Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(2): 1119–1133.
DOI: 10.3934/era.2023056
Received: 08 September 2022
Revised: 03 December 2022
Accepted: 07 December 2022
Published: 20 December 2022

Research article

A priori estimates of solutions to nonlinear fractional Laplacian equation

Tao Zhang[1] **and Tingzhi Cheng**[2,*]

[1] School of Mathematics and Information Science, Yantai University, Yantai 264005, China
[2] School of Mathematics and Statistics Science, Ludong University, Yantai 264025, China

* **Correspondence:** Email: chengtingzhi1989@163.com.

Abstract: In this paper, we focus on the research of a priori estimates of several types of semi-linear fractional Laplacian equations with a critical Sobolev exponent. Employing the method of moving planes, we can achieve a priori estimates which are closely connected to the existence of solutions to nonlinear fractional Laplacian equations. Our result can extend a priori estimates of the second order elliptic equation to the fractional Laplacian equation and we believe that the method used here will be applicable to more general nonlocal problems.

Keywords: a priori estimates; fractional Laplacian; nonlocal problems

1. Introduction

The objective of this paper is to investigate the following semi-linear fractional Laplacian equation

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(u(x)), \quad u(x) > 0, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where $0 < \alpha < 2$, $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a smooth bounded domain. We will focus on the research of a priori estimates of the solutions to Problem (1.1), which plays a crucial role in the existence theory of solutions to Problem (1.1).

It is well known that Problem (1.1) originates from its local case $\alpha = 2$, i.e., the following semilinear second order elliptic equation:

$$\begin{cases} -\Delta u(x) = f(u(x)), \quad u(x) > 0, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.2}$$

During the past few decades, Eq (1.2) has arisen in many fields of mathematics. For example, in the branch of conformal geometry, the famous Nirenberg problem is stated as follows: does there exist a positive function $K(x)$ on a standard sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$, $n \geq 2$, such that the scalar curvature $\mathbf{R}_g$ of another conformal metric $g = e^w g_{\mathbb{S}^n}$ $(w(x), x \in \mathbb{S}^n)$ equals exactly to itself on the sphere $\mathbb{S}^n$? We know that the problem can be equivalently transformed to solve the following semi-linear second order elliptic equations:

$$-\Delta_{g_{\mathbb{S}^n}} w(x) + 1 = K(x)e^{2w(x)}, \ x \in \mathbb{S}^2, \tag{1.3}$$

and

$$-\Delta_{g_{\mathbb{S}^n}} v(x) + c(n)\mathbf{R}_0 v(x) = c(n)K(x)v^{\frac{n+2}{n-2}}(x), \ x \in \mathbb{S}^n, \ n \geq 3, \tag{1.4}$$

where $c(n) = \frac{n-2}{4(n-1)}$, $\mathbf{R}_0 = n(n-1)$ is the scalar curvature of $(\mathbb{S}^n, g_{\mathbb{S}^n})$ and $v(x) = e^{\frac{n-2}{4}w(x)}$. Since their significant contribution to the research on the Nirenberg problem, a priori estimates have attracted a large amount of attention. In a seminal series of papers, Chang et al. [1], Han [2] and Schoen-Zhang [3] obtained some interesting qualitative analysis on the solutions of (1.3) ($n = 2$) and (1.4) ($n = 3$). They have demonstrated that a sequence of solutions cannot blow up at more than one point. For higher dimensions ($n \geq 4$), Li [4, 5] has proved that a sequence of solutions can blow up at more than one point, which is quite different from the case in lower dimensions.

Apart from the Nirenberg problem, in another classical paper, Gidas and Spruck [6] investigated the priori estimates of the following semi-linear second order elliptic equation which arises in the variational problems

$$\begin{cases} -\Delta u(x) = u^p(x), \quad u(x) > 0, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $1 < p < \frac{n+2}{n-2}$, $\Omega \subset \mathbb{R}^n$ and $n \geq 3$ is a smooth bounded domain. They proved that there exists a constant $C(n, p, \Omega)$, such that

$$\|u\|_{L^\infty(\Omega)} \leq C. \tag{1.6}$$

Furthermore, one can refer to [7–12] and the references therein to see more details about a priori estimates of the classical elliptic equations.

Motivated by the above results, we mainly consider the priori estimates of Problem (1.1). The operator $(-\Delta)^{\frac{\alpha}{2}}$ is the well known fractional Laplacian, which arises in many models related to diverse physical phenomena, such as turbulence, quasi-geostrophic flows, water waves, anomalous diffusion and molecular dynamics (see [13, 14] and the references therein). The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}(0 < \alpha < 2)$ is a nonlocal operator which can be well defined on smooth functions

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy,$$

where $P.V.$ stands for the Cauchy principle value. In addition, this operator can also be defined as

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+\alpha}} dy,$$

where the integral is absolutely convergent for sufficiently smooth functions. Define

$$\mathcal{L}_\alpha = \left\{ u : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+\alpha}} dy < +\infty \right\}.$$

Then it is easy to see that for $u \in C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_\alpha$, $(-\Delta)^{\frac{\alpha}{2}} u(x)$ is well defined for all $x \in \Omega$.

However, compared to the priori estimates for Laplacian equations, fewer results concerning the priori estimates of fractional Laplacian equations have been acquired to date. For example, Jin et al. [15] considered the following fractional Nirenberg problem (1.7):

$$c_{n,-\sigma}\int_{\mathbb{S}^n}\frac{v(\xi)-v(\eta)}{|\xi-\eta|^{n+2\sigma}}dvol_{g_{\mathbb{S}^n}}+\mathbf{R}^g_\sigma v(\xi)=c_{n,\sigma}K(\xi)v^{\frac{n+2\sigma}{n-2\sigma}}(\xi),\ \xi\in\mathbb{S}^n, \tag{1.7}$$

where $c_{n,-\sigma}=\frac{2^{2\sigma}\Gamma(\frac{n+2\sigma}{2})}{\pi^{\frac{n}{2}}\Gamma(1-\sigma)}$, $0<\sigma<1$, $\mathbf{R}^g_\sigma$ represents the Q_σ curvature (see [16, 17] for details) and $v\in C^2(\mathbb{S}^n)$. Under some constraints on $K(\xi)$, they obtained a priori estimates of the solutions by employing the blow up analysis method. However, we should mention that their problem is defined on a sphere which is a special domain without a boundary. In [18], the authors have pointed that the boundary estimates of solutions to fractional Laplacian equations cannot be good enough to proceed the blow up analysis. So the method used in [15] fails for the general domain whose boundary set is not empty.

In addition to Problem (1.7), a recent paper [19] analyzed the following fractional Laplacian equation

$$(-\Delta)^{\frac{\alpha}{2}}u(x)=u^p(x),\quad x\in\Omega, \tag{1.8}$$

where $1<p<\frac{n+\alpha}{n-\alpha}$ and $\Omega\subset\mathbb{R}^n$ is a smooth bounded domain. They derived that there exists a uniform constant $C(n,\alpha,p,\Omega)$ such that $\|u\|_{L^\infty(\Omega)}\le C$. Their result extended the famous results of Gidas and Spruck [6] to the fractional Laplacian equation, however, only covering subcritical nonlinearities. Then a natural question arises: what will happen for the critical case?

In this paper, we will concentrate on the analysis of semi-linear fractional Laplacian equations with a critical exponent on general domains whose boundary sets are not empty. We derive a priori estimates of solutions near the boundary which can partially solve the priori estimates for fractional Laplacian equations with a critical exponent.

Because of the nonlocal property, it is often hard to investigate the fractional Laplacian. In order to overcome this difficulty, Caffarelli and Silvestre [20] introduced an extension method which transformed the nonlocal operator to the local case in higher dimensions. According to the extension method, the fractional Laplacian can be equivalently defined as follows: for a function $u:\mathbb{R}^n\to\mathbb{R}$, consider $U:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ satisfying

$$\begin{cases} div(y^{1-\alpha}\nabla U)=0, & (x,y)\in\mathbb{R}^n\times[0,\infty),\\ U(x,0)=u(x). \end{cases} \tag{1.9}$$

Then

$$(-\triangle)^{\frac{\alpha}{2}}u=-C_{n,\alpha}\lim_{y\to0^+}y^{1-\alpha}\frac{\partial U}{\partial y}.$$

In the last decades, many researchers have applied the extension method to deal with the nonlinear equations involving a fractional Laplacian operator (see [15, 18, 21–28] and the references therein). For instance, Jin et al. [15] studied Problem (1.7) with the help of spherical projection transformation which switches the domain of Problem (1.7) to the whole space; they used the extension method to deal with a nonlinear fractional Laplacian equation which is defined in the whole space. Unfortunately, the extension method will lose its magic when the domain of the equation is not the whole space $\mathbb{R}^n$.

More precisely, we will study a priori estimates of the semi-linear fractional Laplacian equation with more general nonlinearities

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u(x) = f(x,u), \quad u(x) > 0, & x \in \Omega, \\ u(x) \equiv 0, & x \in \mathbb{R}^n \backslash \Omega,\end{cases} \tag{1.10}$$

where $\alpha \in (0,2)$ and $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain. If $f(x,u) = u^p$, $p = \frac{n+\alpha}{n-\alpha}$, then (1.10) can cover the critical case of (1.8). The lack of a Liouville type theorem and sufficient boundary regularity makes it hard for us to use the blow-up analysis, so here we search another new method to overcome this setback.

The strategy we will follow is the method of moving planes. The method of moving planes was initiated by Aleksandrov [29] and has been further developed to study the monotonicity and radial symmetry properties of the positive solutions to many elliptic equations; the details can be seen in Serrin [30], Gidas et al. [31,32], Berestycki and Nirenberg [33,34], Chen and Li [35], Li [36,37], and so on. However, due to the nonlocality of the fractional operator, for the fractional Laplacian equation (1.10), it is difficult to use the method of moving planes directly.

Recently, Chen et al. [38] systematically developed a direct method of moving planes to deal with the nonlocal problems. In [38], they proved some maximum principles for antisymmetric functions and applied the method of moving planes to nonlocal operators. They considered the following problem

$$(-\Delta)^{\frac{\alpha}{2}}u(x) = f(u), \quad x \in \mathbb{R}^n. \tag{1.11}$$

Under some suitable assumptions on f, with the help of the Kelvin transform and these maximum principles, they obtained some interesting results on the classifications of solutions to Eq (1.11).

In this paper, with the help of this direct method of moving planes for the fractional Laplacian, we will investigate a priori estimates of the positive solutions to the equations with a nonlocal operator. Our results can exclude the possibility of solutions from being blown up near the boundary. In particular, in a strictly convex domain, we will obtain a priori bounds of the solutions to a larger family of semi-linear fractional Laplacian equations like type (1.8), which has the exponent p that lies in $(1,\infty)$.

Our first result concerns the case in a strictly convex domain.

Theorem 1.1. *Let Ω be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$. Suppose that $u \in C^{1,1}_{loc}(\Omega) \cap C(\mathbb{R}^n)$ is a positive solution which solves the following equation*

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u(x) = u^p(x), & x \in \Omega, \\ u(x) \equiv 0, & x \in \mathbb{R}^n \backslash \Omega,\end{cases} \tag{1.12}$$

where $0 < \alpha < 2$ and $p > 1$. Then there exists a positive number δ and a constant C (independent of u) such that $\|u\|_{L^\infty(\Omega_\delta)} \leq C$ with $\Omega_\delta = \{x | dist(x, \partial\Omega) \leq \delta\}$.

The assumptions that $u \in C^{1,1}_{loc}(\Omega) \cap C(\mathbb{R}^n)$ and u is compact supported is essential to the definition of the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}u$.

A similar result can also be proved for Eq (1.10) with more general nonlinearities.

Corollary 1.2. *Let Ω be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$. Suppose that $u \in C^{1,1}_{loc}(\Omega) \cap C(\mathbb{R}^n)$ is a positive solution that solves the following equation*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x,u), & x \in \Omega, \\ u(x) \equiv 0, & x \in \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.13}$$

where $0 < \alpha < 2$ and $f(x,u)$ satisfies the following conditions:

(*i*) $\forall x_0 \in \Omega, \exists \delta_{x_0}$ *such that for $u \geq 0, |x - x_0| < \delta_{x_0}, x \in \partial\Omega, h(t) = f(x + t\nu(x_0), u)$ is a nondecreasing function of t and $t \in [0, \delta_{x_0}]$, where $\nu(x_0)$ is the unit inner normal vector;*
(*ii*) *$f(x,u)$ is locally Lipschitz continuous for u;*
(*iii*) *$f(x,u) \geq c_1 u^p$, $\forall x \in \mathbb{R}^n, 1 < p < \infty$ and $c_1 > 0$.*

Then there exists a positive number δ' and a constant C' (independent of u) such that $\|u\|_{L^\infty(\Omega_{\delta'})} \leq C'$ with $\Omega_{\delta'} = \{x | dist(x, \partial\Omega) \leq \delta'\}$.

However, when the domain is not strictly convex, new complexity arises since the method of moving planes can not be applied to the equation directly. In order to overcome this difficulty, we first make use of the smoothness of the domain to pull in a uniform ball which is tangent to the boundary of the domain. Consequently, by the technique of the Kelvin transform, we manage to bring back the strict convexity. With these minor adjustments, we can use the method of moving planes again. In this way, we can also get a priori estimates of the solutions to the equations whose domains are not strictly convex.

Theorem 1.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Suppose that $u \in C^{1,1}_{loc}(\Omega) \cap C(\mathbb{R}^n)$ is a positive solution which solves* (1.13) *and $f(x,u) \in C^1(\mathbb{R}^n \times \mathbb{R})$ satisfies*

(*i*) *$|\nabla_x f(x,u)| \leq C_1 f(x,u)$ for $x \in \Sigma = \{x | dist(x, \partial\Omega) < \delta_1\}$;*
(*ii*) *$f(x,u) \geq c_1 u^p$, $\forall x \in \mathbb{R}^n, p > 1$ and $c_1 > 0$;*
(*iii*) *$\frac{f(x,t)}{t^{\overline{p}}}$ is nonincreasing for t, $1 \leq \overline{p} < \frac{n+\alpha}{n-\alpha}$.*

Then there exists a positive number δ_2 and a constant C_2 (independent of u) such that $\|u\|_{L^\infty(\Omega_{\delta_2})} \leq C_2$, with $\Omega_{\delta_2} = \{x | dist(x, \partial\Omega) \leq \delta_2\}$.

Condition (*iii*) in Theorem 1.3 implies that the order of u belongs to the subcritical case. Moreover, we can also treat the critical case in the domain which may not be strictly convex.

Theorem 1.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Suppose that $u \in C^{1,1}_{loc}(\Omega) \cap C(\mathbb{R}^n)$ is a positive solution which solves the following equation*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = u^p(x), & x \in \Omega, \\ u(x) \equiv 0, & x \in \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.14}$$

where $0 < \alpha < 2$ and $p = \frac{n+\alpha}{n-\alpha}$. Then there exists a positive number δ and a constant C_3 (independent of u) such that $\|u\|_{L^\infty(\Omega_\delta)} \leq C_3$ with $\Omega_\delta = \{x | dist(x, \partial\Omega) \leq \delta\}$.

Theorem 1.4 improves the results in [19] for the critical exponent case near the boundary of the domain.

This paper is organized as follows. In Section 2, we will provide a new type of maximum principle for λ-antisymmetric functions which is similar to the one discovered by Chen et al. in [38]. We will use this maximum principle to carry on the method of moving planes in Sections 3 and 4 to get a priori estimates near the boundary in strictly convex domains and non-strictly convex domains respectively.

Remark 1.5. For $f(x,u) = Q(x)u^p(x)$, the result of this paper can also be applied to the well known Q-curvature problem which is an important topic in conformal geometry; more details about the topic of Q-curvature can be seen in [16, 17].

Remark 1.6. The condition (i) in Corollary 1.2 seems to be artificial; however, we can point out that for $f(x,u) = R(x)h(u) + g(u)$ with the conditions $R(x) \in C^1(\overline{\Omega})$ and $\frac{\partial R}{\partial \nu}|_{\partial\Omega} < 0$, where ν is the unit outer normal vector of $\partial\Omega$, it is easy to check that $f(x,u)$ can satisfy Condition (i).

Remark 1.7. We should point out that the Condition (i) in Theorem 1.3 eliminates the monotonicity condition of the nonlinear term $f(x,u)$ near the boundary which is exactly the Condition (i) in Corollary 1.2 (similar conditions to Condition (i) in Corollary 1.2 have been mentioned in [39]). The improvement in this condition has expanded the range of the nonlinear term $f(x,u)$. For example, $\Omega = B_R(0)$ and $f(x,u) = e^{C_1|x|}u^p$. In this case, $f(x,u)$ can not satisfy the Condition (i) in Corollary 1.2; however, it can be contained in the class which is attached to Theorem 1.3.

Remark 1.8. In fact, we can reduce the assumption on Ω to be $C^{2,\alpha}$. Here we demand the smoothness of Ω for simplicity.

Remark 1.9. *Particularly, we can derive the priori estimates for Eq (*1.14*) in the whole domain by combining the results in [15] and Theorem* 1.4 *together.*

Apart from the semi-linear fractional Laplacian equation, we believe that this method can also be beneficial to some types of fully nonlinear fractional Laplacian equations as long as we have the corresponding a priori integral estimates; hence, the key to obtaining a priori estimates of fully nonlinear equations is to derive some types of integral estimates.

Open problem: Can we prove that there exists some uniform constant C such that all solutions of the equations

$$\begin{cases} F_\alpha u(x) = u^p(x), \quad u(x) > 0, & x \in \Omega, \\ u(x) \equiv 0, & x \in \mathbb{R}^n \backslash \Omega, \end{cases}$$

with

$$\begin{cases} F_\alpha(u(x)) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{G(u(x)-u(y))}{|x-y|^{n+\alpha}}, \ 0 < \alpha < 2, \\ G(0) = 0, G'(\cdot) \geq \sigma > 0, \end{cases}$$

satisfy that $\int_{\Omega'} u(x)dx \leq C, \Omega' \subset\subset \Omega$?

2. Preliminary lemma

In order to apply the method of moving planes directly to the fractional Laplacian, we will introduce the maximum principle which is similar to the ones discovered by Chen et al. [38].

Lemma 2.1. *(Narrow Region Maximum Principle) Let D be a bounded domain which is contained in $H_\lambda = \{x|\lambda - l < x_1 < \lambda\}$ with a small l. Consider $u \in C^{1,1}_{loc}(D) \cap \mathcal{L}_\alpha$ which is a λ-antisymmetric function being lower semi-continuous on $\bar{D}$ that satisfies $u(x) \geq 0$ in $\Sigma_\lambda \backslash D$, $\Sigma_\lambda = \{(x_1, x') \in \mathbb{R}^n | x_1 < \lambda\}$.*

Suppose $u(x_0) = \min\limits_{x \in \overline{D}} u(x)$, $x_0 \in D$ and

$$(-\triangle)^{\alpha/2} u(x_0) + c(x_0) u(x_0) \geq 0,$$

where $c(x_0)$ is bounded from below. Then, for a sufficiently small l, we have

$$u(x) \geq 0, \ \ x \in D. \tag{2.1}$$

The details of the proof is similar to the proof of Theorem 2 in [38], so we omit it here.

3. Strictly convex domain

Before proving the main theorems, we need to define some notations which will be frequently used throughout this paper:

$$\begin{aligned}
&x = (x_1, x') \in \mathbb{R}^n, x_1 \in \mathbb{R}, x' \in \mathbb{R}^{n-1}, x^\lambda = (2\lambda - x_1, x'), T_\lambda = \{(x_1, x') \in \mathbb{R}^n | x_1 = \lambda\},\\
&\Sigma_\lambda = \{(x_1, x') \in \mathbb{R}^n | x_1 > \lambda\}, \quad \widetilde{\Sigma}_\lambda = \{(x_1, x') \in \mathbb{R}^n | x_1 < \lambda\}.
\end{aligned}$$

Proof of Theorem 1.1: For any $y^\star \in \partial\Omega$, we prove that

$$u(x) \leq C, \quad x \in B_\delta(y^\star) \cap \Omega,$$

for some positive number δ depending only on Ω. It is worth pointing out that we only need to treat the case in which $B_\delta(y^\star) \cap \Omega$ contains only one connected component because of the convexity condition of Ω and the assumption that $y^\star \in \partial\Omega$.

Since the operator $(-\Delta)^{\frac{\alpha}{2}}$ is invariant under translation and orthogonal transformation, without loss of generality, we can assume that $\nu^0 = (-1, 0 \cdots, 0)$ is the inner unit normal vector of Ω at $y^\star$. Due to the fact that Ω is a bounded domain, we may assume that $\Omega \subset \{x \in \mathbb{R}^n || x_1| \leq 1\}$ and $\partial\Omega \cap \{x \in \mathbb{R}^n | x_1 = 1\} \neq \emptyset$. Set

$$u_\lambda(x) = u(x^\lambda), \quad w_\lambda(x) = u_\lambda(x) - u(x), \quad \forall x \in \Sigma_\lambda \cap \Omega.$$

Claim: There exists a small enough l such that

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda \cap \Omega, \quad \forall \lambda \in [1 - l, 1]. \tag{3.1}$$

Suppose not. Then there exists x_0 such that

$$w_\lambda(x_0) = \min_{x \in \overline{\Sigma_\lambda \cap \Omega}} w_\lambda(x) < 0.$$

By $0 \leq u(x) \in C(\mathbb{R}^n)$, $u(x) \equiv 0, x \in \mathbb{R}^n \backslash \Omega$, we can notice that $w_\lambda(x) \geq 0$ on $\partial(\Omega \cap \Sigma_\lambda)$, so x_0 must be an interior point in $\Omega \cap \Sigma_\lambda$.

Now we can achieve that

$$(-\Delta)^{\frac{\alpha}{2}} w_\lambda(x_0) = u^p_\lambda(x_0) - u^p(x_0) \geq p u^{p-1}(x_0) w_\lambda(x_0).$$

Compared to Lemma 2.1, here $c(x_0) = u^{p-1}(x_0)$ is uniformly bounded since $u(x) \in C(\mathbb{R}^n)$ which has compact support. Now by Lemma 2.1, we can get that, for l small enough, $w_\lambda(x) \geq 0, x \in \Sigma_\lambda \cap \Omega$; this yields a contradiction and proves (3.1).

With the help of (3.1), $u(\overline{y} + t\nu^0(\overline{y}))$ is increasing for $t \in [0, t_0]$, $\overline{y} \in B_{t_0}(y^\star) \cap \partial\Omega$, $t_0 < l$ and $\nu^0(\overline{y})$ denotes the inner unit normal vector. By the smoothness and strictly convex property of $\partial\Omega$ near the point $y^\star$, we can derive that the inner normal directions at $B_{t_0}(y^\star) \cap \partial\Omega$ can form a uniform cone centered at some point $b = (1 - b', 0)$ with $0 < b' \leq k_0 < \frac{t_0}{2}$ and a positive angle θ_0 depending only on Ω. Let

$$I = \left\{ \nu \,\middle|\, \nu \cdot \nu^0 \geq |\nu| \cos\theta_0, |\nu| \leq \frac{t_0}{2} \right\}$$

be a typical piece of the cone.

For any $b \in B_{\frac{t_0}{2}}(y^\star) \cap \Omega$, we can denote the cone which is centered at b by

$$I_b = \{ b + \nu \,|\, \nu \in I \}.$$

The above arguments simply imply that

$$u(y) \geq u(b), \quad b \in B_{\frac{t_0}{2}}(y^\star) \cap \Omega, y \in I_b \cap \Omega. \tag{3.2}$$

Refer to the potential theory stated in Section IV, Chapter IV of [40]; since the domain Ω is smooth, the Green function $G(x, y)$ exists and $u(x)$ can be represented as

$$u(x) = \int_\Omega G(x, y) u^p(y) dy. \tag{3.3}$$

Thus by using (3.3), we can deduce that

$$\begin{aligned} u(b) &\geq \int_{I_b} G(b, y) u^p(y) dy \\ &\geq \int_{I_b} G(b, y) dy \cdot u^p(b) \end{aligned}$$

$$\Rightarrow u^{p-1}(b) \leq C = \frac{1}{\int_{I_b} G(b, y) dy}.$$

We should notice that $|I_b|$ has a uniform lower bound $L_1 > 0$ and $G(b, y) \geq L_2$, $y \in I_b$ with L_2 independent of b, so a constant C is independent of b. Combined with (3.1), we have that

$$u(x) \leq C, \quad x \in B_{\frac{t_0}{2}}(y^\star) \cap \Omega. \tag{3.4}$$

The estimate (3.4) is valid for any point $y^\star \in \partial\Omega$; thus, we can have

$$\|u\|_{L^\infty(\Omega_\delta)} \leq C \tag{3.5}$$

for $\delta = \delta(\Omega)$ and $C = C(n, \Omega, p, \alpha)$. This completes the proof of Theorem 1.1.

The proof of Corollary 1.2 is similar to the proof of Theorem 1.1; for the sake of completeness, we give a proof here.

Proof of Corollary 1.2: Similar to the proof of Theorem 1.1, we also need to prove that there exists $\delta'(\Omega)$ such that for any $y^\star \in \partial\Omega$, $\|u\|_{L^\infty(B_{\delta'}(y^\star)\cap\Omega)} \leq C'$.

By completely using the notation introduced in the proof above, we also need to show the monotonicity property of the solutions along the direction of each inner normal vector at $y \in B_{\delta'}(y^\star) \cap \partial\Omega$. By the compactness of $\overline{\Omega}$, we can easily deduce that there exists a uniform δ' which is independent of $y \in B_{\delta'}(y^\star) \cap \partial\Omega$ such that $h(t) = f(\overline{y} + t\nu(y))$ is nondecreasing for $t \in [0, \delta']$, where $\nu(y)$ denotes the inner normal vector at the point y, and $|\overline{y} - y| < \delta'$. Similar to the proof in Theorem 1.1, we pick out the typical x_1 direction to show that $w_\lambda(x) = u_\lambda(x) - u(x) \geq 0$ for $x \in \Sigma_\lambda \cap \Omega$ when $\lambda \in [1 - l, 1]$ with $l < \delta'$ sufficiently small. If not, there exists x_0 such that

$$w(x_0) = \underline{\min}_{x\in\Sigma_\lambda\cap\Omega} w_\lambda(x) < 0. \tag{3.6}$$

Similar to the analysis mentioned above, x_0 can only be located in the interior part of Ω. We will concentrate on the analysis at the point x_0 again by making use of Condition (i) and (ii) imposed on $f(x, u)$:

$$\begin{aligned}
&(-\Delta)^{\frac{\alpha}{2}} w_\lambda(x_0)\\
=&f(x_0^\lambda, u_\lambda(x_0)) - f(x_0, u(x_0))\\
=&f(x_0^\lambda, u_\lambda(x_0)) - f(x_0, u_\lambda(x_0)) + f(x_0, u_\lambda(x_0)) - f(x_0, u(x_0))\\
\geq&f(x_0, u_\lambda(x_0)) - f(x_0, u(x_0))\\
=&\frac{f(x_0, u_\lambda(x_0)) - f(x_0, u(x_0))}{u_\lambda(x_0) - u(x_0)}(u_\lambda(x_0) - u(x_0)).
\end{aligned}$$

By the locally Lipschitz continuous condition of $f(x, u)$ on u, here $c(x_0) = \frac{f(x_0,u_\lambda(x_0))-f(x_0,u(x_0))}{u_\lambda(x_0)-u(x_0)}$ is bounded from below; then, by Lemma 2.1 we get $w_\lambda(x) \geq 0, x \in \Sigma_\lambda \cap \Omega$ for a sufficiently small l, which is a contradiction of (3.6).

With the above argument, we can have the desired monotonicity property of u along the direction of the inner normal vector. With the help of the Green representation formula, we have

$$u(x) = \int_\Omega G(x, y)f(y, u(y))dy. \tag{3.7}$$

Repeating the process which is similar to the proof of Theorem 1.1, we can derive the following results

$$\begin{aligned}
u(b) &\geq \int_{I_b} G(b, y)f(y, u(y))dy\\
&\geq c_1 \int_{I_b} G(b, y)dy \cdot u^p(b),
\end{aligned}$$

for $b \in B_{\frac{t_0}{2}}(y^\star) \cap \Omega$ with t_0 the same definition as mentioned in the proof of Theorem 1.1. Then it is easy to see that $u(b) \leq C'$. By the monotonicity property we have proved already, we can get that $\|u\|_{L^\infty(\Omega_{\delta'})} \leq C'(p, n, \alpha, c_1, \Omega)$.

4. Generalized domain

In this section, we will turn our attention to a priori estimates for the case of more generalized domains. The main difficulties lie in the lack of a convexity condition, which brings tremendous difficulties to the application of the method of moving planes. In order to deal with this setback, we shall rely on the Kelvin transform to help us recover the convexity condition. After that, we can apply the method of moving planes again to deal with the problem of a priori estimates.

Proof of Theorem 1.3: Since Ω is a smooth domain, there exists a uniform ball with a radius equal to ε_0 $(0 < \varepsilon_0 < 1)$. Set $\varepsilon_1 = \min\left\{\varepsilon_0, \varepsilon_0'\right\}$, where ε_0' is a constant which will be confirmed later. For any $x^0 \in \partial\Omega$, we also want to prove that

$$u(x) \le C, \quad x \in B_{\delta_2}(x^0) \cap \Omega,$$

for some positive number δ_2 depending on $\Omega, \delta_1, n, \alpha, \overline{p}$ and C_1. Similar to the proof of Theorem 1.1, we only also need to consider that $B_{\delta_2}(x^0) \cap \Omega$ contains only one connected component. Set $x^1 = x^0 + \varepsilon_1 \nu^0$ where ν^0 is the unit outward normal direction of Ω at the point x^0. We may assume that $\nu^0 = \{-1, 0, \cdots, 0\}$. It is easy to see that the ball $B_{\varepsilon_1}(x^1)$ is tangent to Ω at x^0 and $B_{\varepsilon_1}(x^1) \cap \Omega = \varnothing$. Then we transform Ω to Ω^* via the conformal transformation

$$x \to y = x^1 + \frac{x - x^1}{|x - x^1|^2} = T(x), \quad x \in \Omega. \tag{4.1}$$

An elementary calculation can show that $\Omega^* \subset B_{\frac{1}{\varepsilon_1}}(x^1)$ and Ω^* is tangent to $B_{\frac{1}{\varepsilon_1}}(x^1)$ at the point $x^\star = T(x^0)$. Without loss of generality, we may assume x^1 to be the original point O, so $x^\star$ can be represented as $x^\star = \left\{\frac{1}{\varepsilon_1}, 0, \cdots, 0\right\}$.

Let $v(x) = \frac{1}{|x-x^1|^{n-\alpha}} u(x^1 + \frac{x-x^1}{|x-x^1|^2})$. By a direct calculation, for $v(x)$, we have the following equations

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} v(x) = \frac{1}{|x|^{n+\alpha}} f\left(\frac{x}{|x|^2}, |x|^{n-\alpha} v(x)\right), & x \in \Omega^*, \\ v(x) \equiv 0, & x \in \mathbb{R}^n \backslash \Omega^*, \end{cases} \tag{4.2}$$

or

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} v(x) = f^\star(x, v), & x \in \Omega^*, \\ v(x) \equiv 0, & x \in \mathbb{R}^n \backslash \Omega^*, \end{cases} \tag{4.3}$$

where $0 < \alpha < 2$ and

$$f^\star(x, v) = \frac{1}{|x|^{n+\alpha-\overline{p}(n-\alpha)}} \frac{f\left(\frac{x}{|x|^2}, t\right)}{t^{\overline{p}}} v^{\overline{p}}, \ t = |x|^{n-\alpha} v(x). \tag{4.4}$$

The outward normal direction of $\partial\Omega^*$ at $x^\star$ is $\overline{\nu}^\star = (1, 0, \cdots, 0)$ and Ω^* is strictly convex around $x^\star$; then, the convexity condition of the domain is retrieved after the Kelvin transform. So we can apply the method of moving planes near the neighborhood of $x^\star$ provided that $f^\star$ is monotone decreasing along the outward normal direction for x in a small neighborhood of $x^\star$. Next, we will prove that $f^\star$ has desired monotone properties near the point $x^\star$.

Define

$$\Omega_\delta^* = \{x | x \in \Omega^*, dist(x, \partial\Omega^*) \le \delta\}; \tag{4.5}$$

we can see that $\Omega^*_{\delta_3} \subset T(\Omega_{\delta_2})$ for a δ_3 sufficiently small. For $x = (x_1, 0, \cdots, 0) \in \Omega^*_{\delta_3}$ with $|x - x^\star| \leq \frac{1}{2\varepsilon_1}$ and $\nu^\star = (\nu_1^\star, \nu_2^\star, \cdots, \nu_n^\star)$ which is the unit outward normal vector whose direction is sufficiently close to the direction of the vector $\overline{\nu}^\star$ with $\nu_1^\star \geq \frac{9}{10}$, we set $g^\star = \frac{t^{\overline{p}} f^\star}{v^{\overline{p}}}$ (t and v can be regarded as variables independent of x). Then

$$
\begin{aligned}
\frac{\partial g^\star}{\partial \nu^\star}(x, v) &= \frac{\partial}{\partial \nu^\star}\left(\frac{1}{|x|^{n+\alpha-\overline{p}(n-\alpha)}}\right) f + \frac{1}{|x|^{(n+\alpha)-\overline{p}(n-\alpha)}} \nabla_x f \cdot \nabla\left(\frac{x}{|x|^2}\right) \cdot \nu^\star \\
&\leq -((n+\alpha) - \overline{p}(n-\alpha)) \frac{1}{|x|^{(n+\alpha)-\overline{p}(n-\alpha)+1}} \frac{\nu_1^\star x_1 + \nu_2^\star x_2 + \cdots + \nu_n^\star x_n}{|x|} f \\
&+ \frac{n(2n+2)C_1}{|x|^{(n+\alpha)-\overline{p}(n-\alpha)+2}} f \\
&\leq -\frac{f}{|x|^{(n+\alpha)-\overline{p}(n-\alpha)+1}} \left(\frac{7(n+\alpha-\overline{p}(n-\alpha))}{20} - \frac{2n(n+1)C_1}{|x|}\right) \\
&\leq -\frac{f}{|x|^{(n+\alpha)-\overline{p}(n-\alpha)+1}} \left(\frac{7(n+\alpha-\overline{p}(n-\alpha))}{20} - 4n(n+1)C_1\varepsilon_0'\right) \\
&< 0;
\end{aligned}
$$

here we may choose ε_0' sufficiently small to ensure that $\frac{7(n+\alpha-\overline{p}(n-\alpha))}{20} - 4n(n+1)C_1\varepsilon_0' > 0$. Since $t, v > 0$, we can get that $\frac{\partial f^\star}{\partial \nu^\star} < 0$.

Now we will use the method of moving planes to the new function $v(x)$. Set

$$
v_\lambda(x) = v(x^\lambda), \quad w_\lambda(x) = v_\lambda(x) - v(x), \quad \forall x \in \Sigma_\lambda \cap \Omega^*.
$$

We prove that there exists l small enough such that

$$
w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda \cap \Omega^*, \quad \forall \lambda \in \left[\frac{1}{\varepsilon_1} - l, \frac{1}{\varepsilon_1}\right]. \tag{4.6}
$$

Suppose not, there exists x_0 such that

$$
w_\lambda(x_0) = \underline{\min}_{x \in \Sigma_\lambda \cap \Omega^*} w_\lambda(x) < 0. \tag{4.7}
$$

Similar to the previous argument, we will try to get a contradiction on the negative minimal point x_0:

$$
\begin{aligned}
&(-\Delta)^{\frac{\alpha}{2}} w_\lambda(x_0)\\
=&\frac{1}{|x_0^\lambda|^{n+\alpha-\overline{p}(n-\alpha)}} \frac{f\left(\frac{x_0^\lambda}{|x_0^\lambda|^2}, |x_0^\lambda|^{n-\alpha} v(x_0^\lambda)\right)}{(|x_0^\lambda|^{n-\alpha} v(x_0^\lambda))^{\overline{p}}} v_\lambda^{\overline{p}}(x_0) - \frac{1}{|x_0|^{n+\alpha-\overline{p}(n-\alpha)}} \frac{f\left(\frac{x_0}{|x_0|^2}, |x_0|^{n-\alpha} v(x_0)\right)}{(|x_0|^{n-\alpha} v(x_0))^{\overline{p}}} v^{\overline{p}}(x_0)\\
=&\frac{1}{|x_0^\lambda|^{n+\alpha-\overline{p}(n-\alpha)}} \frac{f\left(\frac{x_0^\lambda}{|x_0^\lambda|^2}, |x_0^\lambda|^{n-\alpha} v(x_0^\lambda)\right)}{(|x_0^\lambda|^{n-\alpha} v(x_0^\lambda))^{\overline{p}}} v_\lambda^{\overline{p}}(x_0) - \frac{1}{|x_0|^{n+\alpha-\overline{p}(n-\alpha)}} \frac{f\left(\frac{x_0}{|x_0|^2}, |x_0^\lambda|^{n-\alpha} v(x_0^\lambda)\right)}{(|x_0^\lambda|^{n-\alpha} v(x_0^\lambda))^{\overline{p}}} v_\lambda^{\overline{p}}(x_0)\\
+&\frac{1}{|x_0|^{n+\alpha-\overline{p}(n-\alpha)}} \frac{f\left(\frac{x_0}{|x_0|^2}, |x_0^\lambda|^{n-\alpha} v(x_0^\lambda)\right)}{(|x_0^\lambda|^{n-\alpha} v(x_0^\lambda))^{\overline{p}}} v_\lambda^{\overline{p}}(x_0) - \frac{1}{|x_0|^{n+\alpha-\overline{p}(n-\alpha)}} \frac{f\left(\frac{x_0}{|x_0|^2}, |x_0|^{n-\alpha} v(x_0)\right)}{(|x_0|^{n-\alpha} v(x_0))^{\overline{p}}} v_\lambda^{\overline{p}}(x_0)\\
+&\frac{1}{|x_0|^{n+\alpha-\overline{p}(n-\alpha)}} \frac{f\left(\frac{x_0}{|x_0|^2}, |x_0|^{n-\alpha} v(x_0)\right)}{(|x_0|^{n-\alpha} v(x_0))^{\overline{p}}} (v_\lambda^{\overline{p}}(x_0) - v^{\overline{p}}(x_0))\\
\geq&\frac{1}{|x_0|^{n+\alpha-\overline{p}(n-\alpha)}} \frac{f\left(\frac{x_0}{|x_0|^2}, |x_0|^{n-\alpha} v(x_0)\right)}{(|x_0|^{n-\alpha} v(x_0))^{\overline{p}}} (v_\lambda^{\overline{p}}(x) - v^{\overline{p}}(x_0))\\
\geq&\frac{\overline{p}}{|x_0|^{n+\alpha-\overline{p}(n-\alpha)}} \frac{f\left(\frac{x_0}{|x_0|^2}, |x_0|^{n-\alpha} v(x_0)\right)}{(|x_0|^{n-\alpha} v(x_0))^{\overline{p}}} v^{\overline{p}-1}(x_0) w_\lambda(x_0),
\end{aligned}
$$

where the first inequality was deduced by using the property $\frac{\partial f^\star}{\partial v^\star} < 0$ and the condition (iii) imposed on $f(x,u)$. By Lemma 2.1, we can get a contradiction, so (4.6) is true. By applying (4.6) to different inner normal directions $\nu(\overline{x})$ at the point $\overline{x} \in \partial\Omega^*$ with $|\overline{x} - x^\star| \leq t_0$, we can find that $v(\overline{x} + t\nu(\overline{x}))$ is increasing for $t \in [0, t_0]$ for some positive number t_0 depending on $n, \alpha, \overline{p}, \Omega, \delta_1$ and C_1. The smoothness and the strict convexity of Ω^* near $x^\star$ imply that the inner normal directions at $B_{t_0}(x^\star) \cap \Omega^*$ can form a cone with a positive angle θ_1 depending on Ω. Repeating the process of proof conducted in Theorem 1.1, we can also prove that $\|v\|_{L^\infty(\Omega^*_{\delta_3})} \leq C$ for $\Omega^*_{\delta_3} \subset \Omega^* \cap B_{t_0}(x^\star)$. By the definition of the Kelvin transform $u(x) = \frac{1}{|x-x^1|^{n-\alpha}} v\left(x^1 + \frac{x-x^1}{|x-x^1|^2}\right)$, we can get that $\|u\|_{L^\infty(\Omega_{\delta_2})} \leq C_2$.

The proof of Theorem 1.4 is entirely similar to the proof of Theorem 1.3, so we omit the details here.

Acknowledgements

The work of the first author was partially supported by the National Natural Science Foundation of China (No. 12001472) and Natural Science Foundation of Shandong Province of China (No. ZR2020QA015). The work of the second author was partially supported by the Natural Science Foundation of Shandong Province of China (No. ZR2021QA070) and Shanghai Frontier Research Center of Modern Analysis.

Conflict of interest

The authors declare that there is no conflicts of interest.

References

1. A. Chang, M. J. Gursky, P. Yang, The scalar curvature equation on 2- and 3-spheres, *Calc. Var. Partial Differ. Equations*, **1** (1993), 205–229. https://doi.org/10.1007/BF01191617
2. Z. C. Han, Prescribing Gaussian curvature on $\mathbb{S}^2$, *Duke Math. J.*, **61** (1990), 679–703. https://doi.org/10.1215/S0012-7094-90-06125-3
3. R. Schoen, D. Zhang, Prescribed scalar curvature on then-sphere, *Calc. Var. Partial Differ. Equations*, **4** (1996), 1–25. https://doi.org/10.1007/bf01322307
4. Y. Y. Li, Prescribing scalar curvature on $\mathbb{S}^n$ and related problems, Part I, *J. Differ. Equations*, **120** (1995), 319–410. https://doi.org/10.1006/jdeq.1995.1115
5. Y. Y. Li, Prescribing scalar curvature on $\mathbb{S}^n$ and related problems, Part II: existence and compactness, *Commun. Pure Appl. Math.*, **49** (1996), 541–597. https://doi.org/10.1002/(SICI)1097-0312(199606)49:6<541::AID-CPA1>3.0.CO;2-A
6. B. Gidas, J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, *Commun. Partial Differ. Equations*, **6** (1981), 883–901. https://doi.org/10.1080/03605308108820196
7. C. C. Chen, C. S. Lin, Estimates of the conformal scalar curvature equation via the method of moving planes, *Commun. Pure Appl. Math.*, **50** (1997), 971–1017. https://doi.org/10.1002/(SICI)1097-0312(199710)50:10<971::AID-CPA2>3.0.CO;2-D
8. C. C. Chen, C. S. Lin, Estimates of the conformal scalar curvature equation via the method of moving planes, II, *J. Differ. Geom.*, **49** (1998), 115–178. https://doi.org/10.4310/jdg/1214460938
9. W. Chen, C. Li, A priori estimates for prescribing scalar curvature equations, *Ann. Math.*, **145** (1997), 547–564. https://doi.org/10.2307/2951844
10. Y. Y. Li, L. Zhang, Compactness of solutions to the Yamabe problem, *C.R. Math.*, **338** (2004), 693–695. https://doi.org/10.1016/j.crma.2004.02.018
11. Y. Y. Li, L. Zhang, Compactness of solutions to the Yamabe problem, II, *Calc. Var. Partial Differ. Equations*, **24** (2005), 185–237. https://doi.org/10.1007/s00526-004-0320-7
12. Y. Y. Li, L. Zhang, Compactness of solutions to the Yamabe problem, III, *J. Funct. Anal.*, **245** (2007), 438–474. https://doi.org/10.1016/j.jfa.2006.11.010
13. A. Carpinteri, F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, in *CISM International Centre for Mechanical Sciences*, Courses and Lectures, Springer-Verlag, **378** (1997). https://doi.org/10.1007/978-3-7091-2664-6
14. A. A. Lokshin, Y. V. Suvorova, Mathematical theory of wave propagation in media with memory, *Moskov. Gos. Univ.*, Moscow, 1982.
15. T. L. Jin, Y. Y. Li, J. G. Xiong, On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions, *J. Eur. Math. Soc.*, **16** (2014), 1111–1171. https://doi.org/10.4171/JEMS/456
16. A. Chang, M. González, Fractional Laplacian in conformal geometry, *Adv. Math.*, **226** (2011), 1410–1432. https://doi.org/10.1016/j.aim.2010.07.016
17. C. R. Graham, M. Zworski, Scattering matrix in conformal geometry, *Invent. Math.*, **152** (2003), 89–118. https://doi.org/10.1007/s00222-002-0268-1

18. X. Ros-Oton, J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, *J. Math. Pures Appl.*, **101** (2014), 275–302. https://doi.org/10.1016/j.matpur.2013.06.003

19. W. Chen, C. Li, Y. Li, A direct blow up and rescaling agrument on nonlocal elliptic equation, *Int. J. Math.*, **27** (2016), 1650064. https://doi.org/10.1142/S0129167X16500646

20. L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Commun. Partial Differ. Equations*, **32** (2007), 1245–1260. https://doi.org/10.1080/03605300600987306

21. X. Cabré, Y. Sire, Nonlinear equations for fractional Laplacians, I: regularity, maximum principles, and Hamiltonian estimates, *Ann. Inst. Henri Poincare C*, **31** (2014), 23–53. https://doi.org/10.1016/J.ANIHPC.2013.02.001

22. X. Cabré, J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, *Adv. Math.*, **224** (2010), 2052–2093. https://doi.org/10.1016/j.aim.2010.01.025

23. A. Capella, J. Dávila, L. Dupaigne, Y. Sire, Regularity of radial extremal solutions for some non-local semilinear equations, *Commun. Partial Differ. Equations*, **36** (2011), 1353–1384. https://doi.org/10.1080/03605302.2011.562954

24. S. Dipierro, G. Palatucci, E. Valdinoci, Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian, preprint, arXiv:1202.0576.

25. R. Servadei, E. Valdinoci, Mountain Pass solutions for nonlocal elliptic operators, *J. Math. Anal. Appl.*, **389** (2012), 887–898. https://doi.org/10.1016/j.jmaa.2011.12.032

26. L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Commun. Pure Appl. Math.*, **60** (2007), 67–112. https://doi.org/10.1002/cpa.20153

27. L. Zhang, C. Li, W. Chen, T. Cheng, A Liouville theorem for α-harmonic functions in $\mathbb{R}^n_+$, *Discrete Contin. Dyn. Syst.*, **36** (2016), 1721–1736. https://doi.org/10.3934/dcds.2016.36.1721

28. R. Zhuo, W. Chen, X. Cui, Z. Yuan, Symmetry and non-existence of solutions for a nonlinear system involving the fractional Laplacian, *Discrete Contin. Dyn. Syst.*, **36** (2016), 1125–1141. https://doi.org/10.3934/dcds.2016.36.1125

29. A. D. Aleksandrov, A charcteristic property of spheres, *Ann. Mat. Pura Appl.*, **58** (1962), 303–315. https://doi.org/10.1007/BF02413056

30. J. Serrin, A symmetry problem in potential theory, *Arch. Ration. Mech. Anal.*, **43** (1971), 304–318. https://doi.org/10.1007/BF00250468

31. B. Gidas, W. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, *Commun. Math. Phys.*, **68** (1979), 209–243. https://doi.org/10.1007/BF01221125

32. B. Gidas, W. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *Adv. Math. Suppl. Stud. A*, **7** (1981), 369–402.

33. H. Berestycki, L. Nirenberg, Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations, *J. Geom. Phys.*, **5** (1988), 237–275. https://doi.org/10.1016/0393-0440(88)90006-X

34. H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method, *Bol. Soc. Bras. Mat.-Bull. Braz. Math. Soc.*, **22** (1991), 1–37. https://doi.org/10.1007/BF01244896

35. W. Chen, C. Li, Classifications of solutions of some nonlinear elliptic equations, *Duke Math. J.*, **63** (1991), 615–622. https://doi.org/10.1215/S0012-7094-91-06325-8

36. C. Li, Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on bounded domains, *Commun. Partial Differ. Equations*, **16** (1991), 491–526. https://doi.org/10.1080/03605309108820766

37. C. Li, Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains, *Commun. Partial Differ. Equations*, **16** (1991), 585–615. https://doi.org/10.1080/03605309108820770

38. W. Chen, C. Li, Y. Li, A direct method of moving planes for the fractional laplacian, *Adv. Math.*, **308** (2017), 404–437. https://doi.org/10.1016/j.aim.2016.11.038

39. D. G. de Figueiredo, P. L. Lions, R. D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, *J. Math. Pures Appl.*, **61** (1982), 41–63. https://doi.org/10.1007/978-3-319-02856-9_11

40. N. S. Landkoff, *Foundations of Modern Potential Theory*, Springer-Verlag, New York-Heidelberg, 1972.