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Research article

Generalized tilting modules and Frobenius extensions

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Abstract: Let A/S be a ring extension with S commutative. We prove that $\omega \otimes_S A_A$ is a generalized tilting module if ω_S is a generalized tilting module. In this case, we obtain that ${}^{\perp}\omega$-resol.dim$_S(M)$ and ${}^{\perp}(\omega{\otimes_S} A)$-resol.dim$_A(M)$ are identical for any A-module M. As an application, we show that S satisfies gorenstein symmetric Conjecture if and only if so does A. Furthermore, we introduce the concept of ${}^{\perp}\omega$-Gorenstein projective modules, and we obtain that the relative Gorenstein projectivity is invariant under Frobenius extensions.

Keywords: generalized tilting modules; Frobenius extensions; separable extensions; ${}^{\perp}\omega$-dimensions; ${}^{\perp}\omega$-Gorenstein projective modules

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unital right modules unless stated otherwise. For a ring S, we denote the category of all right S-modules (resp. finitely generated right S-modules) by Mod-S (resp. mod-S). We use $\mathrm{pd}_S(M)$ (resp. $\mathrm{id}_S(M)$) to denote projective dimension (resp. injective dimension) of M_S.

The generalized tilting modules were firstly introduced as a generalization of tilting modules by T. Wakamatsu in [1]. Sometimes, it is also called the Wakamatsu tilting module, see [2].

Definition 1.1. Let S be a ring. An S-module $\omega_S \in$mod-S is called a *generalized tilting module* (it is also called a *Wakamatsu tilting module*) if it is self-orthogonal, that is $\mathrm{Ext}_S^i(\omega_S, \omega_S) = 0$ for any $i \geq 1$, and there is an exact sequence

$$0 \to S_S \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} T_i \to \cdots$$

such that: (1) $T_i \in$addω_S for any $i \geq 0$, where addω_S is the full subcategory of mod-S that consisting of all modules isomorphic to direct summands of finite direct sum of copies of ω_S, and (2) after applying by $\mathrm{Hom}_S(-, \omega_S)$ the sequence is still exact.

About the generalized tilting module, there is a famous homological conjecture in the representation theory of Artin algebras, which is called the Wakamatsu tilting conjecture (WTC). This conjecture states that every generalized tilting module with finite projective dimension is tilting, or equivalently, every generalized tilting module with finite injective dimension is cotilting (see [3]). The homological conjecture is closely related to other homological conjectures. For example, the validity of finitistic dimension conjecture (FDC) implies the validity of WTC, and the validity of WTC implies the validity of the Gorenstein symmetric conjecture (GSC) and the Generalized Nakayama conjecture (GNC) (see [3, 4]). Hence, the generalized tilting modules are studied widely, see [3–6].

The notion of Frobenius extensions was firstly introduced by Kasch in [7] as a generalization of Frobenius algebras. They play an important role in topological quantum field theories in dimension 2 and even 3 (see [8]) and in representation theory and knot theory (see [9–11]). Also, each Frobenius extension with base ring commutative provides us with a series of solutions to classical Yang-Baxter equation (see [10]). The fundamental example of Frobenius extensions is the group algebras induced by a finite index subgroup. There are other examples of Frobenius extensions include Hopf subalgebras, finite extensions of enveloping algebras of Lie super-algebras and finite extensions of enveloping algebras of Lie coloralgebras etc [12, 13].

Separable extensions were firstly defined by Hirata and Sugano in [14] as a generalization of separable algebras, and they made a thorough study of these connection with Galois theory for noncommutative rings and generalizations of Azumaya algebras. If a ring extension is both separable extension and Frobenius extension, then it is called a *separable Frobenius extension*. Sugano proved that the central projective separable extensions are Frobenius extensions in [15]. More examples of separable Frobenius extensions can be found in Example 2.4. We refer to [10] for more details.

It is well-known that many homological properties are preserved under change of rings, especially excellent extension and Frobenius extension (see [16–20]). In this paper, we will consider some homological modules and homological dimension related a generalized tilting module under Frobenius (or separable Frobenius) extensions.

For a generalized tilting module ω_S, we denote the *left orthogonal class* of ω by ${}^{\perp}\omega_S = \{X \in$Mod-$S \,|\, \mathrm{Ext}^i_S(X_S, \omega_S) = 0$, for any $i \geq 1\}$. The ${}^{\perp}\omega$-resolution dimension of a module is defined as follows.

Definition 1.2. Let M be an S-module. The *${}^{\perp}\omega$-resolution dimension* of M, denoted by ${}^{\perp}\omega$-resol.dim$_S(M)$, is defined as ${}^{\perp}\omega$-resol.dim$_S(M) = \inf\{n \mid \exists {}^{\perp}\omega_S$-resolution $0 \to T_n \to \cdots T_1 \to T_0 \to M \to 0\}$. We set ${}^{\perp}\omega$-resol.dim$_S(M) = \infty$ if no such integer exists.

For the homological dimension above, we have the following result.

Theorem A. Let A/S be a Frobenius extension with S commutative. For any A-module M, we have ${}^{\perp}\omega$-resol.dim$_S(M) = {}^{\perp}(\omega \otimes_S A)$-resol.dim$_A(M)$.

As an application, we get the following corollary.

Corollary B. Let S and A be both two-sided Noetherian rings and A/S be a Frobenius extension. Then S is a Gorenstein ring if and only if so is A. Furthermore, S satisfies GSC if and only if so does A.

Relating to a generalized tilting module ω_S, ${}^{\perp}\omega$-Gorenstein projective module is defined as a kind of relative Gorenstein homological module.

Definition 1.3. Let ω be a generalized tilting S-module. A right S-module G is called ${}^{\perp}\omega$-*Gorenstein projective* if there exists an exact sequence

$$\mathbf{P} := \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective right S-modules which is still exact after applying $\mathrm{Hom}_S(-, X)$ for any module $X \in {}^{\perp}\omega_S$ and $G = \mathrm{Ker}(P_0 \to P^0)$. Furthermore, the exact sequence $\mathbf{P}$ is called an ${}^{\perp}\omega$-complete projective resolution of G.

We denoted by ${}^{\perp}\omega$-$\mathcal{GP}(S)$ the full subcategory of Mod-S consisting of all ${}^{\perp}\omega$-Gorenstein projective modules.

For the relative Gorenstein projectivity, we prove it is preserved under Frobenius (or separable Frobenius) extensions.

Theorem C. Let A/S be a Frobenius extension with S commutative and ω a generalized tilting S-module. Then G_S is ${}^{\perp}\omega$-Gorenstein projective if and only if $G \otimes_S A_A$ is ${}^{\perp}(\omega \otimes_S A)$-Gorenstein projective.

Theorem D. Let A/S be a Frobenius extension with S commutative, ω a generalized tilting S-module and G an A-module. If G_A is ${}^{\perp}(\omega \otimes_S A)$-Gorenstein projective, then G_S is ${}^{\perp}\omega$-Gorenstein projective. Furthermore, if the ring extension A/S is also separable, then G_A is ${}^{\perp}(\omega \otimes_S A)$-Gorenstein projective if and only if G_S is ${}^{\perp}\omega$-Gorenstein projective.

The paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we will study the ${}^{\perp}\omega$-resolution dimension under Frobenius extensions; the main result Theorem A is proved, see Theorem 3.5. As a corollary, we show that for a Frobenius extension of two-sided Noetherian rings, the base ring is Gorenstein if and only if so is the extension ring, see Corollary 3.12. In Section 4, ${}^{\perp}\omega$-Gorenstein projectivity and ${}^{\perp}\omega$-Gorenstein projective dimension are studied, the main results Theorem C and Theorem D are proved, see Theorems 4.4 and 4.5, respectively. Furthermore, we get some applications concerning some classical homological dimensions in this section.

2. Preliminaries

A *ring extension* A/S is a ring homomorphism $S \xrightarrow{l} A$. A ring extension is an algebra if S is commutative and l factors $S \to Z(A) \hookrightarrow A$ where $Z(A)$ is the center of A. The natural bimodule structure of ${}_SA_S$ is given by $s \cdot a \cdot s' := l(s) \cdot a \cdot l(s')$. Similarly, we can get some other module structures, for example A_S, ${}_SA_A$ and ${}_AA_S$, etc.

For a ring extension A/S, there is a *restriction functor* R : Mod-A $\to$ Mod-S sending $M_A \mapsto M_S$, given by $m \cdot s := m \cdot l(s)$. In the opposite direction, there are two natural functors as follows:

(1) $T = - \otimes_S A_A$: Mod-S $\to$ Mod-A is given by $M_S \mapsto M \otimes_S A_A$.

(2) $H = \mathrm{Hom}_S({}_AA_S, -)$: Mod-S $\to$ Mod-A is given by $M_S \mapsto \mathrm{Hom}_S({}_AA_S, M_S)$.

It is easy to check that both (T, R) and (R, H) are adjoint pairs.

Definition 2.1. (see reference [10, Theorem 1.2]) A ring extension A/S is a *Frobenius extension*, provided that one of the following equivalent conditions holds:

(1) The functors T and H are naturally equivalent.

(2) ${}_SA_A \cong \mathrm{Hom}_S({}_AA_S, {}_SS_S)$ and A_S is finitely generated projective.

(3) ${}_AA_S \cong \mathrm{Hom}_{S^{\mathrm{op}}}({}_SA_A, {}_SS_S)$ and ${}_SA$ is finitely generated projective.

(4) There exist $E \in \mathrm{Hom}_{S-S}(A, S)$, $x_i, y_i \in A$ such that for $\forall a \in A$, one has $\sum_i x_i E(y_i a) = a$ and $\sum_i E(a x_i) y_i = a$.

Definition 2.2. A ring extension A/S is a *separable extension* if and only if

$$\mu : A \otimes_S A \to A, \quad a \otimes b \mapsto ab,$$

is a split epimorphism of A-A-bimodules. If a ring extension A/S is both Frobenius extension and separable extension, then it is called a *separable Frobenius extension*.

Let A/S be a ring extension and M an A-module. Then M_S is a right S-module. There is a natural surjective map $\pi : M \otimes_S A \to M$ given by $m \otimes a \mapsto ma$ for any $m \in M$ and $a \in A$. It is easy to check that π is split as an S-module homomorphism. However, π is not split as an A-homomorphism in general. The following lemma is a characterization of separable extensions.

Lemma 2.3. (see [21]) *A ring extension A/S is separable if and only if for every module M_A and A-homomorphism $M \to N$, the natural epimorphism $M \otimes_S A_A \to M_A$ is a split A-epimorphism and natural with respect to $M \to N$.*

There are some other examples of Frobenius extensions or separable Frobenius extensions.

Example 2.4. (1) Let S be any ring and $A = S[x]/(x^2)$ be the quotient ring of $S[x]$. Then A/S is a Frobenius extension (see [22, Lemma 3.1]).

(2) Let A be a ring and n a positive integer. Then $M_n(A)$ is a separable Frobenius extension of $S_n(A)$, where $M_n(A)$ is the full $n \times n$ matrix ring over A and $S_n(A)$ is the centrosymmetric matrix ring over A, see [11, Theorem]

(3) Let S be a commutative algebra and A an *Azumaya algebra* over S. Then A/S is a separable Forbenius extension. See [10, Chapter 5] for more details.

(4) Every strongly separable extension is a separable Frobenius extension. Some examples of strongly separable extensions can be found in [9].

(5) Every excellent extension is a Frobenius extension. Furthermore, for an excellent extension, if the base ring is commutative, then it is also a separable extension. More examples of excellent extension can be found in [17, Example 2.2].

For an S-module W, we denote the left orthogonal class of W by ${}^{\perp}W = \{X \in \text{Mod-}S \,|\, \mathrm{Ext}^i_S(X, W) = 0,$ for any $i \geq 1\}$. Recall that a class of modules is called projectively resolving if it contains all projective modules and closed under extensions and kernel of epimorphisms. We have the following lemma.

Lemma 2.5. *The class ${}^{\perp}W$ is projectively resolving and closed under direct sums and summands.*

Remark 2.6. The condition "(2)" in Definition 1.1 can be replaced by "$\mathrm{Coker} f_i \in {}^{\perp}\omega_S$ for any $i \geq 0$" .

By Lemma 2.5, the following result is a direct consequence of [23, Lemma 3.12].

Proposition 2.7. *Let M be an S-module and consider two exact sequences,*

$$0 \to K_n \to T_{n-1} \to \cdots \to T_0 \to M \to 0,$$

$$0 \to K'_n \to T'_{n-1} \to \cdots \to T'_0 \to M \to 0,$$

where $T_0, \cdots, T_{n-1}$ and $T'_0, \cdots, T'_{n-1}$ are in ${}^{\perp}W$. Then K_n is in ${}^{\perp}W$ if and only if so is K'_n.

3. $^\perp\omega$-dimensions under Frobenius extensions

In this section, ω is always a generalized tilting S-module. We prove that if A/S is a Frobenius extension with base ring S commutative, then $\omega \otimes_S A_A$ is also a generalized tilting module over A and $^\perp\omega$-resol.dim$_S(M)$ and $^\perp(\omega \otimes_S A)$-resol.dim$_A(M)$ are identical for any A-module M.

The following two lemmas show that orthogonal classes with respect to a generalized tilting module are preserved under Frobenius extensions.

Lemma 3.1. *Let A/S be a Frobenius extension of rings and X a right A-module. Then $X_A \in {}^\perp(\omega\otimes_S A)_A$ if and only if $X_S \in {}^\perp\omega_S$.*

Proof. Since A/S is a Frobenius extension, we have $T(M_S) = M \otimes_S A_A \cong \mathrm{Hom}_S({}_AA_S, M_S) = H(M_S)$ for any $M_S \in$ Mod-S. By the adjoint isomorphism,

$$\begin{aligned}\mathrm{Ext}^i_S(X_S, \omega_S) &\cong \mathrm{Ext}^i_S(X \otimes_A A_S, \omega_S)\\ &\cong \mathrm{Ext}^i_A(X_A, \mathrm{Hom}_S({}_AA_S, \omega_S))\\ &\cong \mathrm{Ext}^i_A(X_A, \omega \otimes_S A_A)\end{aligned}$$

for any $i \geq 1$. Consequently, $\mathrm{Ext}^i_S(X_S, \omega_S) = 0$ for any $i \geq 1$ if and only if $\mathrm{Ext}^i_A(X_A, \omega \otimes_S A_A) = 0$ for any $i \geq 1$. □

Lemma 3.2. *Let A/S be a Frobenius extension with S commutative. For any S-module X, $X_S \in {}^\perp\omega_S$ if and only if $X \otimes_S A_A \in {}^\perp(\omega \otimes_S A)_A$.*

Proof. By the adjoint isomorphism, for any $i \geq 1$,

$$\begin{aligned}\mathrm{Ext}^i_A(X \otimes_S A, \omega \otimes_S A) &\cong \mathrm{Ext}^i_S(X, \mathrm{Hom}_A({}_SA, \omega \otimes_S A))\\ &\cong \mathrm{Ext}^i_S(X, \omega \otimes_S A).\end{aligned}$$

Since A/S is a Frobenius extension, ${}_SA$ is a finitely generated projective S-module. And so $\omega \otimes_S A_S \in \mathrm{add}\omega_S$. If $X_S \in {}^\perp\omega_S$, then $\mathrm{Ext}^i_S(X, \omega\otimes_S A) = 0$ for any $i \geq 1$. It follows that $\mathrm{Ext}^i_A(X\otimes_S A, \omega\otimes_S A) = 0$ for any $i \geq 1$, that is, $X \otimes_S A_A \in {}^\perp(\omega \otimes_S A)_A$.

Conversely, if $X \otimes_S A_A \in {}^\perp(\omega \otimes_S A)_A$, then $X \otimes_S A_S \in {}^\perp\omega_S$ by Lemma 3.1. Since X_S is a direct summand of $X \otimes_S A_S$, we know that X_S is also in ${}^\perp\omega_S$. □

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is called a *Frobenius functor* if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that both (F, G) and (G, F) are adjoint pairs (see [24]). By Definition 2.1, we know that the functors $- \otimes_S A_A \cong \mathrm{Hom}_S({}_AA_S, -)$ induced by the Frobenius bimodule ${}_AA_S$ are Frobenius functors.

Proposition 3.3. *Let A/S be a Frobenius extension with S commutative. Then the Frobenius bimodule ${}_AA_S$ induces a Frobenius functor from ${}^\perp\omega_S$ to ${}^\perp(\omega \otimes_S A)_A$.*

Proof. By assumption, $T(= - \otimes_S A_A) \cong H(= \mathrm{Hom}_S({}_AA_S, -))$: Mod-S $\to$ Mod-A is a Frobenius functor with the restriction functor R as a left and right adjoint at same time.

By Lemmas 3.1 and 3.2, we get $T\mid_{{}^\perp\omega_S} \subseteq {}^\perp(\omega \otimes_S A)_A$ and $R\mid_{{}^\perp(\omega\otimes_S A)_A} \subseteq {}^\perp\omega_S$, respectively. Hence T is a Frobenius functor from ${}^\perp\omega_S$ to ${}^\perp(\omega \otimes_S A)_A$. □

The following proposition shows that we can get generalized tilting modules over an extension ring from generalized tilting modules over a base ring when the ring extension is Frobenius.

Proposition 3.4. *Let A/S be a Frobenius extension with S commutative. If ω_S is a generalized tilting module, then $\omega \otimes_S A_A$ is also a generalized tilting A-module.*

Proof. Since ω_S is finitely generated as an S-module, we know that $\omega \otimes_S A_A$ is also finitely generated as an A-module. For any $i \geq 1$,

$$\begin{aligned}\mathrm{Ext}^i_A(\omega \otimes_S A_A, \omega \otimes_S A_A) &\cong \mathrm{Ext}^i_S(\omega_S, \mathrm{Hom}_A({}_SA_A, \omega \otimes_S A_A))\\ &\cong \mathrm{Ext}^i_S(\omega_S, \omega \otimes_S A_S).\end{aligned}$$

By assumption, ${}_SA$ is finitely generated projective. Then $\omega \otimes_S A_S \in \mathrm{add}\omega_S$. It follows that $\mathrm{Ext}^i_A(\omega \otimes_S A_A, \omega \otimes_S A_A) = 0$ for $i \geq 1$ from that ω_S is self-orthogonal.

On the other hand, since ω_S is a generalized tilting S-module, there exists an exact sequence

$$\mathbf{T} := 0 \to S_S \to T_0 \to T_1 \to \cdots \to T_i \to \cdots$$

with $T_i \in \mathrm{add}\omega_S$ for any $i \geq 0$ and $\mathrm{Hom}_S(\mathbf{T}, \omega_S)$ is still exact. Applying by the functor $- \otimes_S A_A$, we get the following sequence

$$\mathbf{T} \otimes_S A_A := 0 \to S \otimes_S A_A \cong A_A \to T_0 \otimes_S A_A \to T_1 \otimes_S A_A \to \cdots \to T_i \otimes_S A_A \to \cdots$$

with each $T_i \otimes_S A_A \in \mathrm{add}(\omega \otimes_S A)_A$, which is exact because ${}_SA$ is finitely generated projective. Since $\omega \otimes_S A_S$ is in $\mathrm{add}\omega_S$, $\mathrm{Hom_S}(\mathbf{T}, \omega \otimes_S \mathrm{A_S})$ is exact. Considering the following complex isomorphisms

$$\begin{aligned}\mathrm{Hom}_A(\mathbf{T} \otimes_S A_A, \omega \otimes_S A_A) &\cong \mathrm{Hom}_S(\mathbf{T}, \mathrm{Hom}_A({}_SA_A, \omega \otimes_S A_A))\\ &\cong \mathrm{Hom}_S(\mathbf{T}, \omega \otimes_S A_S),\end{aligned}$$

we know that $\mathrm{Hom}_A(\mathbf{T} \otimes_S A_A, \omega \otimes_S A_A)$ is also exact.

By Definition 1.1, $\omega \otimes_S A_A$ is a generalized tilting A-module. □

Let $\mathcal{X}$ be a subcategory of Mod-S and M an S-module. If there exists an exact sequence $\cdots \to X_n \to \cdots X_1 \to X_0 \to M \to 0$ in Mod-S with $X_i \in \mathcal{X}$ for any $i \geq 0$, then we define the *$\mathcal{X}$-resolution dimension* of M, denoted by $\mathcal{X}$-resol.dim$_S(M)$, as $\mathcal{X}$-resol.dim$_S(M)$=inf$\{n \mid \exists$ an exact sequence $0 \to T_n \to \cdots T_1 \to T_0 \to M \to 0$ with each $T_i \in \mathcal{X}$ for $0 \leq i \leq n$ $\}$. We set $\mathcal{X}$-resol.dim$_S(M) = \infty$ if no such integer exists.

Theorem 3.5. *Let A/S be a Frobenius extension with S commutative. For any A-module M, we have* ${}^{\perp}\omega$−resol.dim$_S(M)$ = ${}^{\perp}(\omega \otimes_S A)$−resol.dim$_A(M)$.

Proof. Without loss of generality, we assume that ${}^{\perp}(\omega \otimes_S A)$-resol.dim$_A(M) = n < \infty$. There is an exact sequence $0 \to T_n \to \cdots T_1 \to T_0 \to M_A \to 0$ with $T_i \in {}^{\perp}(\omega \otimes_S A)$ in Mod-A for $0 \leq i \leq n$. Applying the restriction functor $- \otimes_A A_S$, we get the following exact sequence $0 \to T_n \to \cdots T_1 \to T_0 \to M_S \to 0$ in Mod-S. By Lemma 3.1, $T_i \in {}^{\perp}\omega_S$ as S-modules for $0 \leq i \leq n$. Then ${}^{\perp}\omega$-resol.dim$_S(M) \leq n$.

Conversely, we assume that ${}^{\perp}\omega$-resol.dim$_S(M) = m < \infty$. As an A-module M, there is an exact sequence $0 \to K_m \to G_{m-1} \to \cdots \to G_0 \to M_A \to 0$ in Mod-A with G_i in ${}^{\perp}(\omega \otimes_S A)_A$ for any $0 \leq i \leq m-1$. Applying by restriction functor, we get the following exact sequence $0 \to K_m \to G_{m-1} \to \cdots \to G_0 \to M_S \to 0$ in Mod-S with G_i in ${}^{\perp}\omega_S$ for any $0 \leq i \leq m-1$ by Lemma 3.1. Since ${}^{\perp}\omega$-resol.dim$_S(M) = m$, K_m is also in ${}^{\perp}\omega_S$ as an S-module by Proposition 2.7. It follows from

Lemma 3.1 that K_m is in ${}^{\perp}(\omega \otimes_S A)_A$ as an A-module. Hence ${}^{\perp}(\omega \otimes_S A)$-resol.dim$_A(M) \leq m = {}^{\perp}\omega$-resol.dim$_S(M)$. □

Similar to some classical homological dimensions, we have the following property of ${}^{\perp}\omega$-resolution dimensions.

Proposition 3.6. *If $(M_i)_{i\in I}$ is a family of S-modules, then*

$$ {}^{\perp}\omega\text{-resol.dim}(\bigoplus_{i\in I} M_i) = \sup\{{}^{\perp}\omega\text{-resol.dim}(M_i) \mid i \in I\}. $$

Proof. It is easy to see that ${}^{\perp}\omega$-resol.dim$(\oplus_{i\in I} M_i) \leq \sup\{{}^{\perp}\omega\text{-resol.dim}(M_i) \mid i \in I\}$ since ${}^{\perp}\omega$ is closed under direct sums.

For the converse inequality "≥", it suffices to show that ${}^{\perp}\omega$-resol.dim$(M') \leq {}^{\perp}\omega$-resol.dim$(M)$ for any direct summand M' of M. Without loss of generality, we assume that ${}^{\perp}\omega$-resol.dim$(M) = n$ is finite. We use induction on n.

If $n = 0$, M is in ${}^{\perp}\omega$, then so is M'.

Now, we assume that $n > 0$. We write that $M = M' \oplus M''$ for some S-module M''. Taking exact sequences $0 \to K' \to P' \to M' \to 0$ and $0 \to K'' \to P'' \to M'' \to 0$, where P' and P'' are projective. By the Horseshoe Lemma, we get the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & K' \oplus K'' & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P' & \longrightarrow & P' \oplus P'' & \longrightarrow & P'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with exact columns and split exact rows. It follows from Proposition 2.7 that ${}^{\perp}\omega$-resol.dim$(K' \oplus K'') = n - 1$. Hence the inductive hypothesis yields that ${}^{\perp}\omega$-resol.dim$(K') \leq n - 1$. Therefore, we have ${}^{\perp}\omega$-resol.dim$(M') \leq n$ by Proposition 2.7 again. □

Theorem 3.7. *Let A/S be a Frobenius extension with S commutative. For any S-module M, we have ${}^{\perp}\omega$-resol.dim$_S(M) = {}^{\perp}(\omega \otimes_S A)$-resol.dim$_A(M \otimes_S A)$.*

Proof. Firstly, we claim that ${}^{\perp}(\omega \otimes_S A)$-resol.dim$_A(M \otimes_S A) \leq^{\perp} \omega$-resol.dim$_S(M)$. We can assume that ${}^{\perp}\omega$-resol.dim$_S(M) = n < \infty$.

By Definition 1.2, there is an exact sequence $0 \to T_n \to \cdots T_1 \to T_0 \to M_S \to 0$ with $T_i \in {}^{\perp}\omega_S$ in Mod-S for $0 \leq i \leq n$. By Lemma 3.2, $T_i \otimes_S A_A \in {}^{\perp}(\omega \otimes_S A)_A$ for $0 \leq i \leq n$. Applying by the functor $- \otimes_S A_A$, we get the following sequence

$$ 0 \to T_n \otimes_S A_A \to \cdots \to T_1 \otimes_S A_A \to T_0 \otimes_S A_A \to M_S \otimes_S A_A \to 0 $$

in Mod-A, which is also exact because $_SA$ is finitely generated projective. Then $^{\perp}(\omega \otimes_S A)$-resol.dim$_A(M \otimes_S A) \leq n = {}^{\perp}\omega$-resol.dim$_S(M)$.

Conversely, there is a natural surjective map $\pi : M \otimes_S A \rightarrow M$ given by $\pi(m \otimes a) = ma$ for any $m \in M$ and $a \in A$ which is split as an S-module homomorphism. Therefore, we have that M_S is a direct summand of $M \otimes_S A_S$, and $^{\perp}\omega$-resol.dim$_S(M) \leq {}^{\perp}\omega$-resol.dim$_S(M \otimes_S A)$ by Proposition 3.6. It follows from Theorem 3.5 that $^{\perp}\omega$-resol.dim$_S(M) \leq {}^{\perp}\omega$-resol.dim$_S(M \otimes_S A) \leq {}^{\perp}(\omega \otimes_S A)$-resol.dim$_A(M \otimes_S A)$.

Therefore, we get $^{\perp}\omega$-resol.dim$_S(M) = {}^{\perp}(\omega \otimes_S A)$-resol.dim$_A(M \otimes_S A)$. □

We define r.Global$^{\perp}\omega$-resol.dim$(S) = \sup\{{}^{\perp}\omega$-resol.dim$(M) \mid M$ is any right S-module $\}$, and call it right global $^{\perp}\omega$-resolution dimension of S.

Proposition 3.8. *Let A/S be a Frobenius extension with S commutative. Then* r.Global$^{\perp}\omega$-resol.dim$(S) = r$.Global$^{\perp}(\omega \otimes_S A)$-resol.dim$(A)$.

Proof. It follows from Theorem 3.5 that r.Global$^{\perp}(\omega \otimes_S A)$-resol.dim$(A) \leq r$.Global$^{\perp}\omega$-resol.dim$(S)$. And Theorem 3.7 shows that r.Global$^{\perp}\omega$-resol.dim$(S) \leq r$.Global$^{\perp}(\omega \otimes_S A)$-resol.dim$(A)$. □

Lemma 3.9. (see *[4, Propositon 3.1]*) *Let ω_S be a generalized tilting S-module and n a non-negative integer. Then* r.id$_S(\omega) \leq$ n *if and only if* $^{\perp}\omega$-resol.dim$_S(M) \leq n$ *for any module M in* mod-S.

Theorem 3.10. *Let A/S be a Frobenius extension with S commutative. Then* r.id$_S(\omega) = r$.id$_A(\omega \otimes_S A_A)$.

Proof. Using the fact that the functors R, T and H preserve the finiteness of modules, we get the assertion by Proposition 3.8 and Lemma 3.9. □

Corollary 3.11. *Let A/S be a Frobenius extension of rings. Then* r.id$_S(S) = r$.id$_A(A)$.

Proof. Put $\omega_S = S_S$, and the corollary follows from the Theorem 3.10. □

Recall that a two sided Noetherian ring S is called a Gorenstein ring if l.id$_S(S)$ and r.id$_S(S)$ are finite. A famous homological conjecture is called Gorenstein symmetric conjecture (GSC), which states that the left injective and right injective dimensions of a two sided Noetherian ring are identical. It is well-known that l.id$_S(S)$ and r.id$_S(S)$ are identical provided that both of them are finite (see [25]). By the corollary above, we have the following valuable corollary.

Corollary 3.12. *Let S and A be two-sided Noetherian rings and A/S be a Frobenius extension. Then S is a Gorenstein ring if and only if so is A. Furthermore, S satisfies* GSC *if and only if so does A.*

Proof. By the"symmetry" of Frobenius extension and Corollary 3.11 , we get l.id$_S(S) = l$.id$_A(A)$. Thus, r.id$_S(S) = r$.id$_A(A)$ and l.id$_S(S) = l$.id$_A(A)$ if A/S is a Frobenius extension. □

Example 3.13. (1) Let A be a two-sided Noetherian ring and $M_n(A)$ the $n \times n$ matrix ring over A. Then $M_n(A)$ is an excellent extension of A. It follows that A is Gorenstein if and only if so is $M_n(A)$ from Corollary 3.12.

(2) Let A be an Artin ring and $Q = A[x]/(x^2)$ is the quotient of the polynomial ring, where x is a variable which is supposed to commute with all the elements of A. Then A is Gorenstein if and only if so is Q.

(3) Let A be a central separable Artin algebra over center C. Then A is a strong separable extension of C. By Corollary 3.12, A is Gorenstein if and only if so is C.

4. $^{\perp}\omega$-Gorenstein projective modules

In this section, we will consider the $^{\perp}\omega$-Gorenstein projectivity and $^{\perp}\omega$-Gorenstein projective dimension under Frobenius extension, where ω_S is a generalized tilting S-module. Furthermore, some corollaries related to classical homological dimensions are obtained.

For the $^{\perp}\omega$-Gorenstein projective modules (see Definition 1.3), we have the following facts.

Remark 4.1. (1) The $^{\perp}\omega$-Gorenstein projective module is a special case of $\mathfrak{X}$-Gorenstein projective module for $\mathfrak{X} = {}^{\perp}\omega$. (See [26, Definition 2.1]).

(2) Every projective module is $^{\perp}\omega$-Gorenstein projective. And every $^{\perp}\omega$-Gorenstein projective module is Gorenstein projective.

(3) The class $^{\perp}\omega$-$\mathcal{GP}(S)$ is projectively resolving and closed under direct summands and direct sums.(See [26, Proposition 2.6]).

The following assertion is a direct consequence of [23, Lemma 3.12].

Proposition 4.2. *Let M be an S-module. Suppose that*

$$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$$

and

$$0 \to K'_n \to G'_{n-1} \to \cdots \to G'_0 \to M \to 0$$

are two exact sequences, where $G_0, \cdots, G_{n-1}$ and $G'_0, \cdots, G'_{n-1}$ are in $^{\perp}\omega$-$\mathcal{GP}(S)$. Then K_n is in $^{\perp}\omega$-$\mathcal{GP}(S)$ if and only if so is K'_n.

It is easy to get the following equivalent condition of $^{\perp}\omega$-Gorenstein projectivity by Definition 1.3.

Proposition 4.3. (see *[26, Propostion 2.4]) Let G be a right S-module. Then the followings are equivalent.*

(1) *G is $^{\perp}\omega$-Gorenstein projective.*

(2) i) $\mathrm{Ext}^i_S(G, X) = 0$ *for any $X \in {}^{\perp}\omega$ and $i > 0$;*

ii) *There exists an exact sequence* $\mathbf{Q} := 0 \to G \to P^0 \to P^1 \to \cdots$ *in* Mod-S *with P^i projective for every $i \geq 0$ such that* $\mathrm{Hom}(\mathbf{Q}, X)$ *is still exact for any $X \in {}^{\perp}\omega$.*

(3) *There exists a short exact sequence of S-modules* $0 \to G \to P \to G' \to 0$, *where P is projective and G' is $^{\perp}\omega$-Gorenstein projective.*

The following results show that $^{\perp}\omega$-Gorenstein projectivity is preserved under Frobenius extensions.

Theorem 4.4. *Let A/S be a Frobenius extension with S commutative. For an S-module G, G is $^{\perp}\omega$-Gorenstein projective if and only if $G \otimes_S A_A$ is $^{\perp}(\omega \otimes_S A)$-Gorenstein projective.*

Proof.(⇒): It suffices to show $G \otimes_S A_A$ satisfying the condition (2) in the Proposition 4.3 when G_S is an $^{\perp}\omega$-Gorenstein projective module. For any $X \in {}^{\perp}(\omega \otimes_S A)_A$, it follows from Lemma 3.1 that $X_S \in {}^{\perp}\omega_S$. Then, for any $i \geq 1$,

$$\begin{aligned}\mathrm{Ext}^i_A(G \otimes_S A_A, X_A) &\cong \mathrm{Ext}^i_S(G_S, \mathrm{Hom}_A({}_SA_A, X_A)) \\ &\cong \mathrm{Ext}^i_S(G_S, X_S) \\ &= 0.\end{aligned}$$

Since G_S is ${}^{\perp}\omega$-Gorenstein projective, there is an exact sequence $\mathbf{Q} := 0 \to G_S \to P^0 \to P^1 \to \cdots$ in Mod-S with P^i projective for any $i \geq 0$ and $\mathrm{Hom}_S(\mathbf{Q}, X)$ is still exact for any $X \in {}^{\perp}\omega_S$. By the assumption, ${}_SA$ is finitely generated projective, and we get the following sequence

$$\mathbf{Q} \otimes_S A_A := 0 \to G \otimes_S A_A \to P^0 \otimes_S A_A \to P^1 \otimes_S A_A \to \cdots$$

is still exact with $P^i \otimes_S A_A$ projective in Mod-A for any $i \geq 0$. For any $X_A \in {}^{\perp}(\omega \otimes_S A)_A$, then $X_S \in {}^{\perp}\omega_S$. Thus

$$\begin{aligned}\mathrm{Hom}_A(\mathbf{Q} \otimes_S A_A, X_A) &\cong \mathrm{Hom}_S(\mathbf{Q}, \mathrm{Hom}_A({}_SA_A, X_A))\\ &\cong \mathrm{Hom}_S(\mathbf{Q}, X_S)\end{aligned}$$

is exact. Therefore, $G \otimes_S A_A$ is ${}^{\perp}(\omega \otimes_S A)$-Gorenstein projective by Proposition 4.3.

($\Leftarrow$): We claim that $G \otimes_S A_S$ is ${}^{\perp}\omega$-Gorenstein projective when $G \otimes_S A_A$ is ${}^{\perp}(\omega \otimes_S A)_A$-Gorenstein projective. For any $Y_S \in {}^{\perp}\omega_S$, $Y \otimes_S A_A \in {}^{\perp}(\omega \otimes_S A)_A$ by Lemma 3.2. Then, for any $i \geq 1$,

$$\begin{aligned}\mathrm{Ext}^i_S(G \otimes_S A_S, Y_S) &\cong \mathrm{Ext}^i_S(G \otimes_S A \otimes_A A_S, Y_S)\\ &\cong \mathrm{Ext}^i_A(G \otimes_S A_A, \mathrm{Hom}_S({}_AA_S, Y_S))\\ &\cong \mathrm{Ext}^i_A(G \otimes_S A_A, Y \otimes_S A_A)\\ &= 0\end{aligned}$$

because $G \otimes_S A_A$ is ${}^{\perp}(\omega \otimes_S A)_A$-Gorenstein projective.

By assumption, there is an exact sequence $\mathbf{P} := 0 \to G \otimes_S A_A \to P^0 \to P^1 \to \cdots$ in Mod-A with P^i projective for any $i \geq 0$ and $\mathrm{Hom}_A(\mathbf{P}, X)$ is still exact for any $X \in {}^{\perp}(\omega \otimes_S A)_A$. After applying the restriction functor $R = - \otimes_A A_S$, we get the following exact sequence $\mathbf{P} := 0 \to G \otimes_S A_S \to P^0 \to P^1 \to \cdots$ with P^i projective in Mod-S for any $i \geq 0$. And, for any $Y_S \in {}^{\perp}\omega$, the complex

$$\begin{aligned}\mathrm{Hom}_S(\mathbf{P}_S, Y_S) &\cong \mathrm{Hom}_S(\mathbf{P} \otimes_A A_S, Y_S)\\ &\cong \mathrm{Hom}_A(\mathbf{P}, \mathrm{Hom}_S({}_AA_S, Y_S))\\ &\cong \mathrm{Hom}_A(\mathbf{P}, Y \otimes_S A_A)\end{aligned}$$

is exact because $Y \otimes_S A_A \in^{\perp} (\omega \otimes_S A)_A$. Thus $G \otimes_S A_S$ is ${}^{\perp}\omega$-Gorenstein projective.

It is well-known that G_S is a direct summand of $G \otimes_S A_S$ and the class ${}^{\perp}\omega_S$-$\mathcal{GP}(S)$ is closed under direct summands. Therefore, G_S is ${}^{\perp}\omega$-Gorenstein projective. □

Theorem 4.5. *Let A/S be a Frobenius extension with S commutative. For any A-module G, if G_A is ${}^{\perp}(\omega \otimes_S A)$-Gorenstein projective, then G_S is ${}^{\perp}\omega$-Gorenstein projective. Furthermore, if the ring extension A/S is also separable, then G_A is ${}^{\perp}(\omega \otimes_S A)$-Gorenstein projective if and only if G_S is ${}^{\perp}\omega$-Gorenstein projective.*

Proof. The first assertion follows from the proof of sufficiency of Theorem 4.4.

If the ring extension A/S is also separable, then G_A is a direct summand of $G \otimes_S A_A$ by Lemma 2.3. By Theorem 4.4, $G \otimes_S A_A$ is ${}^{\perp}(\omega \otimes_S A)_A$-Gorenstein projective if G_S is ${}^{\perp}\omega_S$-Gorenstein projective. And we have that G_A is ${}^{\perp}(\omega \otimes_S A)_A$-Gorenstein projective since the class ${}^{\perp}(\omega \otimes_S A)_A$-$\mathcal{GP}(A)$ is closed under direct summands. □

By Theorems 4.4 and 4.5, similar to the proof of Proposition 3.3, we have

Corollary 4.6. *Let A/S be a Frobenius extension with S commutative. Then the Frobenius bimodule ${}_AA_S$ induces a Frobenius functor from ${}^\perp\omega$-$\mathcal{GP}(S)$ to ${}^\perp(\omega\otimes_S A)$-$\mathcal{GP}(A)$.*

Similar to the classical homological dimensions, we define the ${}^\perp\omega$-Gorenstein projective dimension of modules and the global ${}^\perp\omega$-Gorenstein projective dimension of rings as follows.

Definition 4.7. Let M be an S-module. The ${}^\perp\omega$-Gorenstein projective dimension of M, denoted by ${}^\perp\omega$-$\mathrm{Gpd}_S(M)$, is defined as ${}^\perp\omega$-$\mathrm{Gpd}_S(M) = \inf\{n \mid \exists {}^\perp\omega$-Gorenstein projective resolution $0 \to G_n \to \cdots G_1 \to G_0 \to M \to 0\}$. We set ${}^\perp\omega$-$\mathrm{Gpd}_S(M) = \infty$ if no such integer exists.

We define $r.{}^\perp\omega$-Ggldim$(S) = \sup\{{}^\perp\omega$-$\mathrm{Gpd}_S(M) \mid M$ is any right S-module $\}$, and call it right global ${}^\perp\omega$-Gorenstein projective dimension of S.

Proposition 4.8. *Let A/S be a Frobenius extension with S commutative. For each S-module M, we have ${}^\perp\omega$-$\mathrm{Gpd}_S(M) = {}^\perp(\omega\otimes_S A)$-$\mathrm{Gpd}_A(M\otimes_S A)$.*

Proof. The proof is similar to that of Theorem 3.7, for the sake of completeness, we give the proof as follows.

Assume that ${}^\perp\omega$-$\mathrm{Gpd}_S(M) = n < \infty$, there is an exact sequence $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ in Mod-S with G_i being ${}^\perp\omega$-Gorenstein projective for $0 \le i \le n$. Applying by the functor $T = -\otimes_S A_A$, we get the following exact sequence

$$0 \to G_n\otimes_S A_A \to G_{n-1}\otimes_S A_A \to \cdots \to G_0\otimes_S A_A \to M\otimes_S A_A \to 0$$

in Mod-A with $G_i\otimes_S A_A$ being ${}^\perp(\omega\otimes_S A)$-Gorenstein projective for $0 \le i \le n$ by Theorem 4.4. Then ${}^\perp(\omega\otimes_S A)$-$\mathrm{Gpd}_A(M\otimes_S A) \le n =^\perp \omega$-$\mathrm{Gpd}_S(M)$.

Conversely, we can assume that ${}^\perp(\omega\otimes_S A)$-$\mathrm{Gpd}_A(M\otimes_S A) = m < \infty$. As an S-module M, there is an exact sequence $0 \to K_m \to G_{m-1} \to \cdots \to G_0 \to M \to 0$ in Mod-S with G_i projective for $0 \le i \le m-1$. Since ${}_SA$ is Projective as an S-module, applying the functor $T = -\otimes_S A_A$, we obtain the following exact sequence

$$0 \to K_m\otimes_S A_A \to G_{m-1}\otimes_S A_A \to \cdots \to G_0\otimes_S A_A \to M\otimes_S A_A \to 0,$$

where $G_i\otimes_S A_A$ is ${}^\perp(\omega\otimes_S A)$-Gorenstein projective by Theorem 4.4 for $0 \le i \le m-1$. Thus $K_m\otimes_S A_A$ is also ${}^\perp(\omega\otimes_S A)$-Gorenstein projective by Proposition 4.2. Again by Theorem 4.4, K_m is an ${}^\perp\omega$-Gorenstein projective S-module. Thus ${}^\perp\omega$-$\mathrm{Gpd}_S(M) \le m = {}^\perp(\omega\otimes_S A)$-$\mathrm{Gpd}_A(M\otimes_S A)$.

Therefore, ${}^\perp\omega$-$\mathrm{Gpd}_S(M) = {}^\perp(\omega\otimes_S A)$-$\mathrm{Gpd}_A(M\otimes_S A)$. □

Corollary 4.9. *Let S be a commutative Artin ring and A/S a Frobenius extension. For each S-module M_S, we have* $\mathrm{pd}_S(M) = \mathrm{pd}_A(M\otimes_S A)$.

Proof. Since S is a commutative Artin ring, there exists some generalized tilting module ω_S with $\mathrm{id}_S(\omega) = 0$ (in fact, an injective cogenerator is such generalized tilting module). For any M_A ∈Mod-A and $i \ge 1$,

$$\begin{aligned}\mathrm{Ext}^i_A(M_A, \omega\otimes_S A_A) &\cong \mathrm{Ext}^i_A(M_A, \mathrm{Hom}_S({}_AA_S, \omega_S))\\ &\cong \mathrm{Ext}^i_S(M\otimes_A A_S, \omega_S)\\ &\cong \mathrm{Ext}^i_A(M_S, \omega_S)\end{aligned}$$

$$= 0$$

Hence $\omega \otimes_S A_A$ is also a generalized tilting A-module with $\mathrm{id}_A(\omega \otimes_S A_A) = 0$. In this case, the ${}^{\perp}\omega$-Gorenstein projective S-module is same to projecitive S-module and the ${}^{\perp}\omega$-Gorenstein projective dimension coincides with the classical projective dimension. The assertion follows from Proposition 4.8 . □

Proposition 4.10. *Let A/S be a Frobenius extension with S commutative and ω_S a generalized tilting S-module and M a right A-module. Then* ${}^{\perp}\omega\text{-}\mathrm{Gpd}_S(M) \leq {}^{\perp}(\omega \otimes_S A)\text{-}\mathrm{Gpd}_A(M)$*. Furthermore, if the ring extension A/S is also separable, then* ${}^{\perp}\omega\text{-}\mathrm{Gpd}_S(M) = {}^{\perp}(\omega \otimes_S A)\text{-}\mathrm{Gpd}_A(M)$.

Proof. It is trivial for the case of ${}^{\perp}(\omega \otimes_S A)\text{-}\mathrm{Gpd}_A(M) = \infty$. We assume that ${}^{\perp}(\omega \otimes_S A)\text{-}\mathrm{Gpd}_A(M) = n < \infty$, there is an exact sequence $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M_A \to 0$ in Mod-A with G_i being ${}^{\perp}(\omega \otimes_S A)_A$-Gorenstein projective for $0 \leq i \leq n$. Applying the restriction functor $R = - \otimes_A A_S$, we have an exact sequence $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M_S \to 0$ in Mod-S with G_i being ${}^{\perp}\omega$-Gorenstein projective for $0 \leq i \leq n$ by Theorem 4.5. Therefore, ${}^{\perp}\omega\text{-}\mathrm{Gpd}_S(M) \leq {}^{\perp}\omega \otimes_S A\text{-}\mathrm{Gpd}_A(M)$.

Conversely, we can assume that ${}^{\perp}\omega\text{-}\mathrm{Gpd}_S(M) = m$. There is an exact sequence $0 \to G_m \to G_{m-1} \to \cdots \to G_0 \to M \to 0$ in Mod-S with G_i being ${}^{\perp}\omega_S$-Gorenstein projective for $0 \leq i \leq n$. By Theorem 4.4, the following sequence

$$0 \to G_m \otimes_S A_A \to G_{m-1} \otimes_S A_A \to \cdots \to G_0 \otimes_S A_A \to M \otimes_S A_A \to 0,$$

in Mod-A is exact with $G_i \otimes_S A_A$ being ${}^{\perp}(\omega \otimes_S A)$-Gorenstein projective. And so ${}^{\perp}(\omega \otimes_S A)\text{-}\mathrm{Gpd}_A(M \otimes_S A) \leq m =^{\perp} \omega\text{-}\mathrm{Gpd}_S(M)$.

If the ring extension A/S is separable, then M_A is a direct summand of $M \otimes_S A_A$. It follows from [26, Proposition 3.4] that ${}^{\perp}\omega \otimes_S A\text{-}\mathrm{Gpd}_A(M) \leq {}^{\perp}(\omega \otimes_S A)\text{-}\mathrm{Gpd}_A(M \otimes_S A) \leq m$. □

The following result maybe is well-known. In fact, we have known that : for a Frobenius extension A/S and an A-module M, if $\mathrm{pd}_A(M) < \infty$, then one has $\mathrm{pd}_A(M) = \mathrm{pd}_S(M)$, see [27, Theorem 8].

Corollary 4.11. *Let S be a commutative Artin ring and A/S be a Frobenius extension. For each right A-module M,* $\mathrm{pd}_S(\mathrm{M}) \leq \mathrm{pd}_A(\mathrm{M})$*. Furthermore, if the ring extension A/S is also separable, then* $\mathrm{pd}_S(\mathrm{M}) = \mathrm{pd}_A(\mathrm{M})$.

Proof. The proof is similar to that of the Corollary 4.9. □

Corollary 4.12. *Let A/S be a Frobenius extension with S commutative and ω_S a generalized tilting S-module. Then* $r.{}^{\perp}\omega\text{-}\mathrm{Ggldim}(S) \leq r.{}^{\perp}(\omega \otimes_S A)\text{-}\mathrm{Ggldim}(A)$*. Furthermore, if the ring extension A/S is also separable, then* $r.{}^{\perp}\omega\text{-}\mathrm{Ggldim}(S) = r.{}^{\perp}\omega \otimes_S A\text{-}\mathrm{Ggldim}(A)$.

Proof. The first assertion follows from Proposition 4.8. Furthermore, if A/S is separable, M_A is a direct summand of $M \otimes_S A_A$ for any $M \in$Mod-A by Lemma 2.3. And the second assertion follows from Proposition 4.10. □

Corollary 4.13. *Let S be a commutative Artin ring and A/S a Frobenius extension. Then* $\mathrm{gldim}(S) \leq \mathrm{gldim}(A)$*. Furthermore, if the ring extension A/S is also separable, then* $\mathrm{gldim}(S) = \mathrm{gldim}(A)$.

Proof. It follows from Corollarys 4.9 and 4.11. □

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Conflict of interest

This work does not have any conflicts of interest.

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