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Research article

S-asymptotically ω-periodic solutions in distribution for a class of stochastic fractional functional differential equations

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Abstract: In this paper, we introduce the concept of an S-asymptotically ω-periodic process in distribution for the first time, and by means of the successive approximation and the Banach contraction mapping principle, respectively, we obtain sufficient conditions for the existence and uniqueness of the S-asymptotically ω-periodic solutions in distribution for a class of stochastic fractional functional differential equations.

Keywords: square-mean S-asymptotically ω-periodic process; S-asymptotically ω-periodic solution in distribution; fractional differential equation

1. Introduction

Periodicity is such an important phenomenon; people have studied it for a long time, and many researchers have investigated the properties about asymptotically almost automorphic, almost periodic, asymptotically almost periodic and S-asymptotically ω-periodic solutions of various determinate differential systems (see, e.g., [1–4]). The differential equations of fractional order have received great attention in recent years (see, e.g., [1, 5–8]). Cuevas and de Souza [9] considered the S-asymptotically ω-periodic solution of the semilinear integro-differential equation of fractional order given by

$$\begin{cases} x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ax(s)\mathrm{d}s + f(t, x(t)), \\ x(0) = c_0. \end{cases}$$

Moreover, in [10], Cuevas and de Souza considered the S-asymptotically ω-periodic solutions of the following form:

$$\begin{cases} x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ax(s)\mathrm{d}s + f(t, x_t), \\ x(0) = \psi_0 \in \mathcal{B}, \end{cases}$$

where $\mathcal{B}$ is some abstract phase space. Shu et al. [11] considered the existence of the S-asymptotically ω-periodic solutions of the following Caputo fractional differential equations with infinite delay

$$\begin{cases} D_t^{\alpha}(x(t) + F(t, x_t)) + A(x(t)) = G(t, x_t), \ t \geq 0 \\ x(0) = \psi_0 \in \mathcal{B}, \end{cases}$$

where $0 < \alpha < 1$. Recently, stochastic perturbations have been taken into consideration in the mathematical systems [12]; specially, Liu and Sun [13] made an initial contribution to the concepts of Poisson square-mean almost automorphy and almost automorphy in distribution and got the existence results of the almost automorphic in distribution solutions for a kind of stochastic differential equations with Lévy noise. Additionally, Li [14] considered the weighted pseudo almost automorphic solutions for nonautonomous stochastic partial differential equations driven by Lévy noise. Ma et al. [15] investigated the existence of almost periodic solutions for a class of fractional impulsive neutral stochastic differential equations with infinite delay.

Inspired by the work mentioned above, we investigate the existence of the mild solutions and the S-asymptotically ω-periodic solutions in distribution in an abstract space for a class of stochastic fractional functional differential equations driven by Lévy noise of the form

$$\begin{cases} \mathrm{d}D(t, x_t) & = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} AD(s, x_s)\mathrm{d}s\mathrm{d}t + f(t, x_t)\mathrm{d}t \\ & + g(t, x_t)\mathrm{d}w(t) + \int_{|u|_U<1} F(t, x(t^-), u)\tilde{N}(\mathrm{d}t, \mathrm{d}u) \\ & + \int_{|u|_U\geq 1} G(t, x(t^-), u)N(\mathrm{d}t, \mathrm{d}u), x_0 = \phi \in C^b_{\mathcal{F}_0}([-\tau, 0], \mathbb{X}), \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, $D(t, \varphi) = \varphi(0) + h(t, \varphi)$, the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear densely defined operator of sectorial type on a Banach space $\mathbb{X}$, and f, g, F and G are functions subject to some additional conditions. The convolution integral in (1.1) is known as the Riemann-Liouville fractional integral [16]. We introduce the concept of a Poisson square-mean S-asymptotically ω-periodic solution for (1.1) in order to correspond to the effect of the Lévy noise. We make an initial consideration of the S-asymptotically ω-periodic solution in distribution in an abstract space C for (1.1).

In Section 2, we review and introduce some concepts about the square mean S-asymptotically ω-periodic process and some of their basic properties. We show the existence and uniqueness of the mild solution and the S-asymptotically ω-periodic solution in distribution to (1.1) in Sections 3 and 4, respectively. An example and the conclusions of this paper are given in the last two sections.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which satisfies the usual conditions, and $(H, |\cdot|)$ and $(U, |\cdot|)$ are real separable Hilbert spaces. $\mathcal{L}(U, H)$ denote the space of all bounded linear operators from U to H which with the usual operator norm $\|\cdot\|_{\mathcal{L}(U,H)}$ is a Banach space. $L^2(P, H)$ is the space of all H-valued random variables X such that $\mathbb{E}|X|^2 = \int_\Omega |X|^2 \mathrm{d}P < \infty$. For $X \in L^2(P, H)$, let $\|X\| := (\int_\Omega |X|^2 \mathrm{d}P)^{1/2}$; it is well known that $(L^2(P, H), \|\cdot\|)$ is a Hilbert space. We denote $\mathcal{M}^2([0, T], H)$ for the collection of stochastic processes $x(t) : [0, T] \to L^2(P, H)$ such that $\mathbb{E} \int_0^T |x(s)|\mathrm{d}s < \infty$. Let $\tau > 0$ and we introduce the space.

$C := C([-\tau, 0]; H)$ denotes the family of all right-continuous functions with left hand limits φ from $[-\tau, 0]$ to H. If $x(t) \in \mathcal{M}^2([-\tau, T], H)$ with right-continuous functions and with left-hand limits;

then, $x_t(\theta) = x(t+\theta) \in C$ for each $\theta \in [-\tau, 0]$. The space C is assumed to be equipped with the norm $\|\varphi\|_C = \sup_{-\tau\le\theta\le 0} |\varphi(\theta)|_H$. $C^b_{\mathcal{F}_0}([-\tau,0];H)$ denotes the family of all almost surely bounded $\mathcal{F}_0$-measurable, C-valued random variables. $\{x_t\}_{t\in\mathbf{R}}$ is regarded as a C-valued stochastic process. In the following discussion, we always consider the Lévy processes that are U-valued.

2.1. Lévy process

Let L be a Lévy process on U; we write $\Delta L(t) = L(t) - L(t^-)$ for all $t \ge 0$. We define a counting Poisson random measure N on $(U - \{0\})$ through

$$N(t, O) = \sharp\{0 \le s \le t : \ \Delta L(s)(\omega') \in O\} = \Sigma_{0\le s\le t}\chi_O(\Delta L(s)(\omega'))$$

for any Borel set O in $(U - \{0\})$, where χ_O is the indicator function. We write $\nu(\cdot) = E(N(1,\cdot))$ and call it the intensity measure associated with L. We say that a Borel set O in $(U - \{0\})$ is bounded below if $0 \in \bar{O}$, where $\bar{O}$ is a closure of O. If O is bounded below, then $N(t, O) < \infty$ almost surely for all $t \ge 0$ and $(N(t,O),\ t \ge 0)$ is a Poisson process with the intensity $\nu(O)$. So, N is called a Poisson random measure. For each $t \ge 0$ and O bounded below, the associated compensated Poisson random measure $\tilde{N}$ is defined by $\tilde{N}(t, O) = N(t, O) - t\nu(O)$ (see [17, 18]).

Proposition 2.1. *(see [17]) (Lévy-Itô decomposition). If L is a U-valued Lévy process, then there exist $a \in U$, a U-valued Wiener process w with the covariance operator Q, the so-called Q-wiener process and an independent Poisson random measure N on* $\mathbf{R}^+ \times (U - \{0\})$ *such that, for each* $t \ge 0$,

$$L(t) = at + w(t) + \int_{|u|_U<1} u\tilde{N}(t, \mathrm{d}u) + \int_{|u|_U\ge 1} uN(t, \mathrm{d}u), \tag{2.1}$$

where the Poisson random measure N has the intensity measure ν which satisfies $\int_U (|y|^2_U \wedge 1)\nu(\mathrm{d}y) < \infty$ *and $\tilde{N}$ is the compensated Poisson random measure of N.*

For more properties of the Lévy process and Q-Wiener processes, we refer the readers to [19] and [20]. We assume the covariance operator Q of w is of the trace class, i.e., $TrQ < \infty$ and the Lévy process L is defined on the filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathbf{R}^+})$ in this paper. We also denote $b := \int_{|y|_U\ge 1} \nu(\mathrm{d}x)$ throughout the paper.

2.2. Poisson square-mean S-asymptotically ω-periodic process

In [13], if for any $s \in \mathbf{R}$, $\lim_{t\to s} \|x(t) - x(s)\| = 0$, then the stochastic process $x : \mathbf{R} \to L^2(P, H)$ is said to be L^2-continuous; if $\|x\|_\infty = \sup_{t\in\mathbf{R}} \|x(t)\| < \infty$, the stochastic process $x : \mathbf{R} \to L^2(P, H)$ is said to be L^2-bounded. Denote by $C_b(\mathbf{R}; L^2(P, H))$ the Banach space of all L^2-bounded and L^2-continuous mappings from $\mathbf{R}$ to $L^2(P, H)$ endowed with the norm $\|\cdot\|_\infty$.

Definition 2.1. *1) An L^2-continuous stochastic process $x : \mathbf{R}^+ \to L^2(P, H)$ is said to be square-mean S-asymptotically ω-periodic if there exists $\omega > 0$ such that* $\lim_{t\to\infty} \|x(t+\omega) - x(t)\| = 0$. *The collection of all S-asymptotically ω-periodic stochastic processes $x : \mathbf{R}^+ \to L^2(P, H)$ is denoted by $SAP_\omega(L^2(P, H))$.*

2) A function $g : \mathbf{R}^+ \times C \to \mathcal{L}(U, L^2(P, H))$, $(t, \varphi) \mapsto g(t, \varphi)$ is said to be square-mean S-asymptotically ω-periodic in t for each $\varphi \in C$ if g is continuous in the following sense:

$$\mathbb{E}\|(g(t,\phi) - g(t',\varphi))Q^{1/2}\|^2_{\mathcal{L}(U,L^2(P,H))} \to 0 \ as \ (t',\varphi) \to (t,\phi)$$

and

$$\lim_{t\to\infty}\mathbb{E}\|(g(t+\omega,\phi)-g(t,\phi))Q^{1/2}\|^2_{\mathcal{L}(U,L^2(P,H))}=0$$

for each $\phi\in C$.

3) *A function* $F:\mathbf{R}^+\times C\times U\to L^2(P,H)$, $(t,\phi,u)\mapsto F(t,\phi,u)$ *with* $\int_U\|F(t,\phi,u)\|^2\nu(\mathrm{d}u)<\infty$ *is said to be Poisson square-mean S-asymptotically* ω*-periodic in* t *for each* $\phi\in C$ *if* F *is continuous in the following sense:*

$$\int_U\|F(t,\phi,u)-F(t',\varphi,u)\|^2\nu(\mathrm{d}u)\to 0\ as\ (t',\varphi)\to(t,\phi),$$

and that

$$\lim_{t\to\infty}\int_U\|F(t+\omega,\phi,u)-F(t,\phi,u)\|^2\nu(\mathrm{d}u)=0$$

for each $\phi\in C$.

Remark 2.1. *Any square-mean S-asymptotically* ω*-periodic process* $x(t)$ *is* L^2*-bounded and, by [3],* $SAP_\omega(L^2(P,H))$ *is a Banach space when it is equipped with the norm*

$$\|x\|_\infty:=\sup_{t\in\mathbf{R}^+}\|x(t)\|=\sup_{t\in\mathbf{R}^+}(\mathbb{E}|x(t)|^2)^{\frac{1}{2}}.$$

For the sequel, we introduce some definitions about square-mean S-asymptotically ω-periodic functions with parameters.

Definition 2.2. 1) *A function* $f:\mathbf{R}^+\times C\to L^2(P,H)$ *is said to be uniformly square-mean S-asymptotically* ω*-periodic in* t *on bounded sets if for every bounded set* K *of* C*, we have* $\lim_{t\to\infty}\|f(t+\omega,\phi)-f(t,\phi)\|=0$ *uniformly on* $\phi\in C$.

2) *A function* $g:\mathbf{R}^+\times C\to\mathcal{L}(U,L^2(P,H))$ *is said to be uniformly square-mean S-asymptotically* ω*-periodic on bounded sets if for every bounded set* K *of* C*, we have*

$$\lim_{t\to\infty}\mathbb{E}\|(g(t+\omega,\phi)-g(t,\phi))Q^{1/2}\|^2_{\mathcal{L}(U,L^2(P,H))}=0$$

uniformly on $\phi\in K$.

3) *A function* $F:\mathbf{R}^+\times C\times U\to L^2(P,H)$ *with* $\int_U\|F(t,\phi,u)\|^2\nu(\mathrm{d}u)<\infty$ *is said to be uniformly Poisson square-mean S-asymptotically* ω*-periodic in* t *on bounded sets if for every bounded set* K *of* C,

$$\lim_{t\to\infty}\int_U\|F(t+\omega,\phi,u)-F(t,\phi,u)\|^2\nu(\mathrm{d}u)=0$$

uniformly on $\phi\in K$.

Lemma 2.1. *Let* $f:\mathbf{R}^+\times C\to L^2(P,H)$, $(t,\phi)\mapsto f(t,\phi)$ *be uniformly square-mean S-asymptotically* ω*-periodic in* t *on bounded sets of* C*, and assume that* f *satisfies the Lipschitz condition in the sense* $\|f(t,\phi)-f(t,\varphi)\|^2\le L\|\phi-\varphi\|^2_C$ *for all* $\phi,\varphi\in C$ *and* $t\in\mathbf{R}$*, where* L *is independent of* t*. Then for any square-mean S-asymptotically* ω*-periodic process* $Y:\mathbf{R}\to L^2(P,H)$*, the stochastic process* $F:\mathbf{R}\to L^2(P,H)$ *given by* $F(t):=f(t,Y_t)$ *is square-mean S-asymptotically* ω*-periodic.*

Proof. Since $Y(t) \in SAP_\omega(L^2(P,H))$, the range of $Y(t)$ is a bounded set in $L^2(P,H)$, which means that $\{Y_t\}_{t\in\mathbf{R}^+}$ is also a bounded set in C. Then, $\lim_{t\to\infty}\|F(t+\omega, Y_{t+\omega}) - F(t, Y_{t+\omega})\| = 0$. For any $\epsilon > 0$, $\exists\, T(\epsilon)$ such that $\|F(t+\omega, Y_{t+\omega}) - F(t, Y_{t+\omega})\| < \epsilon/2$ and $\|Y_{t+\omega} - Y_t\| < \frac{\epsilon}{2L}$. We get

$$
\begin{aligned}
\|F(t+\omega) - F(t)\| &\leq \|F(t+\omega, Y_{t+\omega}) - F(t, Y_t)\| \\
&= \|F(t+\omega, Y_{t+\omega}) - F(t, Y_{t+\omega})\| \\
&+ \|F(t+\omega, Y_{t+\omega}) - F(t, Y_t)\| \\
&\leq \epsilon.
\end{aligned}
$$

□

Lemma 2.2. *Let $F : \mathbf{R}^+ \times C \times U \to L^2(P,H)$ be uniformly Poisson square-mean S-asymptotically ω-periodic in t on bounded sets of C and F satisfy the Lipschitz condition in the sense*

$$\int_U \|F(t,\phi,u) - F(t,\varphi,u)\|^2 \nu(\mathrm{d}u) \leq L\|\phi - \varphi\|_C^2$$

for all $\phi, \varphi \in C$ and $t \in \mathbf{R}$, where L is independent of t. Then, for any square-mean S-asymptotically ω-periodic process $Y(t) : \mathbf{R} \to L^2(P,H)$, the stochastic process $\tilde{F} : \mathbf{R} \times U \to L^2(P,H)$ given by $\tilde{F}(t,u) := F(t, Y_t, u)$ is Poisson square-mean S-asymptotically ω-periodic.

Proof. Since F is uniformly Poisson square-mean S-asymptotically ω-periodic in t on bounded sets of C and $Y(t) \in SAP_\omega(L^2(P,H)$, the range $\mathcal{R}(Y)$ of $Y(t)$ is a bounded set in $L^2(P,H)$, namely, $\{Y_t\}_{t\in\mathbf{R}}$ is bounded in C; we get $\lim_{t\to\infty}\int_U \|F(t+\omega,\phi,u) - F(t,\phi,u)\|^2\nu(\mathrm{d}u) = 0$ uniformly for $\phi \in \{Y_t\}_{t\in\mathbf{R}^+}$. For any $\epsilon > 0$, there is a $T(\epsilon) > 0$ such that for any $t \geq T(\epsilon)$, the inequalities $\int_U \|F(t+\omega,\phi,u) - F(t,\phi,u)\|^2\nu(\mathrm{d}u) \leq \epsilon/4$, $\forall\, \phi \in \{Y_t\}_{t\in\mathbf{R}^+}$ and $\|Y_{t+\omega} - Y_t\|^2 < \frac{\epsilon}{2L}$ hold. Note that

$$\tilde{F}(t+\omega,u) - \tilde{F}(t,u) = F(t+\omega, Y_{t+\omega}, u) - F(t+\omega, Y_t, u) + F(t+\omega, Y_t, u) - F(t, Y_t, u);$$

so, for the above ϵ, when $t \geq T(\epsilon)$,

$$
\begin{aligned}
\int_U \|\tilde{F}(t+\omega,u) - \tilde{F}(t,u)\|^2\nu(\mathrm{d}u) &\leq 2\int_U \|F(t+\omega, Y_{t+\omega}, u) - F(t, Y_{t+\omega}, u)\|^2\nu(\mathrm{d}u) \\
&+ 2\int_U \|F(t, Y_{t+\omega}, u) - F(t, Y_t, u)\|^2\nu(\mathrm{d}u) \\
&\leq \epsilon/2 + 2L\|Y_{t+\omega} - Y_t\|^2 \\
&\leq \epsilon.
\end{aligned}
$$

We deduce that

$$\lim_{t\to\infty}\int_U \|\tilde{F}(t+\omega,u) - \tilde{F}(t,u)\|^2\nu(\mathrm{d}u) = 0,$$

which means that $\tilde{F}(t,u)$ is Poisson square-mean S-asymptotically ω-periodic. □

Let $\mathcal{P}(C)$ be the space of Borel probability measures on C; for $P_1,\ P_2 \in \mathcal{P}(C)$, denote the metric $d_{\mathfrak{L}}$ as follows:

$$d_{\mathfrak{L}}(P_1,P_2) = \sup_{f\in\mathfrak{L}} \Big|\int_C f(\sigma)P_1(\mathrm{d}\sigma) - \int_C f(\sigma)P_2(\mathrm{d}\sigma)\Big|,$$

where

$$\mathfrak{L} = \{f : C \to \mathbf{R} :\ |f(\phi) - f(\varphi)| \leq \|\phi - \varphi\|_C \text{ and } |f(\cdot)| \leq 1\}.$$

Definition 2.3. *A stochastic process $x_t : \mathbf{R} \to C$ is said to be S-asymptotically ω-periodic in distribution if the law $\mu(t)$ of x_t is a $\mathcal{P}(C)$-valued S-asymptotically ω-periodic mapping, i.e., there is a positive number ω such that*

$$\lim_{t\to\infty} d_{\mathfrak{L}}(\mu(t+\omega), \mu(t)) = 0.$$

Lemma 2.3. *Any square-mean S-asymptotically ω-periodic solution of (1.1) is necessarily S-asymptotically ω-periodic in distribution.*

Proof. Let $x(t) \in SPA_\omega(L^2(P, H))$ be a solution of (1.1), then, $\exists\ \omega > 0$,

$$\lim_{t\to\infty} \|x(t+\omega) - x(t)\| = 0. \tag{2.2}$$

We need to show that the law $\mu(t)$ of x_t satisfies

$$\lim_{t\to\infty} d_{\mathfrak{L}}(\mu(t+\omega), \mu(t)) = 0,$$

which is equivalent to show that, $\forall\ \epsilon > 0$, $\exists\ T > 0$,

$$\sup_{f\in\mathfrak{L}} \Big| \int_C f(\sigma)\mu(t+\omega)(\mathrm{d}\sigma) - \int_C f(\sigma)\mu(t)(\mathrm{d}\sigma)\Big| \le \epsilon,\ \forall t \ge T.$$

For any $f \in \mathfrak{L}$,

$$|\mathbb{E}(f(x_t^{x_\omega})) - \mathbb{E}(f(x_t^{\phi}))| \le \mathbb{E}(2 \wedge \|x_t^{x_\omega} - x_t^{\phi}\|_C).$$

According to (2.2), there is a $T_\epsilon > 0$ satisfying $\mathbb{E}\|x_t^{x_\omega} - x_t^{\phi}\|_C^2 \le \epsilon^2$. For the arbitrary $f \in \mathfrak{L}$, we get

$$\sup_{f\in\mathfrak{L}} \Big| \int_C f(\sigma)\mu(t+\omega)(\mathrm{d}\sigma) - \int_C f(\sigma)\mu(t)(\mathrm{d}\sigma)\Big| \le \epsilon,\ \forall t \ge T.$$

The proof is completed. □

2.3. *Sectorial operators*

The definitions and the properties about sectorial operators have been studied well in the past decades; for details, see [21, 22].

Definition 2.4. *Let $\mathbb{X}$ be an Banach space; $A : D(A) \subseteq \mathbb{X} \to \mathbb{X}$ is a closed linear operator. A is said to be a sectorial operator of the type μ and angle θ if there exist $0 < \theta < \pi/2$, $M > 0$ and $\mu \in \mathbf{R}$ such that the resolvent $\rho(A)$ of A exists outside of the sector $\mu + S_\theta = \{\mu + \lambda : \lambda \in \mathbf{C}, |arg(-\lambda)| < \theta\}$ and $\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda-\mu|}$ when λ does not belong to $\mu + S_\theta$.*

Definition 2.5. *(see [23]) Let A be a closed and linear operator with domain the $D(A)$ defined on a Banach space $\mathbb{X}$. We call A the generator of a solution operator if there exist $\mu \in \mathbf{R}$ and a strongly continuous function $S_\alpha : \mathbf{R}^+ \to \mathcal{L}(\mathbb{X}, \mathbb{X})$ such that $\{\lambda^\alpha : Re(\lambda) > \mu\} \subset \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)\mathrm{d}t$, where $Re(\lambda) > \mu$, $x \in \mathbb{X}$. In this case, $S_\alpha(\cdot)$ is called the solution operator generated by A.*

If A is a sectorial operator of the type μ with $1 < \theta < \pi(1 - \frac{\alpha}{2})$, then A is the generator of a solution operator given by $S_\alpha(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\mathrm{d}\lambda$, where γ is a suitable path lying outside of the sector $\mu + S_\theta$. Cuesta and Mendizabal [24] showed that, if A is a sectorial operator of the type $\mu < 0$, for some $M > 0$ and $0 < \theta < \pi(1 - \frac{\pi}{2})$, there is $C > 0$ such that

$$\|S_\alpha(t)\| \le \frac{CM}{1 + |\mu|t^\alpha},\ t \ge 0. \tag{2.3}$$

3. Existence of mild solution

Definition 3.1. *An $\mathcal{F}_t$-progressively measurable stochastic process $\{x(t)\}_{t\in\mathbf{R}}$ is called a mild solution of (1.1) if it satisfies the corresponding stochastic integral equation:*

$$\begin{aligned} x(t) &= S_\alpha(t)(\phi(0)+h(0,\phi)) - h(t,x_t) + \int_0^t S_\alpha(t-s)f(s,x_s)\mathrm{d}s \\ &+ \int_0^t S_\alpha(t-s)g(s,x_s)\mathrm{d}w(s) \\ &+ \int_0^t\int_{|u|<1} S_\alpha(t-s)F(s,x(s^-),u)\tilde{N}(\mathrm{d}s,\mathrm{d}u) \\ &+ \int_0^t\int_{|u|\geq 1} S_\alpha(t-s)G(s,x(s^-),u)N(\mathrm{d}s,\mathrm{d}u), \\ x_0 &= \phi(\cdot)\in C^b_{\mathcal{F}_0}([-\tau,0];H) \end{aligned} \tag{3.1}$$

for all $t\geq 0$.

In the following discussion, we impose the following conditions.

(H1) A is a sectorial operator of the type $\mu<0$ and angle θ with $0\leq\theta\leq\pi(1-\alpha/2)$.

(H2) $h(t,0)=0$ and, for all $\varphi,\psi\in C$ there exists a constant $k_0\in(0,1)$ such that $|h(t,\varphi)-h(t,\psi)|\leq k_0\|\varphi-\psi\|_C$.

(H3) $f:\mathbf{R}^+\times C\to L^2(P,H)$, $g:\ \mathbf{R}^+\times C\to\mathcal{L}(U,L^2(P,H))$, $f(t,0)=0$, $g(t,0)=0$, $F:\mathbf{R}^+\times C\times U\to L^2(P,H)$, $G:\mathbf{R}^+\times C\times U\to L^2(P,H)$, $F(t,0,u)=0$ and $G(t,0,u)=0$. For all $t\in\mathbf{R}^+$,

$$\begin{aligned} \|f(t,\phi)-f(t,\varphi)\|^2 &\leq L\|\phi-\varphi\|_C^2, \\ \mathbb{E}\|(g(t,\phi)-g(t,\varphi))Q^{1/2}\|^2_{\mathcal{L}(U,L^2(P,H))} &\leq L\|\phi-\varphi\|_C^2, \\ \int_{|u|_U<1}\|F(t,\phi,u)-F(t,\varphi,u)\|^2\nu(\mathrm{d}u) &\leq L\|\phi-\varphi\|_C^2, \\ \int_{|u|_U\geq 1}\|G(t,\phi,u)-G(t,\varphi,u)\|^2\nu(\mathrm{d}u) &\leq L\|\phi-\varphi\|_C^2, \end{aligned}$$

where the constant $L>0$ is independent of t.

Theorem 3.1. *If (H1)–(H2) hold, then the Cauchy problem (1.1) has a unique mild solution.*

Proof. Define $x_0^0=\phi$, and $x^0(t)=S_\alpha(t)(\phi(0)+f(0,\phi))$ for $t\geq 0$.

Set $x_0^n=\phi$; for $n=1,2,...,\ \forall\ T\in(0,\infty)$, we define the sequence of successive approximations to (1.1) as follows:

$$\begin{aligned} x^n(t)+h(t,x_t^n) &= S_\alpha(t)(\phi(0)+h(0,\phi)) + \int_0^t S_\alpha(t-s)f(s,x_s^{n-1}))\mathrm{d}s \\ &+ \int_0^t S_\alpha(t-s)g(s,x_s^{n-1})\mathrm{d}w(s) + \int_0^t\int_{|u|<1} S_\alpha(t-s)F(s,x_{s^-}^{n-1},u)\tilde{N}(\mathrm{d}s,\mathrm{d}u) \\ &+ \int_0^t\int_{|u|\geq 1} S_\alpha(t-s)G(s,x_{s^-}^{n-1},u)N(\mathrm{d}s,\mathrm{d}u), \end{aligned} \tag{3.2}$$

for $t \in [0,T]$. Obviously, $x^0(\cdot) \in C_b(\mathbf{R}, L^2(P,H))$ and $\|x^0(\cdot)\|_\infty^2 \leq c'$, where $c' = 2(CM)^2(1+k_0^2)\|\phi\|_C^2$ is a positive constant.

$$\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t}|x^n(t)+h(t,x_t^n)|^2 &= 5\mathbb{E}\sup_{0\leq s\leq t}|S_\alpha(t)(\phi(0)+h(0,\phi))|^2 \\
&+ 5\mathbb{E}\sup_{0\leq s\leq t}|\int_0^s S_\alpha(s-r)f(r,x_r^{n-1})\mathrm{d}r|^2 \\
&+ 5\mathbb{E}\sup_{0\leq s\leq t}|\int_0^s S_\alpha(s-r)g(r,x_r^{n-1})\mathrm{d}w(r)|^2 \\
&+ 5\mathbb{E}\sup_{0\leq s\leq t}|\int_0^s\int_{|u|<1} S_\alpha(s-r)F(r,x_{r^-}^{n-1},u)\tilde{N}(\mathrm{d}r,\mathrm{d}u)|^2 \\
&+ 5\mathbb{E}\sup_{0\leq s\leq t}|\int_0^s\int_{|u|\geq 1} S_\alpha(s-r)G(r,x_{r^-}^{n-1},u)N(\mathrm{d}r,\mathrm{d}u)|^2 \\
&= \sum_{i=1}^5 I_i.
\end{aligned}$$

Obviously, $I_1 \leq 5c'$, and

$$\begin{aligned}
I_2 &= 5\mathbb{E}\sup_{0\leq s\leq t}|\int_0^s S_\alpha(s-r)f(r,x_r^{n-1})\mathrm{d}r|^2 \\
&\leq 5\mathbb{E}\sup_{0\leq s\leq t}\int_0^s S_\alpha(s-r)\mathrm{d}r\int_0^s S_\alpha(s-r)|f(r,x_r^{n-1})|^2\mathrm{d}r \\
&\leq 5CM\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}L\mathbb{E}\int_0^t S_\alpha(t-r)\|x_r^{n-1}\|_C^2\mathrm{d}r \\
&\leq 5(CM)^2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}L\mathbb{E}\int_0^t\|x_r^{n-1}\|_C^2\mathrm{d}r.
\end{aligned}$$

It follows from Itô's isometry that

$$\begin{aligned}
I_3 &= 5\mathbb{E}\sup_{0\leq s\leq t}|\int_0^s S_\alpha(s-r)g(r,x_r^{n-1})\mathrm{d}w(r)| \\
&\leq 5(CM)^2\mathbb{E}\int_0^t\|g(r,x_r^{n-1})Q^{1/2}\|_{\mathcal{L}(U,L^2(P,H))}^2\mathrm{d}r \\
&\leq 5(CM)^2L\mathbb{E}\int_0^t\|x_r^{n-1}\|_C^2\mathrm{d}r.
\end{aligned}$$

By using the properties of integrals for Poisson random measures, we get

$$\begin{aligned}
I_4 &= \mathbb{E}\sup_{0\leq s\leq t}|\int_0^s\int_{|u|<1} S_\alpha(t-r)F(r,x_{r^-}^{n-1},u)\tilde{N}(\mathrm{d}r,\mathrm{d}u)|^2 \\
&\leq 5(CM)^2L\mathbb{E}\int_0^t\|x_r^{n-1}\|_C^2\mathrm{d}r,
\end{aligned}$$

$$\begin{aligned}
I_5 &= 5\mathbb{E}\sup_{0\le s\le t}|\int_0^s\int_{|u|\ge 1} S_\alpha(s-r)G(r,x_{r^-}^{n-1},u)N(\mathrm{d}r,\mathrm{d}u)|^2\\
&\le 10(CM)^2[\int_0^t\mathbb{E}\int_{|u\ge 1|}(\frac{1}{1+|\mu|(t-s)^\alpha})^2|G(r,x_{r^-}^{n-1},u)|^2\nu(\mathrm{d}u)\mathrm{d}s\\
&+ \int_0^t\frac{1}{1+|\mu|(t-r)^\alpha}\int_{|u|\ge 1}\nu(\mathrm{d}u)\mathrm{d}s\mathbb{E}\int_0^t(\int_{|u|\ge 1}|G(r,x_{r^-}^{n-1},u)|^2\nu(\mathrm{d}u))\mathrm{d}s]\\
&\le 10(CM)^2L(1+b\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})\mathbb{E}\int_0^t\|x_r^{n-1}\|_C^2\mathrm{d}r.
\end{aligned}$$

Note that

$$\begin{aligned}
|x^n(t)|^2 &= |x^n(t)+h(t,x_t^n)-h(t,x_t^n)|^2\\
&\le \frac{1}{1-k_0}(|x^n(t)+h(t,x_t^n)|^2+k_0(1-k_0)\|x_t^n\|_C^2);
\end{aligned}$$

by taking the expectation on both sides of the above inequality, we get

$$\mathbb{E}\sup_{0\le s\le t}|x^n(s)|^2\le\frac{1}{1-k_0}\mathbb{E}\sup_{0\le s\le t}|x^n(s)+h(s,x_s^n)|^2+k_0\mathbb{E}\sup_{0\le s\le t}\|x_s^n\|_C^2.$$

Combining the estimations for I_1-I_5, we get

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}|x^n(s)|^2 &\le \frac{1}{1-k_0}5(CM)^2[(1+k_0^2)\|\phi\|_C^2]+\frac{1}{1-k_0}5(CM)^2L\\
&\times (\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+2+2(1+b\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}))\mathbb{E}\int_0^t\|x_r^{n-1}\|_C^2\mathrm{d}r\\
&= c_1+c_2\mathbb{E}\sup_{0\le s\le t}\int_0^s\|x_r^{n-1}\|_C^2\mathrm{d}r.
\end{aligned}$$

Then, for any arbitrary positive integer $\tilde{k}$, we have

$$\max_{1\le n\le\tilde{k}}\mathbb{E}\sup_{0\le s\le t}|x^n(s)|^2\le c_1+c_2\int_0^t\|\phi\|_C^2\mathrm{d}r+c_2\int_0^t\sup_{0\le r\le t}\max_{1\le n\le\tilde{k}}\mathbb{E}\|x^{n-1}(r)\|_C^2\mathrm{d}r.$$

By the Gronwall inequality, we get

$$\max_{1\le n\le\tilde{k}}\mathbb{E}\sup_{0\le s\le t}|x^n(s)|^2\le(c_1+c_2\|\phi\|^2T)e^{c_3t}.$$

Due to the arbitrary $\tilde{k}$, we have

$$\mathbb{E}\sup_{0\le s\le T}|x^n(s)|^2\le(c_1+c_2\|\phi\|_C^2T)e^{c_3T}. \tag{3.3}$$

So, $x^n(t)\in\mathcal{M}^2([0,T],H)$.

Obviously, we have

$$\begin{aligned}\mathbb{E}(\sup_{0\le s\le t}|x^{n+1}(s)-x^n(s)|^2) &\le \frac{1}{1-k_0}\mathbb{E}(\sup_{0\le s\le t}(|x^{n+1}(s)-x^n(s)+h(s,x_s^{n+1})-h(s,x_s^n)|^2\\ &+ k_0\mathbb{E}(\sup_{0\le s\le t}|x^{n+1}(s)-x^n(s)|^2),\end{aligned}$$

namely,

$$\mathbb{E}(\sup_{0\le s\le t}|x^{n+1}(s)-x^n(s)|^2)\le\frac{1}{(1-k_0)^2}\mathbb{E}\sup_{0\le s\le t}|x^{n+1}(s)-x^n(s)+h(s,x_s^{n+1})-h(s,x_s^n)|^2,$$

and

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|[x^{n+1}(s)-x^n(s)]+[h(t,x_s^{n+1})-h(t,x_s^n)]|^2 &\le 4\mathbb{E}\sup_{0\le s\le t}|\int_0^s S_\alpha(s-r)[f(r,x_r^n))-f(r,x_r^{n-1}))]\mathrm{d}r|^2\\ &+ 4\mathbb{E}\sup_{0\le s\le t}|\int_0^s S_\alpha(s-r)[g(r,x_r^n)-g(r,x_r^{n-1})]\mathrm{d}w(r)|^2\\ &+ 4\mathbb{E}\sup_{0\le s\le t}|\int_0^t\int_{|u|<1}S_\alpha(s-r)[F(r,x_{r^-}^n,u)\\ &- F(r,x_{r^-}^{n-1},u)]\tilde{N}(\mathrm{d}r,\mathrm{d}u)|^2\\ &+ 4\mathbb{E}\sup_{0\le s\le t}|\int_0^s\int_{|u|\ge1}S_\alpha(s-r)[G(r,x_{r^-}^n,u)\\ &- G(r,x_{r^-}^{n-1},u)]N(\mathrm{d}r,\mathrm{d}u)|^2.\end{aligned}$$

By the fact that

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|\int_0^s S_\alpha(s-r)[f(r,x_r^n))-f(r,x_r^{n-1}))]\mathrm{d}r|^2 &\le L\mathbb{E}\sup_{0\le s\le t}\int_0^s S_\alpha(s-r)\mathrm{d}r\int_0^s S_\alpha(s-r)\|x_r^n-x_r^{n-1}\|_C^2\mathrm{d}r\\ &\le L(CM)^2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\mathbb{E}\int_0^t\|x_r^n-x_r^{n-1}\|_C^2\mathrm{d}r,\end{aligned}$$

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|\int_0^s S_\alpha(s-r)[g(r,x_r^n)-g(r,x_r^{n-1})]\mathrm{d}w(r)|^2 &\le \mathbb{E}\int_0^t S_\alpha^2(t-r)\|[g(r,x_r^n)-g(r,x_r^{n-1})]Q^{1/2}\|^2_{\mathcal{L}(U,L^2(P,H))}\mathrm{d}r\\ &\le L(CM)^2\mathbb{E}\int_0^t\|x_r^n-x_r^{n-1}\|_C^2\mathrm{d}r,\end{aligned}$$

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}\int_0^t\int_{|u|<1}S_\alpha(s-r)[F(r,x_{r^-}^n,u)-F(r,x_{r^-}^{n-1},u)]\tilde{N}(\mathrm{d}r,\mathrm{d}u)|^2 &\le L\mathbb{E}\int_0^t S_\alpha^2(t-r)\|x_{r^-}^n-x^{n-1}r^-\|_C^2\mathrm{d}r\\ &\le L(CM)^2\mathbb{E}\int_0^t\|x_{r^-}^n-x^{n-1}r^-\|_C^2\mathrm{d}r\end{aligned}$$

and

$$\mathbb{E}\sup_{0\le s\le t}|\int_0^s\int_{|u|\ge1}S_\alpha(s-r)[G(r,x^n_{r^-},u)-G(r,x^{n-1}_{r^-},u)]N(\mathrm{d}r,\mathrm{d}u)|^2$$
$$\le 2CM\int_0^t S_\alpha(t-r)\mathrm{d}r\int_{|u|\ge1}\nu(\mathrm{d}u)\mathbb{E}\int_0^t\int_{|u|\ge1}|G(r,x^n_{r^-},u)-G(r,x^{n-1}_{r^-},u)|^2\nu(\mathrm{d}u)\mathrm{d}r$$
$$+2(CM)^2\int_0^t\mathbb{E}\int_{|u|\ge1}|G(r,x^n_{r^-},u)-G(r,x^{n-1}_{r^-},u)|^2\nu(\mathrm{d}u)\mathrm{d}s$$
$$\le(2(CM)^2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}b+2(CM)^2)L\mathbb{E}\int_0^t\|x^n_{r^-}-x^{n-1}r^-\|^2_C\mathrm{d}r,$$

we get

$$\mathbb{E}(\sup_{0\le s\le t}|x^{n+1}(s)-x^n(s)|^2)\le c\int_0^t\mathbb{E}(\sup_{0\le r\le s}|x^n(r)-x^{n-1}(r)|^2)\mathrm{d}r,$$

where $c=\frac{4L(CM)^2}{(1-k_0)^2}(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+4+2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}b)L$. Note that

$$\mathbb{E}\sup_{0\le s\le t}|x^1(s)-x^0(s)|^2\le c_0,$$

where $c_0=5[k_0^2c'+Lc'(1+2b)(CM\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2+4Lc'(CM)^2\frac{|\mu|^{-\frac{1}{2\alpha}}\pi}{2\alpha\sin(\pi/2\alpha)}]$ is a positive number; by induction, we get

$$\mathbb{E}(\sup_{0\le s\le t}|x^{n+1}(s)-x^n(s)|^2)\le c_0\frac{(c_6t)^n}{n!}.\tag{3.4}$$

Taking $t=T$ in (3.4), we have

$$\mathbb{E}(\sup_{0\le t\le T}|x^{n+1}(t)-x^n(t)|^2)\le c_0\frac{(c_6T)^n}{n!}.\tag{3.5}$$

Hence

$$P\{\sup_{0\le t\le T}|x^{n+1}(t)-x^n(t)|>\frac{1}{2^n}\}\le\frac{c_0[c_6T]^n}{n!}.$$

Note that $\sum_{n=0}^\infty\frac{c_0[c_6T]^n}{n!}<\infty$; by using the Borel-Cantelli lemma, we can get a stochastic process $x(t)$ on $[0,T]$ such that $x_n(t)$ uniformly converges to $x(t)$ as $n\to\infty$ almost surely.

It is easy to check that $x(t)$ is a unique mild solution of (1.1). The proof of the theorem is complete. □

4. Existence of S-asymptotically ω-periodic solution in distribution

Lemma 4.1. *If* $x(t)\in SAP_\omega(L^2(P,H)$ *and* $T(t-s)\in\mathcal{L}(\mathbf{R},\mathbf{R})$ *then* $\Gamma_1(t)=\int_0^t T(t-s)x(s)\mathrm{d}s\in SAP_\omega(L^2(P,H))$.

The proof process is similar to that of Lemma 1 in [25], so we omit it.

Lemma 4.2. *If $x(t) \in SAP_\omega(L^2(P,H))$, then*

$$\Gamma_2(t) = \int_0^t S_\alpha(t-s)x(s)\mathrm{d}w(s) \in SAP_\omega(L^2(P,H)).$$

Proof. It is obvious that $\Gamma_2(t)$ is L^2-continuous since

$$\begin{aligned}
\|\Gamma_2(t+\omega) - \Gamma_2(t)\|^2 &= * \quad \|\int_0^{t+\omega} S_\alpha(t+\omega-s)x(s)\mathrm{d}w(s) - \int_0^t S_\alpha(t-s)x(s)\mathrm{d}w(s)\|^2 \\
&= \quad 2\|\int_0^{\omega} S_\alpha(t+\omega-s)x(s)\mathrm{d}w(s)\|^2 \\
&+ \quad 2\|\int_\omega^{t+\omega} S_\alpha(t+\omega-s)x(s)\mathrm{d}w(s) - \int_0^t S_\alpha(t-s)x(s)\mathrm{d}w(s)\|^2 \\
&\leq \quad 2\|\int_0^{\omega} S_\alpha(t+\omega-s)x(s)\mathrm{d}w(s)\|^2 \\
&+ \quad 2\|\int_0^{t} S_\alpha(t-s)(x(s+\omega)-x(s))\mathrm{d}w(s)\|^2.
\end{aligned}$$

Since $x(t) \in SAP_\omega(L^2(P,H))$, for any $\epsilon > 0$, we can choose $T_\epsilon > 0$ such that when $t > T_\epsilon$, $\|x(t+\omega) - x(t)\| < \epsilon$. For the above ϵ, we have

$$\begin{aligned}
2\|\int_0^{t} S_\alpha(t-s)(x(s+\omega)-x(s))\mathrm{d}w(s)\|^2 &\leq 4\int_0^{T_\epsilon} \|S_\alpha(t-s)\|^2\|x(s+\omega)-x(s)\|^2\mathrm{d}s \\
&+ 4\int_{T_\epsilon}^{t} \|S_\alpha(t-s)\|^2\|x(s+\omega)-x(s)\|^2\mathrm{d}s.
\end{aligned}$$

Note that

$$2\|\int_0^{\omega} S_\alpha(t+\omega-s)x(s)\mathrm{d}w(s)\|^2 \leq 2\frac{(CM)^2}{1+|\mu|^2t^{2\alpha}}\int_0^\omega \|x(s)\|^2\mathrm{d}s \to 0,\ t\to\infty,$$

that

$$\int_0^{T_\epsilon} \|S_\alpha(t-s)\|^2\|x(s+\omega)-x(s)\|^2\mathrm{d}s \leq 4\frac{(CM)^2}{1+|\mu|^2(t-T_\epsilon)^{2\alpha}}\|x\|_\infty T_\epsilon \to 0,\ t\to\infty$$

and that

$$\int_{T_\epsilon}^{t} \|S_\alpha(t-s)\|^2\|x(s+\omega)-x(s)\|^2\mathrm{d}s \leq \epsilon^2\frac{(CM)^2|\mu|^{-2/\alpha}\pi}{2\alpha\sin(\pi/2\alpha)};$$

we get $\lim_{t\to\infty}\|\Gamma_2(t+\omega) - \Gamma_2(t)\| = 0$. So $\Gamma_2(t) \in SAP_\omega(L^2(P,H))$. □

The following lemma is made obvious by using Lemmas 2.1 and 2.2 and a similar discussion as that for Lemma 4.2.

Lemma 4.3. *If $x(t) \in SAP_\omega(L^2(P,H))$ and $F : \mathbf{R}^+ \times C \times U \to L^2(P,H)$ is uniformly Poisson square-mean S-asymptotically ω-periodic in t on bounded sets of C, then*

$$\Gamma_3(t) = \int_0^t \int_{|u|_U<1} S_\alpha(t-s)F(s,x_s,u)\tilde{N}(\mathrm{d}u,\mathrm{d}s) \in SAP_\omega(L^2(P,H))$$

and

$$\Gamma_4(t) = \int_0^t \int_{|u|_U\geq 1} S_\alpha(t-s)G(s,x_s,u)N(\mathrm{d}u,\mathrm{d}s) \in SAP_\omega(L^2(P,H)).$$

Theorem 4.1. *Assume that (H1)–(H3) are satisfied and h, f and g are uniformly square-mean S-asymptotically ω-periodic on bounded sets of $\mathcal{C}$. F and G are uniformly Poisson square-mean S-asymptotically ω-periodic on bounded sets of $\mathcal{C}$. Then, (1.1) has a unique S-asymptotically ω-periodic solution in distribution if*

$$5k_0^2 + 5(CM)^2 L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2(1+b) + 20L(CM)^2\frac{|\mu|^{-1/2\alpha}\pi}{2\alpha\sin(\pi/2\alpha)} < 1.$$

Proof. Let us first show the existence of the square-mean S-asymptotically ω-periodic solution of (1.1); so, we consider the operator Φ acting on the Banach space $SAP_\omega(L^2(P,H))$ given by

$$\begin{aligned}\Phi x(t) &= S_\alpha(t)(\phi(0)+h(0,\phi)) - h(t,x_t) + \int_0^t S_\alpha(t-s)f(s,x_s)\mathrm{d}s \\ &+ \int_0^t S_\alpha(t-s)g(s,x_s)\mathrm{d}w(s) + \int_0^t\int_{|u|<1} S_\alpha(t-s)F(s,x_{s^-},u)\tilde{N}(\mathrm{d}s,\mathrm{d}u) \\ &+ \int_0^t\int_{|u|\geq 1} S_\alpha(t-s)G(s,x_{s^-},u)N(\mathrm{d}s,\mathrm{d}u).\end{aligned} \tag{4.1}$$

From a previous assumption one can easily see that $\Phi x(t)$ is well defined and L^2-continuous. Moreover, from Lemma 2.1, Lemma 4.1, Lemma 4.2, and Lemma 4.3 we infer that Φ maps $SAP_\omega(L^2(P,H))$ into itself. Next, we prove that Φ is a strict contraction on $SAP_\omega(L^2(P,H))$. Indeed, for $x,\ \tilde{x} \in SAP_\omega(L^2(P,H))$, we get

$$\begin{aligned}\|\Phi\bar{x}(t) - \Phi\tilde{x}(t)\|^2 &\leq 5k_0^2\|\bar{x}_t - \tilde{x}_t\|_C^2 + 5(CM)^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2\|\bar{x}(t)-\tilde{x}(t)\|_\infty^2 \\ &+ 20L(CM)^2\frac{|\mu|^{-1/2\alpha}\pi}{2\alpha\sin(\pi/2\alpha)}\|\bar{x}(t)-\tilde{x}(t)\|_\infty^2 + 10L(CM)^2(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2 b\|\bar{x}(t)-\tilde{x}(t)\|_\infty^2 \\ &= [5k_0^2 + 5(CM)^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2(1+b) + 20L(CM)^2\frac{|\mu|^{-1/2\alpha}\pi}{2\alpha\sin(\pi/2\alpha)}] \times \|\bar{x}(t)-\tilde{x}(t)\|_\infty^2.\end{aligned}$$

Since $5k_0^2 + 5(CM)^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2(1+b) + 20L(CM)^2\frac{|\mu|^{-1/2\alpha}\pi}{2\alpha\sin(\pi/2\alpha)} < 1$, it follows that Φ is a contraction mapping on $SAP_\omega(L^2(P,H))$. By the classical Banach fixed-point principle, there exists a unique $x \in SAP_\omega(L^2(P,H))$ such that $\Phi x = x$, which is the unique square-mean S-asymptotically ω-periodic solution of (1.1). By Lemma 2.3, we deduce that (1.1) has a unique S-asymptotically ω-periodic solution in distribution. The proof is now complete. □

5. An example

In this section, an example is provided to illustrate the results obtained in previous sections. Let $H = L^2([0,\pi])$ and $w(t)$ be an H-valued Wiener process; given $\tau > 0$, we consider the following initial problem

$$\begin{cases} d[x(t,\xi)-\frac{\sin t}{8}x(t-\tau,\xi)] = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{\partial^2}{\partial\xi^2}[x(s,\xi)-\frac{\sin s}{8}x(t-\tau,\xi)]\mathrm{d}s\mathrm{d}t \\ \qquad + \frac{1}{8}(\sin\ln(t+1)+cost)x(t-\tau,\xi)\mathrm{d}t \\ \qquad + \frac{1}{8}(\sin\ln(t+1)+cost)x(t-\tau,\xi)\mathrm{d}w(t) \\ \qquad + \int_{|u|_U<1}\frac{1}{8}(\cos t+\frac{\ln(t+1)}{t})x(t-\tau,\xi)\tilde{N}(\mathrm{d}t,\mathrm{d}u), \\ x(t,0)=x(t,\pi)=0, \\ x_0(\theta,\xi)=\phi(\theta,\xi)\in C^b_{\mathcal{F}_0}([-\tau,0],H),\theta\in[-\tau,0],\xi\in[0,\pi]. \end{cases} \tag{5.1}$$

The operator $A: H\to H$ by $A=\frac{\partial^2}{\partial x^2}$ and $D(A)=\{z\in H:\ z''\in H,\ z(0)=z(\pi)=0\}$ is the infinitesimal generator of a strongly continuously cosine family [26]. Based on the estimates on the norms of the operators of Theorems 3.3 and 3.4 in [27], the operators $S_q(t)$ in the mild solution of (5.1) satisfy $\|S_q(t)\|_{\mathcal{L}(H,H)}\le 3$, and $h(t,\varphi)=\frac{\sin t}{8}x(t-\tau,\xi)$; obviously, the function h satisfies (H2) and $k_0=\frac{1}{8}$.

$$f(t,\phi)=\frac{1}{8}(\sin\ln(t+1)+cost)\phi(t-\tau), g(t,\phi)=\frac{1}{8}(\sin\ln(t+1)+cost)\phi(t-\tau),$$

$$F(t,\phi,u)=\frac{1}{8}(\cos t+\frac{\ln(t+1)}{t})\phi(t-\tau)$$

satisfy (H3), and L can be chosen as $L=\frac{1}{4}$. According to Theorem 3.1 in Section 3, the Cauchy problem (5.1) has a mild solution . Moreover, if

$$\begin{aligned} &5k_0^2+5(CM)^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2(1+b)+20L(CM)^2\frac{|\mu|^{-1/2\alpha}\pi}{2\alpha\sin(\pi/2\alpha)} \\ &\le \frac{5}{64}+45(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2(1+b)+90\frac{|\mu|^{-1/2\alpha}\pi}{2\alpha\sin(\pi/2\alpha)}<1, \end{aligned}$$

the Cauchy problem (5.1) has a unique S-asymptotically ω-periodic solution in distribution.

6. Conclusions

In this work, inspired by the idea in [13], we established the concept of a Poisson square-mean S-asymptotically ω-periodic solution for (1.1) in order to correspond to the effect of Lévy noise. Furthermore, we made an initial consideration of the S-asymptotically ω-periodic solution in distribution in an abstract space C for (1.1). We established the existence and uniqueness of a mild solution of a class of stochastic fractional differential evolution equations with delay and Piosson jumps. First, the existence and the uniqueness of the mild solution for this type of equation are derived by means of the successive approximation under Lipschitz conditions. We also obtained sufficient conditions for the existence and the uniqueness of the S-asymptotically ω-periodic solution in distribution. To the best of our knowledge, this is the first attempt to discuss this property for these kinds of stochastic fractional functional differential equations.

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Conflict of interest

The authors declare that there is no conflict of interest.

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