigma\rangle}\right\|_2&=\max_{1\leq i\leq m}\left[\mathbf{E}\left|\mathbf{p}_{i,k}^{\langle\varsigma\rangle}\right|^2\right]^{\frac{1}{2}}\\&=\max_{1\leq i\leq m}\Bigg\{\mathbf{E}\Bigg|\prod_{s=k_0}^{k-1}e^{-c_{i,s}h}\mathbf{p}_{i,k_0}^{\langle\varsigma\rangle}+\sum_{v=k_0}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-c_{i,s}h}(1-e^{-c_{i,v}h})}{c_{i,v}}\\&\quad\times\Big[\sum_{q=1}^{n}\nu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{p}_{i,v}^{\langle\varsigma\rangle}+d_{i,v}\mathbf{m}_{i,v}^{\langle\varsigma\rangle}+\sum_{j=1}^{m}\varpi_{ij,v}\eta_j(\mathbf{m}_{j,v}^{\langle\varsigma\rangle})w_{2j,v}\Bigg|^2\Bigg\}^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&\leq \|\rho\|_{\mathbb{X}} + \max_{1\leq i\leq m} \frac{1-e^{-\underline{c}_i h}}{\underline{c}_i} \sum_{v=k_0}^{k-1} e^{-\underline{c}_i h(k-v-1)} \Bigg\{ \mathbf{E}\Bigg[ \sum_{q=1}^{n} |\nu_{iq}| \Big|\Delta^2_{\hbar_q} \mathbf{p}^{\langle\varsigma\rangle}_{i,v}\Big| \\
&\quad + \bar{d}_i \Big|\mathbf{m}^{\langle\varsigma\rangle}_{j,v}\Big| + \sum_{j=1}^{m} \bar{\varpi}_{ij} L^{\eta}_j \Big|\mathbf{m}^{\langle\varsigma\rangle}_{j,v}\Big| \Big|w_{2j,v}\Big| \Bigg]^2 \Bigg\}^{\frac{1}{2}} \\
&\leq \|\rho\|_{\mathbb{X}} + \max_{1\leq i\leq m} \frac{1}{\underline{c}_i} \Bigg[ \sum_{q=1}^{n} \frac{2|\nu_{iq}|}{\hbar^2} \mathcal{P}_K + \bar{d}_i \mathcal{M}_K + h^{-\frac{1}{2}} \sum_{j=1}^{m} \bar{\varpi}_{ij} L^{\eta}_j \mathcal{M}_K \Bigg] \\
&\leq \|\rho\|_{\mathbb{X}} + \max_{1\leq i\leq m} \frac{1}{\underline{c}_i} \Bigg[ \sum_{q=1}^{n} \frac{2|\nu_{iq}|}{\hbar^2} + \bar{d}_i + h^{-\frac{1}{2}} \sum_{j=1}^{m} \bar{\varpi}_{ij} L^{\eta}_j \Bigg] \mathcal{Z}_K \\
&\leq \lambda_0 + \lambda_1 \mathcal{Z}_K, \quad \forall (\varsigma, k) \in \mho_\varsigma \times [k_0, K]_{\mathbb{Z}}.
\end{aligned} \tag{3.3}
$$

Combining (3.2) and (3.3), it follows from $(G_2)$ that

$$
\mathcal{Z}_K \leq \lambda_0 + \lambda_1 \mathcal{Z}_K \Longrightarrow \mathcal{Z}_K \leq \frac{\lambda_0}{1-\lambda_1},
$$

which induces from the arbitrariness of $K$ that

$$
\max_{(\varsigma,k)\in\mho_\varsigma\times[k_0,\infty)_{\mathbb{Z}}} \Big\|\mathbf{m}^{\langle\varsigma\rangle}_k\Big\|_2 \leq \frac{\lambda_0}{1-\lambda_1}, \quad \max_{(\varsigma,k)\in\mho_\varsigma\times[k_0,\infty)_{\mathbb{Z}}} \Big\|\mathbf{p}^{\langle\varsigma\rangle}_k\Big\|_2 \leq \frac{\lambda_0}{1-\lambda_1}.
$$

This completes the proof.

**Remark 3.1.** *The assumption* $\min\{a_\diamond, c_\diamond\} > 0$ *in* $(G_2)$*, in which* $a_{i,\cdot} := a^*_{i,\cdot} + 2\sum_{q=1}^{n} \frac{\mu_{iq}}{\hbar^2}$ *and* $c_{i,\cdot} := c^*_{i,\cdot} + 2\sum_{q=1}^{n} \frac{\nu_{iq}}{\hbar^2}$ *for* $i = 1, 2, \dots, m$*, implies that the sum of the diffusions' intensities of the ith mRNA or ith protein is greater than -*$\frac{1}{2}\hbar^2$ *times the decay rates of the ith mRNA or ith protein, respectively,* $i = 1, 2, \dots, m$*. Furthermore, if* $\mu_{i\cdot}$ *and* $\nu_{i\cdot}$ *are nonnegative constants, then* $\lambda_1 < 1$ *in* $(G_2)$ *is equal to*

$$
a^*_i > \sum_{j=1}^{m} \bar{b}_{ij} L^f_j + h^{-\frac{1}{2}} \sum_{j=1}^{m} \bar{\gamma}_{ij} L^{\sigma}_j \text{ and } c^*_i > \bar{d}_i + h^{-\frac{1}{2}} \sum_{j=1}^{m} \bar{\varpi}_{ij} L^{\eta}_j, \quad i = 1, 2, \dots, m.
$$

*This indicates that the spatial diffusions with nonnegative intensive coefficients have no influence on global mean square boundedness of GRNs (2.1).*

**Remark 3.2.** *To assure the validity of condition* $(G_2)$*, the following aspects should be paid attention in the application.*

(**i**) *The coefficients of GRNs (2.1), except for* $a_{i,k}$ *and* $c_{i,k}$*, should be better to choose lesser constants, instead,* $a_{i,k}$ *and* $c_{i,k}$ *should be selected biggish positive constants for any* $i = 1, 2, \dots, m$*,* $k \in \mathbb{Z}$*.*

(**ii**) *Generally, small positive constants are selected for the time and space steps' length* $h$ *and* $\hbar$*.*

(**iii**) *The activation functions* $f_j$*,* $\sigma_j$ *and* $\eta_j$ *of GRNs (2.1) are best to select some small enough positive constants* $L^f_j$*,* $L^{\sigma}_j$ *and* $L^{\eta}_j$ *for any* $i = 1, 2, \dots, m$*.*

**Remark 3.3.** *Remarkably, the assumptions* $f_j(0) = \sigma_j(0) = \eta_j(0) = 0$ *in* $(G_1)$ *are not obligatory. We can remove this assumption, but it will increase the computational difficulties of this discussion.*

**Remark 3.4.** *If $h, \hbar \to 0$ in assumption $(G_2)$, then $\lambda_1 < 1$ is hard to be valid. So it is a disadvantage of the proposed method in this discussion. We hope it can be improved in the future works.*

When $\gamma_{ij} = \varpi_{ij} = 0$ in GRNs (2.1) for $i, j = 1, 2, \ldots, m$, then it is turned into a determined networks as noted below

$$\begin{cases} \mathbf{m}_{i,k+1}^{\langle\varsigma\rangle} = e^{-a_{i,k}h}\mathbf{m}_{i,k}^{\langle\varsigma\rangle} + \dfrac{1-e^{-a_{i,k}h}}{a_{i,k}}\Big[\sum_{q=1}^{n}\mu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{m}_{i,k}^{\langle\varsigma\rangle} + \sum_{j=1}^{m} b_{ij,k}f_j(\mathbf{p}_{j,k}^{\langle\varsigma\rangle}) + I_{i,k}\Big], \\ \mathbf{p}_{i,k+1}^{\langle\varsigma\rangle} = e^{-c_{i,k}h}\mathbf{p}_{i,k}^{\langle\varsigma\rangle} + \dfrac{1-e^{-c_{i,k}h}}{c_{i,k}}\Big[\sum_{q=1}^{n}\nu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{p}_{i,k}^{\langle\varsigma\rangle} + d_{i,k}\mathbf{m}_{i,k}^{\langle\varsigma\rangle}\Big], \quad \forall(\varsigma,k)\in\mho_\varsigma\times\mathbb{Z}, \end{cases} \tag{3.4}$$

where $i = 1, 2, \ldots, m$.

Further, let $\mu_{i\cdot} = \nu_{i\cdot} = 0$ in GRNs (3.4), it is changed to the following discrete-time networks

$$\begin{cases} \mathbf{m}_{i,k+1}^{\langle\varsigma\rangle} = e^{-a_{i,k}h}\mathbf{m}_{i,k}^{\langle\varsigma\rangle} + \dfrac{1-e^{-a_{i,k}h}}{a_{i,k}}\Big[\sum_{j=1}^{m} b_{ij,k}f_j(\mathbf{p}_{j,k}^{\langle\varsigma\rangle}) + I_{i,k}\Big], \\ \mathbf{p}_{i,k+1}^{\langle\varsigma\rangle} = e^{-c_{i,k}h}\mathbf{p}_{i,k}^{\langle\varsigma\rangle} + \dfrac{1-e^{-c_{i,k}h}}{c_{i,k}}d_{i,k}\mathbf{m}_{i,k}^{\langle\varsigma\rangle}, \quad \forall(\varsigma,k)\in\mho_\varsigma\times\mathbb{Z}, \end{cases} \tag{3.5}$$

where $i = 1, 2, \ldots, m$.

**Corollary 3.1.** *Supposing that $(G_1)$ and the following condition hold.*

$(G_3)$ $\min\{a_\diamond, c_\diamond\} > 0$ *and*

$$\lambda_2 := \max_{1\le i\le m}\left\{\frac{1}{\underline{a}_i}\Big[\sum_{q=1}^{n}\frac{2|\mu_{iq}|}{\hbar^2} + \sum_{j=1}^{m}\bar{b}_{ij}L_j^f\Big], \frac{1}{\underline{c}_i}\Big[\sum_{q=1}^{n}\frac{2|\nu_{iq}|}{\hbar^2} + \bar{d}_i\Big]\right\} < 1.$$

*Then the solution of GRNs (3.4) is global mean square boundedness, i.e.,*

$$\max_{(\varsigma,k)\in\mho_\varsigma\times[k_0,\infty)_{\mathbb{Z}}}\left\{\left\|\mathbf{m}_k^{\langle\varsigma\rangle}\right\|_2, \left\|\mathbf{p}_k^{\langle\varsigma\rangle}\right\|_2,\right\} \le \frac{\lambda_0}{1-\lambda_2}.$$

**Corollary 3.2.** *Supposing that $(G_1)$ and the following condition hold.*

$(G_4)$ $\min\{a_\diamond, c_\diamond\} > 0$ *and* $\lambda_3 := \max_{1\le i\le m}\left\{\frac{1}{\underline{a}_i}\Big[\sum_{j=1}^{m}\bar{b}_{ij}L_j^f\Big], \frac{1}{\underline{c}_i}\bar{d}_i\right\} < 1.$

*Then the solution of GRNs (3.5) is global mean square boundedness, i.e.,*

$$\max_{(\varsigma,k)\in\mho_\varsigma\times[k_0,\infty)_{\mathbb{Z}}}\left\{\left\|\mathbf{m}_k^{\langle\varsigma\rangle}\right\|_2, \left\|\mathbf{p}_k^{\langle\varsigma\rangle}\right\|_2,\right\} \le \frac{\lambda_0}{1-\lambda_3}.$$

**Remark 3.5.** *The existing literatures [24–27] had reported various dynamical explorations of discrete-time GRNs (3.5). However, for discrete spatial diffusions networks (2.1), almost no paper involves.*

## 4. Global exponential stability and optimal convergent rate

This section is chiefly concerned with global exponential stability in mean square sense to GRNs (2.1) via the theory of stochastic calculus and some inequalities' skills. Firstly, the definition of global exponential stability in mean square sense with the exponential convergent rate to GRNs (2.1) is described as a mathematical expression, which displays the asymptotic relationships of two arbitrary solutions of GRNs (2.1) with different initial values. Subsequently, global exponential stability in mean square sense to GRNs (2.1) is achieved and one optimization problem under nonlinear constraints is created to gain the optimal convergent rate. At last, to acquire the optimal solution of convergent rate in a feasible region, an enforceable algorithm is proposed to solve this regard.

Let $(\mathbf{m},\mathbf{p})^T$ and $(\tilde{\mathbf{m}},\tilde{\mathbf{p}})^T$ with $\mathbf{m} = (\mathbf{m}_1,\dots,\mathbf{m}_m)^T$, $\tilde{\mathbf{m}} = (\tilde{\mathbf{m}}_1,\dots,\tilde{\mathbf{m}}_m)^T$, $\mathbf{p} = (\mathbf{p}_1,\dots,\mathbf{p}_m)^T$ and $\tilde{\mathbf{p}} = (\tilde{\mathbf{p}}_1,\dots,\tilde{\mathbf{p}}_m)^T$ be two arbitrary solutions of GRNs (2.1) with initial values

$$\mathbf{m}_{k_0}^{\langle\varsigma\rangle} = \delta^{\langle\varsigma\rangle} = (\delta_1^{\langle\varsigma\rangle},\dots,\delta_m^{\langle\varsigma\rangle})^T, \quad \tilde{\mathbf{m}}_{k_0}^{\langle\varsigma\rangle} = \tilde{\delta}^{\langle\varsigma\rangle} = (\tilde{\delta}_1^{\langle\varsigma\rangle},\dots,\tilde{\delta}_m^{\langle\varsigma\rangle})^T,$$

$$\mathbf{p}_{k_0}^{\langle\varsigma\rangle} = \rho^{\langle\varsigma\rangle} = (\rho_1^{\langle\varsigma\rangle},\dots,\rho_m^{\langle\varsigma\rangle})^T, \quad \tilde{\mathbf{p}}_{k_0}^{\langle\varsigma\rangle} = \tilde{\rho}^{\langle\varsigma\rangle} = (\tilde{\rho}_1^{\langle\varsigma\rangle},\dots,\tilde{\rho}_m^{\langle\varsigma\rangle})^T,$$

where $\tilde{\delta}^{\langle\varsigma\rangle}$ and $\tilde{\rho}^{\langle\varsigma\rangle}$ are $\mathscr{F}_{k_0}$-adaptive and $\mathscr{F}_{k_0+1}$-adaptive, respectively; $\varsigma \in \mho_\varsigma$.

Let $\mathbf{e} = (\mathbf{e}_1,\dots,\mathbf{e}_m)^T$, $\tilde{\mathbf{e}} = (\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_m)^T$, $\hat{\delta} = (\hat{\delta}_1,\dots,\hat{\delta}_m)^T$ and $\hat{\rho} = (\hat{\rho}_1,\dots,\hat{\rho}_m)^T$, where $\mathbf{e}_i = \mathbf{m}_i - \tilde{\mathbf{m}}_i$, $\tilde{\mathbf{e}}_i = \mathbf{p}_i - \tilde{\mathbf{p}}_i$, $\hat{\delta}_i = \delta_i - \tilde{\delta}_i$ and $\hat{\rho}_i = \rho_i - \tilde{\rho}_i$, $i = 1,2,\dots,m$. From (2.4), it gets

$$\begin{cases} \mathbf{e}_{i,k}^{\langle\varsigma\rangle} = \displaystyle\prod_{s=k_0}^{k-1} e^{-a_{i,s}h}\hat{\delta}_i^{\langle\varsigma\rangle} + \sum_{v=k_0}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_{i,s}h}(1-e^{-a_{i,v}h})}{a_{i,v}} \\ \qquad\times\left[\displaystyle\sum_{q=1}^{n}\mu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{e}_{i,v}^{\langle\varsigma\rangle} + \sum_{j=1}^{m} b_{ij,v}\tilde{f}_j(\tilde{\mathbf{e}}_{j,v}^{\langle\varsigma\rangle}) + \sum_{j=1}^{m}\gamma_{ij,v}\tilde{\sigma}_j(\tilde{\mathbf{e}}_{j,v}^{\langle\varsigma\rangle})w_{1j,v}\right], \\ \tilde{\mathbf{e}}_{i,k}^{\langle\varsigma\rangle} = \displaystyle\prod_{s=k_0}^{k-1} e^{-c_{i,s}h}\hat{\rho}_i^{\langle\varsigma\rangle} + \sum_{v=k_0}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-c_{i,s}h}(1-e^{-c_{i,v}h})}{c_{i,v}} \\ \qquad\times\left[\displaystyle\sum_{q=1}^{n}\nu_{iq}\tilde{\Delta}_{\hbar_q}^2\tilde{\mathbf{e}}_{i,v}^{\langle\varsigma\rangle} + d_{i,v}\mathbf{e}_{i,v}^{\langle\varsigma\rangle} + \sum_{j=1}^{m}\varpi_{ij,v}\tilde{\eta}_j(\mathbf{e}_{j,v}^{\langle\varsigma\rangle})w_{2j,v}\right], \end{cases} \tag{4.1}$$

where $(\varsigma,k) \in \mho_\varsigma \times [k_0,\infty)_{\mathbb{Z}}$ with some initial point $k_0 \in \mathbb{Z}$,

$$\tilde{f}_j(\tilde{\mathbf{e}}_{j,k}^{\langle\varsigma\rangle}) = f_j(\mathbf{p}_{j,k}^{\langle\varsigma\rangle}) - f_j(\tilde{\mathbf{p}}_{j,k}^{\langle\varsigma\rangle}), \quad \tilde{\sigma}_j(\tilde{\mathbf{e}}_{j,k}^{\langle\varsigma\rangle}) = \sigma_j(\mathbf{p}_{j,k}^{\langle\varsigma\rangle}) - \sigma_j(\tilde{\mathbf{p}}_{j,k}^{\langle\varsigma\rangle}), \quad \tilde{\eta}_j(\mathbf{e}_{j,k}^{\langle\varsigma\rangle}) = \eta_j(\mathbf{m}_{j,k}^{\langle\varsigma\rangle}) - \eta_j(\tilde{\mathbf{m}}_{j,k}^{\langle\varsigma\rangle}),$$

$i, j = 1,2,\dots,m$. Besides, it holds $\mathbf{e}_k^{\langle\varsigma\rangle}\Big|_{\varsigma\in\partial\mho_\varsigma} = 0$ and $\tilde{\mathbf{e}}_k^{\langle\varsigma\rangle}\Big|_{\varsigma\in\partial\mho_\varsigma} = 0$ for all $k \in [k_0,\infty)_{\mathbb{Z}}$.

**Definition 4.1.** *GRNs (2.1) is said to be globally mean-square $\kappa$-exponential convergent if it exists $\mathscr{L} > 0$ and $0 < \kappa < 1$ such that*

$$\max_{\varsigma\in\mho_\varsigma}\left\{\left\|\mathbf{e}_k^{\langle\varsigma\rangle}\right\|_2, \left\|\tilde{\mathbf{e}}_k^{\langle\varsigma\rangle}\right\|_2\right\} \le \mathscr{L}e^{-\kappa(k-k_0)h}\max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\}, \quad \forall k \in [k_0,\infty)_{\mathbb{Z}}.$$

*Hereon, $\kappa$ is called the convergent rate of GRNs (2.1).*

**Theorem 4.1.** *Let* $(G_1)$*–*$(G_2)$ *hold. Then GRNs (2.1) is globally mean-square exponential convergent with the best convergent rate* $\kappa$ *with respect to a two-tuples* $(k_0, \mathscr{L})$*, which can be addressed by solving the following optimization problem under nonlinear constraints*

$$\min_{(\kappa,\mathscr{L}')^T}(-\kappa) \text{ subject to } \begin{cases} 0<\mathscr{L}'<1, \\ 0<\kappa\le \min\limits_{1\le i\le m}\{\underline{a}_i,\underline{c}_i\}, \\ \dfrac{(1-e^{-\min_{1\le i\le m}\{\underline{a}_i,\underline{c}_i\}h})e^{\kappa h}}{1-e^{-(\min_{1\le i\le m}\{\underline{a}_i,\underline{c}_i\}-\kappa)h}}\lambda_1<1-\mathscr{L}', \end{cases} \tag{4.2}$$

*where* $\mathscr{L}' := \frac{1}{\mathscr{L}}$.

**Proof.** Owing to $(G_2)$, it has $\mathscr{L} > 1$ and $0 < \kappa < \min_{1\le i\le m}\{\underline{a}_i, \underline{c}_i\}$ ensuring

$$\upsilon_\kappa := \frac{(1-e^{-\min_{1\le i\le m}\{\underline{a}_i,\underline{c}_i\}h})e^{\kappa h}}{1-e^{-(\min_{1\le i\le m}\{\underline{a}_i,\underline{c}_i\}-\kappa)h}}\lambda_1 < 1-\frac{1}{\mathscr{L}}. \tag{4.3}$$

In the light of the error system (4.1), it yields

$$\begin{aligned}\left|\mathbf{e}_{i,k}^{\langle\varsigma\rangle}\right| \le{}& e^{-\underline{a}_i h(k-k_0)}\left|\hat{\delta}_i^{\langle\varsigma\rangle}\right| + \frac{1-e^{-\underline{a}_i h}}{\underline{a}_i}\sum_{v=k_0}^{k-1} e^{-\underline{a}_i h(k-v-1)} \\ &\times\Big[\sum_{q=1}^{n}|\mu_{iq}|\left|\tilde{\Delta}^2_{\hbar_q}\mathbf{e}_{i,v}^{\langle\varsigma\rangle}\right| + \sum_{j=1}^{m}\bar{b}_{ij}L_j^f\left|\tilde{\mathbf{e}}_{j,v}^{\langle\varsigma\rangle}\right| + \sum_{j=1}^{m}\bar{\gamma}_{ij}L_j^\sigma\left|\tilde{\mathbf{e}}_{j,v}^{\langle\varsigma\rangle}\right|\left|w_{1j,v}\right|\Big],\end{aligned}$$

where $(\varsigma, k) \in \mho_\varsigma \times [k_0, \infty)_\mathbb{Z}$, $i = 1, 2, \dots, n$. By Minkowski, $C_p$ inequalities and Lemma 2.3, one has

$$\begin{aligned}\left\|\mathbf{e}_k^{\langle\varsigma\rangle}\right\|_2 ={}& \max_{1\le i\le m}\left\{\mathbf{E}\left|\mathbf{e}_{i,k}^{\langle\varsigma\rangle}\right|^2\right\}^{\frac{1}{2}} \\ \le{}& \max_{1\le i\le m} e^{-\underline{a}_i h(k-k_0)}\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2 + \max_{1\le i\le m}\frac{1-e^{-\underline{a}_i h}}{\underline{a}_i}\sum_{v=k_0}^{k-1} e^{-\underline{a}_i h(k-v-1)} \\ &\times\Big[\sum_{q=1}^{n}\frac{2|\mu_{iq}|}{\hbar^2}\left\|\mathbf{e}_v^{\langle\varsigma\rangle}\right\|_2 + \sum_{j=1}^{m}\bar{b}_{ij}L_j^f\left\|\tilde{\mathbf{e}}_v^{\langle\varsigma\rangle}\right\|_2 + h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\gamma}_{ij}L_j^\sigma\left\|\tilde{\mathbf{e}}_v^{\langle\varsigma\rangle}\right\|_2\Big],\end{aligned} \tag{4.4}$$

where $(\varsigma, k) \in \mho_\varsigma \times [k_0, \infty)_\mathbb{Z}$, $i = 1, 2, \dots, n$.

Similarly,

$$\begin{aligned}\left\|\tilde{\mathbf{e}}_k^{\langle\varsigma\rangle}\right\|_2 \le{}& \max_{1\le i\le m} e^{-\underline{c}_i h(k-k_0)}\left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2 + \max_{1\le i\le m}\frac{1-e^{-\underline{c}_i h}}{\underline{c}_i}\sum_{v=k_0}^{k-1} e^{-\underline{c}_i h(k-v-1)} \\ &\times\Big[\sum_{q=1}^{n}\frac{2|\nu_{iq}|}{\hbar^2}\left\|\tilde{\mathbf{e}}_v^{\langle\varsigma\rangle}\right\|_2 + \bar{d}_i\left\|\mathbf{e}_{i,v}^{\langle\varsigma\rangle}\right\|_2 + h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\varpi}_{ij}L_j^\eta\left\|\tilde{\mathbf{e}}_v^{\langle\varsigma\rangle}\right\|_2\Big],\end{aligned} \tag{4.5}$$

where $(\varsigma, k) \in \mho_\varsigma \times [k_0, \infty)_\mathbb{Z}$, $i = 1, 2, \dots, n$.

Subsequently, we will use the proof by contradiction. Supposing that

$$\max_{\varsigma\in\mho_\varsigma}\left\{\left\|\mathbf{e}_k^{\langle\varsigma\rangle}\right\|_2, \left\|\tilde{\mathbf{e}}_k^{\langle\varsigma\rangle}\right\|_2\right\} \le \mathscr{L}e^{-\kappa(k-k_0)h}\max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\}, \quad \forall k\in[k_0,\infty)_\mathbb{Z}. \tag{4.6}$$

If not, it must exist $k' \in (k_0, \infty)_{\mathbb{Z}}$ ensuring that

$$\max_{\varsigma\in\mho_\varsigma}\left\{\left\|\mathbf{e}_k^{\langle\varsigma\rangle}\right\|_2, \left\|\tilde{\mathbf{e}}_k^{\langle\varsigma\rangle}\right\|_2\right\} \leq \mathscr{L} e^{-\kappa(k-k_0)h} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\}, \quad \forall k \in [k_0, k'-1]_{\mathbb{Z}} \tag{4.7}$$

and

$$\max_{\varsigma\in\mho_\varsigma}\left\{\left\|\mathbf{e}_{k'}^{\langle\varsigma\rangle}\right\|_2, \left\|\tilde{\mathbf{e}}_{k'}^{\langle\varsigma\rangle}\right\|_2\right\} > \mathscr{L} e^{-\kappa(k'-k_0)h} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\}. \tag{4.8}$$

In line with (4.4), (4.7) and (4.3) in turn, it induces

$$\begin{aligned}
\max_{\varsigma\in\mho_\varsigma}\left\|\mathbf{e}_{k'}^{\langle\varsigma\rangle}\right\|_2 \leq\ & \max_{1\leq i\leq m} e^{-\underline{a}_i h(k'-k_0)} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} + \max_{1\leq i\leq m} \frac{1-e^{-\underline{a}_i h}}{\underline{a}_i} \sum_{v=k_0}^{k'-1} e^{-\underline{a}_i h(k'-v-1)} \\
& \times\left[\sum_{q=1}^{n}\frac{2|\mu_{iq}|}{\hbar^2} + \sum_{j=1}^{m}\bar{b}_{ij}L_j^f + h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\gamma}_{ij}L_j^\sigma\right]\mathscr{L} e^{-\kappa(v-k_0)h} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} \\
\leq\ & \max_{1\leq i\leq m} e^{-\underline{a}_i h(k'-k_0)} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} + \max_{1\leq i\leq m} \frac{1-e^{-\underline{a}_i h}}{\underline{a}_i} \sum_{v=k_0}^{k'-1} e^{-(\underline{a}_i-\kappa)(k'-v-1)h} \\
& \times\left[\sum_{q=1}^{n}\frac{2|\mu_{iq}|}{\hbar^2} + \sum_{j=1}^{m}\bar{b}_{ij}L_j^f + h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\gamma}_{ij}L_j^\sigma\right]e^{\kappa h}\mathscr{L} e^{-\kappa(k'-k_0)h} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} \\
\leq\ & \max_{1\leq i\leq m} e^{-\underline{a}_i h(k'-k_0)} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} + \max_{1\leq i\leq m} \frac{1-e^{-\underline{a}_i h}}{\underline{a}_i(1-e^{-(\underline{a}_i-\kappa)h})} \\
& \times\left[\sum_{q=1}^{n}\frac{2|\mu_{iq}|}{\hbar^2} + \sum_{j=1}^{m}\bar{b}_{ij}L_j^f + h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\gamma}_{ij}L_j^\sigma\right]e^{\kappa h}\mathscr{L} e^{-\kappa(k'-k_0)h} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} \\
=\ & \left\{\frac{1}{\mathscr{L}}\max_{1\leq i\leq m} e^{-(\underline{a}_i-\kappa)(k'-k_0)h} + \upsilon_\kappa\right\}\mathscr{L} e^{-\kappa(k'-k_0)h} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} \\
\leq\ & \mathscr{L} e^{-\kappa(k'-k_0)h} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\}.
\end{aligned} \tag{4.9}$$

This induces a confliction with (4.8) and (4.6) is valid.

In the light of (4.5), (4.7) and (4.3) in turn, it results in

$$\begin{aligned}
\max_{\varsigma\in\mho_\varsigma}\left\|\tilde{\mathbf{e}}_{k'}^{\langle\varsigma\rangle}\right\|_2 \leq\ & \max_{1\leq i\leq m} e^{-\underline{c}_i h(k'-k_0)} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} + \max_{1\leq i\leq m} \frac{1-e^{-\underline{c}_i h}}{\underline{c}_i} \sum_{v=k_0}^{k'-1} e^{-\underline{c}_i h(k'-v-1)} \\
& \times\left[\sum_{q=1}^{n}\frac{2|\nu_{iq}|}{\hbar^2} + \bar{d}_i + h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\varpi}_{ij}L_j^\eta\right]\mathscr{L} e^{-\kappa(v-k_0)h} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} \\
\leq\ & \max_{1\leq i\leq m} e^{-\underline{c}_i h(k'-k_0)} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} + \max_{1\leq i\leq m} \frac{1-e^{-\underline{c}_i h}}{\underline{c}_i(1-e^{-(\underline{c}_i-\kappa)h})} \\
& \times\left[\sum_{q=1}^{n}\frac{2|\nu_{iq}|}{\hbar^2} + \bar{d}_i + h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\varpi}_{ij}L_j^\eta\right]e^{\kappa h}\mathscr{L} e^{-\kappa(k'-k_0)h} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\} \\
=\ & \left\{\frac{1}{\mathscr{L}}\max_{1\leq i\leq m} e^{-(\underline{c}_i-\kappa)(k'-k_0)h} + \upsilon_\kappa\right\}\mathscr{L} e^{-\kappa(k'-k_0)h} \max_{\varsigma\in\mho_\varsigma}\left\{\left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2\right\}
\end{aligned}$$

$$\leq \mathscr{L} e^{-\kappa(k'-k_0)h} \max_{\varsigma\in\mho_\varsigma} \left\{ \left\|\hat{\delta}^{\langle\varsigma\rangle}\right\|_2, \left\|\hat{\rho}^{\langle\varsigma\rangle}\right\|_2 \right\}. \tag{4.10}$$

This induces a confliction with (4.8) and (4.6) is valid. As a consequence, GRNs (2.1) is globally mean-square exponential convergent. The proof is finished.

In view of assumption $(G_2)$, the feasible region of optimization problem (4.2) is not empty. However, the optimal solution to optimization problem (4.2) may not exist. To ensure the existence of optimal solution, we can simplify optimization problem (4.2) according to the following algorithm.

**Algorithm 1** Optimal exponential convergent rate of GRNs (2.1)

**1)** Initial the coefficients of GRNs (2.1) and compute $\lambda_1$.

**2)** Take the value of $\mathscr{L}'$ in interval $(0, 1-\lambda_1)$.

**3)** Solve the following optimization problem under nonlinear constraints:

$$\min_{\kappa}(-\kappa) \text{ subject to } \begin{cases} 0 < \kappa \leq \min\limits_{1\leq i\leq m}\{\underline{a}_i, \underline{c}_i\}, \\ \dfrac{(1-e^{-\min_{1\leq i\leq m}\{\underline{a}_i,\underline{c}_i\}h})e^{\kappa h}}{1-e^{-(\min_{1\leq i\leq m}\{\underline{a}_i,\underline{c}_i\}-\kappa)h}}\lambda_1 < 1-\mathscr{L}'. \end{cases}$$

Similar to the arguments as those in Corollaries 3.1 and 3.2, it directly obtains the following two corollaries according to Theorem 4.1.

**Corollary 4.1.** *Supposing that $(G_1)$ and $(G_3)$ hold. Then GRNs (3.4) is globally mean-square exponential convergent with the best convergent rate $\kappa$ with respect to a two-tuples $(k_0, \mathscr{L})$, which can be addressed by solving the following optimization problem under nonlinear constraints*

$$\min_{(\kappa,\mathscr{L}')^T}(-\kappa) \textit{ subject to } \begin{cases} 0 < \mathscr{L}' < 1, \\ 0 < \kappa \leq \min\limits_{1\leq i\leq m}\{\underline{a}_i, \underline{c}_i\}, \\ \dfrac{(1-e^{-\min_{1\leq i\leq m}\{\underline{a}_i,\underline{c}_i\}h})e^{\kappa h}}{1-e^{-(\min_{1\leq i\leq m}\{\underline{a}_i,\underline{c}_i\}-\kappa)h}}\lambda_2 < 1-\mathscr{L}'. \end{cases}$$

**Corollary 4.2.** *Supposing that $(G_1)$ and $(G_4)$ hold. Then GRNs (3.5) is globally mean-square exponential convergent with the best convergent rate $\kappa$ with respect to a two-tuples $(k_0, \mathscr{L})$, which can be addressed by solving the following optimization problem under nonlinear constraints*

$$\min_{(\kappa,\mathscr{L}')^T}(-\kappa) \textit{ subject to } \begin{cases} 0 < \mathscr{L}' < 1, \\ 0 < \kappa \leq \min\limits_{1\leq i\leq m}\{\underline{a}_i, \underline{c}_i\}, \\ \dfrac{(1-e^{-\min_{1\leq i\leq m}\{\underline{a}_i,\underline{c}_i\}h})e^{\kappa h}}{1-e^{-(\min_{1\leq i\leq m}\{\underline{a}_i,\underline{c}_i\}-\kappa)h}}\lambda_3 < 1-\mathscr{L}'. \end{cases}$$

**Remark 4.1.** *Under the assumptions in* Theorem 4.1 *and as analysed in* Remark 3.1*, the spatial diffusions with nonnegative intensive coefficients have no influence on global exponential convergence of GRNs (2.1) as well.*

**Remark 4.2.** *Global exponential stability of GRNs is an important researching subject in the discussions of the past decades, because it not only depends on the scale of the initial values, but also tends to the equilibrium state with a constant convergence rate, and literatures [5, 7, 9–11] had been made extensive research on this subject. However, it exists few papers focusing on the optimal convergence rate to global exponential stability of GRNs, and* Theorem 4.1*, Corollaries 4.1 and 4.2 in this article accomplishes this work with respect to discrete space and time GRNs (2.1), (3.4) and (3.5), respectively.*

## 5. Random $\tau$-periodic solution

Define $\blacktriangle := \{(k,s) \in \mathbb{Z}\times\mathbb{Z}, s \le k\}$, coordinate function $\mathbb{B}_{pj,k}(\omega) := \mathbb{B}_{pj}(kh,\omega) = \omega_{pj,k}$ and $\vartheta$ : $\mathbb{Z}\times\Omega \to \Omega \subseteq \mathbb{R}^{2m}$ by

$$\vartheta_k\omega(s) = \Big(\omega_{11,k+s} - \omega_{11,k}, \ldots, \omega_{1m,k+s} - \omega_{1m,k}, \omega_{21,k+s} - \omega_{21,k}, \ldots, \omega_{2m,k+s} - \omega_{2m,k}\Big)^T,$$

where $\omega = (\omega_{11}, \ldots, \omega_{1m}, \omega_{21}, \ldots, \omega_{2m})^T \in \Omega$, $k, s \in \mathbb{Z}$, $p = 1, 2$, $j = 1, 2, \ldots, m$. In view of reference [38], $(\Omega, \mathcal{F}, \mathbf{P}, (\vartheta_k)_{k\in\mathbb{Z}})$ is a metric dynamical system. Considering a stochastic periodic semi-flow $X : \blacktriangle \times \Omega \times \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ of period $\tau \in \mathbb{Z}$, which fulfils the semi-flow relationship

$$X(k,r,\omega) = X(k,s,\omega) \circ X(s,r,\omega)$$

and the periodic property

$$X(k+\tau, s+\tau, \omega) = X(k,s,\vartheta_\tau\omega), \quad \forall r \le s \le k, r, s, k \in \mathbb{Z}.$$

**Definition 5.1.** ( [12]) *Random periodic path of period $\tau \in \mathbb{Z}$ of the semi-flow $X : \blacktriangle \times \Omega \times \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is an $\mathcal{F}$-measurable sequence $Y : \mathbb{Z}\times\Omega \to \mathbb{R}^n$ ensuring*

$$X(k,s,\omega)Y(s,\omega) = Y(k,\omega), \quad Y(s+\tau,\omega) = Y(s,\vartheta_\tau\omega), \quad \forall (k,s) \in \blacktriangle, \omega \in \Omega, \ a.e.$$

In this section, we use $\mathbf{w}^{\langle\cdot\rangle}_{\cdot}(k_0,\varphi) = (\mathbf{m}^{\langle\cdot\rangle}_{\cdot}(k_0,\varphi), \mathbf{p}^{\langle\cdot\rangle}_{\cdot}(k_0,\varphi))^T$ to denote the solution of GRNs (2.1) with initial value $\varphi = (\delta,\rho)^T$ starting from time $k_0$, where $\delta$ and $\rho$ are defined as that in Section 3. Consequently, $X^{\langle\cdot\rangle}(k,s) : \Omega\times\mathbb{R}^{2m} \to \mathbb{R}^{2m}$ defined by $X^{\langle\cdot\rangle}(k,s)\varphi = \mathbf{w}^{\langle\cdot\rangle}_k(s,\varphi)$ becomes a semi-flow, $\forall k \in [s,\infty)_{\mathbb{Z}}$, where $s \in \mathbb{Z}$ is the starting point here.

**Theorem 5.1.** *Let assumptions $(G_1)$–$(G_2)$ and the following condition hold.*

$(G_5)$ *$a_{i,k}$, $b_{i,k}$, $c_{i,k}$, $d_{i,k}$, $\gamma_{ij,k}$, $\varpi_{ij,k}$ and $I_{i,k}$ are $\tau$-periodic sequences with respect to (w.r.t.) time variable $k \in \mathbb{Z}$, $i, j = 1, 2, \ldots, m$.*

*Then a random $\tau$-periodic process $\mathbf{w}^{\langle\cdot\rangle}_{*,k} = \Big(\mathbf{m}^{\langle\cdot\rangle}_{*,k}, \mathbf{p}^{\langle\cdot\rangle}_{*,k}\Big)^T \in L^2(\Omega,\mathbb{R}^{2m})$ $(k \in \mathbb{N})$ solves GRNs (2.1).*

**Proof.** According to Theorem 3.2, $\mathbf{w}^{\langle t\rangle}_{\cdot}(k_0,\cdot) : L^2(\Omega,\mathbb{R}^{2m}) \to L^2(\Omega,\mathbb{R}^{2m})$. Let $\varphi = (\delta,\rho)^T \in L^2(\Omega,\mathbb{R}^{2m})$. By the property of semi-flow, for any integers $k, k', k'' \ge 0$,

$$\mathbf{w}^{\langle\cdot\rangle}_k(-k'\tau - k''\tau,\varphi) = \mathbf{w}^{\langle\cdot\rangle}_k(-k'\tau) \circ \mathbf{w}^{\langle\cdot\rangle}_{-k'\tau}(-k'\tau - k''\tau,\varphi).$$

In accordance with Theorem 4.1, for any $\epsilon > 0$, it has $k_* > 0$ such that

$$\left\|\mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau - k''\tau, \varphi) - \mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \varphi)\right\|_2 < \epsilon, \quad k' > k_*, k'' \in \mathbb{N}, k \in \mathbb{Z}.$$

So it exists $k_{**} > 0$ such that for any $k', k''' \geq k_{**}$,

$$\left\|\mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \varphi) - \mathbf{w}_k^{\langle\cdot\rangle}(-k'''\tau, \varphi)\right\|_2 < \epsilon,$$

i.e., $\left\{\mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \varphi) : k' \in \mathbb{N}\right\}$ is a Cauchy sequence and

$$\mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \varphi) \longrightarrow \mathbf{w}_{*,k}^{\langle\cdot\rangle} \text{ in } L^2(\Omega, \mathbb{R}^{2m}) \text{ as } k' \to \infty, k \in \mathbb{N}.$$

Define a semi-flow $X^{\langle\cdot\rangle}(k, s, \varphi) := \mathbf{w}_k^{\langle\cdot\rangle}(k, s, \varphi)$ for all $s, k \in \mathbb{Z}$. It attains

$$X^{\langle\cdot\rangle}(k, s) \circ \mathbf{w}_s^{\langle\cdot\rangle}(-k'\tau, \varphi) \longrightarrow X^{\langle\cdot\rangle}(k, s) \circ \mathbf{w}_{*,s}^{\langle\cdot\rangle} \text{ as } k' \to \infty, \text{ where } k \in \mathbb{N},$$

which is addressed by the continuity of $\mathbf{w}_k^{\langle\cdot\rangle}(s, \cdot) : L^2(\Omega, \mathbb{R}^{2m}) \to L^2(\Omega, \mathbb{R}^{2m})$, $\forall s, k \in \mathbb{Z}$. Besides,

$$X^{\langle\cdot\rangle}(k, s) \circ \mathbf{w}_s^{\langle\cdot\rangle}(-k'\tau, \varphi) = \mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \varphi) \to \mathbf{w}_{*,k}^{\langle\cdot\rangle} \text{ in } L^2(\Omega, \mathbb{R}^m), \text{ as } k' \to \infty, k \in \mathbb{N}.$$

As a result,

$$X^{\langle\cdot\rangle}(k, s) \circ \mathbf{w}_{*,s}^{\langle\cdot\rangle} = \mathbf{w}_{*,k}^{\langle\cdot\rangle}, \ \mathbf{P}-\ a.s., k \in \mathbb{N}.$$

Let $\tilde{\varphi} = (\tilde{\delta}, \tilde{\rho})^T \in L^2(\Omega, \mathbb{R}^{2m})$ be another initial values of GRNs (2.1). From Theorem 4.1, it has

$$\left\|\mathbf{w}_{*,k}^{\langle\cdot\rangle} - \mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \tilde{\varphi})\right\|_2 \leq \left\|\mathbf{w}_{*,k}^{\langle\cdot\rangle} - \mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \varphi)\right\|_2 + \left\|\mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \varphi) - \mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \tilde{\varphi})\right\|_2$$

tends to 0 as $k' \to \infty$ for $k \in \mathbb{N}$. So the convergence is not related to the initial value.

Finally, random $\tau$-periodicity should be demonstrated. Based on Eq (2.4), it gains

$$\begin{aligned}
\mathbf{m}_{i,k+\tau}^{\langle\varsigma\rangle}(-k'\tau + \tau, \varphi) = {} & e^{-a_{i,s}(k+k'\tau)h}\delta_i^{\langle\varsigma\rangle} + \sum_{v=-k'\tau}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_{i,s}h}(1 - e^{-a_{i,v}h})}{a_{i,v}} \\
& \times\Big[\sum_{q=1}^{n}\mu_{iq}\tilde{\Delta}_{\hbar_q}^2\mathbf{m}_{i,v+\tau}^{\langle\varsigma\rangle}(-k'\tau + \tau, \varphi) + \sum_{j=1}^{m}b_{ij,v}f_j(\mathbf{p}_{j,v+\tau}^{\langle\varsigma\rangle}(-k'\tau + \tau, \varphi)) \\
& + \sum_{j=1}^{m}\gamma_{ij,v}\sigma_j(\mathbf{p}_{j,v+\tau}^{\langle\varsigma\rangle}(-k'\tau + \tau, \varphi))\frac{1}{h}\Delta\tilde{w}_{1j,v} + b_{i,v}\mathbf{p}_{i,v+\tau}^{\langle\varsigma\rangle}(-k'\tau + \tau, \phi) + I_{i,v}\Big]
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{p}_{i,k+\tau}^{\langle\varsigma\rangle}(-k'\tau + \tau, \varphi) = {} & e^{-c_{i,s}(k+k'\tau)h}\rho_i^{\langle\iota\rangle} + \sum_{v=-k'\tau}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-c_{i,s}h}(1 - e^{-c_{i,v}h})}{c_{i,v}} \\
& \times\Big[\sum_{q=1}^{n}\nu_{iq}\Delta_{\hbar_q}^2\mathbf{p}_{i,v+\tau}^{\langle\varsigma\rangle}(-k'\tau + \tau, \varphi) + d_{i,v}\mathbf{m}_{i,v+\tau}^{\langle\varsigma\rangle}(-k'\tau + \tau, \varphi) \\
& + \sum_{j=1}^{m}\varpi_{ij,v}\eta_j(\mathbf{m}_{j,v+\tau}^{\langle\varsigma\rangle}(-k'\tau + \tau, \varphi))\frac{1}{h}\Delta\tilde{w}_{2j,v}\Big], \quad \forall k' \in \mathbb{Z}, \varsigma \in \mho_\varsigma,
\end{aligned}$$

where $\tilde{w}_k = (\tilde{w}_{1,k}, \tilde{w}_{2,k})^T = \vartheta_\tau\omega(k)$ with $\tilde{w}_{1,k} = (\tilde{w}_{11,k}, \ldots, \tilde{w}_{1m,k})^T$ and $\tilde{w}_{2,k} = (\tilde{w}_{21,k}, \ldots, \tilde{w}_{2m,k})^T$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, m$. In addition,

$$\begin{aligned}\vartheta_\tau \mathbf{m}_{i,k}^{\langle\varsigma\rangle}(-k'\tau, \phi) = {} & e^{-a_{i,s}(k+k'\tau)h}\vartheta_\tau\delta_i^{\langle\varsigma\rangle} + \sum_{v=-k'\tau}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_{i,s}h}(1-e^{-a_{i,v}h})}{a_{i,v}} \\ & \times\Big[\sum_{q=1}^{n}\mu_{iq}\tilde{\Delta}_{\hbar_q}^2\vartheta_\tau\mathbf{m}_{i,v}^{\langle\varsigma\rangle}(-k'\tau,\varphi) + \sum_{j=1}^{m} b_{ij,v}f_j(\vartheta_\tau\mathbf{p}_{j,v}^{\langle\varsigma\rangle}(-k'\tau,\varphi)) \\ & + \sum_{j=1}^{m}\gamma_{ij,v}\sigma_j(\vartheta_\tau\mathbf{p}_{j,v}^{\langle\varsigma\rangle}(-k'\tau,\varphi))\frac{1}{h}\Delta\tilde{w}_{1j,v} + b_{i,v}\vartheta_\tau\mathbf{p}_{i,v}^{\langle\varsigma\rangle}(-k'\tau,\phi) + I_{i,v}\Big]\end{aligned}$$

and

$$\begin{aligned}\vartheta_\tau \mathbf{p}_{i,k}^{\langle\varsigma\rangle}(-k'\tau, \phi) = {} & e^{-c_{i,s}(k+k'\tau)h}\vartheta_\tau\rho_i^{\langle\varsigma\rangle} + \sum_{v=-k'\tau}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-c_{i,s}h}(1-e^{-c_{i,v}h})}{c_{i,v}}\Big[\sum_{q=1}^{n}\nu_{iq}\Delta_{\hbar_q}^2\vartheta_\tau\mathbf{p}_{i,v}^{\langle\varsigma\rangle}(-k'\tau,\varphi) \\ & + d_{i,v}\vartheta_\tau\mathbf{m}_{i,v}^{\langle\varsigma\rangle}(-k'\tau,\varphi) + \sum_{j=1}^{m}\varpi_{ij,v}\eta_j(\vartheta_\tau\mathbf{m}_{j,v}^{\langle\varsigma\rangle}(-k'\tau,\varphi))\frac{1}{h}\Delta\tilde{w}_{2j,v}\Big],\end{aligned}$$

where $k, k' \in \mathbb{Z}$, $\vartheta \in \mho_\varsigma$, $i = 1, 2, \ldots, m$.

By pathwise uniqueness of the solution to GRNs (2.1), it gets

$$\begin{aligned}\mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \varphi(\vartheta_\tau\omega))\,(w.r.t.\,\vartheta_\tau\omega) &= \vartheta_\tau\mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \varphi(\omega))\,(w.r.t.\,\omega) \\ &= \mathbf{w}_{k+\tau}^{\langle\cdot\rangle}(-k'\tau+\tau, \varphi(\vartheta_\tau\omega))\,(w.r.t.\,\omega), \quad \forall k, k' \in \mathbb{Z}.\end{aligned}$$

Consequently,

$$\mathbf{w}_{k+\tau}^{\langle\cdot\rangle}(-k'\tau+\tau, \varphi(\vartheta_\tau\omega))\,(w.r.t.\,\omega) \xrightarrow{L^2} \mathbf{w}_{*,k+\tau}^{\langle\cdot\rangle}\,(w.r.t.\,\omega),$$

$$\mathbf{w}_k^{\langle\cdot\rangle}(-k'\tau, \varphi(\vartheta_\tau\omega))\,(w.r.t.\,\vartheta_\tau\omega) \xrightarrow{L^2} \mathbf{w}_{*,k}^{\langle\cdot\rangle}\,(w.r.t.\,\vartheta_\tau\omega),$$

as $k' \to \infty$, where $k \in \mathbb{N}$. Thus, $\mathbf{w}_{*,k+\tau}^{\langle\cdot\rangle}(\omega) = \mathbf{w}_{*,k}^{\langle\cdot\rangle}(\vartheta_\tau\omega)$, $\mathbf{P}$-a.s., $\forall k \in \mathbb{N}$. This completes the proof.

In allusion to determined GRNs (3.4) and (3.5), we have the following results.

**Corollary 5.1.** *Let assumptions* $(G_1)$, $(G_3)$ *and* $(G_5)$ *hold. Then GRNs (3.4) possesses a random* $\tau$*-periodic solution.*

**Corollary 5.2.** *Let assumptions* $(G_1)$, $(G_4)$ *and* $(G_5)$ *hold. Then GRNs (3.5) possesses a random* $\tau$*-periodic solution.*

**Remark 5.1.** *According to assumptions in Theorems 4.1 and 5.1, the rules concluded in* Remark 3.1 *are also applicative for global exponential stability and random periodicity of GRNs (2.1). In the case of stochastic neural networks, periodic dynamics is an important behavior among various dynamical performances. By employing semi-flow relationship of stochastic models, the existence of random periodic solutions of some nonlinear stochastic systems had been studied in the published literatures [12–15]. It should be noted that the research of random periodic solutions to stochastic GRNs, let alone stochastic discrete-time GRNs with discrete spatial diffusions, has not been addressed to the authors' knowledge.*

## 6. Experimental illustrations

This section gives an experimental example to verify the feasibility of main results for discrete space and time stochastic GRNs, which have been addressed in the former sections of this article.

Considering the following discrete-time stochastic GRNs with discrete spatial diffusions

$$\begin{cases} \begin{pmatrix} \mathbf{m}_{1,k+1}^{\langle\varsigma\rangle} \\ \mathbf{m}_{2,k+1}^{\langle\varsigma\rangle} \end{pmatrix} = \begin{pmatrix} e^{-9h} & 0 \\ 0 & e^{-12h} \end{pmatrix} \begin{pmatrix} \mathbf{m}_{1,k}^{\langle\varsigma\rangle} \\ \mathbf{m}_{2,k}^{\langle\varsigma\rangle} \end{pmatrix} + \begin{pmatrix} \frac{1-e^{-9h}}{9} & 0 \\ 0 & \frac{1-e^{-12h}}{12} \end{pmatrix} \Bigg[ 0.12\tilde{\Delta}_{\hbar}^{2} \begin{pmatrix} \mathbf{m}_{1,k}^{\langle\varsigma\rangle} \\ \mathbf{m}_{2,k}^{\langle\varsigma\rangle} \end{pmatrix} \\ \qquad - \begin{pmatrix} 1.5 & 0.5\sin(k\pi + \frac{\pi}{5}) \\ 0.8 & 0.2\cos(k\pi + \frac{\pi}{5}) \end{pmatrix} \begin{pmatrix} f_1(\mathbf{p}_{1,k}^{\langle\varsigma\rangle}) \\ f_2(\mathbf{p}_{2,k}^{\langle\varsigma\rangle}) \end{pmatrix} + \begin{pmatrix} 0.1 & 0 \\ 0.2 & 0.05 \end{pmatrix} \\ \qquad \times \begin{pmatrix} \sigma_1(\mathbf{p}_{1,k}^{\langle\varsigma\rangle}) w_{11,k} \\ \sigma_2(\mathbf{p}_{2,k}^{\langle\varsigma\rangle}) w_{12,k} \end{pmatrix} + \begin{pmatrix} 1.5 + 0.5\sin(k\pi + \frac{\pi}{5}) \\ 0.8 + 0.2\cos(k\pi + \frac{\pi}{5}) \end{pmatrix} \Bigg], \\ \begin{pmatrix} \mathbf{p}_{1,k+1}^{\langle\varsigma\rangle} \\ \mathbf{p}_{2,k+1}^{\langle\varsigma\rangle} \end{pmatrix} = \begin{pmatrix} e^{-14h} & 0 \\ 0 & e^{-10h} \end{pmatrix} \begin{pmatrix} \mathbf{p}_{1,k}^{\langle\varsigma\rangle} \\ \mathbf{p}_{2,k}^{\langle\varsigma\rangle} \end{pmatrix} + \begin{pmatrix} \frac{1-e^{-14h}}{14} & 0 \\ 0 & \frac{1-e^{-10h}}{10} \end{pmatrix} \Bigg[ 0.15\tilde{\Delta}_{\hbar}^{2} \begin{pmatrix} \mathbf{p}_{1,k}^{\langle\varsigma\rangle} \\ \mathbf{p}_{2,k}^{\langle\varsigma\rangle} \end{pmatrix} \\ \qquad + \begin{pmatrix} \sin(k\pi + \frac{\pi}{3}) & 0.1 \\ 0 & \cos(k\pi + \frac{2\pi}{5}) \end{pmatrix} \begin{pmatrix} \eta_1(\mathbf{m}_{1,k}^{\langle\varsigma\rangle}) w_{21,k} \\ \eta_2(\mathbf{m}_{2,k}^{\langle\varsigma\rangle}) w_{22,k} \end{pmatrix} + 0.1 \begin{pmatrix} |\mathbf{m}_{1,k}^{\langle\varsigma\rangle}| \\ |\mathbf{m}_{2,k}^{\langle\varsigma\rangle}| \end{pmatrix} \Bigg], \end{cases} \tag{6.1}$$

where $(\varsigma, k) \in (0, 10) \times \mathbb{Z}_0$,

$$\mathbf{m}_{i,k}^{\langle\varsigma\rangle}\Big|_{\varsigma=0} = \mathbf{m}_{i,k}^{\langle\varsigma\rangle}\Big|_{\varsigma=10} = 0, \quad \mathbf{p}_{i,k}^{\langle\varsigma\rangle}\Big|_{\varsigma=0} = \mathbf{p}_{i,k}^{\langle\varsigma\rangle}\Big|_{\varsigma=10} = 0, \quad \forall k \in \mathbb{Z}_0, i = 1, 2.$$

Taking $h = 0.1$ and $\hbar = 0.5$. Corresponding to GRNs (2.1),

$$a_{1,k} = 9, \quad a_{2,k} = 12, \quad c_{1,k} = 14, \quad c_{2,k} = 10, \quad \mu_{11} = \mu_{22} = 0.12, \quad \mu_{12} = \mu_{21} = 0,$$

$$\nu_{11} = \nu_{22} = 0.15, \quad \nu_{12} = \nu_{21} = 0, \quad b_{11,k} = 1.5, \quad b_{22,k} = 0.2\cos(k\pi + \frac{\pi}{5}),$$

$$b_{12,k} = 0.5\sin(k\pi + \frac{\pi}{5}), \quad b_{21,k} = 0.8, \quad \gamma_{11,k} = 0.1, \quad \gamma_{22,k} = 0.05, \quad \gamma_{21,k} = 0.2,$$

$$\varpi_{11,k} = \sin(k\pi + \frac{\pi}{3}), \quad \varpi_{22,k} = \cos(k\pi + \frac{2\pi}{5}), \quad \varpi_{12,k} = 0.1, \quad \gamma_{12,k} = \varpi_{21,k} = 0,$$

$$I_{1,k} = 15 + 0.5\sin(k\pi + \frac{\pi}{5}), \quad I_{2,k} = 8 + 0.2\cos(k\pi + \frac{\pi}{5}), \quad d_{1,k} = d_{2,k} = 0.1,$$

$$f_i(\mathbf{p}_{i,k}^{\langle\varsigma\rangle}) = \frac{\left(\frac{\mathbf{p}_{i,k}^{\langle\varsigma\rangle}}{10}\right)^2}{1 + \left(\frac{\mathbf{p}_{i,k}^{\langle\varsigma\rangle}}{10}\right)^2}, \quad \sigma_i(\mathbf{p}_{i,k}^{\langle\varsigma\rangle}) = \frac{\left(\frac{\mathbf{p}_{i,k}^{\langle\varsigma\rangle}}{20}\right)^2}{1 + \left(\frac{\mathbf{p}_{i,k}^{\langle\varsigma\rangle}}{20}\right)^2}, \quad \eta_i(\mathbf{m}_{i,k}^{\langle\varsigma\rangle}) = \frac{\left(\frac{\mathbf{m}_{i,k}^{\langle\varsigma\rangle}}{15}\right)^2}{1 + \left(\frac{\mathbf{m}_{i,k}^{\langle\varsigma\rangle}}{15}\right)^2}, \quad i = 1, 2, \forall k \in \mathbb{Z}_0.$$

Obviously, $L_1^f = L_2^f = 0.1$, $L_1^\sigma = L_2^\sigma = 0.05$, $L_1^\eta = L_2^\eta = \frac{1}{15}$. With the help of MATLAB toolbox, it gains $\lambda_1 = 0.2168 < 1$ and all assumptions of Theorem 4.1 are satisfied. Thus, by Theorem 4.1, GRNs (6.1) is globally mean-square exponential convergent. Additionally, taking $\mathscr{L} = 2$, it computes the optimal convergence rate $\kappa = 7.0859$ by solving the optimization problem under nonlinear constraints displayed in Algorithm 1. The trajectories of global exponential stability of GRNs (6.1) with optimal convergence rate $\kappa = 7.0859$ in 3-dimensional and 2-dimensional spaces have been showed in Figures 1–4. Besides, by Theorem 5.1, GRNs (6.1) possesses a random 2-periodic oscillation, which is displayed in Figures 5 and 6.

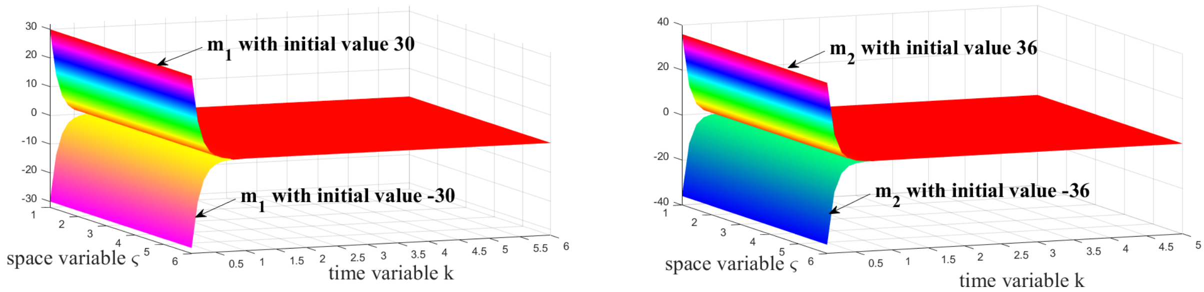


**Figure 1.** Global exponential stability of $\mathbf{m}_{1,k}^{\langle\varsigma\rangle}$ and $\mathbf{m}_{2,k}^{\langle\varsigma\rangle}$ with optimal convergence rate $\kappa = 7.0859$.

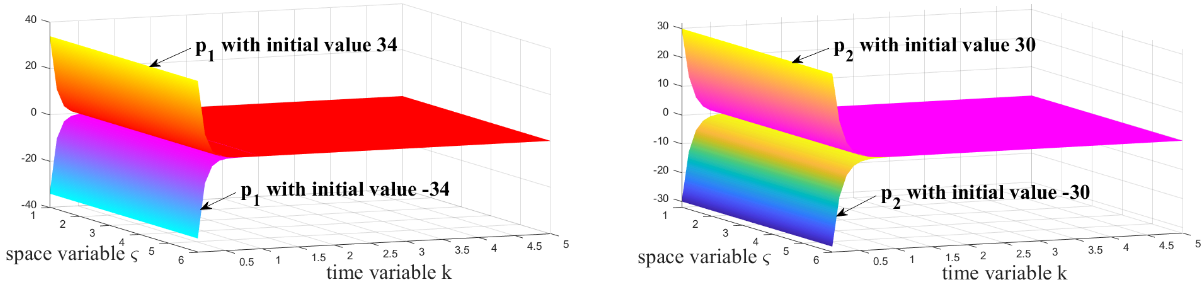


**Figure 2.** Global exponential stability of $\mathbf{p}_{1,k}^{\langle\varsigma\rangle}$ and $\mathbf{p}_{2,k}^{\langle\varsigma\rangle}$ with optimal convergence rate $\kappa = 7.0859$.

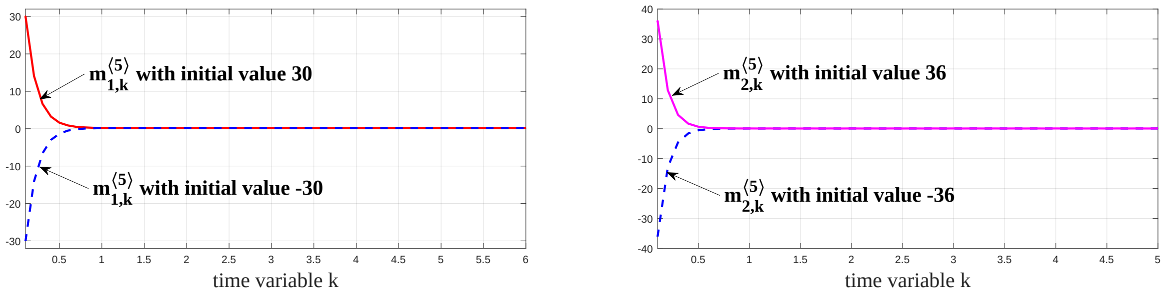


**Figure 3.** Global exponential stability of $\mathbf{m}_{1,k}^{\langle 5\rangle}$ and $\mathbf{m}_{2,k}^{\langle 5\rangle}$ with optimal convergence rate $\kappa = 7.0859$.

**Remark 6.1.** *Literatures [24–27] only researched the time discrete GRNs, e.g., state estimation [25], global exponential stability [26], bifurcations and chaos [27]. Compared with literatures [24–27], this discussion has the following advantages: 1) discrete-time GRNs with discrete spatial diffusions is considered; 2) random periodicity is studied. Thus, the current research expands the works in literatures [24–27].*

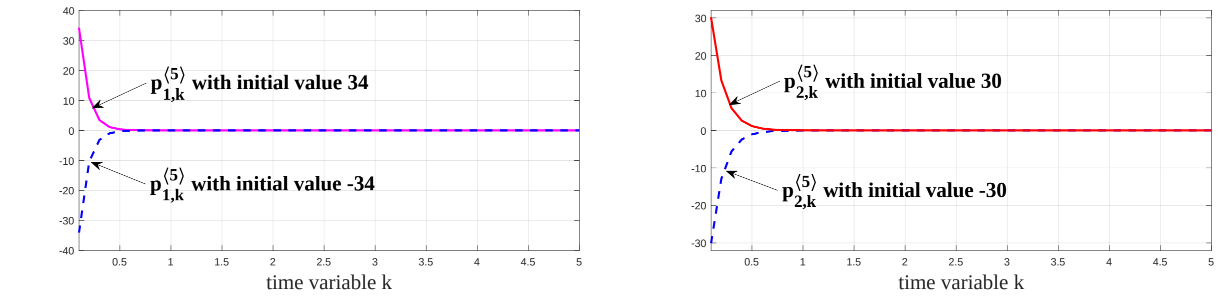


**Figure 4.** Global exponential stability of $\mathbf{p}_{1,k}^{\langle 5\rangle}$ and $\mathbf{p}_{2,k}^{\langle 5\rangle}$ with optimal convergence rate $\kappa = 7.0859$.

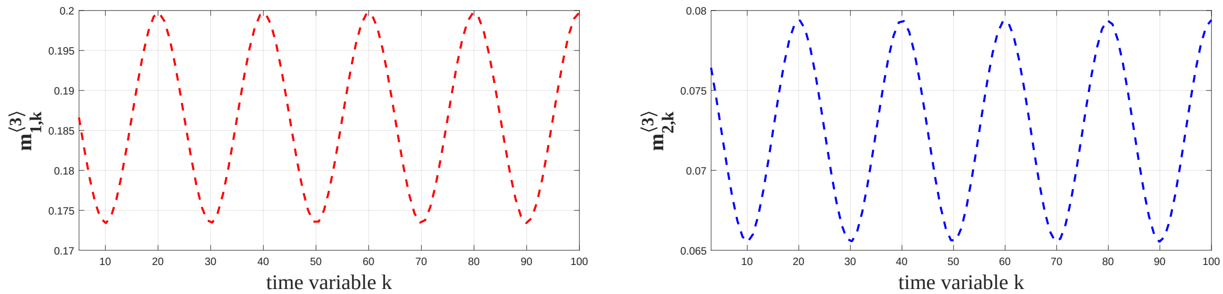


**Figure 5.** Random 2-periodic oscillation of $\mathbf{m}_{1,k}^{\langle 3\rangle}$. and $\mathbf{m}_{2,k}^{\langle 3\rangle}$.

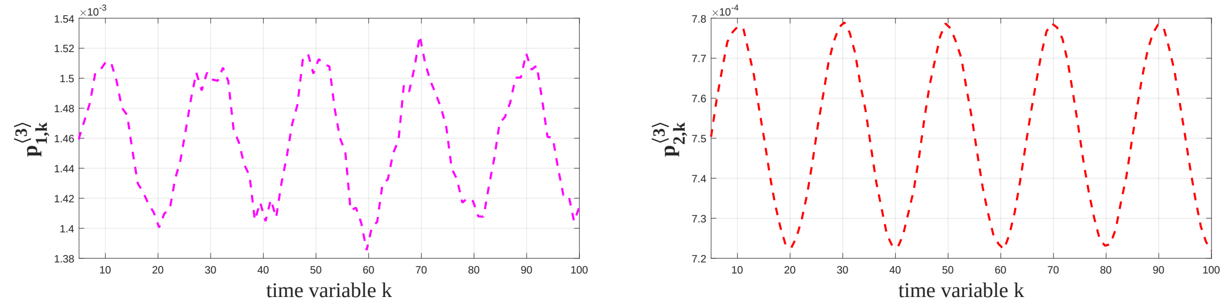


**Figure 6.** Random 2-periodic oscillation of $\mathbf{p}_{1,k}^{\langle 3\rangle}$ and $\mathbf{p}_{2,k}^{\langle 3\rangle}$.

## 7. Conclusions and perspectives

A discrete-time and discrete-space stochastic genetic regulatory networks is proposed, which can be regarded as a fully discretized configuration of stochastic genetic regulatory networks with reaction-diffusions. Based on the constant variational formulation in discrete form of Lemma 2.1, global existence, mean-square boundedness, global exponential stability with optimal convergence speed, and random periodic solutions of this discrete-time stochastic genetic regulatory networks are discussed. In addition, several important inequalities at the end of Section 2 are essential for the discussion in this paper, such as Minkowski inequality. It is worth noting that the work in this paper will open up the study of qualitative problems of discrete-time genetic regulatory networks and lay the theoretical and practical foundation for future work in this field.

In the future, several open topics can be considered further as follows.

- Considerations of the effects of time delays [5].
- Considerations of Markovian jumps [22].
- Considerations of the issues of controls, e.g., synchronization [8], $H_\infty$ states' estimations [25], etc.
- Researches of almost periodic sequences [5] and almost automorphism [28], etc.
- Researches of fractional models [39, 40].
- Discussions of other stabilities, e.g., finite-time and fixed-time stability [41], etc.

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## Conflict of interest

The authors declare there is no conflicts of interest.

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