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Research article

Harnack inequality for a p-Laplacian equation with a source reaction term involving the product of the function and its gradient

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Abstract: A p-Laplacian type problem with a source reaction term involving the product of the function and its gradient is considered in this paper. A Harnack inequality is proved, and the main idea is based on de Giorgi-Nash-Moser iteration and Moser's iteration technique. As a consequence, *Hölder* continuity and boundness for the solution of this problem also are obtained.

Keywords: p-Laplace; Harnack inequality; de Giorgi-Nash-Moser iteration; Moser's iteration

1. Introduction

In this paper, we consider a p-Laplacian Dirichlet boundary problem with a source reaction term involving the product of the function and its gradient as follows:

$$\begin{cases} -\Delta_p u = u^{\alpha_1}|\nabla u|^{\alpha_2}, \\ u > 0, u \in \mathcal{H}_0^p(\Omega). \end{cases} \tag{1.1}$$

We mainly focus on the Harnack inequality for nonnegative solutions of problem (1.1), when Ω is a bounded domain in $\mathbb{R}^N$, $N > 2$, and $0 < \alpha_1, \alpha_2 < p - 1$ are positive exponents.

Particularly, if $p = 2, \alpha_1 = 1$, and $\alpha_2 = 2$, the equation in problem (1.1) becomes

$$-\Delta u = u|\nabla u|^2,$$

which is associated with the Euler-Lagrange equation of the Dirichlet energy of mappings between two Riemannian manifolds; see [1]. From this point of view, we call it a p-Harmonic Mappings type equation. If $\alpha_1 = 0$, we recall the equation in problem (1.1) is the Hamilton-Jacobi equation:

$$-\Delta_p u = |\nabla u|^{\alpha_2}.$$

In [2], the author proved that any C^1 solution of the Hamilton-Jacobi equation in an arbitrary domain $\Omega \subset \mathbb{R}^N$ with $N \geq p > 1$ and $\alpha_2 > p - 1$ satisfies

$$|\nabla u| \leq C_{N,p,\alpha_2}(dist(x, \partial\Omega))^{-\frac{1}{\alpha_2 - p + 1}}.$$

Similar estimates and related results can be found in [3–7], and see also the references therein. If $\alpha_2 = 0$, then problem (1.1) reduces to the subcase Lane-Emden equation,

$$\begin{cases} -\Delta_p u = u^{\alpha_1}, \\ u > 0, u \in \mathcal{H}_0^p(\Omega), \end{cases} \tag{1.2}$$

which has been widely studied in the literature [8–11]. For the generalized case of problem (1.1), when $p = 2, \alpha_1 + \alpha_2 > 1, \alpha_1 \geq 0$, and $0 \leq \alpha_2 \leq 2$, the authors in [12] proved a local Harnack inequality and nonexistence of positive solutions in $\mathbb{R}^N$ under the condition $\alpha_1(N-2) + \alpha_2(N-1) < 1$. In addition, the local and global estimates of the solution are also obtained in [12]. For $p = 2, \alpha_1 > 0$, and $\alpha_2 > 2$, Liouville type theorems of problem (1.1) were studied in [13], and the proof technique is based on monotonicity properties for the spherical averages of sub- and super-harmonic functions, combined with a gradient bound obtained by a local Bernstein argument.

As is known to all, the Harnack inequality plays an important role in the theory of regularity for elliptic and parabolic partial differential equations, and it was originally defined for harmonic functions in the plane and much later became an important tool in the general theory of harmonic functions. In [14], the German mathematician C-G. Axel von Harnack formulated and proved the following theorem in the case $N = 2$.

Theorem. A ***(Harnack inequality [14])*** *Let u be a nonnegative harmonic function in an open set $E \subset \mathbb{R}^N$. Then, for all $x \in B_r(x_0) \subset B_R(x_0) \subset E$,*

$$(\frac{R}{R+r})^{N-2}\frac{R-r}{R+r}u(x_0) \leq u(x) \leq (\frac{R}{R-r})^{N-2}\frac{R+r}{R-r}u(x_0),$$

where $B_R(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < R\}$.

The estimate above is scale invariant in the sense that it does not change for various choices of R, with $r = mR, m \in (0, 1)$ fixed. Furthermore, it depends neither on the position of the ball $B_R(x_0)$ nor on u itself. In its modern version, the currently used Harnack inequality for harmonic functions is given in the following form:

Theorem. B ***(Harnack inequality [15])*** *Let $N \geq 2$ and $E \subset \mathbb{R}^N$ be an open set. Then, there exists a constant $C > 1$, dependent only on the dimension N, such that*

$$C^{-1} \sup_{B_r(x_0)} u \leq u(x_0) \leq C \inf_{B_r(x_0)} u,$$

for every nonnegative harmonic function $u : E \to \mathbb{R}$ and for every ball $B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$ such that $B_{2r}(x_0)$ is contained in E.

Generally speaking, the Harnack inequality asserts that the upper and lower bounds of the solution for a specific initial-boundary value problem can be locally estimated by each other. Although such an estimate seems trivial according to its two forms in Theorems A and B, it plays a fundamental role in many other aspects, such as in the proof of the Liouville Theorem (see [16, 17]), in the construction of solutions to the Dirichlet problem for the Laplace equation using Perron's method (see [18] for more details) and so on. It is worth pointing out that one can also refer to [19, 20] for the applications of Harnack inequalities in geometry analysis and the estimates of heat kernel on manifolds.

We will use an iteration technique to obtain a Harnack inequality for solutions of problem (1.1) in this paper; it was originally developed by de Giorgi [21]-Nash [22]-Moser [23] to deal with the regularity of solutions for general linear elliptic and parabolic equations. More specifically, this method originally derived from the study of regularity of calculus of variations problems, which mainly focus on the *Hölder* continuity for solutions of linear uniformly elliptic equations. Here, we follow the method developed by Moser in [23], in which the following type weak Harnack inequality was proved:

Theorem. C *(**Moser's Harnack inequality [23]**) Let $u \in \mathcal{H}^1(\Omega)$ be a non-negative weak solution to the linear equation*

$$div(a_{i,j}(x)u(x)) = 0 \ \ in\ \Omega$$

with measurable coefficients $a_{i,j}(x) : \Omega \to \mathbb{R}^{N\times N}$ being elliptic and bounded. Then, for every ball $B_{\sqrt{N}R}(x_0) \subset \Omega \subset \mathbb{R}^N$, there holds

$$\inf_{B_{R/2}(x_0)} u \geq C \sup_{B_{R/2}(x_0)} u$$

where C is a constant depending on N and the coefficients $a_{i,j}(x)$. In addition, the Hölder continuity is obtained as a direct correspondence.

Nevertheless, it should be pointed out that this iteration method can also deal with the regularity of quasilinear operators, especially for the p-Laplacian operator. Furthermore, the iteration technique developed by Moser can overcome the difficulties caused by the degeneracy of the p-Laplacian operator, and more details about this issue can be found in [24, 25]. The standard techniques of this iteration method associated with p-Laplacian operators in the homogeneous case

$$div(|\nabla u|^{p-2})\nabla u = 0$$

can be found in [26].

Now, we state the main result in this paper.

Theorem 1.1. *(**Harnack inequality**) If $u \in \mathcal{H}_0^p(\Omega)$ is a solution of problem (1.1), then for any ball $B_r(x) \subset B_R(x) \subset\subset \Omega, 0 < r < R, x \in \Omega$, there exists $C = C(N, \varepsilon, p)$ such that*

$$\sup_{B_r(x)} u \leq C \inf_{B_R(x)} u.$$

Remark 1.1. *Throughout this paper, we assume that $\varepsilon > 0$ is small enough.*

This paper is organized as follows. We first give some lemmas and preliminaries in Section 2 which will be used in the proof of Theorem 1.1. Then, the Harnack inequality for the solution of problem (1.1) is addressed in Section 3 using de Giorgi-Nash-Moser iteration and Moser's iteration technique. In addition, *Hölder* continuity and boundness for the solution of problem (1.1) also are obtained in Section 3.

2. Preliminaries

In this section, we give some preliminaries and lemmas needed in the proof of the main result.

Definition 2.1. *We say $u \in \mathcal{H}_0^p(\Omega)$ is a sub-solution (sup-solution) of problem (1.1) if u is positive a.e. in Ω and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx \le (\ge) \int_\Omega u^{\alpha_1}|\nabla u|^{\alpha_2}\varphi dx$$

for any $\varphi \in \mathcal{H}_0^p(\Omega)$, $\varphi \ge 0$ in Ω.

Lemma 2.1. *(**John-Nirenberg Lemma [27,28]**) Suppose $w \in L^1_{loc}(\Omega)$ satisfies*

$$\int_{B_r(x)} |w(x) - w_{B_r}|dx \le Kr^N \quad for\ any\ B_r(x) \in \Omega,$$

where $w_{B_r} = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy$. Then, there holds, for any $B_r(x) \subset \Omega$,

$$\int_{B_r(x)} e^{p_0|w(x)-w_{B_r}|/K}dx \le Cr^N$$

for some positive p_0 and C depending only on N.

Lemma 2.2. *([28]) Let ω and σ be nondecreasing functions in an interval $(0, R]$. Suppose there holds*

$$\omega(\tau r) \le \gamma\omega(r) + \sigma(r)$$

for all $r \le R$ and some $0 < \gamma, \tau < 1$. Then, for any $\mu \in (0, 1)$ and $r \le R$, we have

$$\omega(r) \le C\left\{\left(\frac{r}{R}\right)^\alpha \omega(R) + \sigma\left(r^\mu R^{1-\mu}\right)\right\}$$

where $C = C(\gamma, \tau)$ and $\alpha = \alpha(\gamma, \tau, \mu)$ are positive constants. In fact, $\alpha = (1 - \mu)\log\gamma/\log\tau$.

Lemma 2.3. *(**Leray-Schauder's principle [29]**) Assume that X is a real Banach space, Ω is an open bounded subset of X, and $\Phi : [a, b]\times\bar{\Omega} \longrightarrow X$ is given by $\Phi(\lambda, u) = u - T(\lambda, u)$ with T being a compact map. Suppose also that*

$$\Phi(\lambda, u) = u - T(\lambda, u) \ne 0, \quad \forall(\lambda, u) \in [a, b] \times \partial\Omega.$$

We define $\Phi_a = I - T(a, u)$, where I denotes the identity map in X. If $\deg(\Phi_a, \Omega, 0) \ne 0$, then,
(i) $\Phi(\lambda, u) = u - T(\lambda, u) = 0$ has a solution in Ω for every $a \le \lambda \le b$.
(ii) Furthermore, define $\Sigma = \{(\lambda, u) \in [a, b] \times \bar{\Omega} : \Phi(\lambda, u) = 0\}$ and $\Sigma_\lambda = \{u \in \bar{\Omega} : (\lambda, u) \in \Sigma\}$. Thus, there exists a compact connected set $Q \subset \Sigma$ such that

$$Q \cap (\{a\} \times \Sigma_a) \ne \emptyset \text{ and } Q \cap (\{b\} \times \Sigma_b) \ne \emptyset.$$

3. Harnack inequality

In this section, we give a Harnack inequality for the solution of problem (1.1) by de Giorgi-Nash-Moser iteration and Moser's iteration technique. First, we deduce the following lemmas.

Lemma 3.1. *Suppose $u \in \mathcal{H}_0^p(\Omega)$ is a super-solution of problem (1.1). Then, for any ball $B_r(x) \subset B_R(x) \subset\subset \Omega, 0 < r < R < 1, p-1 < \beta \le p$, there holds,*

$$\inf_{B_r(x)} u \ge C(N,\varepsilon)\{\frac{1}{(R-r)^p}\int_{B_R(x)} u^{-\beta}dx\}^{-1/\beta},$$

where $C(N,\varepsilon)$ is a constant depending on ε and N, and $\varepsilon > 0$ is small enough.

Proof. Choose a test function as follows: $\varphi = \eta^p u^s$, where $\eta \in C_0^\infty(B_R(x)), \eta \equiv 1$ *in* $B_r(x), 0 < \eta \le 1, |\nabla\eta| \le \frac{1}{R-r}$. It is not difficult to calculate that

$$\nabla\varphi = p\eta^{p-1}u^s\nabla\eta + s\eta^p u^{s-1}\nabla u.$$

Multiplying on both sides of the first equation for problem (1.1) by φ and integrating by parts, since u is a super-solution of problem (1.1), we get that

$$p\int_{B_R(x)}\eta^{p-1}u^s|\nabla u|^{p-2}\nabla u\cdot\nabla\eta dx + s\int_{B_R(x)}\eta^p u^{s-1}|\nabla u|^p dx \ge \int_{B_R(x)} u^{\alpha_1+s}|\nabla u|^{\alpha_2}\eta^p dx.$$

Hence, by *Hölder* inequality and Young's inequality with ε, we obtain that

$$\begin{aligned}
&-s\int_{B_R(x)}\eta^p u^{s-1}|\nabla u|^p dx\\
&\le p\int_{B_R(x)}\eta^{p-1}u^s|\nabla u|^{p-2}\nabla u\cdot\nabla\eta dx - \int_{B_R(x)} u^{\alpha_1+s}|\nabla u|^{\alpha_2}\eta^p dx\\
&\le p\int_{B_R(x)}\eta^{p-1}u^s|\nabla u|^{p-2}\nabla u\cdot\nabla\eta dx\\
&\le p\int_{B_R(x)}\eta^{p-1}|\nabla u|^{p-1}u^{(s-1)\frac{p-1}{p}}u^{s-(s-1)\frac{p-1}{p}}|\nabla\eta| dx\\
&\le p(\int_{B_R(x)}\eta^p u^{s-1}|\nabla u|^p dx)^{\frac{p-1}{p}}(\int_{B_R(x)} u^{-\beta}|\nabla\eta|^p dx)^{\frac{1}{p}}\\
&\le \varepsilon p^{\frac{p}{p-1}}\int_{B_R(x)}\eta^p u^{s-1}|\nabla u|^p dx + C(\varepsilon)\int_{B_R(x)} u^{-\beta}|\nabla\eta|^p dx,
\end{aligned} \tag{3.1}$$

where $s = -\beta-(p-1)$. Choose ε small enough and consider the definition of φ, and we can get that

$$\begin{aligned}
\int_{B_R(x)}\eta^p u^{s-1}|\nabla u|^p dx &\le C(\varepsilon)\int_{B_R(x)} u^{-\beta}|\nabla\eta|^p dx\\
&\le C(\varepsilon)\frac{1}{(R-r)^p}\int_{B_R(x)} u^{-\beta}dx.
\end{aligned} \tag{3.2}$$

Also, we define $\kappa = \frac{N}{N-p}$ and deduce the following conclusion by Sobolev inequality:

$$\begin{aligned}(\int_{B_r(x)} u^{-\kappa\beta}dx)^{\frac{1}{\kappa}} &= (\int_{B_r(x)} u^{-\frac{N\beta}{N-p}}dx)^{\frac{N-p}{N}}\\ &\leq C(N)\int_{B_r(x)} |\nabla(u^{-\frac{\beta}{p}})|^p dx\\ &= C(N)\int_{B_r(x)} |u^{-\frac{\beta}{p}-1}\nabla u|^p dx\\ &= C(N)\int_{B_r(x)} u^{s-1}|\nabla u|^p dx\\ &\leq C(N,\varepsilon)\frac{1}{(R-r)^p}\int_{B_R(x)} u^{-\beta}dx,\end{aligned} \tag{3.3}$$

where $C(N)$ is a constant depending on N, $C(N,\varepsilon)$ is a constant depending on ε and N, and the last inequality is based on the conclusion of (3.2). Define $r_i = r + 2^{-i}(R-r), \beta_i = -\kappa^i\beta, \mathcal{A}_{\beta_i,r_i} = \int_{B_{r_i}(x)} u^{\beta_i}dx$, $i = 1, 2, ...n...$; and owing to (3.3), we have

$$\begin{aligned}\mathcal{A}_{\beta_i,r_i}^{\frac{1}{\kappa^i}} &\leq \{C(N,\varepsilon)\frac{1}{(r_{i-1}-r_i)^p}\}^{\frac{1}{\kappa^{i-1}}}\mathcal{A}_{\beta_{i-1},r_{i-1}}^{\frac{1}{\kappa^{i-1}}}\\ &\leq \cdots\cdots\\ &\leq \{C(N,\varepsilon)\frac{2^p}{(R-r)^p}\}^{\frac{1}{\kappa^{i-1}}+\frac{1}{\kappa^{i-2}}+\cdots+\frac{1}{\kappa^0}}\mathcal{A}_{\beta_0,r_0}.\end{aligned} \tag{3.4}$$

Let $i \longrightarrow \infty$, and it is obvious that

$$\sup_{B_r(x)} u^{-\beta} \leq C(N,\varepsilon)\frac{1}{(R-r)^p}\int_{B_R(x)} u^{-\beta}dx. \tag{3.5}$$

Thus,

$$\inf_{B_r(x)} u \geq C(N,\varepsilon)\{\frac{1}{(R-r)^p}\int_{B_R(x)} u^{-\beta}dx\}^{-\frac{1}{\beta}}.$$

This completes the proof.

Lemma 3.2. *Suppose $u \in \mathcal{H}_0^p(\Omega)$ is a sub-solution of problem (1.1). Then, for any ball $B_r(x) \subset B_R(x) \subset\subset \Omega, 0 < r < R < 1, p-1 < \beta \leq p$, there exists a constant $C = C(N,\varepsilon)$ such that,*

$$\sup_{B_r(x)} u \leq \begin{cases} C\{\frac{1}{(R-r)^p}\int_{B_R(x)} u^\beta dx\}^{\frac{1}{\beta}}, & if \int_{B_R(x)} u^\beta dx \geq 1,\\ C(\frac{1}{R-r})^{\frac{p}{\beta}}, & if \int_{B_R(x)} u^\beta dx < 1.\end{cases}$$

Proof. Choose a test function $\phi = \eta^p u^t$, where $\eta \in C_0^\infty(B_R(x))$, $\eta \equiv 1$ in $B_r(x)$, $0 \leq \eta \leq 1$, $|\nabla\eta| \leq \frac{1}{R-r}$ and $t = \beta-(p-1)$. Multiplying by ϕ on both sides of the first equation in problem (1.1) and integrating by parts, noting that u is a sub-solution, one obtains that

$$\begin{aligned} & p\int_{B_R(x)} \eta^{p-1}u^t|\nabla u|^{p-2}\nabla u\cdot\nabla\eta dx + t\int_{B_R(x)} \eta^p u^{t-1}|\nabla u|^p dx \\ & \leq \int_{B_R(x)} u^{\alpha_1+t}|\nabla u|^{\alpha_2}\eta^p dx. \end{aligned} \tag{3.6}$$

By *Hölder* inequality and Young's inequality with ε, we infer that

$$\begin{aligned} & t\int_{B_R(x)} \eta^p u^{t-1}|\nabla u|^p dx \\ & \leq p\int_{B_R(x)} \eta^{p-1}u^t|\nabla\eta||\nabla u|^{p-1}dx + \int_{B_R(x)} u^{\alpha_1+t}|\nabla u|^{\alpha_2}\eta^p dx \\ & \leq \varepsilon p^{\frac{p}{p-1}}\int_{B_R(x)} \eta^p u^{t-1}|\nabla u|^p dx + C(\varepsilon)\int_{B_R(x)} u^\beta|\nabla\eta|^p dx \\ & + \int_{B_R(x)} u^{\alpha_1+t}|\nabla u|^{\alpha_2}\eta^p dx. \end{aligned} \tag{3.7}$$

Choose ε small enough, and we obtain that

$$\begin{aligned} & \int_{B_R(x)} \eta^p u^{t-1}|\nabla u|^p dx \\ & \leq C(\varepsilon)\int_{B_R(x)} u^\beta|\nabla\eta|^p dx + \int_{B_R(x)} u^{\alpha_1+t}|\nabla u|^{\alpha_2}\eta^p dx. \end{aligned} \tag{3.8}$$

Again, by *Hölder* inequality and Young's inequality with ε, we have

$$\begin{aligned} & \int_{B_R(x)} u^{\alpha_1+t}|\nabla u|^{\alpha_2}\eta^p dx \\ & = \int_{B_R(x)} \eta^p|\nabla u|^{\alpha_2}u^{(t-1)\frac{\alpha_2}{p}}u^{\alpha_1+t-(t-1)\frac{\alpha_2}{p}}dx \\ & \leq (\int_{B_R(x)} \eta^{\frac{p^2}{\alpha_2}}u^{t-1}|\nabla u|^p dx)^{\frac{\alpha_2}{p}}(\int_{B_R(x)} u^{\frac{p(\alpha_1+t)-(t-1)\alpha_2}{p-\alpha_2}}dx)^{1-\frac{\alpha_2}{p}} \\ & \leq \varepsilon\int_{B_R(x)} \eta^{\frac{p^2}{\alpha_2}}u^{t-1}|\nabla u|^p dx + C(\varepsilon)\int_{B_R(x)} u^{\frac{p(\alpha_1+t)-(t-1)\alpha_2}{p-\alpha_2}}dx \\ & \leq \varepsilon\int_{B_R(x)} \eta^{\frac{p^2}{\alpha_2}}u^{t-1}|\nabla u|^p dx + C(\varepsilon)(\int_{B_R(x)} u^\beta dx)^{\frac{\beta-\delta}{\beta}}, \end{aligned} \tag{3.9}$$

and the last inequality holds owing to $0 < \frac{p(\alpha_1+t)-(t-1)\alpha_2}{p-\alpha_2} < \beta$. Thus, we define $\frac{p(\alpha_1+t)-(t-1)\alpha_2}{p-\alpha_2} \triangleq \beta-\delta$, where $\delta > 0$. Obviously, due to (3.7) and (3.9), we claim the following result

$$\int_{B_r(x)} u^{t-1}|\nabla u|^p dx \leq \begin{cases} C(\varepsilon)\frac{1}{(R-r)^p}\int_{B_R(x)} u^\beta dx, & if \int_{B_R(x)} u^\beta dx \geq 1, \\ C(\varepsilon)(\int_{B_R(x)} u^\beta dx)^{\frac{\beta-\delta}{\beta}}, & if \int_{B_R(x)} u^\beta dx < 1. \end{cases} \tag{3.10}$$

We assume $\kappa = \frac{N}{N-p}$ as in Lemma 3.1. Therefore, Sobolev inequality and (3.10) imply that

$$\begin{aligned}
&(\int_{B_r(x)} u^{\kappa\beta} dx)^{\frac{1}{\kappa\beta}} \\
&= \{\int_{B_r(x)} (u^{\frac{\beta}{p}})^{\frac{Np}{N-p}} dx\}^{\frac{N-p}{N\beta}} \\
&\leq \{C(N) \int_{B_r(x)} |\nabla(u^{\frac{\beta}{p}})|^p dx\}^{\frac{1}{\beta}} \\
&= \{C(N) \int_{B_r(x)} u^{t-1} |\nabla u|^p dx\}^{\frac{1}{\beta}} \\
&\leq \begin{cases} C(N,\varepsilon)\{\frac{1}{(R-r)^p} \int_{B_R(x)} u^\beta dx\}^{\frac{1}{\beta}}, & if \int_{B_R(x)} u^\beta dx \geq 1, \\ C(N,\varepsilon)(\int_{B_R(x)} u^\beta dx)^{\frac{\beta-\delta}{\beta^2}}, & if \int_{B_R(x)} u^\beta dx < 1, \end{cases}
\end{aligned} \tag{3.11}$$

where $C(N)$ is a constant depending on N, and $C(N,\varepsilon)$ is a constant depending on N and ε. Similarly, consider r_i, β_i as $r_i = r + 2^{-i}(R-r), \beta_i = \kappa^i \beta$. We deduce the following claims from (3.11)

$$\|u\|_{\beta_i, r_i} \leq \begin{cases} C^{\frac{1}{\beta_i}}(N,\varepsilon)(\frac{1}{r_i - r_{i-1}})^{\frac{p}{\beta_{i-1}}} \|u\|_{\beta_{i-1}, r_{i-1}}, & if \ \|u\|_{\beta_{i-1}, r_{i-1}} \geq 1, \\ \\ C^{\frac{1}{\beta_i}}(N,\varepsilon)(\frac{1}{r_i - r_{i-1}})^{\frac{p}{\beta_{i-1}}} \|u\|_{\beta_{i-2}, r_{i-2}}, & if \ \|u\|_{\beta_{i-1}, r_{i-1}} < 1, \end{cases} \tag{3.12}$$

where $\|u\|_{\beta_i, r_i} = (\int_{B_{r_i(x)}} u^{\beta_i} dx)^{\frac{1}{\beta_i}}$. Actually, we assume that $\|u\|_{\beta_{i-2}, r_{i-2}} \geq 1$. Nevertheless, the conclusion still holds because we can omit the intermediate terms if they are less than 1. Therefore, we have the following conclusions by iterating (3.12)

$$\|u\|_{\beta_i, r_i} \leq C^{\sum_{m=0}^{i-1} \frac{1}{\beta^m}}(N,\varepsilon) 2^{p \sum_{m=0}^{i-1} \frac{m}{\kappa^m}} (\frac{1}{R-r})^{p \sum_{m=0}^{i-1} \frac{1}{\kappa^m}} \|u\|_{\beta_0, r_0}$$

Let $i \longrightarrow +\infty$, and we have

$$\sup_{B_r(x)} u \leq \begin{cases} C(N,\varepsilon)\{\frac{1}{(R-r)^p} \int_{B_R(x)} u^\beta dx\}^{\frac{1}{\beta}}, & if \ \int_{B_R(x)} u^\beta dx \geq 1, \\ \\ C(N,\varepsilon)(\frac{1}{R-r})^{\frac{p}{\beta}}, & if \int_{B_R(x)} u^\beta dx < 1. \end{cases} \tag{3.13}$$

The proof is complete.

Lemma 3.3. *Suppose $u \in \mathcal{H}_0^p(\Omega)$ is a super-solution of problem (1.1). Then for any ball $B_r(x) \subset B_R(x) \subset\subset \Omega, 0 < r < R < 1, x \in \Omega, 0 < \beta \leq p-1$, there exist $p-1 \leq \widehat{\beta} < p^*$ and constant $C(N,\varepsilon)$ depending on ε and N such that*

$$(\int_{B_r(x)} u^{\widehat{\beta}} dx)^{1/\widehat{\beta}} \leq C(N,\varepsilon)\{\frac{1}{(R-r)^p} \int_{B_R(x)} u^\beta dx\}^{1/\beta}.$$

Proof. Again, pick the same test function $\phi = \eta^p u^t$, where η is a cutoff function in $B_R(x)$ as in Lemma 3.2, and let $t = \beta - (p-1)$. Multiplying the first equation of problem (1.1) by ϕ and integrating by parts, noticing that u is a super-solution of problem (1.1), the following result can be obtained:

$$\begin{aligned} -t\int_{B_R(x)} \eta^p u^{t-1}|\nabla u|^p dx &\leq p\int_{B_R(x)} \eta^{p-1}u^t|\nabla\eta||\nabla u|^{p-1}dx \\ &\leq \varepsilon\int_{B_R(x)} \eta^p u^{t-1}|\nabla u|^p dx + C(\varepsilon)p\int_{B_R(x)} u^\beta|\nabla\eta|^p dx. \end{aligned} \tag{3.14}$$

Thus,

$$\begin{aligned} (\int_{B_r(x)} u^{\kappa\beta})^{\frac{1}{\kappa\beta}} &\leq C(N)\{\int_{B_r(x)} |\nabla u^{\frac{\beta}{p}}|^p\}^{\frac{1}{\beta}} \\ &= C(N)\{\int_{B_R(x)} \eta^p u^{t-1}|\nabla u|^p\}^{\frac{1}{\beta}} \\ &\leq C(N,\varepsilon)\{\frac{1}{(R-r)^p}\int_{B_R(x)} u^\beta\}^{\frac{1}{\beta}}. \end{aligned} \tag{3.15}$$

The proof can be completed by iterating finite steps over (3.15). For $\kappa > 1$, the desired $p-1 \leq \widehat{\beta} < p^*$ exists such that

$$(\int_{B_r(x)} u^{\widehat{\beta}}dx)^{1/\widehat{\beta}} \leq C(N,\varepsilon)\{\frac{1}{(R-r)^p}\int_{B_R(x)} u^\beta dx\}^{1/\beta}.$$

Lemma 3.4. *Suppose $u \in \mathcal{H}_0^p(\Omega)$ is a super-solution of problem (1.1). Then for any ball $B_R(x) \subset\subset \Omega, x \in \Omega, 0 < R < 1$, there exists $\varepsilon_0 > 0$ small enough such that*

$$(\int_{B_R(x)} u^{\varepsilon_0})^{1/\varepsilon_0} \leq C(N,\varepsilon,p)(\int_{B_R(x)} u^{-\varepsilon_0})^{-1/\varepsilon_0},$$

where $C(N,\varepsilon,p)$ is a constant depending on N, p and ε.

Proof. 1) Suppose $w = \log u$ satisfies

$$\int_{B_R(x)} |w(x) - w_{B_r}|dx \leq CR^N \quad for\ any\ B_R(x) \in \Omega,$$

where $w_{B_r} = ⨍_{B_r(x)} wdx = \frac{1}{|B_r(x)|}\int_{B_r(x)} wdx$. According to Lemma 2.1, the following result holds:

$$(⨍_{B_R(x)} e^{w\varepsilon_0}dx)\cdot(⨍_{B_R(x)} e^{-w\varepsilon_0}dx) \leq C(N),$$

which can be rewritten as

$$(\int_{B_R(x)} u^{\varepsilon_0}dx)\cdot(\int_{B_R(x)} u^{-\varepsilon_0}dx) \leq C(N).$$

Consequently, the desired inequality follows immediately.

2) Now, we prove that w does belong to $BMO(\Omega)$. We choose the similar test function $\psi = \eta^p u^{1-p}$ and multiply both sides of the first equation in problem (1.1) by ψ, and we integrate by parts, with

$\eta \in C_0^\infty(\Omega)$, $\eta \equiv 1$ in $B_R(x)$, $0 \le \eta \le 1$. Observing that u is a super-solution of problem (1.1), we have that

$$\begin{aligned}(p-1)\int_{B_R(x)} \eta^p u^{-p}|\nabla u|^p dx &\le p\int_{B_R(x)} \eta^{p-1}u^{1-p}|\nabla\eta||\nabla u|^{p-1}\\ &\le p(\int_{B_R(x)} u^{-p}|\nabla u|^p\eta^p)^{1-\frac{1}{p}}(\int_{B_R(x)}|\nabla\eta|^p)^{\frac{1}{p}}\\ &\le \varepsilon p\int_{B_R(x)}\eta^p u^{-p}|\nabla u|^p + C(\varepsilon)p\int_{B_R(x)}|\nabla\eta|^p.\end{aligned} \tag{3.16}$$

This means that

$$\begin{aligned}\int_{B_R(x)}|\nabla\log u|^p &\le C(\varepsilon,p)\int_{B_R(x)}|\nabla\eta|^p\\ &\le C(\varepsilon,p)R^{-p}|B_R(x)|\\ &\le C(N,\varepsilon,p)R^{N-p}.\end{aligned} \tag{3.17}$$

Thanks to the Poincar*é* inequality, the following result can be claimed:

$$\begin{aligned}\int_{B_R(x)}|\log u-(\log u)_{x,R}| &\le C(N)R^p\int_{B_R(x)}|\nabla\log u|^p\\ &\le C(N,\varepsilon,p)R^N.\end{aligned} \tag{3.18}$$

This completes the proof of Lemma 3.4.

Proof of Theorem 1.1. Without loss of generality, we may assume $\varepsilon_0 < p$. From Lemmas 3.1–3.4, there holds

$$\begin{aligned}\sup_{B_r(x)} u &\le C(N,\varepsilon)\{\frac{1}{(R-r)^p}\int_{B_R(x)}u^p dx\}^{\frac{1}{p}} \quad (\textit{Lemma } 3.2)\\ &\le C(N,\varepsilon)\{\frac{1}{(R-r)^p}\int_{B_R(x)}u^{\varepsilon_0} dx\}^{\frac{1}{\varepsilon_0}} \quad (\textit{Lemma } 3.3)\\ &\le C(N,\varepsilon,p)\{\frac{1}{(R-r)^p}\int_{B_R(x)}u^{-\varepsilon_0} dx\}^{-1/\varepsilon_0} \quad (\textit{Lemma } 3.4)\\ &\le C(N,\varepsilon,p)\inf_{B_R(x)} u \quad (\textit{Lemma } 3.1).\end{aligned} \tag{3.19}$$

As a consequence of the Harnack inequality, we can obtain the following existence and regularity results.

Corollary 3.1. *If u is a solution of problem (1.1) in the space $\mathcal{H}_0^p(\Omega)$, then u locally belongs to the Hölder space $C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$.*

Proof. For $B_r(x) \subset\subset \Omega$, define the oscillation of u as $w = M(r) - m(r)$, where $M(r) = \underset{B_r(x)}{sup}\, u, m(r) = \underset{B_r(x)}{inf}\, u$. According to Theorem 1.1, we can make the following claim:

$$\begin{aligned} M(r) - m(\frac{r}{2}) &= \underset{B_{\frac{r}{2}}(x)}{sup}\{M(r) - u\} \\ &\leq C(N, \varepsilon, p) \underset{B_{\frac{r}{2}}(x)}{inf}\{M(r) - u\} \\ &= C(N, \varepsilon, p)\{M(r) - M(\frac{r}{2})\}. \end{aligned} \tag{3.20}$$

Similarly, we have the following result:

$$M(\frac{r}{2}) - m(r) \leq C(N, \varepsilon, p)\{m(\frac{r}{2}) - m(r)\}. \tag{3.21}$$

Obviously, by adding up (3.20) and (3.21), the following claims can be guaranteed:

$$w(\frac{r}{2}) \leq \gamma w(r), for\ \gamma = \frac{C-1}{C+1} \in (0, 1). \tag{3.22}$$

Using Lemma 2.1 and (3.22), we can obtain the following result:

$$w(\frac{r}{2}) \leq C(N, \varepsilon, p) r^{\alpha} w(R), \alpha \in (0, 1).$$

The proof is completed.

Corollary 3.2. *There exists a solution u of problem (1.1) in the space $\mathcal{H}_0^p(\Omega)$ with $|u|_{C_0(\Omega)} \leq C$, for some constant $C > 0$.*

Proof. 1) Owing to Corollary 3.1, we can obtain that if u is a solution of problem (1.1) in $\mathcal{H}_0^p(\Omega)$, then there exists a constant $C(N, \varepsilon, \Omega, p)$, such that $||u||_{\mathcal{H}_0^p(\Omega)} \leq C(N, \varepsilon, \Omega, p)$.

2) Define an operator $T : \mathcal{H}_0^p(\Omega) \longrightarrow \mathcal{H}_0^p(\Omega), T(t, u) = u - \Phi(t, u)$, with $\Phi(t, u) = -\Delta_p^{-1} \circ A(t, u)$, with $A(t, u) = tu^{\alpha_1}|\nabla u|^{\alpha_2} + (1 - t)$. Consider the problem as follows:

$$\begin{cases} -\Delta_p u = tu^{\alpha_1}|\nabla u|^{\alpha_2} + (1 - t), \\ u > 0, u \in \mathcal{H}_0^p(\Omega). \end{cases} \tag{3.23}$$

Obviously, we have the following result for the ball $\mathbf{B} = \{u \in \mathcal{H}_0^p(\Omega) : ||u||_{\mathcal{H}_0^p(\Omega)} \leq C(N, \varepsilon, \Omega, p) + 1\}$ similar to the estimate above, this means that $\Phi(t, u)$ is compact on $\mathbf{B}$, for $A(t, u)$ is bounded on $\mathbf{B}$. In addition, $T(t, u) \neq 0$ on $\partial\mathbf{B}$; thus, Lemma 2.3 holds that

$$\deg(T(0, u), \mathbf{B}, 0) = \deg(T(1, u), \mathbf{B}, 0).$$

This completes the proof of Corollary 3.2.

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Conflict of interest

The authors declare there is no conflicts of interest.

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