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Research article

Persistence of the heteroclinic loop under periodic perturbation

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Abstract: We consider an autonomous ordinary differential equation that admits a heteroclinic loop. The unperturbed heteroclinic loop consists of two degenerate heteroclinic orbits γ_1 and γ_2. We assume the variational equation along the degenerate heteroclinic orbit γ_i has $d_i\,(d_i > 1, i = 1, 2)$ linearly independent bounded solutions. Moreover, the splitting indices of the unperturbed heteroclinic orbits are s and $-s$ $(s \geq 0)$, respectively. In this paper, we study the persistence of the heteroclinic loop under periodic perturbation. Using the method of Lyapunov-Schmidt reduction and exponential dichotomies, we obtained the bifurcation function, which is defined from $\mathbb{R}^{d_1+d_2+2}$ to $\mathbb{R}^{d_1+d_2}$. Under some conditions, the perturbed system can have a heteroclinic loop near the unperturbed heteroclinic loop.

Keywords: heteroclinic orbit; heteroclinic loop; bifurcation; Lyapunov-Schmidt reduction; exponential dichotomies

1. Introduction

The problems in homoclinic or heteroclinic bifurcation are critical in dynamic systems because they may have some complex dynamic behavior, such as chaotic motions [1]. Homoclinic and heteroclinic orbits are important invariant sets. The homoclinic orbit tends asymptotically to the same hyperbolic equilibrium along stable and unstable manifolds. However, the heteroclinic orbit tends asymptotically to two different hyperbolic equilibria along the stable and unstable manifolds. A heteroclinic loop consists of two saddles connecting two heteroclinic orbits. A numerical simulation reveals that the Lorenz equation has a heteroclinic loop when $\sigma = 10$, $r \approx 40.375$ and $b \approx 2.623$ [2]. The heteroclinic loop is equidimensional if the two saddles have the same dimension of the unstable manifold. Otherwise, it is heterodimensional loop [3]. This elementary phenomenon occurs in any dimension larger than two, and is one of the primary mechanisms for non-hyperbolicity. In addition, the existence of the heteroclinic loop is often related to the traveling wave solutions of the reaction-diffusion equation.

In [4], Han et al. considered quadratic Hamiltonian systems with a heteroclinic loop under polynomial perturbations. Using the Melnikov function, the authors found three limit cycles near the heteroclinic loop. Later, Sun, Han, and Yang extended the theory for a heteroclinic loop with a cusp in [5]. Chen, Oksasoglu, and Wang considered a heteroclinic loop under periodic perturbation on the plane [6]. They proved three types of dynamic behavior near the heteroclinic loop under periodic perturbation. One of which with strange attractors admitting SRB measures representing chaos. More complicated dynamic behavior, such as strange attractors and horseshoes near the heteroclinic loop with periodic perturbation see, [7] and [8].

Chow, Deng, and Terman [9] investigated the homoclinic or periodic orbit bifurcated from a heteroclinic loop based on the method developed by Shilnikov. In 1998, Zhu and Xia [10] established a coordinate system in a neighborhood of a heteroclinic loop. They studied the bifurcation of the heteroclinic loop using the coordinate systems near the heteroclinic loop. Moreover, Rademacher [11] studied the homoclinic orbit bifurcated from a codimension 1 and 2 heteroclinic loops by Lin's method [12]. In [13], Geng, Wang, and Liu investigated the bifurcation of a heterodimensional loop using the local coordinate system. They assumed the unperturbed equation has a heteroclinic loop in $\mathbb{R}^4$ that the splitting indices of the unperturbed heteroclinic orbits are 1 and -1. They obtained the persistence condition for the heterodimensional loop. For more research results regarding the bifurcation of the heteroclinic loop see [14].

We let $d, d \geq 1$, denote the number of the bounded solutions of the variational equation along the heteroclinic orbit. If $d = 1$, the homoclinic or heteroclinic orbit is nondegenerate; otherwise, it is degenerate [15], which means, along the orbit, the intersection of the spaces tangent to the stable and unstable manifolds of the equilibrium has a d dimensional subspace. Hence, parameter d describes the degeneration of the homoclinic or heteroclinic orbit.

The primary purpose of this paper is to extend the theory of [13,14] for heteroclinic loop bifurcation. We consider an autonomous ordinary differential equation that admits a heteroclinic loop in $\mathbb{R}^n$. The unperturbed heteroclinic loop consists of two degenerate heteroclinic orbits. Furthermore, the splitting index of the unperturbed heteroclinic orbits can be arbitrary. We investigate the bifurcation of the heterodimensional loop under periodic perturbation using the Lyapunov-Schmidt reduction method. We start with the following equation:

$$\dot{x}(t) = f(x(t)), \tag{1.1}$$

and its periodic perturbed equation is as follows:

$$\dot{x}(t) = f(x(t)) + \Sigma_{j=1}^{2}\mu_j g_j(x(t), \mu, t), \tag{1.2}$$

where $x \in \mathbb{R}^n$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$, and we make the following assumptions:

(H1) $f \in C^3$.

(H2) p_+ and p_- are the two distinct hyperbolic equilibria of Eq (1.1). Namely, $f(p_\pm) = 0$ and the eigenvalues of $Df(p_\pm)$ lie off the imaginary axis, where D denotes the derivative operator.

(H3) Equation (1.1) has two heteroclinic solutions $\gamma_1(t)$ and $\gamma_2(t)$, which are asymptotic to the equilibrium p_+ and p_-, respectively. That is, $\dot{\gamma}_i(t) = f(\gamma_i(t))$, $i = 1, 2$, and

$$\lim_{t\to+\infty} \gamma_1(t) = p_+, \lim_{t\to-\infty} \gamma_1(t) = p_-,$$

$$\lim_{t\to+\infty} \gamma_2(t) = p_-, \lim_{t\to-\infty} \gamma_2(t) = p_+.$$

(H4) $g_j \in C^3$, $g_j(p_\pm, \mu, t) = 0$, $g_j(x, 0, t) = 0$ and $g_j(x, \mu, t+2) = g_j(x, \mu, t)$.

(H5) $dim(W^s(p_+)) = d_+$ and $dim(W^s(p_-)) = d_-$, where $W^s(p_+)$ and $W^s(p_-)$ are the stable manifold of the equilibrium p_+ and p_-, respectively.

(H6)
$$dim(T_{\gamma_1(0)}W^s(p_+) \bigcap T_{\gamma_1(0)}W^u(p_-)) = d_1$$
and
$$dim(T_{\gamma_2(0)}W^s(p_-) \bigcap T_{\gamma_2(0)}W^u(p_+)) = d_2,$$
where $T_{\gamma_i(0)}W^{s/u}(p_\pm)$ is the tangent spaces of the corresponding invariant manifolds at $\gamma_i(0)$ and $d_i > 1$, $i = 1, 2$.

By $(H3)$ and $(H6)$, we know unperturbed Eq (1.1) has a heteroclinic loop Γ (see Figure 1), where

$$\Gamma = \{p_-\} \cup \{\gamma_1(t) : t \in \mathbb{R}\} \cup \{p_+\} \cup \{\gamma_2(t) : t \in \mathbb{R}\}$$

.

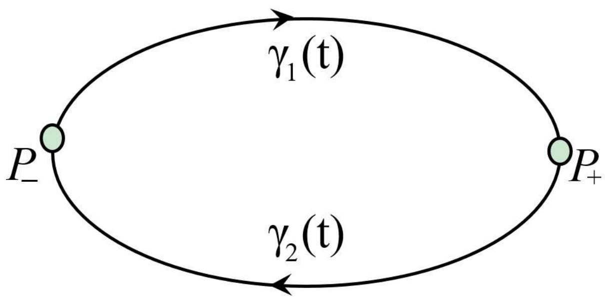

Figure 1. Heteroclinic loop Γ.

By $(H5)$, we know that d_+ and d_- can be arbitrary. Thus, the unperturbed Eq (1.1) has a heterodimensional loop. We provide conditions for the persistence of the heterodimensional loop under periodic perturbation. The structure of the paper is as follows. We present some background on the Lyapunov-Schmidt reduction and Lin's method in Section 2. Section 3 details the notations for the fundamental matrix of the variational equation along the heteroclinic orbit $\gamma_i(t)$ and the main result. Section 4 provides proof of the main result. The bifurcation function is obtained using the functional analytic method. We construct some solutions near the unperturbed heteroclinic loop, which can have a gap at $t = 0$, and glue those solutions at $t = 0$. Thus, the bifurcation function can be obtained. Hence, under some conditions, some solutions near the unperturbed heteroclinic loop can constitute a heteroclinic loop for a perturbed system.

2. Preliminaries

Many problems in bifurcation theory can be changed by solving the zeros of an operator equation in some Banach space. Sometimes, the corresponding operator is not invertible, making it difficult to solve. However, this problem can equivalently transform the operator equation into an equation in a low-dimensional space using the Lyapunov-Schmidt reduction method (see [16]). Therefore, this method is very effective, especially in studying homoclinic or heteroclinic bifurcation.

Lin's method [17] is an implementation of the Lyapunov-Schmidt reduction method to construct solutions near the unperturbed heteroclinic orbit. The idea of Lin's method originated from the work by Chow, Hale, and Mallet-Paret [18] using the function space approach to construct piecewise continuous solutions approximating the unperturbed homoclinic orbit. The bifurcation function can be obtained using these solutions, and the zeros of the bifurcation function correspond to solutions in the homoclinic or heteroclinic bifurcation problems. Later, Palmer [19], Hale and Lin [20] extended the methods to $\mathbb{R}^n$ and the functional differential equation. Lin used the function space approach to construct solutions near the heteroclinic chain [12]. He assumed that heteroclinic orbits in the chain all have the same index. In the 1990s, Gruendler [21, 22] generalized the method to the case of degenerate homoclinic bifurcation problems.

Next, we introduce an application of the Lyapunov-Schmidt reduction method, known as the Fredholm alternative property for linear differential equations. We consider the following nonhomogeneous linear differential equation:

$$\dot{y}(t) = A(t)y(t) + h(t), \tag{2.1}$$

where $y \in \mathbb{R}^n$, $A(t)$ vary continuously with $t \in \mathbb{R}$ and $h(t)$ is bounded and continuous on $t \in \mathbb{R}$. We assume that the homogeneous differential equation $\dot{y}(t) = A(t)y(t)$ has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. Then, $M > 0$, $K_0 > 0$, and projections P and Q exist, such that

$$\begin{aligned} &|U(t)PU^{-1}(s)| \leq K_0 e^{2M(s-t)},\ 0 \leqslant s \leqslant t, \\ &|U(t)(I-P)U^{-1}(s)| \leq K_0 e^{2M(t-s)},\ 0 \leqslant t \leqslant s, \\ &|U(t)(I-Q)U^{-1}(s)| \leq K_0 e^{2M(t-s)},\ t \leqslant s \leqslant 0, \\ &|U(t)QU^{-1}(s)| \leq K_0 e^{2M(s-t)},\ s \leqslant t \leqslant 0, \end{aligned} \tag{2.2}$$

where $U(t)$ is the fundamental matrix. We define the Banach spaces as follows:

$$\mathcal{Z}^r = \{z \in C^r(\mathbb{R}, \mathbb{R}^n) : \max_{0 \leq j \leq r}\{\sup_{t \in \mathbb{R}} |D^j z(t)| e^{M|t|}\} < \infty\},$$

with the norm $\|z\|_r = \max_{0 \leq j \leq r}\{\sup_{t \in \mathbb{R}} |D^j z(t)| e^{M|t|}\}$, $|D^0 z(t)|$ indicates $|z(t)|$. We let the linear operator $L : \mathcal{Z}^1 \to \mathcal{Z}^0$ be defined by

$$L(y) := \dot{y} - A(t)y. \tag{2.3}$$

The adjoint operator for L is

$$L^*(\psi) := \dot{\psi} + (A(t))^T \psi, \tag{2.4}$$

where $(A(t))^T$ denotes the transpose of matrix $A(t)$). By the definition of the linear operator L and the exponential dichotomy, we know that

$$\begin{aligned} &dimKer(L) = dim(Ran(P) \cap Ran(I - Q)), \\ &dimKer(L^*) = dim(Ran(I - P^T) \cap Ran(Q^T)). \end{aligned}$$

If $dimKer(L^*) = d$ and $\psi_1(t), ..., \psi_d(t)$ are the orthonormal unit bases of $Ker(L^*)$, we define a projection operator $\Pi : \mathcal{Z}^0 \to \mathcal{Z}^0$ as follows

$$\Pi(h)(t) = \sum_{i=1}^{d} \psi_i(t) \int_{-\infty}^{\infty} \langle \psi_i^T(t), h(t) \rangle dt. \tag{2.5}$$

By the method of the Lyapunov-Schmidt reduction, Eq (2.1) is equivalent to the following system

$$\dot{y} = A(t)y + (I - \Pi)h(t), \tag{2.6}$$

$$\Pi h(t) = 0. \tag{2.7}$$

By the definition of Π, $Ran(I - \Pi) = RanL$. We can first solve Eq (2.6) for $y \in \mathcal{Z}^1$, and the bifurcation equations are obtained by Eq (2.7). That is,

$$\sum_{i=1}^{d} \psi_i(t) \int_{-\infty}^{\infty} \langle \psi_i^T(t), h(t) \rangle dt = 0, \text{for all } \psi_i \in Ker(L^*). \tag{2.8}$$

Thus, Eq (2.1) has a bounded solution $y(t)$ if and only if Eq (2.8) holds.

3. Notation and main result

The variational equation of (1.1) along the heteroclinic orbit γ_i is:

$$\dot{u}(t) = Df(\gamma_i(t))u(t). \tag{3.1}$$

From $(H6)$, we know that Eq (3.1) has $d_i (d_i > 1)$ linearly independent bounded solutions, $i = 1, 2$. Based on Sacker's definition [23], we can define the splitting index $S(\gamma_i)$ for the unperturbed heteroclinic orbit γ_i, as follows:

$$S(\gamma_1) = d_+ - d_- = s, S(\gamma_2) = d_- - d_+ = -s. \tag{3.2}$$

By $(H3)$ and the exponential dichotomy roughness theorem, we know that the variational Eq (3.1) has two-side exponential dichotomies. We let U_i be the fundamental matrix of Eq (3.1). Then, $M > 0$, $K_0 > 0$, projections P_i and Q_i exist, such that

$$\begin{aligned} &|U_i(t)P_iU_i^{-1}(s)| \leq K_0 e^{2M(s-t)},\ 0 \leqslant s \leqslant t, \\ &|U_i(t)(I - P_i)U_i^{-1}(s)| \leq K_0 e^{2M(t-s)},\ 0 \leqslant t \leqslant s, \\ &|U_i(t)(I - Q_i)U_i^{-1}(s)| \leq K_0 e^{2M(t-s)},\ t \leqslant s \leqslant 0, \\ &|U_i(t)Q_iU_i^{-1}(s)| \leq K_0 e^{2M(s-t)},\ s \leqslant t \leqslant 0, \end{aligned} \tag{3.3}$$

where I is the $n \times n$ unit matrix. We let the linear operator $L_i : \mathcal{Z}^1 \to \mathcal{Z}^0$ be defined by

$$L_i(u) := \dot{u} - Df(\gamma_i(t))u. \tag{3.4}$$

Further, the adjoint operator for L_i is

$$L_i^*(\psi) := \dot{\psi} + (Df(\gamma_i(t)))^T \psi. \tag{3.5}$$

We let U_i^{-1} denote the inverse of U_i. Then we have $U_i^{-1}U_i = I$. Differentiating $U_i^{-1}(t)U_i(t) = I$ with respect to t, we obtain

$$U_i^{-1}\dot{U}_i + \dot{U}_i^{-1}U_i = 0$$

hence,

$$\dot{U}_i^{-1} = -U_i^{-1}\dot{U}_iU_i^{-1} = -U_i^{-1}Df(\gamma_i).$$

Therefore, we have

$$(\dot{U}_i^{-1})^T = -Df(\gamma_i)^T(U_i^{-1})^T.$$

We know that $(U_i^{-1})^T$ is a matrix solution of the adjoint equation of (3.1). Taking the transpose in Eq (3.3), it is apparent that the adjoint equation of (3.1) also has exponential dichotomy on $\mathbb{R}^+$ with projection $I - P_i^T$, and on $\mathbb{R}^-$ with projection $I - Q_i^T$, respectively.

By the definition of the linear operator L_i and the exponential dichotomy, we know that

$$\begin{aligned}
dimKer(L_1) &= dim(Ran(P_1) \cap Ran(I - Q_1)) \\
&= dim(T_{\gamma_1(0)}W^s(p_+) \cap T_{\gamma_1(0)}W^u(p_-)) \\
&= d_1, \\
dimKer(L_2) &= dim(Ran(P_2) \cap Ran(I - Q_2)) \\
&= dim(T_{\gamma_2(0)}W^s(p_-) \cap T_{\gamma_2(0)}W^u(p_+)) \\
&= d_2, \\
dimKer(L_i^*) &= dim(Ran(I - P_i^T) \cap Ran(Q_i^T)).
\end{aligned}$$

From the theory of homoclinic bifurcation, the linear operators L_1 and L_2 are Fredholm operators, and the index of the Fredholm operator L_i is

$$indexL_i = dimKer(L_i) - codimRan(L_i).$$

If $dimKer(L_i^*) = d_i^*, i = 1, 2$, then we have

$$\begin{aligned}
indexL_1 &= d_1 - d_1^* = d_+ - d_- = S(\gamma_1) = s, \\
indexL_2 &= d_2 - d_2^* = d_- - d_+ = S(\gamma_2) = -s.
\end{aligned}$$

In addition, if $u_1^i(t), ..., u_{d_i-1}^i(t), \dot{\gamma}_i(t)$ are the orthonormal unit bases of $Ker(L_i)$, $\varphi_1(t), ..., \varphi_{d_1-s}(t)$ are the orthonormal unit bases of $Ker(L_1^*)$ and $\psi_1(t), ..., \psi_{d_2+s}(t)$ are the orthonormal unit bases of $Ker(L_2^*)$, then define

$$\begin{aligned}
a_{i,k}^1(\alpha_1) &= \int_{-\infty}^{+\infty} \langle \psi_i^T(s), g_k(\gamma_1(s), \mu, s + \alpha_1)\rangle ds, \\
b_{i,pq}^1 &= \int_{-\infty}^{+\infty} \langle \psi_i^T(s), D_{11}f(\gamma_1(s))u_p^1(s)u_q^1(s)\rangle ds,
\end{aligned}$$

where $i = 1, ..., d_1 - s$, $p, q = 1, ..., d_1 - 1$, and $k = 1, 2$. Moreover,

$$a_{j,k}^2(\alpha_2) = \int_{-\infty}^{+\infty} \langle \varphi_i^T(s), g_k(\gamma_2(s), \mu, s + \alpha_2)\rangle dt,$$

$$b^2_{j,mn} = \int_{-\infty}^{+\infty} \langle \varphi_i^T(s), D_{11} f(\gamma_2(s)) u_m^2(s) u_n^2(s) \rangle ds,$$

where $j = 1, ..., d_2 + s$, $m, n = 1, ..., d_2 - 1$, and $k = 1, 2$. Using those notations, we let $M^1 : \mathbb{R}^{d_1-1} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^{d_1-s}$ be given by

$$M^1(\beta^1, \mu, \alpha_1) = (M^1_1(\beta^1, \mu, \alpha_1), ..., M^1_{d_1-s}(\beta^1, \mu, \alpha_1)),$$

and

$$M^1_i(\beta^1, \mu, \alpha_1) = \sum_{k=1}^{2} a^1_{i,k}(\alpha_1)\mu_k + \frac{1}{2} \sum_{p=1}^{d_1-1} \sum_{q=1}^{d_1-1} b^1_{i,pq} \beta^1_p \beta^1_q,$$

where $i = 1, ..., d_1 - s, \beta^1 = (\beta^1_1, ..., \beta^1_{d_1-1})$.

We let $M^2 : \mathbb{R}^{d_2-1} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^{d_2+s}$ be given by

$$M^2(\beta^2, \mu, \alpha_2) = (M^2_1(\beta^2, \mu, \alpha_2), ..., M^2_{d_2+s}(\beta^2, \mu, \alpha_2)),$$

and

$$M^2_j(\beta^2, \mu, \alpha_2) = \sum_{k=1}^{2} a^2_{j,k}(\alpha_2)\mu_k + \frac{1}{2} \sum_{m=1}^{d_2-1} \sum_{n=1}^{d_2-1} b^2_{j,mn} \beta^2_m \beta^2_n,$$

where $j = 1, ..., d_2 + s$ and $\beta^2 = (\beta^2_1, ..., \beta^2_{d_2-1})$. Further, we let $M : \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{d_1-s} \times \mathbb{R}^{d_2+s}$ be given by

$$M(\beta, \mu, \alpha) = (M^1(\beta^1, \mu, \alpha_1), M^2(\beta^2, \mu, \alpha_2)), \tag{3.6}$$

where $\beta = (\beta^1, \beta^2), \alpha = (\alpha_1, \alpha_2)$.

We can state the main result as follows:

Theorem 1. *Assume that* $(H1) - (H5)$ *hold. Let* $M(\beta, \mu, \alpha)$ *be as in Eq (3.6). If there are some points* $(\beta_0, \mu_0, \alpha_0) \in \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2$*, such that*

$$M(\beta_0, \mu_0, \alpha_0) = 0$$

and

$$D_{(\beta,\mu)} M(\beta_0, \mu_0, \alpha_0)$$

is a nonsingular $(d_1 + d_2) \times (d_1 + d_2)$ *matrix, then there exists an open interval* I *containing origin, the* C^1 *function* $\kappa_2 : I \to \mathbb{R}^2$*, and the heteroclinic solutions* $x_1(\varepsilon, t)$*,* $x_2(\varepsilon, t)$ *of the Eq (1.2) with* $\mu = \varepsilon^2(\mu_0 + \kappa_2(\varepsilon))$*, where* $\varepsilon \in I \setminus \{0\}$*,* $x_1(\varepsilon, t)$ *and* $x_2(\varepsilon, t)$ *are located near the heteroclinic orbits* γ_1 *and* γ_2*, such that* $x_1(\varepsilon, t)$*,* $x_2(\varepsilon, t)$*,* p_+ *and* p_- *can constitute a heteroclinic loop* Γ_ε*.*

The proof of Theorem 1 is performed in Section 4. The heteroclinic loop Γ_ε as illustrated in Figure 2.

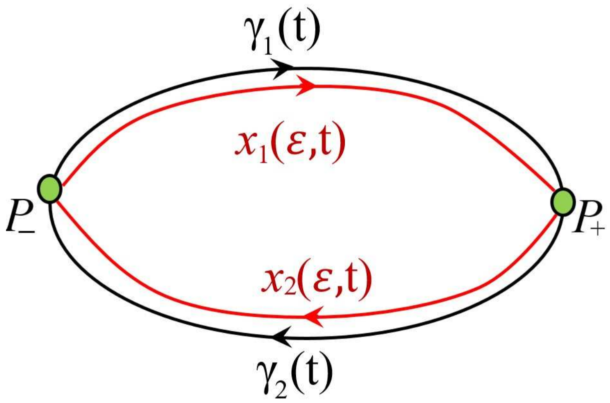

Figure 2. Heteroclinic loop Γ_ε.

4. Proof of Theorem 1

By ($H2$), we know the unperturbed Eq (1.1) has a heteroclinic loop Γ. In this section, we find conditions such that the perturbed Eq (1.2) have a heteroclinic loop Γ_μ with sufficiently small μ. For $i = 1$ or $i = 2$, we suppose $x_i(t)$ is a solution of Eq (1.2). With the change of variable

$$x_i(t + \alpha_i) = \gamma_i(t) + z_i(t), \tag{4.1}$$

Equation (1.2) can be transformed into

$$\dot{z}_i = Df(\gamma_i)z_i + \widetilde{g}(z_i, \mu, \alpha_i), \tag{4.2}$$

where

$$\begin{aligned} \widetilde{g}(z_i, \mu, \alpha_i)(t) =& f(\gamma_i(t) + z_i(t)) - f(\gamma_i(t)) - Df(\gamma_i(t))z_i(t) \\ &+ \Sigma_{j=1}^{2}\mu_j g_j(\gamma_i(t) + z_i(t), \mu, t + \alpha_i). \end{aligned} \tag{4.3}$$

By direct calculation, we have

$$\begin{aligned} &(i)\ \widetilde{g}(0, 0, \alpha_i) = 0;\ D_1\widetilde{g}(0, 0, \alpha_i) = 0; \\ &(ii)\ D_{11}\widetilde{g}(0, 0, \alpha_i) = D_{11}f(\gamma_i); \\ &(iii)\ \frac{\partial \widetilde{g}}{\partial \mu_j}(0, 0, \alpha_i)(t) = g_j(\gamma_i, 0, t + \alpha_i), \end{aligned}$$

where D_i and D_{ij} denote the derivative of the multivariate function concerning its i-th and i and j-th variables, respectively.

Because we only consider the Eq (1.1) under a small periodic perturbed equation, we suppose $\mu \in \overline{B}_1(0, \delta) \subseteq \mathbb{R}^2$, where $\overline{B}_1(0, \delta)$ is a closed set with radius $\delta > 0$ centered at the origin. Moreover, we have the following property regarding the function $\widetilde{g}$.

Lemma 1. *The function* $\widetilde{g}(\cdot, \mu, \alpha_i) : \mathcal{Z}^1 \times \overline{B}_1(0, \delta) \times \mathbb{R} \mapsto \mathcal{Z}^0$.

Proof. For $i = 1$ or $i = 2$, we let $z_i \in \mathcal{Z}^1$ be given. We can choose a closed set B such that $z_i(t), \gamma_i(t), z_i(t) + \gamma_i(t)$ and $p_\pm + z_i(t) + \gamma_i(t)$ are all $\in B$ for $t \in \mathbb{R}$. According to smoothness of $f, g_j \in C^3$ and g_j is periodic about t. We can choose a constant M_1 such that

$$|D_1 f(x)| \leq M_1, |D_1 g_j(x, \mu, t + \alpha_i)| \leq M_1,$$

for $(x, \mu, \alpha_i) \in B \times \overline{B}_1(0, \delta) \times \mathbb{R}$. If $z_i \in \mathcal{Z}^1$, because γ_i is a heteroclinic solution which is heteroclinic to the hyperbolic equilibrium $p_\pm$, we can assign a constant M_2 such that

$$|z_i(t)| \leq M_2 e^{-M|t|}, |z_i(t) + \gamma_i(t) - p_\pm| \leq M_2 e^{-M|t|}.$$

We define $\sigma_1(s) = f(sz_i(t) + \gamma_i(t)) - f(\gamma_i(t)) : [0, 1] \mapsto \mathbb{R}^n$. By the smoothness of f, $\sigma_1 \in C^3$ and for some $s^* \in (0, 1)$,

$$\begin{aligned} f(z_i(t) + \gamma_i(t)) - f(\gamma_i(t)) &= \sigma_1(1) - \sigma_1(0) = \sigma_1'(s^*) \\ &= Df(s^* z_i(t) + \gamma_i(t)) z_i(t). \end{aligned}$$

Therefore,

$$\begin{aligned} |f(z_i(t) + \gamma_i(t)) - f(\gamma_i(t))| &\leq |Df(s^* z_i(t) + \gamma_i(t)) z_i(t)| \\ &\leq M_1 |z_i(t)| \\ &\leq M_1 M_2 e^{-M|t|}. \end{aligned}$$

We define a map $\sigma_2(s) : [0, 1] \mapsto \mathbb{R}^n$ by

$$\sigma_2(s) = g_j(p_\pm + s(z_i(t) + \gamma_i(t) - p_\pm), \mu, t + \alpha_i)) - g_j(p_\pm, \mu, t + \alpha_i)).$$

By $(H4)$, $\sigma_2 \in C^3$,$\sigma_2(1) = g_j(\gamma_i(t) + z_i(t), \mu, t + \alpha_i)$ and $\sigma_2(0) = g_j(p_\pm, \mu, t + \alpha_i) = 0$. For some $s^* \in (0, 1)$, we have

$$\begin{aligned} g_j(\gamma_i(t) + z_i(t), \mu, t + \alpha_i) - g_j(p_\pm, \mu, t + \alpha_i) &= \sigma_2(1) - \sigma_2(0) = \sigma_2'(s^*) \\ = D_1 g_j(p_\pm + s^*(z_i(t) + \gamma_i(t) - p_\pm), \mu, t + \alpha_i))(z_i(t) + \gamma_i(t) - p_\pm). \end{aligned}$$

Therefore,

$$\begin{aligned} &|g_j(\gamma_i(t) + z_i(t), \mu, t + \alpha_i) - g_j(p_\pm, \mu, t + \alpha_i)| \\ &\quad \leq |D_1 g_j(p_\pm + s^*(z_i(t) + \gamma_i(t) - p_\pm), \mu, t + \alpha_i))(z_i(t) + \gamma_i(t) - p_\pm)| \\ &\quad \leq M_1 |(z_i(t) + \gamma_i(t) - p_\pm)| \\ &\quad \leq M_1 M_2 e^{-M|t|}. \end{aligned}$$

For any $\mu \in \mathbb{R}$, $g_j(p_\pm, \mu, t + \alpha_i) = 0$, thus

$$\begin{aligned} \widetilde{g}(z_i, \mu, \alpha_i)(t) =& \widetilde{g}(z_i, \mu, \alpha_i)(t) - \Sigma_{j=1}^2 \mu_j g_j(p_\pm, \mu, t + \alpha_i) \\ =& f(\gamma_i(t) + z_i(t)) - f(\gamma_i(t)) - Df(\gamma_i(t)) z_i(t) \\ &+ \Sigma_{j=1}^2 \mu_j (g_j(\gamma_i(t) + z(t), \mu, t + \alpha_i) - g_j(p_\pm, \mu, t + \alpha_i)). \end{aligned}$$

As a result,

$$\begin{aligned}|\widetilde{g}(z_i,\mu,\alpha_i)(t)| &= |\widetilde{g}(z_i,\mu,\alpha_i)(t) - \Sigma_{j=1}^{2}\mu_j g_j(p_\pm,\mu,t+\alpha_i)| \\ &\le |f(z_i(t)+\gamma_i(t)) - f(\gamma_i(t))| + |Df(\gamma_i(t))z_i(t)| \\ &\quad + |\Sigma_{j=1}^{2}\mu_j(g_j(\gamma_i(t)+z(t),\mu,t+\alpha_i) - g_j(p_\pm,\mu,t+\alpha_i))| \\ &\le (2M_1M_2 + |\mu|M_1M_2)e^{-M|t|},\end{aligned}$$

that is

$$\|\widetilde{g}(z_i,\mu,\alpha_i)\|_0 = \sup_{t\in\mathbb{R}}|\widetilde{g}(z_i,\mu,\alpha_i)(t)|e^{M|t|} \le (2M_1M_2 + \delta M_1M_2).$$

Thus, for any given $z_i \in \mathcal{Z}^1$, $\widetilde{g}(z_i,\mu,\alpha_i) \in \mathcal{Z}^0$. The proof is complete.

From the variable transformation of Eq (4.1), if $\lim_{t\to\pm\infty}|z_i(t)| = 0$, then $x_i(t)$ is a heteroclinic solution which is heteroclinic to the hyperbolic equilibrium p_- and p_+. Hence, the persistence of the heteroclinic loop Γ under the periodic perturbation of Eq (1.1) is equivalent to the search solution $z_i(t)$ of Eq (4.3) in the Banach space $\mathcal{Z}^1$. Next, we use the method of the Lyapunov-Schmidt reduction to solve the operator equations

$$\begin{aligned}L_1(z_1) &= \dot{z}_1 - Df(\gamma_1)z_1 = \widetilde{g}(z_1,\mu,\alpha_1), \\ L_2(z_2) &= \dot{z}_2 - Df(\gamma_2)z_2 = \widetilde{g}(z_2,\mu,\alpha_2),\end{aligned}$$

in the Banach space $\mathcal{Z}^1$.

We define spaces $\widetilde{\mathcal{Z}}_1$ and $\widetilde{\mathcal{Z}}_2$ which are closed linear subspaces of $\mathcal{Z}^0$, as follows

$$\begin{aligned}\widetilde{\mathcal{Z}}_1 &= \{h\in\mathcal{Z}^0 : \int_{-\infty}^{\infty}\langle\varphi_i^T(t),h(t)\rangle dt = 0, i=1,...,d_1-s\}, \\ \widetilde{\mathcal{Z}}_2 &= \{h\in\mathcal{Z}^0 : \int_{-\infty}^{\infty}\langle\psi_i^T(t),h(t)\rangle dt = 0, i=1,...,d_2+s\},\end{aligned} \tag{4.4}$$

where $\varphi_1(t),...,\varphi_{d_1-s}(t)$ are the orthonormal unit bases of $Ker(L_1^*)$ and $\psi_1(t),...,\psi_{d_2+s}(t)$ are the orthonormal unit bases of $Ker(L_2^*)$. We define maps Π_1 and $\Pi_2 : \mathcal{Z}^0 \to \mathcal{Z}^0$ as follows

$$\Pi_1(z)(t) = \sum_{i=1}^{d_1-s}\varphi_i(t)\int_{-\infty}^{\infty}\langle\varphi_i^T(t),z(t)\rangle dt, \tag{4.5}$$

$$\Pi_2(z)(t) = \sum_{i=1}^{d_2+s}\psi_i(t)\int_{-\infty}^{\infty}\langle\psi_i^T(t),z(t)\rangle dt, \tag{4.6}$$

where φ_j^T and ψ_j^T, satisfying $\langle\varphi_i,\varphi_j^T\rangle = \delta_{ij}$ and $\langle\psi_i,\psi_j^T\rangle = \delta_{ij}$, respectively. When $i=j$, $\delta_{ij}=1$, and when $i\neq j$, $\delta_{ij}=0$. By the definition of map Π_1, we have

$$\begin{aligned}(\Pi_1(z))^2(t) &= \Pi_1(\Pi_1(z))(t) \\ &= \sum_{i=1}^{d_1-s}\varphi_i(t)\int_{-\infty}^{\infty}\langle\varphi_i^T(t),\sum_{j=1}^{d_1-s}\varphi_j(t)\int_{-\infty}^{\infty}\langle\varphi_j^T(t),z(t)\rangle dt\rangle dt\end{aligned}$$

$$= \sum_{i=1}^{d_1-s} \varphi_i(t) \sum_{j=1}^{d_1-s} \int_{-\infty}^{\infty} \langle \varphi_j^T(t), z(t) \rangle dt \int_{-\infty}^{\infty} \langle \varphi_i^T(t), \varphi_j(t) \rangle dt$$

$$= \sum_{i=1}^{d_1-s} \varphi_i(t) \int_{-\infty}^{\infty} \langle \varphi_i^T(t), z(t) \rangle dt$$

$$= \Pi_1(z)(t).$$

For map Π_2, we can similarly obtain $(\Pi_2(z))^2(t) = \Pi_2(z)(t)$. Hence, Π_1 and Π_2 are projections. For any $z_i \in \mathcal{Z}^1$, we have

$$\Pi_i(\dot{z}_i - Df(\gamma_i)z_i) = 0.$$

Next, we apply the Lyapunov-Schmidt reduction to solve Eq (4.2). Applying Π_i and $(I - \Pi_i)$ on Eq (4.2), we find that Eq (4.2) is equivalent to the following system

$$\dot{z}_i = Df(\gamma_i)z_i + (I - \Pi_i)\widetilde{g}(z_i, \mu, \alpha_i), \tag{4.7}$$

$$\Pi_i \widetilde{g}(z_i, \mu, \alpha_i) = 0. \tag{4.8}$$

We first solve Eq (4.7) for $z_i \in \mathcal{Z}^1$. Then, the bifurcation equations are obtained by substituting the solution z_i into Eq (4.8).

We can define a bounded linear map $K_i : Ran(L_i) \mapsto \mathcal{Z}^1 \setminus Ker(L_i)$. Thus $K_i(h_i)$ is a solution of the linear operator equation $L_i(u) = \dot{u}(t) - Df(\gamma_i) = h_i$, when $h_i \in Ran(L_i)$. By $(H6)$, we suppose $u_1^i(t), ..., u_{d_i-1}^i(t)$ are the orthonormal unit bases of $Ker(L_i)$. Moreover, we solve Eq (4.7) for $z_i \in \mathcal{Z}^1$.

Lemma 2. *Equation* (4.7) *has a unique solution* $z_i \in \mathcal{Z}^1$ *such that* z_i *satisfies*

$$F_i(z_i, \beta^i, \mu, \alpha_i) = \sum_{j=1}^{d_i-1} \beta_j^i u_j^i + K_i\{(I - \Pi_i)\widetilde{g}(z_i, \mu, \alpha_i)\},$$

where $(\beta^i, \mu, \alpha_i) \in \mathbb{R}^{d_i-1} \times \mathbb{R}^2 \times \mathbb{R}$.

Proof. We define a C^2 map: $F_i : \mathcal{Z}^1 \times \mathbb{R}^{d_i-1} \times \mathbb{R}^2 \times \mathbb{R} \to \mathcal{Z}^1$ as follows:

$$F_i(z_i, \beta^i, \mu, \alpha_i) = \sum_{j=1}^{d_i-1} \beta_j^i u_j^i + K_i\{(I - \Pi_i)\widetilde{g}(z_i, \mu, \alpha_i)\}, \tag{4.9}$$

where $\beta^i = (\beta_1^i, ..., \beta_{d-1}^i) \in \mathbb{R}^{d_i-1}$. By Eq (4.4), we obtain $\widetilde{\mathcal{Z}}_i = Ran(L_i) = Ran(I - \Pi_i)$, $i = 1, 2$. By the definition of the projection operator Π_i and Lemma 3.1, we have $(I - \Pi_i)\widetilde{g}(z_i, \mu, \alpha_i) \in Ran(L_i)$. Thus $K_i\{(I - \Pi_i)\widetilde{g}(z_i, \mu, \alpha_i)\}$ is a solution of the Eq (4.7). And $u_j^i(t) \in Ker(L_i)$, then the fixed points of F_i are the solutions of Eq (4.7). Thus, we must demonstrate that the map F_i has a unique fixed point in the space $\mathcal{Z}^1$.

We let $\overline{B}(0, \delta_1)$, $\overline{B}^i(0, \delta_2)$, and $\overline{B}_1(0, \delta_2)$ be a closed subset with radius $\delta_1 > 0$ and $\delta_2 > 0$ centered at the origins of $\mathcal{Z}^1$, $\mathbb{R}^{d_i-1}$, and $\mathbb{R}^2$. By $\widetilde{g}(0, 0, \alpha_i) = 0$ and the smoothness of f, g_j, we can set δ_1 and δ_2 to be sufficiently small such that

$$\|\widetilde{g}(z_i, \mu, \alpha_i)\|_0 < \delta_2,$$

for $(z_i, \mu, \alpha_i) \in \overline{B}(0, \delta_1) \times \overline{B}_1(0, \delta_2) \times \mathbb{R}$.

Further, $u^i_j \in Ker(L_i)$, K_i and $(I - \Pi_i)$ are bounded linear operators. We can set constants $M_3 > 0, M_4 > 0$ such that

$$\|u^i_j\|_1 \le M_3, \|K_i(I - \Pi_i)\| \le M_4,$$

for any $i = 1, 2$, $j = 1..., d_i - 1$. We let $\delta_2 = \min\{\frac{\delta_1}{2M_3(d_i-1)}, \frac{\delta_1}{2M_4}\}$. For any $(z_i, \beta^i, \mu, \alpha_i) \in \times\overline{B}(0, \delta_1) \times \overline{B}^i(0, \delta_2) \times \overline{B}_1(0, \delta_2) \times \mathbb{R}$, we have

$$\begin{aligned}
\|F_i(z_i, \beta^i, \mu, \alpha_i)\|_1 &= \|\sum_{j=1}^{d_i-1} \beta^i_j u^i_j + K_i\{(I - \Pi_i)\widetilde{g}(z_i, \mu, \alpha_i)\|_1 \\
&\le \|\sum_{j=1}^{d_i-1} \beta^i_j u^i_j\|_1 + \|K_i\{(I - \Pi_i)\widetilde{g}(z_i, \mu, \alpha_i)\|_1 \\
&\le \delta_2(d_i - 1)M_3 + \delta_2 M_4 \\
&\le \delta_1.
\end{aligned}$$

Thus, for any $(\beta^i, \mu, \alpha_i) \in \overline{B}^i(0, \delta_2) \times \overline{B}_1(0, \delta_2) \times \mathbb{R}$, we have

$$F_i(\cdot, \beta^i, \mu, \alpha_i) : \overline{B}(0, \delta_1) \mapsto \overline{B}(0, \delta_1).$$

We let

$$h(z_i)(t) = f(\gamma_i(t) + z_i(t)) - f(\gamma_i(t)) - Df(\gamma_i(t))z_i(t).$$

Then $h(0) = 0, Dh(0) = 0$, so we can choose above δ_1 to be sufficiently small such that $\|Dh(z_i)\| \le \delta_2$, for $z_i \in \overline{B}(0, \delta_1)$. We select a constant $M_5 > 0$ such that $\|D_1 g_j(\gamma_i(t) + z_i(t), \mu, t + \alpha_i)\| \le M_5$, for $(z_i, \mu, \alpha_i) \in \times\overline{B}(0, \delta_1) \times \overline{B}_1(0, \delta_2) \times \mathbb{R}$.

By Eq (4.3), we have

$$\widetilde{g}(z_i, \mu, \alpha_i)(t) = h(z_i)(t) + \Sigma^2_{j=1}\mu_j(g_j(\gamma_i(t) + z_i(t), \mu, t + \alpha_i).$$

For $z^1_i, z^2_i \in \overline{B}(0, \delta_1)$, $(\beta^i, \mu, \alpha_i) \in \overline{B}^i(0, \delta_2) \times \overline{B}_1(0, \delta_2) \times \mathbb{R}$. From Eq (4.3), we obtain the following:

$$\begin{aligned}
&\|F_i(z^1_i, \beta^i, \mu, \alpha_i) - F_i(z^2_i, \beta^i, \mu, \alpha_i)\| \\
&\quad = \|K_i\{(I - \Pi_i)\{\widetilde{g}(z^1_i, \mu, \alpha_i)\} - K_i\{(I - \Pi_i)\widetilde{g}(z^2_i, \mu, \alpha_i)\}\| \\
&\quad = \|K_i\{(I - \Pi_i)\{\widetilde{g}(z^1_i, \mu, \alpha_i) - \widetilde{g}(z^2_i, \mu, \alpha_i)\}\| \\
&\quad = \|K_i\{(I - \Pi_i)\{h(z^1_i(t)) - h(z^2_i(t)) \\
&\qquad + \Sigma^2_{j=1}\mu_j(g_j(\gamma_i(t) + z^1_i(t), \mu, t + \alpha_i) - g_j(\gamma_i(t) + z^2_i(t), \mu, t + \alpha_i))\| \\
&\quad \le \|K_i(I - \Pi_i)\|\{|Dh(z^1_i(t) + s(z^2_i(t) - z^1_i(t)))\||z^1_i(t) - z^2_i(t)| \\
&\qquad + \Sigma^2_{j=1}|\mu_j\|(g_j(\gamma_i(t) + z^1_i(t), \mu, t + \alpha_i) - g_j(\gamma_i(t) + z^2_i(t), \mu, t + \alpha_i))|\} \\
&\quad \le \|K_i(I - \Pi_i)\|\{Dh(z^1_i(t) + s(z^2_i(t) - z^1_i(t))) \\
&\qquad + \Sigma^2_{j=1}|\mu_j|(D_1 g_j(\gamma_i(t) + z^1_i(t) + s(z^1_i(t) - z^2_i(t)), \mu, t + \alpha_i)\}\|z^1_i - z^2_i\|,
\end{aligned}$$

for $s \in (0, 1)$. Thus,

$$\|F_i(z^1_i, \beta^i, \mu, \alpha_i) - F_i(z^2_i, \beta^i, \mu, \alpha_i)\| \le \delta_2(M_4 + 2M_5)\|z^1_i - z^2_i\|.$$

Therefore, if we set $\delta_2 = \min\{\frac{\delta_1}{2M_3(d_i-1)}, \frac{\delta_1}{2M_4}, \frac{1}{2(M_4+2M_5)}\}$, then

$$\|F_i(z_i^1, \beta^i, \mu, \alpha_i) - F_i(z_i^2, \beta^i, \mu, \alpha_i)\| \le \frac{1}{2}\|z_i^1 - z_i^2\|.$$

As a result, F_i is a uniform contraction in $\overline{B}(0, \delta_1)$. By the contraction mapping principle, a unique C^1 map $\omega_i : \overline{B}^i(0, \delta) \times \overline{B}_1(0, \delta) \times \mathbb{R} \mapsto \mathcal{Z}^1$ exists such that

$$\omega_i(\beta^i, \mu, \alpha_i) = \sum_{j=1}^{d_i-1} \beta_j^i u_j^i + K_i\{(I - \Pi_i)\widetilde{g}(\omega_i, \mu, \alpha_i)\}.$$

Moreover, $F_i(0, 0, 0, \alpha_i) = 0$, hence, $\omega_i(0, 0, \alpha_i) = 0$, which implies the desired statement.

Substituting $\omega_i(\beta^i, \mu, \alpha_i)$ into Eq (4.8), we obtain the bifurcation function

$$0 = \Pi_i \widetilde{g}(\omega_i(\beta^i, \mu, \alpha_i), \mu, \alpha_i). \tag{4.10}$$

By the definition of projection Π_i, we have

$$\sum_{i=1}^{d_1-s} \psi_i(t) \int_{-\infty}^{+\infty} \langle \psi_i^T(s), \widetilde{g}(\omega_1(\beta^1, \mu, \alpha_1), \mu, \alpha_1)(s) \rangle ds = 0, \tag{4.11}$$

$$\sum_{i=1}^{d_2+s} \varphi_i(t) \int_{-\infty}^{+\infty} \langle \varphi_i^T(s), \widetilde{g}(\omega_2(\beta^2, \mu, \alpha_2), \mu, \alpha_2)(s) \rangle ds = 0. \tag{4.12}$$

By the linear independence of $\varphi_1, ..., \varphi_{d_1-s}$ and $\psi_1, ..., \psi_{d_2+s}(t)$, Eqs (4.11) and (4.12) are equivalent to

$$H_i^1(\beta^1, \mu, \alpha_1) = \int_{-\infty}^{+\infty} \langle \psi_i^T(s), \widetilde{g}(\omega_1(\beta^1, \mu, \alpha_1), \mu, \alpha_1)(s) \rangle ds = 0, \tag{4.13}$$

$$H_j^2(\beta^2, \mu, \alpha_2) = \int_{-\infty}^{+\infty} \langle \varphi_j^T(s), \widetilde{g}(\omega_2(\beta^2, \mu, \alpha_2), \mu, \alpha_2)(s) \rangle ds = 0, \tag{4.14}$$

where $i = 1, ...d_1 - s$, $j = 1, ..., d_2 + s$. We let

$$H^1(\beta^1, \mu, \alpha_1) = (H_1^1(\beta^1, \mu, \alpha_1), ..., H_{d_1-s}^1(\beta^1, \mu, \alpha_1)),$$
$$H^2(\beta^2, \mu, \alpha_2) = (H_1^2(\beta^2, \mu, \alpha_2), ..., H_{d_2+s}^2(\beta^2, \mu, \alpha_2)).$$

Therefore, by the Lyapunov-Schmidt reduction, we obtained the bifurcation function:

$$H(\beta, \mu, \alpha) = (H^1(\beta^1, \mu, \alpha_1), H^2(\beta^2, \mu, \alpha_2)),$$

where $\beta = (\beta^1, \beta^2)$, $\alpha = (\alpha_1, \alpha_2)$. If there are some parameter values $(\beta, \mu, \alpha) \in \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2$, such that

$$H(\beta, \mu, \alpha) = 0,$$

then $z_i = \omega_i$ is a solution of Eq (4.2). Hence, the perturbed Eq (1.2) has heteroclinic solutions

$$x_1(\beta^1, \mu, \alpha_1)(t) = \gamma_1(t) + \omega_1(\beta^1, \mu, \alpha_1)(t),$$

and

$$x_2(\beta^2,\mu,\alpha_2)(t) = \gamma_2(t) + \omega_2(\beta^2,\mu,\alpha_2)(t),$$

which are asymptotic to the equilibrium p_+ and p_-, that is

$$\lim_{t\to+\infty} x_1(\beta^1,\mu,\alpha_1)(t) = p_+, \lim_{t\to-\infty} x_1(\beta^1,\mu,\alpha_1)(t) = p_-$$

and

$$\lim_{t\to+\infty} x_2(\beta^2,\mu,\alpha_1)(t) = p_-, \lim_{t\to-\infty} x_2(\beta^2,\mu,\alpha_1)(t) = p_+,$$

are uniform for some $(\beta^1,\beta^2,\mu,\alpha_1,\alpha_2)$. Thus, the heteroclinic orbits $x_1(\beta^1,\mu,\alpha_1)(t)$ and $x_2(\beta^2,\mu,\alpha_2)(t)$ and the equilibria p_+, p_- constitute a heteroclinic loop of the perturbed Eq (1.2).

Through direct calculations, the function $H(\beta,\mu,\alpha)$ has the following properties:

$$(i)\ H^1(0,0,\alpha_1) = H^2(0,0,\alpha_2) = 0,\ \ \frac{\partial H_i^1}{\partial \beta_p^1}(0,0,\alpha_1) = \frac{\partial H_j^2}{\partial \beta_q^2}(0,0,\alpha_2) = 0;$$

$$(ii)\ \frac{\partial^2 H_i^1}{\partial \beta_p^1 \partial \beta_q^1}(0,0,\alpha_1) = \int_{-\infty}^{+\infty} \langle \psi_i^T(s), D_{11}f(\gamma_1(s))u_p^1(s)u_q^1(s)\rangle ds;$$

$$(iii)\ \frac{\partial^2 H_j^2}{\partial \beta_p^2 \partial \beta_q^2}(0,0,\alpha_2) = \int_{-\infty}^{+\infty} \langle \varphi_i^T(s), D_{11}f(\gamma_2(s))u_p^2(s)u_q^2(s)\rangle ds;$$

$$(iv)\ \frac{\partial H_i^1}{\partial \mu_k}(0,0,\alpha_1) = \int_{-\infty}^{+\infty} \langle \psi_i^T(s), g_k(\gamma_1(s),\mu,s+\alpha_1)\rangle ds;$$

$$(v)\ \frac{\partial H_j^2}{\partial \mu_k}(0,0,\alpha_2) = \int_{-\infty}^{+\infty} \langle \varphi_i^T(s), g_k(\gamma_2(s),\mu,s+\alpha_2)\rangle dt.$$

We define $M^1 : \mathbb{R}^{d_1-1} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^{d_1-s}$ given by

$$M^1(\beta^1,\mu,\alpha_1) = (M_1^1(\beta^1,\mu,\alpha_1), ..., M_{d_1-s}^1(\beta^1,\mu,\alpha_1)),$$

and

$$M_i^1(\beta^1,\mu,\alpha_1) = \sum_{k=1}^{2} a_{i,k}^1(\alpha_1)\mu_k + \frac{1}{2}\sum_{p=1}^{d_1-1}\sum_{q=1}^{d_1-1} b_{i,pq}^1\beta_p^1\beta_q^1, i = 1, ..., d_1 - s.$$

We define $M^2 : \mathbb{R}^{d_2-1} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^{d_2+s}$ given by

$$M^2(\beta^2,\mu,\alpha_2) = (M_1^2(\beta^2,\mu,\alpha_2), ..., M_{d_2+s}^2(\beta^2,\mu,\alpha_2)),$$

and

$$M_j^2(\beta^2,\mu,\alpha_2) = \sum_{k=1}^{2} a_{j,k}^2(\alpha_2)\mu_k + \frac{1}{2}\sum_{p=1}^{d_2-1}\sum_{q=1}^{d_2-1} b_{j,pq}^2\beta_p^2\beta_q^2, j = 1, ..., d_2 + s.$$

Thus,

$$H^i(\beta^i,\mu,\alpha_i) = M^i(\beta^i,\mu,\alpha_i) + H.O.T.$$

Moreover, we define $M : \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{d_1-s} \times \mathbb{R}^{d_2+s}$ given by

$$M(\beta,\mu,\alpha) = (M^1(\beta^1,\mu,\alpha_1), M^2(\beta^2,\mu,\alpha_2)),$$

hence

$$H(\beta,\mu,\alpha) = M(\beta,\mu,\alpha) + H.O.T.$$

Lemma 3. *If points* $(\beta_0, \mu_0, \alpha_0) \in \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2$ *exists such that* $M(\beta_0, \mu_0, \alpha_0) = 0$*, and* $D_{(\beta,\mu)}M(\beta_0, \mu_0, \alpha_0)$ *is a nonsingular* $(d_1 + d_2) \times (d_1 + d_2)$ *matrix, then an open interval* $I \subset \mathbb{R}$ *exists containing zero and differentiable functions,* $\kappa_1 : I \to \mathbb{R}^{d_1+d_2-2}$ *and* $\kappa_2 : I \to \mathbb{R}^2$*, such that* $\kappa_1(0) = 0$*,* $\kappa_2(0) = 0$*, and* $H(\varepsilon(\beta_0 + \kappa_1(\varepsilon)), \varepsilon^2(\mu_0 + \kappa_2(\varepsilon)), \alpha_0) = 0$ *for* $\varepsilon \in I$*.*

Proof. We define a C^2 function $N : \mathbb{R}^{d_1+d_2-2} \times \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}^{d_1+d_2}$:

$$N(x, y, \varepsilon) = \begin{cases} \frac{1}{\varepsilon^2} H(\varepsilon(\beta_0 + x), \varepsilon^2(\mu_0 + y), \alpha_0), & \text{for } \varepsilon \neq 0, \\ M(\beta_0 + x, \mu_0 + y, \alpha_0), & \text{for } \varepsilon = 0. \end{cases}$$

It is clear that $H = 0$ if and only if $N = 0$ for $\varepsilon \neq 0$. Through direct calculations, we have $N(0, 0, 0) = 0$, and $D_{(x,y)}N(0, 0, 0) = D_{(\beta,\mu)}M(\beta_0, \mu_0, \alpha_0)$ is nonsingular matrix. Using the implicit function theorem, we know an open interval $I \subset \mathbb{R}$ exists containing the zero and differentiable functions, which are $\kappa_1 : I \to \mathbb{R}^{d_1+d_2-2}$ and $\kappa_2 : I \to \mathbb{R}^2$, satisfying $\kappa_1(0) = 0$ and $\kappa_2(0) = 0$, respectively, such that $N(\kappa_1(\varepsilon), \kappa_2(\varepsilon), \varepsilon) = 0$ for $\varepsilon \in I$. Hence, we obtain

$$H(\varepsilon(\beta_0 + \kappa_1(\varepsilon)), \varepsilon^2(\mu_0 + \kappa_2(\varepsilon)), \alpha_0) = 0 \ \text{ for } \ \varepsilon \in I \setminus \{0\}.$$

The proof is complete.

Hence, the perturbed Eq (1.2) has heteroclinic orbits

$$x_1(\varepsilon, t) = \gamma_1(t - \alpha_{1,0}) + \omega_1(\varepsilon(\beta_0^1 + \kappa_1^1(\varepsilon)), \varepsilon^2(\mu_0 + \kappa_2(\varepsilon)), \alpha_{1,0})(t - \alpha_{1,0}),$$

and

$$x_2(\varepsilon, t) = \gamma_2(t - \alpha_{2,0}) + \omega_2(\varepsilon(\beta_0^2 + \kappa_1^2(\varepsilon)), \varepsilon^2(\mu_0 + \kappa_2(\varepsilon), \alpha_{2,0})(t - \alpha_{2,0}),$$

where $\varepsilon \in I \setminus \{0\}, \beta_0 = (\beta_0^1, \beta_0^2), \kappa_1(\varepsilon) = (\kappa_1^1(\varepsilon), \kappa_1^2(\varepsilon)), \alpha_0 = (\alpha_{1,0}, \alpha_{2,0})$. In addition,

$$\lim_{t\to+\infty} x_1(\varepsilon, t) = p_+, \lim_{t\to-\infty} x_1(\varepsilon, t) = p_-,$$

$$\lim_{t\to+\infty} x_2(\varepsilon, t) = p_-, \lim_{t\to-\infty} x_2(\varepsilon, t) = p_+,$$

for some $\varepsilon \in I \setminus \{0\}$. If we let

$$\Gamma_\varepsilon = \{p_-\} \cup \{x_1(\varepsilon, t) : t \in \mathbb{R}\} \cup \{p_+\} \cup \{x_2(\varepsilon, t) : t \in \mathbb{R}\},$$

then some solutions near the unperturbed heteroclinic loop Γ exist which can constitute a heteroclinic loop Γ_ε for perturbed Eq (1.2).

5. Discussion and conclusions

In this paper, we investigated the persistence of a heteroclinic loop under periodic perturbation in $\mathbb{R}^n$. We assumed unperturbed heteroclinic loop is a heterodimensional loop and the unperturbed heteroclinic orbits are degenerate. Using the method of Lyapunov-Schmidt reduction and exponential dichotomies, we obtained the bifurcation function, which is defined by

$$H(\beta, \mu, \alpha) = (H^1(\beta^1, \mu, \alpha_1), H^2(\beta^2, \mu, \alpha_2)),$$

where $\beta = (\beta^1, \beta^2)$, $\alpha = (\alpha_1, \alpha_2)$ and $(\beta^1, \beta^2, \mu, \alpha_1, \alpha_2) \in \mathbb{R}^{d_1-1} \times \mathbb{R}^{d_2-1} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$. Under the condition of Theorem 1, there exist some points such that $H(\beta, \mu, \alpha) = 0$. Hence, there exist heteroclinic solutions $x_1(\varepsilon, t)$, $x_2(\varepsilon, t)$ of the Eq (1.2) with $\mu = \varepsilon^2(\mu_0 + \kappa_2(\varepsilon))$, where $\varepsilon \in I \setminus \{0\}$, $x_1(\varepsilon, t)$ and $x_2(\varepsilon, t)$ are located near the heteroclinic orbits γ_1 and γ_2, such that $x_1(\varepsilon, t)$, $x_2(\varepsilon, t)$, p_+ and p_- can constitute a heteroclinic loop Γ_ε. The heteroclinic tangles is one of the primary mechanisms for non-uniformly hyperbolic dynamics. Our results extended the theory of heteroclinic loop bifurcation.

There are still many interesting and instructive issues worthy of further study. For example, the hyperbolicity of the heteroclinic solution $x_i(\varepsilon, t)$ and chaos motion near the heteroclinic loop Γ_ε.

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Conflict of interest

The authors declare there is no conflicts of interest.

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