Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(2): 1147–1156.
DOI: 10.3934/era.2023058
Received: 30 September 2021
Revised: 09 April 2022
Accepted: 12 April 2022
Published: 21 December 2022

Research article

An infinite semipositone problem with a reversed S-shaped bifurcation curve

Amila Muthunayake[1], Cac Phan[2] and Ratnasingham Shivaji[3,*]

[1] Department of Mathematics, Weber State University, Ogden, UT84403, USA

[2] Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599, USA

[3] Department of Mathematics and Statistics, University of North Carolina at Greensboro, Greensboro, NC 27412, USA

* **Correspondence:** Email: r_shivaj@uncg.edu; Tel: +1-336-334-5836; Fax: +1-336-334-5949.

Abstract: We study positive solutions to the two point boundary value problem:

$$\begin{gathered} Lu = -u'' = \lambda\left\{\frac{A}{u^\gamma} + M[u^\alpha + u^\delta]\right\}\,;\ (0,1) \\ u(0) = 0 = u(1) \end{gathered}$$

where $A < 0$, $\alpha \in (0,1), \delta > 1, \gamma \in (0,1)$ are constants and $\lambda > 0, M > 0$ are parameters. We prove that the bifurcation diagram (λ vs $\|u\|_\infty$) for positive solutions is at least a reversed S-shaped curve when $M \gg 1$. Recent results in the literature imply that for $M \gg 1$ there exists a range of λ where there exist at least two positive solutions. Here, when $M \gg 1$, we prove the existence of a range of λ for which there exist at least three positive solutions and that the bifurcation diagram is at least a reversed S-shaped curve. Further, via a quadrature method and Python computations, for $M \gg 1$, we show that the bifurcation diagram is exactly a reversed S-shaped curve. Also, when the operator L is replaced by a p-Laplacian operator with $p > 1$, as well as p-q Laplacian operator with $p = 4$ and $q = 2$, we show that the bifurcation diagram is again an exactly reversed S-shaped curve when $M \gg 1$.

Keywords: two-point boundary value problems; infinite semipositone reaction terms; positive solutions; multiplicity results; reversed S-shaped bifurcation curves

1. Introduction

We consider the two-point boundary value problem

$$\begin{aligned} Lu = -u'' = \lambda f(u) = \lambda\left\{\frac{A}{u^\gamma} + M[u^\alpha + u^\delta]\right\}\ ;\ (0,1) \\ u(0) = 0 = u(1) \end{aligned} \tag{1.1}$$

where $A < 0$, $\alpha \in (0,1)$, $\delta > 1$, $\gamma \in (0,1)$ are constants and $\lambda > 0$, $M > 0$ are parameters. Note that $\lim_{s\to 0^+} f(s) = -\infty$ and such problems are referred in the literature as infinite semipositone problems (when $f(0) < 0$ and finite they are referred as semipositone problems). Recent results in [1] imply that for $M \gg 1$, there exists $\lambda_i > 0$; $i = 1,2,3$ with $\lambda_1 < \lambda_2 < \lambda_3$ such that (1.1) has a positive solution for $\lambda \in (0,\lambda_3)$ and at least two positive solutions for $\lambda \in (\lambda_1,\lambda_2)$ conjecturing the following bifurcation diagram (λ vs $\|u\|_\infty$) (which turns back to the left).

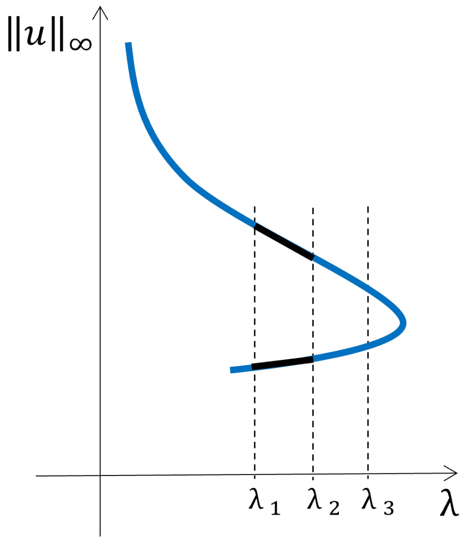

Figure 1. Bifurcation diagram for positive solutions to (1.1) based on the results in [1].

In [1] the authors studied more general classes of such problems (infact, systems of equations with weights in the reaction term). However, here in the autonomous single equation case, we will show that for $M \gg 1$, the bifurcation diagram not only bends back to the left, but will again bend forward to the right (see Figure 1.2). Namely, we prove:

Theorem 1.1. *Let $\alpha \geq \gamma$ and $\alpha\delta < 1$. Then for $M \gg 1$, there exists $\mu_i > 0$; $i = 1,2,3,4$ with $\mu_1 < \mu_2 < \mu_3 < \mu_4$ such that* (1.1) *has a unique positive solution for $\lambda \in (0,\mu_1)$, at least one positive solution for $\lambda \in (0,\mu_4)$, no positive solution for $\lambda > \mu_4$, and at least three positive solutions for $\lambda \in (\mu_2,\mu_3)$.*

Corollary 1.1. *For $M \approx 0$, there exists $\mu^* > 0$ such that* (1.1) *has a unique positive solution for $\lambda \in (0,\mu^*]$ and no positive solutions for $\lambda > \mu^*$.*

Clearly the bifurcation diagrams corresponding to Theorem 1.1 are reversed S-shaped curves as follows:

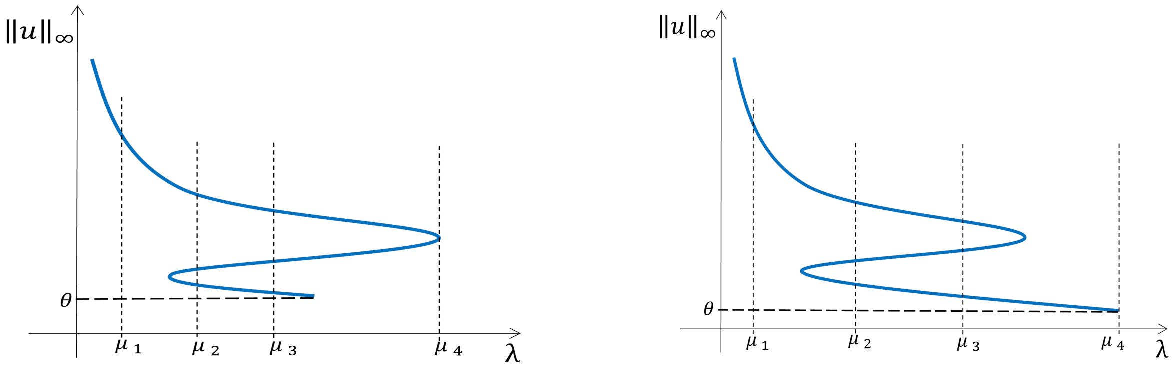

Figure 2. Bifurcation Diagrams for positive solutions to (1.1) based on our results.

We will establish Theorem 1.1 via the quadrature method discussed in the [2, 3]. Further, via Python computations, we note that the reversed S-shaped curves in Figure are infact exact. We also consider the following two problems:

$$\begin{aligned} -(|u'|^{p-2}u')' &= \lambda\left\{\frac{A}{u^\gamma} + M[u^\alpha + u^\delta]\right\};\ (0,1), \qquad p > 1 \\ u(0) &= 0 = u(1) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} -(|u'|^{p-2}u')' - (|u'|^{q-2}u')' &= \lambda\left\{\frac{A}{u^\gamma} + M[u^\alpha + u^\delta]\right\};\ (0,1),\ p = 4,\ q = 2 \\ u(0) &= 0 = u(1). \end{aligned} \tag{1.3}$$

Via quadrature methods and Python computations, we obtained bifurcation diagrams of (1.2) and (1.3) which again are exactly reversed S-shaped for $M \gg 1$. For (2.1), based on our computational results, we conjecture that the critical value of $M = M_c$ beyond which the bifurcation curve is revered S-shaped, is a decreasing function of p. See also [4] where another example with reversed S-shaped bifurcation diagram was discussed via the quadrature method, and [7–9] for results on existence and multiplicity of positive solutions for semilinear elliptic equations with singular non-linearities.
The rest of the paper is organized as follows: In Section 2, we will recall the quadrature method discussed in [2, 3]. In Section 3, we will establish Theorem 1.1 and Corollary 1.1. In Section 4 and 5, we will obtain exact bifurcation diagrams of (1.1) and (1.2) respectively, via the quadrature method and Python computations. Finally, in Section 6, we will obtain exact bifurcation diagrams of (1.3) via the quadrature method described in [5] and Python computations.

2. Preliminaries

Here we recall the quadrature method first introduced in [3]. Note that if u is a positive solution of (1.1) then u must be symmetric about $t = \frac{1}{2}$, $u' > 0$; $(0, \frac{1}{2})$ and $u' < 0$; $(\frac{1}{2}, 1)$. Multiplying the differential equation in (1.1) by u', we obtain

$$-\left[\frac{(u'(t))^2}{2}\right]' = \lambda(F(u(t)))'$$

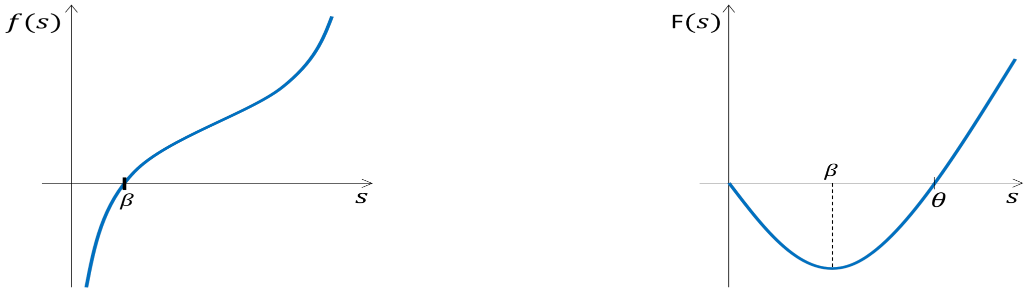

Figure 3. Graphs of functions f and F.

where $F(u) = \int_0^u f(s)ds$. Let β and $\theta > 0$ be the unique positive zeros of f and F respectively. Further, integrating we obtain

$$u'(t) = \sqrt{2\lambda[F(\rho) - F(u(t))]}\ ;\ t \in (0, \tfrac{1}{2}) \tag{2.1}$$

where $\rho = u(\frac{1}{2}) = \|u\|_\infty$. By (2.1) we get $u'(0) = \sqrt{2\lambda F(\rho)}$. This implies $\rho \geq \theta$. Integrating (2.1) again and setting $t \to \frac{1}{2}^-$ we obtain

$$G(\rho) = \sqrt{\lambda(\rho)} = \sqrt{2}\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}}. \tag{2.2}$$

It was established in [3] that G is well defined and continuous on $D = \{\rho > 0 | f(\rho) > 0, F(\rho) > F(s) : s \in [0, \rho)\}$. Further, it was established that if (λ, ρ) satisfy (2.2) then (1.1) has a positive solution with $u(\frac{1}{2}) = \|u\|_\infty = \rho$. Also, in [2], authors proved that $G(\rho)$ is differentiable in D and its derivative is given by

$$\frac{dG(\rho)}{d\rho} = \sqrt{2}\int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{3/2}} dv \tag{2.3}$$

where $H(s) = F(s) - \frac{1}{2} s f(s)$. We will deduce information on the nature of the bifurcation curve by analysing the sign $\frac{dG(\rho)}{d\rho}$. Note that $\frac{dG(\rho)}{d\rho}$ has the same sign as $\frac{d\lambda(\rho)}{d\rho}$. From (2.3), a sufficient condition for $\frac{dG(\rho)}{d\rho}$ to be positive is:

$$H(\rho) > H(s) \text{ for all } s \in [0, \rho)$$

and a sufficient condition for $\frac{dG(\rho)}{d\rho}$ to be negative is:

$$H(\rho) < H(s) \text{ for all } s \in [0, \rho).$$

3. Proofs of Theorem 1.1 and Corollary 1.1

Proof of Theorem 1.1: First we note that $\lim\limits_{\rho\to\infty} \sqrt{\lambda(\rho)} = 0$ since $\lim\limits_{s\to\infty} \frac{f(s)}{s} = \infty$ (superlinear) (see [3,4,6]). Next we note that there exists a unique θ such that $F(\theta) = 0$ since $\lim\limits_{s\to 0^+} f(s) = -\infty$ and $f'(s) = -\frac{A\gamma}{s^{\gamma+1}} + M[\frac{\alpha}{s^{1-\alpha}} + \delta s^{\delta-1}] > 0$, and $D = (\theta, \infty)$. It is sufficient to show that H has the shape as shown in the Figure 4, namely:

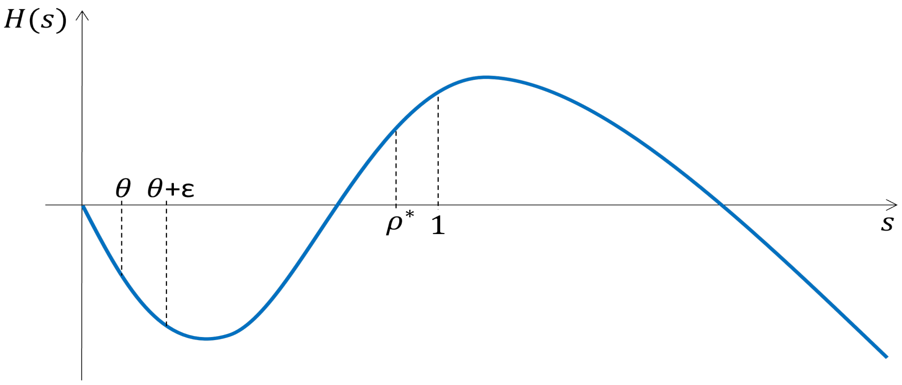

Figure 4. Graph of function H for $M \gg 1$.

(a) If $\alpha \geq \gamma$ then there exists $\epsilon > 0$ such that $H'(s) < 0$ for all $s \in [0, \theta + \epsilon)$ (this implies $\frac{d\sqrt{\lambda(\rho)}}{d\rho} < 0$ for $\rho \in (\theta, \theta + \epsilon)$).

(b) Further, if $\alpha\delta < 1$ then $H(1) > 0$ for $M \gg 1$ and hence there exist some $\rho^* \in (\theta, 1)$ such that $H(\rho^*) > 0$ and $H(\rho^*) > H(s)$ for all $s \in (\theta, \rho^*)$ (this implies $\frac{d\sqrt{\lambda(\rho)}}{d\rho} > 0$ at $\rho = \rho^*$).

(c) There exists a unique solution of (1.1) for $\lambda \approx 0$.

Proof of (a): $F(s) = A\left(\frac{s^{1-\gamma}}{1-\gamma}\right) + M\left(\frac{s^{\alpha+1}}{\alpha+1} + \frac{s^{\delta+1}}{\delta+1}\right) < 0$ on $(0, \theta)$ gives

$$\frac{A}{s^\gamma} < -\frac{(1-\gamma)Ms^\alpha}{(1+\alpha)} - \frac{(1-\gamma)Ms^\delta}{(1+\delta)}. \tag{3.1}$$

By combining (3.1) with

$$H'(s) = \frac{A(1+\gamma)}{2s^\gamma} + \frac{M}{2}[(1-\alpha)s^\alpha + (1-\delta)s^\delta], \tag{3.2}$$

we obtain

$$H'(s) < \frac{Ms^\alpha}{2}\left(\frac{\gamma^2-\alpha^2}{1+\alpha}\right) + \frac{Ms^\delta}{2}\left(\frac{\gamma^2-\delta^2}{1+\delta}\right) < 0 \text{ on } (0, \theta) \text{ for } \alpha \geq \gamma.$$

Note that $\lim_{s\to 0^+} H'(s) = -\infty$. Thus $H'(s) < 0$ on $[0, \theta)$ for $\alpha \geq \gamma$. Next, we will prove that $H'(\theta) < 0$. Assume to the contrary that $H'(\theta) = 0$. Now (3.2) gives

$$A + \frac{M}{1+\gamma}\theta^\gamma[(1-\alpha)\theta^\alpha + (1-\delta)\theta^\delta] = 0 \tag{3.3}$$

and $F(\theta) = 0$ gives

$$A + M(1-\gamma)\,\theta^\gamma\left[\frac{\theta^\alpha}{\alpha+1} + \frac{\theta^\delta}{\delta+1}\right] = 0. \tag{3.4}$$

By (3.3) and (3.4) we obtain

$$M\theta^\gamma\theta^\alpha \underbrace{\left[\frac{(\alpha-\gamma)(\alpha+\gamma)}{(1+\alpha)(1+\gamma)} + \frac{(\delta-\gamma)(\delta+\gamma)}{(1+\gamma)(\delta+1)}\theta^{\delta-\alpha}\right]}_{>0} = 0$$

This is a contradiction since $\theta > 0, M > 0,\ \delta > 1, 0 < \gamma < 1$ and for $\alpha \geq \gamma$. Thus $H'(\theta) < 0$. Hence by continuity, there exists $\epsilon > 0$ such that $H'(s) < 0\ ;\ s \in [0, \theta + \epsilon)$.

Proof of (b): $H(1) = \dfrac{A\,(1+\gamma)}{2(1-\gamma)} + \dfrac{M\,(1-\alpha\delta)}{(\alpha+1)\,(\delta+1)} > 0$ when $\alpha\delta < 1$ and for $M \gg 1$. Now by part (a), we see that $\theta < 1$ for $M \gg 1$. Hence for $M \gg 1$ there exist some $\rho^* \in (\theta, 1)$ such that $H(\rho^*) > 0$ and $H(\rho^*) > H(s)$ for all $s \in [0, \rho^*)$.

Proof of (c): $\lim\limits_{s\to\infty} H'(s) = \lim\limits_{s\to\infty} \dfrac{A(1+\gamma)}{2s^\gamma} + \dfrac{M}{2}[(1-\alpha)s^\alpha + (1-\delta)s^\delta] = -\infty$ implies $\dfrac{d\sqrt{\lambda(\rho)}}{d\rho} < 0$ for $\rho \gg 1$. Thus there exists a unique solution for $\lambda \approx 0$.

Proof of Corollary 1.1: Here it is sufficient to prove that $H'(s) = \dfrac{A(1+\gamma)}{2s^\gamma} + \dfrac{M}{2}[(1-\alpha)s^\alpha - (\delta-1)s^\delta] < 0$ for all $s > 0$ for $M \approx 0$. Let A, γ, α and δ be fixed. Let $p(s) = (1-\alpha)s^\alpha - (\delta-1)s^\delta$. It is easy to see that $p(0) = 0$, $\lim\limits_{s\to\infty} p(s) = -\infty$ and $\lim\limits_{s\to 0^+} p'(s) = \lim\limits_{s\to 0^+} \alpha(1-\alpha)s^{\alpha-1} - \delta(\delta-1)s^{\delta-1} > 0$. Moreover, p is concave since $p''(s) = \alpha(1-\alpha)(\alpha-1)s^{\alpha-2} - \delta(\delta-1)^2 s^{\delta-2} < 0$ for $s > 0$. Thus p achieves unique positive maximum at $s_1 = [\frac{\alpha(1-\alpha)}{\delta(\delta-1)}]^{\frac{1}{\delta-\alpha}}$ and unique zero at $s^* = [\frac{(1-\alpha)}{(\delta-1)}]^{\frac{1}{\delta-\alpha}}$. Since $\delta > \alpha$ it is easy to see that $s^* > s_1$. Now choosing M so that

$$\left|\frac{A(1+\gamma)}{2s^\gamma}\right| > \frac{M}{2} p(s_1)\ ;\ s \in (0, s^*),$$

we conclude that $H'(s) < 0$ for all $s > 0$ for $M \approx 0$. Hence by (2.3) we obtain $\frac{d\sqrt{\lambda(\rho)}}{d\rho} < 0$ for $\rho \in (\theta, \infty)$ when $M \approx 0$ and the corollary is proven.

Remark 3.1. *In [1] it was established that for $M \gg 1$ there exists a range of λ where there exist two positive solutions for (1.1). This implies there must exists a $\rho^* > \theta$ such that $\frac{d\sqrt{\lambda(\rho)}}{d\rho} > 0$ when $\rho = \rho^*$. Combining this fact, $\frac{d\sqrt{\lambda(\rho)}}{d\rho} < 0$ for $\rho \in (\theta, \theta+\epsilon)$ and $\lim\limits_{\rho\to\infty} \sqrt{\lambda(\rho)} = 0$ we can also conclude that the bifurcation diagram is at least a reversed S-shaped curve for $M \gg 1$. This indirect approach (using the result in [1]) does not require $\alpha\delta < 1$ for the bifurcation curve to be at least a reversed S-shaped curve when $M \gg 1$. Note that in the proof Theorem 1.1, $\alpha\delta < 1$ was used only in the proof of part (b).*

4. Computational result of (1.1)

Using (2.2) and Python computations we obtain the following exact bifurcation curves of (1.1) when $\gamma = 0.3$, $\alpha = 0.32$, $\delta = 3.05$ and $A = -4$.

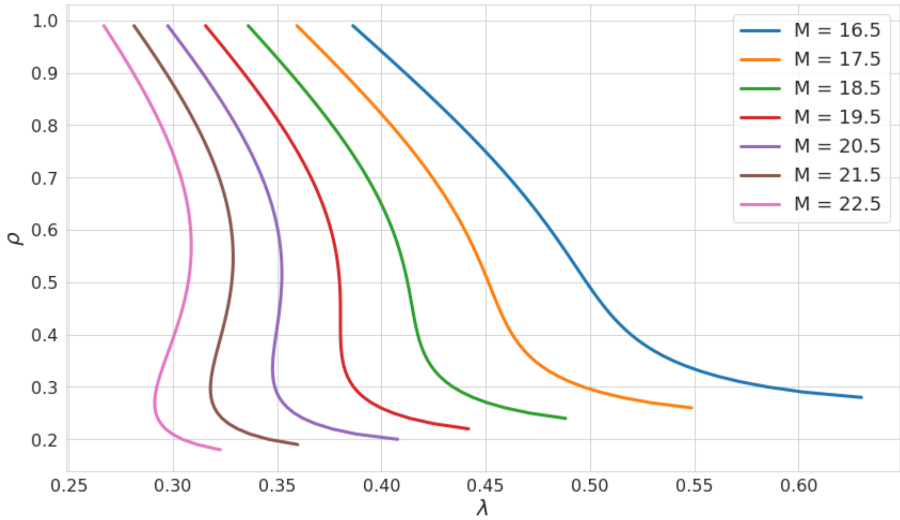

Figure 5. Evolution of bifurcation diagrams for positive solutions to (1.1) when M varies.

5. Computational results of (1.2)

It follows easily that if u is a positive solution of (1.2), then u must be symmetric about $t = \frac{1}{2}$, $u' > 0; (0, \frac{1}{2})$, $u' < 0; (\frac{1}{2}, 1)$ and $u(\frac{1}{2}) = \|u\|_\infty := \rho$. Now we recall the quadrature method described in [4]. Multiplying the differential equation in (1.2) by u' and integrating, we obtain

$$(\frac{p-1}{p})(u'(t))^p = \lambda(F(\rho) - F(u))\ ;\ t \in (0, \frac{1}{2}).$$

where $F(s) = \int_0^s f(z)dz$. Further integrating from 0 to $\frac{1}{2}$, we obtain

$$G_p(\lambda, \rho) := \lambda^{\frac{1}{p}} - 2(\frac{p-1}{p})^{\frac{1}{p}} \int_0^\rho \frac{ds}{(F(\rho) - F(s))^{\frac{1}{p}}} = 0. \tag{5.1}$$

It can be shown that that for $\lambda > 0$ and $\rho \geq \theta$, $G_p(\lambda, \rho)$ is well defined and the bifurcation diagram for positive solutions to (1.2) is given by:

$$S = \{(\lambda, \rho) \mid \lambda > 0, \rho \geq \theta \ \&\ G_p(\lambda, \rho) = 0\}.$$

Using (5.1) and Python computations we obtain the following bifurcation curves for positive solutions for (1.2) when $\gamma = 0.3$, $\alpha = 0.32$, $\delta = 3.05$ and $A = -4$ for multiple values for p and M.

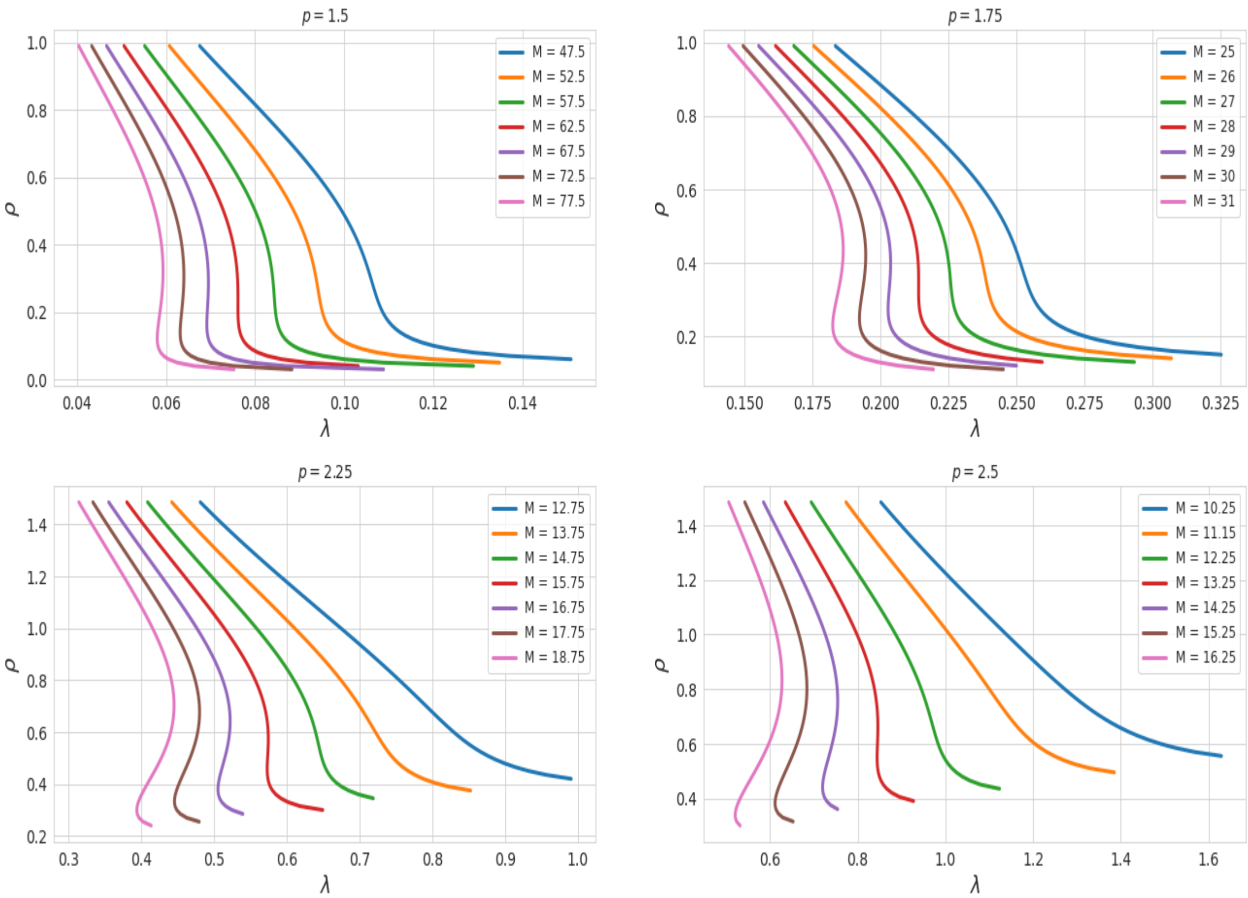

Figure 6. Evolution of bifurcation diagrams of positive solutions to (1.2) when M and p vary.

Remark 5.1. *Our computational results show that the critical value of* $M = M_c$ *beyond which the bifurcation curve is reversed S-shaped, is a decreasing function of* p*.*

6. Computational results of (1.3)

In this section, for the case when $p = 4$ and $q = 2$ of (1.3), namely for the two-point boundary value problem:

$$\begin{gathered} -[(u')^3]' - [(u')]' = \lambda\left\{\frac{A}{u^\gamma} + M[u^\alpha + u^\delta]\right\};\ (0,1) \\ u(0) = 0 = u(1), \end{gathered} \tag{6.1}$$

we compute the bifurcation diagram for positive solutions. It follows easily that if u is a positive solution of (6.1), then u must be symmetric about $t = \frac{1}{2}$, $u' > 0; (0, \frac{1}{2})$, $u' < 0; (\frac{1}{2}, 1)$ and $u(\frac{1}{2}) = \|u\|_\infty := \rho$. Now we recall the quadrature method described in [5]. Multiplying the differential equation in (6.1) by u' and integrating we obtain

$$-\frac{3}{4}[(u')^4]' - \frac{1}{2}[(u')^2]' = \lambda(F(u))' \text{ in } (0,1)$$

where $F(s) = \int_0^s f(z)dz$. Further integrating we obtain

$$3[u'(t)]^4 + 2[u'(t)]^2 = 4\lambda[F(\rho) - F(u(t))]\ ;\ t \in [0, \tfrac{1}{2}]$$

and hence

$$u'(t) = \frac{\sqrt{[1 + 12\lambda(F(\rho) - F(u(t)))]^{\frac{1}{2}} - 1}}{\sqrt{3}}\ ;\ t \in [0, \tfrac{1}{2}]. \tag{6.2}$$

Noting that $u'(0) = \frac{\sqrt{[1+12\lambda F(\rho)]^{\frac{1}{2}}-1}}{\sqrt{3}}$. This implies $\rho \geq \theta$. Integrating (6.2) from 0 to $\frac{1}{2}$, we obtain

$$G(\lambda, \rho) := \int_0^\rho \frac{ds}{\sqrt{[1 + 12\lambda(F(\rho) - F(s))]^{\frac{1}{2}} - 1}} = \frac{1}{2\sqrt{3}}. \tag{6.3}$$

It can be shown that that for $\lambda > 0$ and $\rho \geq \theta$, $G(\lambda, \rho)$ is well defined and the bifurcation diagram for positive solutions to (6.1) is given by:

$$S = \{(\lambda, \rho) \mid \lambda > 0, \rho \geq \theta\ \&\ G(\lambda, \rho) = \tfrac{1}{2\sqrt{3}}\}.$$

Thus using (6.3) and Python computations we obtain the following bifurcation curves of (1.3) when $\gamma = 0.3$, $\alpha = 0.32$, $\delta = 3.05$ and $A = -4$ as M varies.

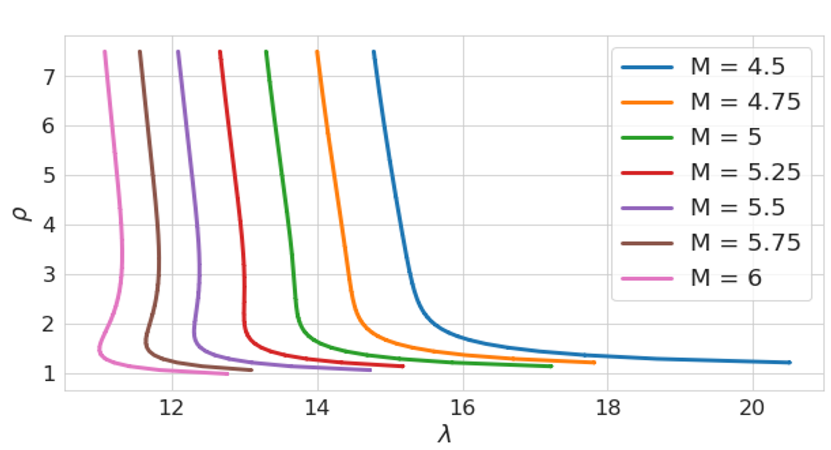

Figure 7. Evolution of bifurcation diagrams of positive solutions to (1.3) when M varies.

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. D. D. Hai, R. Shivaji, Existence and multiplicity of positive radial solutions for singular superlinear elliptic systems in the exterior of a ball, *J. Differ. Equ.*, **266** (2019), 2232–2243. https://doi.org/10.1016/j.jde.2018.08.027

2. K. J. Brown, M. M. A. Ibrahim, R. Shivaji, S-Shaped bifurcation curves, *Nonlinear Anl.*, **5** (1981), 475–486.
3. T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, *Indiana Univ. Math. J.*, **20** (1970), 1–13.
4. E. Ko, E. K. Lee, R. Shivaji, On S-shaped and reversed S-shaped bifurcation curves for singular problems, *Electron. J. Qual. Theo.*, **31** (2011), 1–12. https://doi.org/10.14232/ejqtde.2011.1.31
5. U. Das, A. Muthunayake, R. Shivaji, Existence results for a class of p-q Laplacian semipositone boundary value problems, *Electron. J. Qual. Theo.*, **88** (2020), 1–7. https://doi.org/10.14232/ejqtde.2020.1.88
6. A. Castro, R. Shivaji, Non-negative solutions for a class of non-positone problems, *P. Roy. Soc. Edinb. A*, **108** (1988), 291–302. https://doi.org/10.1017/S0308210500014670
7. A. C. Lazer, P. J. McKenna, On a singular nonlinear elliptic boundary value problem, *P. Am. Math. Soc.*, **111** (1991), 720–730. https://doi.org/10.2307/2048410
8. Z. Zhang, On a Dirichlet problem with a singular nonlinearity, *J. Math. Anal. Appl.*, **194** (1995), 103–113. https://doi.org/10.1006/jmaa.1995.1288
9. Y. Miaoxin, J. Shi, On a singular nonlinear semilinear elliptic problem, *P. Roy. Soc. Edinb. A*, **128** (1998), 1389–1401. https://doi.org/10.1017/S0308210500027384

AIMS Press