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Research article

Existence and multiplicity of nontrivial solutions to discrete elliptic Dirichlet problems

Yuhua Long[1,2,*] **and Huan Zhang**[1,2]

[1] School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China

[2] Guangzhou University, Guangzhou Center for Applied Mathematics, Guangzhou 510006, China

* **Correspondence:** Email: sxlongyuhua@gzhu.edu.cn.

Abstract: In this paper, we study discrete elliptic Dirichlet problems. Applying a variational technique together with Morse theory, we establish several results on the existence and multiplicity of nontrivial solutions. Finally, two examples and numerical simulations are provided to illustrate our theoretical results.

Keywords: discrete elliptic Dirichlet problem; local linking; Morse theory; nontrivial solution

1. Introduction

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. For integers a, b, define $\mathbb{Z}(a,b) := \{a, a+1, ..., b\}$ with $a \leq b$. Given the integers T_1, $T_2 \geq 2$, we write $\Omega := \mathbb{Z}(1,T_1) \times \mathbb{Z}(1,T_2)$. Consider the existence and multiplicity of nontrivial solutions to the following discrete elliptic problem:

$$\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) + f((i,j),u(i,j)) = 0, \qquad (i,j) \in \Omega, \tag{1.1}$$

with Dirichlet boundary conditions

$$u(i,0) = u(i,T_2+1) = 0 \quad i \in \mathbb{Z}(1,T_1), \qquad u(0,j) = u(T_1+1,j) = 0 \quad j \in \mathbb{Z}(1,T_2), \tag{1.2}$$

where Δ_1, Δ_2 are the forward difference operators defined by $\Delta_1 u(i,j) = u(i+1,j) - u(i,j)$, $\Delta_2 u(i,j) = u(i,j+1) - u(i,j)$, and $\Delta^2 u(i,j) = \Delta(\Delta u(i,j))$. Here, $f((i,j),u)$ is continuously differentiable with respect to u and $f((i,j),0) = 0$.

Advances in modern computing devices have made it increasingly convenient to determine the behavior of complex systems through simulations, contributing greatly to the increasing interest in discrete problems. As a result [1–7], difference equations have been widely investigated and numerous results have been obtained [8–16]. With the development of science and technology in modern society and the progress of mathematical research, the study of difference equations has gradually shifted to the study of partial difference equations. For example, [17–19] deal with discrete Kirchhoff-type problems, whereas [20–22] present several results on multiple solutions to partial difference equations.

Equation (1.1) is a partial difference equation involving bivariate sequences with two independent integer variables over Ω. It is elementary, but illustrative of many problems that are of interest in various branches of science, such as chemical reactions, population dynamics with spatial migration, and even the computation and analysis of finite difference equations [23, 24]. Therefore, Eq (1.1) has attracted considerable attention. For example, [24] shows that Eq (1.1) possesses at least two nontrivial solutions, while Zhang [25] studied Eq (1.1) using the extremum principle. Moreover, Eq (1.1) is regarded as a discrete analog of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, u(x, y)) = 0, \qquad (x, y) \in \Omega,$$

which has been extensively studied. Consequently, investigating problem (1.1)–(1.2) is of practical significance.

Morse theory is a very powerful tool for studying the existence of multiple solutions to both differential and difference equations having a variational structure [26–30]. Very recently, Long [18–20] studied partial difference equations via Morse theory and obtained rich results on the existence and multiplicity of nontrivial solutions. This encourages us to consider the existence and multiplicity of nontrivial solutions for problem (1.1)–(1.2) using Morse theory.

The remainder of this paper is organized as follows. In Section 2, the variational structure and the corresponding functional are established according to (1.1)–(1.2). Moreover, we recall some related definitions and propositions that are beneficial to our results. Section 3 displays our main results and the corresponding proofs. Finally, two examples and numerical simulations are provided to illustrate our main results in Section 4.

2. Variational structure and some auxiliary results

Let E be a T_1T_2-dimensional Euclidean space equipped with the usual inner product $(\cdot, \cdot)$ and norm $|\cdot|$. Let

$$\begin{aligned} S =&\{u : \mathbb{Z}(0, T_1 + 1) \times \mathbb{Z}(0, T_2 + 1) \to \mathbb{R} \quad \text{such that} \quad u(i, 0) = u(i, T_2 + 1) = 0, \\ & i \in \mathbb{Z}(0, T_1 + 1) \quad \text{and} \quad u(0, j) = u(T_1 + 1, j) = 0, \quad j \in \mathbb{Z}(0, T_2 + 1)\}. \end{aligned}$$

Define the inner product $\langle \cdot, \cdot \rangle$ on S as

$$\langle u, v\rangle = \sum_{i=1}^{T_1+1} \sum_{j=1}^{T_2} \triangle_1 u(i-1, j) \triangle_1 v(i-1, j) + \sum_{i=1}^{T_1} \sum_{j=1}^{T_2+1} \triangle_2 u(i, j-1) \triangle_2 v(i, j-1), \qquad \forall u, v \in S,$$

and let the induced norm be

$$\|u\| = \sqrt{\langle u, u\rangle} = \left(\sum_{i=1}^{T_1+1} \sum_{j=1}^{T_2} |\triangle_1 u(i-1, j)|^2 + \sum_{i=1}^{T_1} \sum_{j=1}^{T_2+1} |\triangle_2 u(i, j-1)|^2 \right)^{\frac{1}{2}}, \quad \forall u \in S.$$

Then, S is a Hilbert space and is isomorphic to E. Here and hereafter, we take $u \in S$ as an extension of $u \in E$ when necessary.

Consider the functional $I : S \to \mathbb{R}$ in the following form:

$$\begin{aligned} I(u) =&\frac{1}{2}\sum_{i=1}^{T_1+1}\sum_{j=1}^{T_2}|\Delta_1 u(i-1,j)|^2 + \frac{1}{2}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1}|\Delta_2 u(i,j-1)|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i,j),u(i,j)) \\ =&\frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i,j),u(i,j)), \qquad \forall u \in S, \end{aligned} \tag{2.1}$$

where $F((i,j),u) = \int_0^u f((i,j),\tau)d\tau$. Note that $f((i,j),u)$ is continuously differentiable with respect to u. It is clear that $I \in C^2(S,\mathbb{R})$ and solutions of the problem (1.1)–(1.2) are precisely the critical points of $I(u)$. Moreover, for any $u, v \in S$, using the Dirichlet boundary conditions gives

$$\begin{aligned} \langle I'(u), v\rangle =& \sum_{j=1}^{T_2}\sum_{i=1}^{T_1+1}(\Delta_1 u(i-1,j)\cdot\Delta_1 v(i-1,j)) + \sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1}(\Delta_2 u(i,j-1)\cdot\Delta_2 v(i-1,j)) \\ &- \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}(f((i,j),u(i,j))\cdot v(i,j)) \\ =& -\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\{\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) + f((i,j),u(i,j))\}v(i,j). \end{aligned} \tag{2.2}$$

Let Ξ be the discrete Laplacian, which is defined by $\Xi u(i,j) = \Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1)$. According to the conclusion of [31], we know that $-\Xi$ is invertible and the distinct Dirichlet eigenvalues of $-\Xi$ on $[1,T_1]\times[1,T_2]$ can be denoted by $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_{T_1T_2}$. Let $\phi_k = (\phi_k(1), \phi_k(2), \cdots, \phi_k(T_1T_2))^{tr}$, $k \in [1, T_1T_2]$, where each ϕ_k is an eigenvector corresponding to the eigenvalue λ_k. Let $W^- = \text{span}\{\phi_1, ..., \phi_{k-1}\}$, $W^0 = \text{span}\{\phi_k\}$, $W^+ = (W^- \oplus W^0)^\perp$. Then, S can be expressed as

$$S = W^- \oplus W^+ \oplus W^0.$$

For later use, we define another norm as $\|u\|_2 = \left(\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|u(i,j)|^2\right)^{\frac{1}{2}}$. Then, for any $u \in S$, we have that

$$\lambda_1\|u\|_2^2 \le \|u\|^2 \le \lambda_{T_1T_2}\|u\|_2^2.$$

Next, we recall some preliminaries with respect to Morse theory.

We say that the functional I satisfies the Cerami condition ((C) for short) if any sequence $\{u_n\} \subseteq S$ satisfying $I(u_n) \to c$, $(1 + \|u_n\|)\|I'(u_n)\| \to 0$ as $n \to \infty$ has a convergent subsequence. Note that if (C) is satisfied, then the deformation condition ((D) for short) is satisfied [32].

Definition 2.1. *[28, 33] Let u_0 be an isolated critical group of I with $I(u_0) = c \in \mathbb{R}$, and U be a neighborhood of u_0. The group*

$$C_q(I, u_0) := H_q(I^c \cap U, I^c \cap U\backslash u_0), \quad q \in \mathbb{Z}$$

is called the q-th critical group of I at u_0. Let $\kappa = \{u \in S | I'(u) = 0\}$. For all $a \in \mathbb{R}$, each critical point of I is greater than a and $I \in C^2(S, \mathbb{R})$ satisfies (D). Then, the group

$$C_q(I, \infty) := H_q(S, I^a), \quad q \in \mathbb{Z}$$

is called the q-th critical group of I at infinity.

To calculate the critical group at infinity, we need the following auxiliary proposition.

Proposition 2.1. *[34] Suppose that S is a Hilbert space, $\{I_t \in C^2(S, \mathbb{R}) | t \in [0, 1]\}$. I_t' and $\partial_t I_t$ are locally continuous. If I_0 and I_1 satisfy (C), and there exist $a \in \mathbb{R}$ and $\delta > 0$ such that*

$$I_t(u) \leq a \Rightarrow (1 + \|u\|)\|I_t(u)\| \geq \delta, \qquad t \in [0, 1],$$

then

$$C_q(I_0, \infty) = C_q(I_1, \infty), \qquad q \in \mathbb{Z}. \tag{2.3}$$

In particular, if there is some $R > 0$ such that

$$\inf_{t \in [0,1], \|u\| > R} (1 + \|u\|)\|I_t'(u)\| > 0 \tag{2.4}$$

and

$$\inf_{t \in [0,1], \|u\| \leq R} (1 + \|u\|)\|I_t'(u)\| > -\infty, \tag{2.5}$$

then Eq (2.3) is satisfied.

The following three propositions are important in obtaining some nonzero critical points.

Proposition 2.2. *[26] Let S be a real Hilbert space, $I \in C^2(S, \mathbb{R})$. Suppose that u_0 is the isolated critical point of I with limited Morse index $\mu(u_0)$ and null dimension $\nu(u_0)$. $I''(u_0)$ is a Fredholm operator. Moreover, if u_0 is the local minimizer of I, then*

$$C_q(I, u_0) \cong \delta_{q,0}\mathbb{Z}, \quad q = 0, 1, 2, \cdots .$$

Proposition 2.3. *[34] Assume that $I \in C^2(S, \mathbb{R})$ with $S = S^+ \oplus S^-$ and 0 is the isolated critical point of I. If I has a local linking structure at 0 with $k = \dim S^- < \infty$, then*

$$C_q(I, 0) \cong \delta_{q,k}\mathbb{Z}, \qquad k = \mu_0 \quad or \quad k = \mu_0 + \nu_0.$$

Proposition 2.4. *[35] Let $I \in C^2(S, \mathbb{R})$ satisfy (D). Then,*
(J_1) *if $C_q(I, \infty) \not\cong 0$ holds for some q, then I possesses a critical point u such that $C_q(I, u) \not\cong 0$;*
(J_2) *if 0 is the isolated critical point of I and there exists some q such that $C_q(I, \infty) \not\cong C_q(I, 0)$, then I has a nonzero critical point.*

In our proofs, we also require the following Mountain Pass Lemma.

Proposition 2.5. *[33] Let S be a real Banach space and $I \in C^1(S, \mathbb{R})$ satisfy the Palais–Smale condition ((PS) for short). Further, if $I(0) = 0$ and*
(J_3) *there exist constants ρ, $a > 0$ such that $I|_{\partial B_\rho} \geq a$,*
(J_4) *there is some $e \in S \backslash B_\rho$ such that $I(e) \leq 0$.*
Then, I possesses a critical value $c \geq a$ given by

$$c = \inf_{h \in \Gamma} \sup_{x \in [0,1]} I(h(x)),$$

where

$$\Gamma = \{h \in C([0, 1], S) | h(0) = 0, h(1) = e\}.$$

3. Main results and proofs

In this section, we state our main results and provide detailed proofs. First, the following assumptions are required:

(f_1) There exists $k \geq 2$ such that

$$\lambda_k \leq \liminf_{|u|\to\infty} \frac{f((i,j),u)}{u} \leq \limsup_{|u|\to\infty} \frac{f((i,j),u)}{u} \leq \lambda_{k+1}, \qquad (i,j) \in \Omega.$$

(f_2) There exists a subsequence $\{u_{n,k}^{(1)}\} \subseteq \text{span}\{\phi_k\}$ such that $\frac{\|u_{n,k}^{(1)}\|}{\|u_n\|} \to 1$ as $\|u_n\| \to \infty$. Then, there exist $\delta_1, N_1 > 0$ such that

$$\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} \Big(f\big((i,j), u_n(i,j)\big) - \lambda_k u_n(i,j)\Big) u_{n,k}^{(1)}(i,j) \geq \delta_1, \quad n \geq N_1, \qquad (i,j) \in \Omega.$$

(f_3) There exists a subsequence $\{u_{n,k+1}^{(1)}\} \subseteq \text{span}\{\phi_{k+1}\}$ such that $\frac{\|u_{n,k+1}^{(1)}\|}{\|u_n\|} \to 1$ as $\|u_n\| \to \infty$. Then, there exist $\delta_2, N_2 > 0$ such that

$$\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} \Big(\lambda_{k+1} u_n(i,j) - f\big((i,j), u_n(i,j)\big)\Big) u_{n,k+1}^{(1)}(i,j) \geq \delta_2, \quad n \geq N_2, \qquad (i,j) \in \Omega.$$

We are now in a position to state our main results.

Theorem 3.1. *Let* (f_1), (f_2), (f_3) *hold. Moreover, for all* $(i,j) \in \Omega$,
(V_1) $F''((i,j),0) < \lambda_1$,
(V_2) $F''((i,j),u) > 0$ *for all* $u \in \mathbb{R}$
are satisfied. Then, problem (1.1)–(1.2) possesses at least three nontrivial solutions, among which one is positive and one is negative.

Consider the following sign condition:
($F_0^{\pm}$) There exists $\delta > 0$ such that

$$\pm(2F((i,j),u) - \lambda_m u^2) \geq 0, \qquad |u(i,j)| \leq \delta, \qquad (i,j) \in \Omega.$$

Then, we can state the following theorem.

Theorem 3.2. *Suppose* (f_1), (f_2), (f_3) *are satisfied. For all* $(i,j) \in \Omega$, *let*
(V_3) $f'((i,j),0) = \lambda_m$,
(V_4) *there exists* $u_0 \neq 0$ *such that* $f((i,j),u_0) = 0$.
Then, problem (1.1)–(1.2) possesses at least four nontrivial solutions if one of the following conditions is met:
(i) (F_0^+) *with* $2 \leq m \neq k$, **(ii)** (F_0^-) *with* $2 < m \neq k+1$.

Theorem 3.3. *Assume* (f_1), (f_2), (f_3), *and* (V_3) *are true. Further, if* $k = 1$ *and either*
(iii) (F_0^+) *with* $m \neq 1$, **(iv)** (F_0^-) *with* $m \neq 2$,
then problem (1.1)–(1.2) admits at least two nontrivial solutions.

To prove our results, (C) is necessary. First, we present a detailed proof to show that I satisfies (C).

Lemma 3.1. *Let* $(\mathbf{f_1})$*,* $(\mathbf{f_2})$*, and* $(\mathbf{f_3})$ *hold. Then, I satisfies* (C)*.*

Proof. Suppose that $\{u_n\} \subseteq S$ and there exists a constant c such that

$$I(u_n) \to c, \quad (1 + \|u_n\|)\|I'(u_n)\| \to 0, \quad \text{as} \quad n \to \infty.$$

Because S is a T_1T_2-dimensional space, it suffices to show that $\{u_n\}$ is bounded. Otherwise, we can assume that

$$\|u_n\| \to \infty, \qquad \text{as} \quad n \to \infty.$$

Denote $\overline{u}_n = \frac{u_n}{\|u_n\|}$. Then, $\|\overline{u}_n\| = 1$, which means that $\{\overline{u}_n\}$ has some subsequences. Without loss of generality, we set the subsequence to be the sequence. Moreover, there exists $\overline{u} \in S$ with $\|\overline{u}\| = 1$ such that

$$\overline{u}_n \to \overline{u}, \quad \text{as} \quad n \to \infty.$$

For any $\varphi \in S$, we obtain

$$\frac{\langle I'(u_n), \varphi\rangle}{\|u_n\|} = \langle \overline{u}_n, \varphi\rangle - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\left(\frac{f((i,j), u_n(i,j))}{\|u_n\|}, \varphi(i,j)\right).$$

From $(\mathbf{f_1})$, there exist $b \geq \lambda_{k+1}$ and $N > 0$ such that

$$|f((i,j), u_n(i,j))| \leq b(1 + |u_n(i,j)|), \qquad u \in S, \qquad n > N, \qquad (i,j) \in \Omega.$$

Set $b_1 = \frac{b}{\|u_n\|}$ such that, for $n > N$, we have

$$\frac{|f((i,j), u_n(i,j))|}{\|u_n\|} \leq b_1(1 + |\overline{u}_n(i,j)|), \qquad u \in S, \qquad (i,j) \in \Omega. \tag{3.1}$$

If $n \leq N$, then because $f((i,j), \cdot)$ is continuous in $\cdot$, we have

$$\frac{|f((i,j), u_n(i,j))|}{\|u_n\|} \leq \max\{\frac{|f((i,j), u_n(i,j))|}{\|u_n\|}\}, \qquad u \in S, \qquad (i,j) \in \Omega. \tag{3.2}$$

Therefore, Eqs (3.1) and (3.2) ensure that $\{\frac{f((i,j),u_n)}{\|u_n\|}\}$ is bounded. Consequently, $\{\frac{f((i,j),u_n)}{\|u_n\|}\}$ has a convergent subsequence. We still denote this by $\{\frac{f((i,j),u_n)}{\|u_n\|}\}$. Using $(\mathbf{f_1})$ once more, we can assume that there exists some p satisfying $\lambda_k \leq p \leq \lambda_{k+1}$ such that

$$\frac{f((i,j), u_n)}{\|u_n\|} \to p\overline{u}, \qquad \text{as} \quad n \to \infty.$$

Hence,

$$\triangle_1^2 \overline{u}(i-1, j) + \triangle_2^2 \overline{u}(i, j-1) + p\overline{u} = 0, \qquad (i,j) \in \Omega,$$

which means that $\overline{u}$ is the nontrivial solution of

$$\triangle_1^2 u(i-1, j) + \triangle_2^2 u(i, j-1) + pu = 0, \qquad (i,j) \in \Omega$$

with boundary conditions

$$\overline{u}(i,0)=\overline{u}(i,T_2+1)=0,\quad i\in\mathbb{Z}(1,T_1),\qquad \overline{u}(0,j)=\overline{u}(T_1+1,j)=0,\quad j\in\mathbb{Z}(1,T_2).$$

Together with the maximum principle and unique continuation property, this implies that $p\equiv\lambda_k$ or $p\equiv\lambda_{k+1}$ for $\lambda_k\le p\le\lambda_{k+1}$.

If $p\equiv\lambda_k$, then $\overline{u}\in W^0$ and

$$\frac{\|u_{n,k}^{(1)}\|}{\|u_n\|}\to 1,\qquad n\to\infty.$$

In fact, if $\overline{u}\notin W^0$, then $p\overline{u}=0$, which leads to $\|\overline{u}\|=0$. In view of $\|\overline{u}\|=1$, this is a contradiction. Therefore, as $n\to\infty$, we have that

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}(f((i,j),u_n(i,j))-\lambda_k u_n(i,j))u_{n,k}^{(1)}(i,j)=-\langle I'(u_n),u_{n,k}^{(1)}\rangle\le\|u_n\|\|I'(u_n)\|\to 0. \tag{3.3}$$

Obviously, Eq (3.3) is inconsistent with $(\mathbf{f_2})$.

If $p\equiv\lambda_{k+1}$, then $\overline{u}\in\mathrm{span}\{\phi_{k+1}\}$ and

$$\frac{\|u_{n,k+1}^{(1)}\|}{\|u_n\|}\to 1,\qquad n\to\infty.$$

Thus,

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}(\lambda_{k+1}u_n(i,j)-f((i,j),u_n(i,j)))u_{n,k+1}^{(1)}(i,j)=\langle I'(u_n),u_{n,k+1}^{(1)}\rangle\le\|u_n\|\|I'(u_n)\|\to 0$$

as $n\to\infty$, which contradicts $(\mathbf{f_3})$. Therefore, $\{u_n\}$ is bounded.

To calculate critical groups at infinity, we have the following lemma.

Lemma 3.2. *Let $\mu_\infty=\dim(W^0\oplus W^-)$. If $(\mathbf{f_1})$, $(\mathbf{f_2})$, and $(\mathbf{f_3})$ are satisfied, then $C_q(I,\infty)\cong\delta_{q,\mu_\infty}\mathbb{Z}$.*

Proof. First, for $t\in[0,1]$, let $I_t:S\to\mathbb{R}$ be given as

$$\begin{aligned}I_t(u)=&\frac{1}{2}\sum_{i=1}^{T_1+1}\sum_{j=1}^{T_2}|\triangle_1 u(i-1,j)|^2+\frac{1}{2}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1}|\triangle_2 u(i,j-1)|^2-\frac{1-t}{4}\lambda_k\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|u(i,j)|^2\\&-\frac{1-t}{4}\lambda_{k+1}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|u(i,j)|^2-t\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}F((i,j),u(i,j))\\=&\frac{1}{2}\|u\|^2-\frac{1-t}{4}(\lambda_k+\lambda_{k+1})\|u\|_2^2-t\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}F((i,j),u(i,j)).\end{aligned}$$

In the following, we prove that Eq (2.4) in Proposition 2.1 is true. Otherwise, there exists $\{u_n\}\subseteq S$, $t_n\in[0,1]$ such that

$$\|u_n\|\to\infty,\quad (1+\|u_n\|)\|I_t'(u_n)\|\to 0,\qquad n\to\infty.$$

Denote $\overline{u}_n = \frac{u_n}{\|u_n\|}$. Then, $\|\overline{u}_n\| = 1$ and there exists $\overline{u} \in S$ satisfying $\|\overline{u}\| = 1$ such that $\overline{u}_n \to \overline{u}$ as $n \to \infty$. From Lemma 3.1, we know that

$$\frac{f((i,j),u_n)}{\|u_n\|} \to p\overline{u}, \qquad n \to \infty.$$

Suppose that $t_n \to t_0$, whereby is easy to show that $\overline{u}$ is a solution subject to

$$\begin{cases} \triangle_1^2\overline{u}(i-1,j) + \triangle_2^2\overline{u}(i,j-1) - \xi(t_0)\overline{u} = 0, & (i,j) \in \Omega, \\ \overline{u}(i,0) = \overline{u}(i,T_2+1) = 0 \quad i \in \mathbb{Z}(1,T_1), & \overline{u}(0,j) = \overline{u}(T_1+1,j) = 0 \quad j \in \mathbb{Z}(1,T_2), \end{cases}$$

where $\xi(t_0) = \frac{1-t_0}{2}\lambda_k + \frac{1-t_0}{2}\lambda_{k+1} + t_0 p$ and $\lambda_k \le \xi(t_0) \le \lambda_{k+1}$. By the maximum principle and unique continuation property, we find that

$$\xi(t_0) \equiv \lambda_k \quad \text{or} \quad \xi(t_0) \equiv \lambda_{k+1}.$$

Furthermore, $t_0 = 1$ means that $t_n \to 1$ as $n \to \infty$. Therefore,

$$\frac{\|u_{n,k}^{(1)}\|}{\|u_n\|} \to 1 \quad \text{or} \quad \frac{\|u_{n,k+1}^{(1)}\|}{\|u_n\|} \to 1.$$

Note that $(\mathbf{f_2})$ and $(\mathbf{f_3})$ are valid, and so

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\Big(f((i,j),u_n(i,j)) - \lambda_k u_n(i,j)\Big)u_{n,k}^{(1)}(i,j) \ge \delta_1 > 0, \qquad n > N_1$$

or

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\Big(\lambda_{k+1} u_n(i,j) - f((i,j),u_n(i,j))\Big)u_{n,k+1}^{(1)}(i,j) \ge \delta_2 > 0, \qquad n > N_2.$$

Considering $\lambda_k \le \lambda_{k+1}$, we have

$$-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\Big(f((i,j),u_n) - \lambda_k u_n(i,j)\Big)u_{n,k}^{(1)}(i,j) + \frac{1-t_n}{2t_n}(\lambda_k - \lambda_{k+1})\|u_{n,k}^{(1)}\|^2 \to 0, \qquad n \to \infty,$$

which implies that

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\Big(f((i,j),u_n) - \lambda_k u_n(i,j)\Big)u_{n,k}^{(1)}(i,j) + o(1) = \frac{1-t_n}{2t_n}(\lambda_k - \lambda_{k+1})\|u_{n,k}^{(1)}\|^2 \le 0, \qquad n \to \infty.$$

Therefore,

$$\limsup_{n\to\infty}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\Big(f((i,j),u_n) - \lambda_k u_n(i,j)\Big)u_{n,k}^{(1)}(i,j) \le 0$$

or

$$\limsup_{n\to\infty}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\Big(\lambda_{k+1} u_n(i,j) - f((i,j),u_n)\Big)u_{n,k+1}^{(1)}(i,j) \le 0,$$

which guarantees that Eq (2.4) is satisfied.

It is easy to show that Eq (2.5) is satisfied and I_0, I_1 satisfy (C). In fact, because

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}(\lambda_k + \lambda_{k+1})\|u\|_2^2, \qquad u \in S,$$

we have that

$$\triangle_1^2\overline{u}(i-1, j) + \triangle_2^2\overline{u}(i, j-1) + \frac{1}{2}(\lambda_k + \lambda_{k+1})\|\overline{u}\|_2 = 0, \quad (i, j) \in \Omega,$$

which is impossible for $\|\overline{u}\| = 1$. Consequently, I_0 satisfies (C). Moreover,

$$\partial_t I_t = \frac{1}{4}(\lambda_k + \lambda_{k+1})\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i, j), u(i, j))$$

is continuous. According to Proposition 2.1, we have that

$$C_q(I, \infty) = C_q(I_1, \infty) \cong C_q(I_0, \infty), \qquad q \in \mathbb{Z}.$$

Note that $u = 0$ is a unique nondegenerate critical point of I_0 with $\mu_0 = \dim(W^0 \oplus W^-)$. Thus,

$$C_q(I_0, \infty) \cong C_q(I_0, 0) \cong \delta_{q,\mu_\infty}\mathbb{Z}, \qquad \mu_\infty = \mu_0 = \dim(W^0 \oplus W^-).$$

Further, we have that $C_q(I, \infty) \cong \delta_{q,\mu_\infty}\mathbb{Z}$.

To gain some mountain pass-type critical points by applying the cut-technique, we verify the following compactness conditions.

Lemma 3.3. *Let*

$$f^+((i, j), u) = \begin{cases} f((i, j), u), & u \geq 0, \\ 0, & u < 0, \end{cases} \tag{3.4}$$

such that

$$\lambda_k \leq \liminf_{u\to+\infty} \frac{f^+((i, j), u)}{u} \leq \limsup_{u\to+\infty} \frac{f^+((i, j), u)}{u} \leq \lambda_{k+1}.$$

Then, the functional $I^+ : S \to \mathbb{R}$ defined by

$$I^+(u) = \frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F^+((i, j), u(i, j))$$

satisfies (PS), where $F^+((i, j), u) = \int_0^u f^+((i, j), \tau)d\tau$.

Proof. Let $\{u_n\} \subseteq S$ be a $(PS)_c$ sequence, that is,

$$I^+(u_n) \to c, \quad I'^+(u_n) \to 0, \qquad n \to \infty.$$

Similar to Lemma 3.1, assume that $\{u_n\}$ is unbounded. Then, we have

$$|u_n(i, j)| \to \infty, \qquad n \to \infty, \qquad \forall (i, j) \in \Omega. \tag{3.5}$$

Denote $v_n = \frac{u_n}{\|u_n\|}$, so that $\|v_n\| = 1$. As S is a T_1T_2-dimensional Hilbert space, there exists $v \in S$ satisfying $\|v\| = 1$ such that $v_n \to v$ as $n \to \infty$. Hence, for any $\varphi \in S$, it holds that

$$\frac{\langle I'^{+}(u_n), \varphi\rangle}{\|u_n\|} = \langle v_n, \varphi\rangle - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\left(\frac{f^+((i,j), u_n(i,j))}{\|u_n\|}, \varphi(i,j)\right). \tag{3.6}$$

For any $(i, j) \in \Omega$, write $v^+(i, j) = \max\{v(i, j), 0\}$. Then, there exists α satisfying $\lambda_k \le \alpha \le \lambda_{k+1}$ such that

$$\lim_{n\to\infty} \frac{f^+((i,j), u_n(i,j))}{\|u_n\|} = \lim_{n\to\infty} \frac{f^+((i,j), u_n(i,j))}{u_n(i,j)} v_n(i,j) = \alpha v^+(i,j).$$

Together with Eq (3.6), this yields

$$-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\{\Delta_1^2 v(i-1, j) + \Delta_2^2 v(i, j-1) + \alpha v^+(i,j)\}\varphi(i,j) = 0, \qquad n \to \infty,$$

which implies that v is the nontrivial solution of

$$\Delta_1^2 v(i-1, j) + \Delta_2^2 v(i, j-1) + \alpha v^+(i,j) = 0, \qquad (i,j) \in \Omega \tag{3.7}$$

with boundary conditions

$$v(i,0) = v(i, T_2+1) = 0, \quad i \in \mathbb{Z}(1, T_1), \qquad v(0, j) = v(T_1+1, j) = 0, \quad j \in \mathbb{Z}(1, T_2). \tag{3.8}$$

Denote $v(i_0, j_0) := \min\{v(i,j) | (i,j) \in \Omega\}$. We aim to show that $v(i_0, j_0) > 0$. Otherwise, $\alpha v^+(i, j) = 0$ and

$$\Delta_1^2 v(i_0 - 1, j_0) + \Delta_2^2 v(i_0, j_0 - 1) = 0, \qquad (i,j) \in \Omega,$$

which means that $\Delta_1 v(i_0 - 1, j_0) = \Delta_2 v(i_0, j_0 - 1) = 0$. Moreover, $v(i_0 - 1, j_0) = v(i_0, j_0) = v(i_0, j_0 - 1)$. Therefore, $v \equiv 0$ for all $(i, j) \in \Omega$. Additionally, v is the nontrivial solution of Eqs (3.7)–(3.8), which implies that $v(i_0, j_0) > 0$, and so $v(i, j) > 0$. Recall that $\lambda_k \le \alpha \le \lambda_{k+1}$, we so we have that $\{u_n\}$ is bounded.

In the same manner as Lemma 3.3, we can state the following.

Lemma 3.4. *If*

$$\lambda_k \le \liminf_{u\to-\infty} \frac{f^-((i,j), u)}{u} \le \limsup_{u\to-\infty} \frac{f^-((i,j), u)}{u} \le \lambda_{k+1},$$

where

$$f^-((i,j), u) = \begin{cases} f((i,j), u), & u \le 0, \\ 0, & u > 0, \end{cases} \tag{3.9}$$

then the functional $I^- : S \to \mathbb{R}$ *defined by*

$$I^-(u) = \frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F^-((i,j), u(i,j))$$

satisfies (PS)*, where* $F^-((i,j), u) = \int_0^u f^-((i,j), \tau)d\tau$.

Lemma 3.5. *If all conditions of Theorem 3.1 are fulfilled, then I^+ has a critical point $u^+ > 0$ such that $C_q(I^+, u^+) \cong \delta_{q,1}\mathbb{Z}$ and I^- has a critical point $u^- < 0$ such that $C_q(I^-, u^-) \cong \delta_{q,1}\mathbb{Z}$.*

Proof. We prove the case of I^+; the proof of I^- is similar and is omitted for brevity. With the aid of Proposition 2.5, we need only prove that I^+ satisfies $(\mathbf{J_3})$, $(\mathbf{J_4})$. According to $(\mathbf{V_1})$, there exist $\rho, \rho_1 > 0$ such that $F''((i,j),0) < \rho_1 < \lambda_1$ and

$$F((i,j),u) \leq \frac{1}{2}\rho u^2, \quad |u(i,j)| \leq \rho.$$

Then, for any $(i,j) \in \Omega$ and $u \in S$ with $\|u\| \leq \sqrt{\lambda_1}\rho_1$, we have

$$\begin{aligned}
I^+(u) =& \frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F^+((i,j),u(i,j)) = \frac{1}{2}\|u\|^2 - \sum_{u\in U} F((i,j),u(i,j)) \\
\geq& \frac{1}{2}\|u\|^2 - \frac{1}{2}\rho_1 \sum_{u\in U} |u(i,j)|^2 \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\rho_1 \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} |u(i,j)|^2 \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\frac{\rho_1}{\lambda_1}\|u\|^2 \\
>& 0,
\end{aligned}$$

with $U = \{(i,j) \in \Omega | u(i,j) \geq 0\}$, which ensures $(\mathbf{J_3})$ is valid.

Using $(\mathbf{f_1})$, there exist $\gamma > \lambda_{k-1} (\geq \lambda_1)$, $b_2 \in \mathbb{R}$ such that

$$F((i,j),u) \geq \frac{\gamma}{2}u^2 + b_2, \qquad \forall u \in \mathbb{R}.$$

For $t > \rho$, choose $e \in \text{span}\{\phi_1\}$. Then, we have

$$\begin{aligned}
I^+(te) =& \frac{1}{2}\|te\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i,j),te(i,j)) \leq \frac{1}{2}\|te\|^2 - \frac{\gamma\|te\|_2^2}{2} - b_2T_1T_2 \\
=& \frac{1}{2}\|te\|^2 - \frac{\gamma\|te\|^2}{2\lambda_1} - b_2T_1T_2 = \frac{t^2}{2}(1 - \frac{\gamma}{\lambda_1})\|e\|^2 - b_2T_1T_2 \\
\leq& 0.
\end{aligned}$$

Therefore, $(\mathbf{J_4})$ is satisfied.

By the Mountain Pass Lemma, I^+ possesses a critical point $u^+ \neq 0$. Moreover, there exists a sequence $\{u_n^+\}$ such that $I'^+(u_n^+) \to I'^+(u^+)$ as $n \to \infty$. For any $\varphi \in S$, we have

$$\langle I'^+(u_n^+), \varphi\rangle = \langle u_n^+, \varphi\rangle - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} \left(f^+((i,j),u_n^+(i,j)), \varphi(i,j)\right).$$

Letting $n \to \infty$,

$$\lim_{n\to\infty} f^+((i,j),u_n^+(i,j)) = \lim_{n\to\infty} \frac{f^+((i,j),u_n^+(i,j))}{u_n^+(i,j)} u_n^+(i,j) = \alpha u^+(i,j), \quad (i,j) \in \Omega,$$

which leads to

$$\Delta_1^2 u^+(i-1,j) + \Delta_2^2 u^+(i,j-1) + \alpha u^+(i,j) = 0$$

with $\lambda_k \leq \alpha \leq \lambda_{k+1}$ and $u^+(i,j) := \max\{u(i,j), 0\}$. Similar to the proof of Lemma 3.3, u^+ is positive.

We now calculate $C_q(I^+, u^+)$. Recalling Eq (2.1), we find that

$$\langle I''(u^+)v, v\rangle \geq 0, \qquad \forall v \in S$$

and there exists some $v_0 \neq 0$ such that

$$\langle I''(u^+)v_0, v\rangle = 0, \qquad \forall v \in S,$$

which implies that v_0 is a solution of

$$\begin{cases} \Delta_1^2 v_0(i-1,j) + \Delta_2^2 v_0(i,j-1) - F''((i,j),u^+)v_0(i,j) = 0, \qquad (i,j) \in \Omega, \\ v_0(i,0) = v_0(i,T_2+1) = 0, \quad i \in \mathbb{Z}(1,T_1), \qquad v_0(0,j) = v_0(T_1+1,j) = 0, \quad j \in \mathbb{Z}(1,T_2). \end{cases}$$

Combining this with $(\mathbf{V_2})$, it follows that

$$\begin{cases} \Delta_1^2 v(i-1,j) + \Delta_2^2 v(i,j-1) - \lambda F''((i,j),u^+)v(i,j) = 0, \qquad (i,j) \in \Omega, \\ v(i,0) = v(i,T_2+1) = 0, \quad i \in \mathbb{Z}(1,T_1), \qquad v(0,j) = v(T_1+1,j) = 0, \quad j \in \mathbb{Z}(1,T_2) \end{cases}$$

admits an eigenvalue $\lambda = 1$ such that $\dim\ker(I''(u^+)) = 1$. Hence, we conclude that $C_q(I,u^+) \cong C_q(I^+,u^+) \cong \delta_{q,1}\mathbb{Z}$. This completes the proof.

It is time for us to give the detailed proof of Theorem 3.1 using Proposition 2.4.

Proof of Theorem 3.1 Given $I'(0) = 0$ and $(\mathbf{V_1})$, then

$$\langle I''(0)u, u\rangle \geq (1 - \frac{F''((i,j),0)}{\lambda_1})\|u\|^2, \qquad (i,j) \in \Omega,$$

which implies that 0 is the local minimizer of I. From Proposition 2.2, we have

$$C_q(I,0) \cong \delta_{q,0}\mathbb{Z}, \qquad q \in \mathbb{Z}. \tag{3.10}$$

Furthermore, Lemma 3.2 ensures that

$$C_q(I,\infty) \cong \delta_{q,\mu_\infty}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

Then, according to Proposition 2.4, there exists some critical point $u_1 \neq 0$ of I such that

$$C_{\mu_\infty}(I,u_1) \ncong 0. \tag{3.11}$$

Consequently, we conclude that u^+, u^- and u_1 are nontrivial critical points of I with $u^+ > 0$ and $u^- < 0$. The proof of Theorem 3.1 is achieved.

Before verifying Theorem 3.2 using Proposition 2.3, we need the following lemma about local linking.

Lemma 3.6. *Let* $(\mathbf{V_3})$ *and* $(\mathbf{F_0^+})$ *(or* $(\mathbf{F_0^-})$*) hold. Then, I has a local linking at 0 with respect to*

$$S = S^- \oplus S^+,$$

where $S^- = span\{\phi_1, \phi_2, ..., \phi_m\}$ *(or* $S^- = span\{\phi_1, \phi_2, ..., \phi_{m-1}\}$*).*

Proof. Suppose that ($\mathbf{F}_0^+$) is satisfied. Then, there exists $\delta > 0$ such that $|u(i,j)| \le \delta$, $\|u\| \le \delta\sqrt{\lambda_{T_1T_2}}$, and

$$F((i,j),u) \ge \frac{1}{2}\lambda_m u^2.$$

For $u \in S^-$ with $0 < \|u\| \le \delta\sqrt{\lambda_{T_1T_2}}$, we have

$$I(u) = \frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i,j),u(i,j)) \le \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_m\|u\|_2^2 \le \frac{1}{2}(1 - \frac{\lambda_m}{\lambda_m})\|u\|^2 = 0. \tag{3.12}$$

For $u \in S^+$ with $0 < \|u\| < \delta\sqrt{\lambda_{T_1T_2}}$, we obtain

$$I(u) \ge \frac{1}{2}\|u\|^2 - \frac{1}{2}\frac{\lambda_m}{\lambda_{m+1}}\|u\|^2 = \frac{1}{2}(1 - \frac{\lambda_m}{\lambda_{m+1}})\|u\|^2 > 0. \tag{3.13}$$

Obviously, $I(0) = 0$. Combining this with Eqs (3.12) and (3.13), it is clear that I has a local linking at 0.

Proof of Theorem 3.2 Denote $\mu_0 = \dim \operatorname{span}\{\phi_1, ..., \phi_{m-1}\}$, $\nu_0 = \dim \operatorname{span}\{\phi_m\}$. Let ($\mathbf{V_3}$) be true. Then, 0 is degenerate. Taking account of Proposition 2.3, we find that

$$C_q(I,0) \cong \delta_{q,\mu_0+\nu_0}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

In contrast, Lemma 3.2 gives $C_q(I,\infty) \cong \delta_{q,\mu_\infty}\mathbb{Z}$. Therefore, Proposition 2.4 guarantees the existence of u^* such that

$$C_{\mu_\infty}(I,u^*) \not\cong 0.$$

Moreover, $m \ne k$ implies that $\mu_0 + \nu_0 \ne \mu_\infty$. Thus, $u^* \ne 0$.

From ($\mathbf{V_4}$), there exists $u_0 \ne 0$ such that $f((i,j),u_0) = 0$. Without loss of generality, we can assume that $u_0 > 0$. In the sequence, we intend to obtain the local minimizer of I. Define

$$\widetilde{f}((i,j),u) = \begin{cases} 0, & u < 0, \\ f((i,j),u), & u \in [0,u_0], \\ 0, & u > u_0, \end{cases} \tag{3.14}$$

and let

$$\widetilde{I}(u) = \frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} \widetilde{F}((i,j),u(i,j)), \qquad u \in S,$$

where $\widetilde{F}((i,j),u) = \int_0^u \widetilde{f}((i,j),\tau)d\tau$. Then, $\widetilde{I}$ is coercive and continuous. Therefore, there exists a minimizer $\widetilde{u}_0$ of $\widetilde{I}$. From the maximum principle, we deduce that $\widetilde{u}_0 = 0$ or $0 < \widetilde{u}_0(i,j) < u_0$ for all $(i,j) \in \Omega$. Moreover, ($\mathbf{V_3}$) means that 0 is not a minimizer. Consequently, $\widetilde{u}_0 \ne 0$ is a local minimizer of I and $C_q(I,\widetilde{u}_0) \cong \delta_{q,0}\mathbb{Z}$, $q \in \mathbb{Z}$.

Denote $\widehat{F}((i,j),u) = \int_0^u \widehat{f}((i,j),\tau)d\tau$, where

$$\widehat{f}((i,j),v) = f((i,j),v + \widetilde{u}_0) - f((i,j),\widetilde{u}_0), \qquad (i,j) \in \Omega, \qquad v \in S.$$

The corresponding functional is then given by

$$\widehat{I}(v) = \frac{1}{2}\|v\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} \widehat{F}((i,j), v(i,j)), \qquad v \in S.$$

If v is a nonzero critical point of $\widehat{I}$, then $\widetilde{u}_0 + v$ is a critical point of I with

$$C_q(\widehat{I}, v) = C_q(I, \widetilde{u}_0 + v).$$

Define

$$\widehat{f}^+((i,j), v) = \begin{cases} \widehat{f}((i,j), v), & v \geq 0, \\ 0, & v < 0, \end{cases} \tag{3.15}$$

and construct the corresponding functional as

$$\widehat{I}^+(v) = \frac{1}{2}\|v\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} \widehat{F}^+((i,j), v(i,j)), \qquad v \in S,$$

with $\widehat{F}^+((i,j), u) = \int_0^u \widehat{f}^+((i,j), \tau) d\tau$. Then, $\widetilde{u}_0$ is a local minimizer of I for $\widehat{I}^+$ satisfying (PS), which leads to $v = 0$ being a local minimizer of $\widehat{I}^+$. Thus, $(\mathbf{J_3})$ is fulfilled. Applying $(\mathbf{f_1})$ yields

$$\widehat{I}^+(te) \leq 0, \qquad t \to +\infty,$$

which ensures that $(\mathbf{J_4})$ is satisfied. By Lemma 2.5, $\widehat{I}^+$ possesses a critical point $v^+ > 0$. Furthermore, $\widehat{I}$ possesses a critical point v^+ with $C_q(\widehat{I}, v^+) \cong \delta_{q,1}\mathbb{Z}$, $q \in \mathbb{Z}$. As a result, $u^+ = \widetilde{u}_0 + v^+$ is a critical point of I satisfying $C_q(I, u^+) \cong \delta_{q,1}\mathbb{Z}$, $q \in \mathbb{Z}$. Similarly, $u^- < \widetilde{u}_0$ is a critical point of I satisfying $C_q(I, u^-) \cong \delta_{q,1}\mathbb{Z}$. Therefore, u^*, $\widetilde{u}_0$, $u^\pm$ are four nontrivial solutions of I and $\widetilde{u}_0$, u^+ are positive.

For the case $u_0 < 0$, repeating the above steps shows that I has four nontrivial solutions, among which there are two negative solutions. Therefore, I admits four nontrivial solutions and the proof is finished.

Proof of Theorem 3.3 Similar to the proof of Theorem 3.2, we find that $C_q(I, 0) \cong \delta_{q,\mu_0+\nu_0}\mathbb{Z}$, $q \in \mathbb{Z}$. Because $k = 1$, $C_q(I, \infty) \cong \delta_{q,1}\mathbb{Z}$ and there exists some critical point $\overline{u}^*$ such that $C_1(I, \overline{u}^*) \not\cong 0$. Moreover, u^* is the critical point of I satisfying $C_q(I, \overline{u}^*) \cong \delta_{q,1}\mathbb{Z}$. From $m \neq 1$, we conclude that $\overline{u}^* \neq 0$. If $\kappa = \{0, \overline{u}^*\}$, the Morse equality implies that

$$(-1)^{\mu_0+\nu_0} + (-1)^1 = (-1)^1. \tag{3.16}$$

Of course, Eq (3.16) is impossible. Therefore, problem (1.1)–(1.2) possesses at least two nontrivial solutions.

4. Examples

Finally, we present two examples to verify the feasibility of our results.

Example 4.1. *Take $T_1 = 3$, $T_2 = 2$, and consider*

$$\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) + \frac{(\frac{\lambda_1}{2} - \frac{\lambda_k+\lambda_{k+1}}{2})u(i,j)}{1+[u(i,j)]^2} + \frac{\lambda_k+\lambda_{k+1}}{2}u(i,j) = 0 \tag{4.1}$$

with the boundary value conditions of Eq (1.2).

Because $f((i,j),u) = \frac{(\frac{\lambda_1}{2} - \frac{\lambda_k+\lambda_{k+1}}{2})u(i,j)}{1+[u(i,j)]^2} + \frac{\lambda_k+\lambda_{k+1}}{2}u(i,j)$, it follows that

$$F''((i,j),u) = \frac{\lambda_k+\lambda_{k+1}}{2}u^4 + \Big(\frac{3}{2}(\lambda_k+\lambda_{k+1}) - \frac{\lambda_1}{2}\Big)u^2 + \frac{\lambda_1}{2} > 0.$$

It is not difficult to verify that $f((i,j),0) = 0$, $F''((i,j),0) = \frac{\lambda_1}{2} < \lambda_1$, and

$$\lambda_k \le \lim_{|u|\to\infty} \frac{f((i,j),u)}{u} = \frac{\lambda_k+\lambda_{k+1}}{2} \le \lambda_{k+1},$$

which means that $(\mathbf{f_1})$, $(\mathbf{V_1})$ and $(\mathbf{V_2})$ are satisfied. If $\frac{\|u_{n,k}^{(1)}\|}{\|u_n\|} \to 1$ as $\|u_n\| \to \infty$, then we obtain

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{2}\Bigg(\frac{(\frac{\lambda_1}{2} - \frac{\lambda_k+\lambda_{k+1}}{2})u_n(i,j)}{1+[u_n(i,j)]^2} + \frac{\lambda_k+\lambda_{k+1}}{2}u_n(i,j) - \lambda_k u_n(i,j)\Bigg)u_{n,k}^{(1)}(i,j)\\
=&\sum_{i=1}^{3}\sum_{j=1}^{2}\Bigg(\frac{(\frac{\lambda_1}{2} - \frac{\lambda_k+\lambda_{k+1}}{2})u_n(i,j)}{1+[u_n(i,j)]^2} + \frac{\lambda_{k+1}-\lambda_k}{2}u_n(i,j)\Bigg)u_{n,k}^{(1)}(i,j)\\
=&\sum_{i=1}^{3}\sum_{j=1}^{2}\Bigg[\frac{\frac{\lambda_1}{2} - \frac{\lambda_k+\lambda_{k+1}}{2}}{1+\frac{1}{[u_{n,k}^{(1)}(i,j)]^2}} + \frac{\lambda_{k+1}-\lambda_k}{2}[u_{n,k}^{(1)}(i,j)]^2\Bigg] \to +\infty.
\end{aligned}$$

Therefore, $(\mathbf{f_2})$ is satisfied. Similarly, $(\mathbf{f_3})$ is valid. Therefore, Theorem 3.1 guarantees that problem (4.1)–(1.2) admits at least three nontrivial solutions, of which one is positive and one is negative.

Example 4.2. *Take $T_1 = 3$, $T_2 = 2$, and consider*

$$\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) + \frac{2(\lambda_m - \frac{\lambda_k+\lambda_{k+1}}{2})u}{2-u^2} + \frac{\lambda_k+\lambda_{k+1}}{2}u = 0 \tag{4.2}$$

with the boundary value conditions of Eq (1.2).

Denote

$$f((i,j),u) = \frac{2(\lambda_m - \frac{\lambda_k+\lambda_{k+1}}{2})u}{2-u^2} + \frac{\lambda_k+\lambda_{k+1}}{2}u.$$

Then, $f((i,j),0) = 0$ and there exists $u = \pm\sqrt{\frac{4\lambda_m}{\lambda_k+\lambda_{k+1}}} \neq 0$ such that $f((i,j),u) = 0$, which means that $(\mathbf{V_4})$ is satisfied.

A direct computation yields

$$F((i,j),u) = (\frac{\lambda_k+\lambda_{k+1}}{2} - \lambda_m)\ln(2-u^2) + \frac{\lambda_k+\lambda_{k+1}}{4}u^2$$

and

$$f'((i,j),u) = \frac{2(\lambda_m - \frac{\lambda_k+\lambda_{k+1}}{2})(2+u^2)}{(2-u^2)^2} + \frac{\lambda_k + \lambda_{k+1}}{2},$$

and so $f'((i,j),0) = \lambda_m$ and $(\mathbf{V_3})$ is valid.

As $\|u_n\| \to \infty$, if $\frac{\|u_{n,k}^{(1)}\|}{\|u_n\|} \to 1$, then

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{2}\left(\frac{2(\lambda_m - \frac{\lambda_k+\lambda_{k+1}}{2})u_n(i,j)}{2-[u_n(i,j)]^2} + \frac{\lambda_k+\lambda_{k+1}}{2}u_n(i,j) - \lambda_k u_n(i,j)\right)u_{n,k}^{(1)}(i,j)\\
=&\sum_{i=1}^{3}\sum_{j=1}^{2}\left(\frac{2(\lambda_m - \frac{\lambda_k+\lambda_{k+1}}{2})u_n(i,j)}{1+[u_n(i,j)]^2} + \frac{\lambda_{k+1}-\lambda_k}{2}u_n(i,j)\right)u_{n,k}^{(1)}(i,j)\\
=&\sum_{i=1}^{3}\sum_{j=1}^{2}\left[\frac{2\lambda_m - (\lambda_k+\lambda_{k+1})}{1+\frac{1}{[u_{n,k}^{(1)}(i,j)]^2}} + \frac{\lambda_{k+1}-\lambda_k}{2}[u_{n,k}^{(1)}(i,j)]^2\right] \to +\infty.
\end{aligned}$$

Therefore, $(\mathbf{f_2})$ is satisfied. Similar to $(\mathbf{f_2})$, we can show that $(\mathbf{f_3})$ is satisfied.

In the following, we verify $(\mathbf{f_1})$ and $(\mathbf{F_0^+})$. If we write

$$A = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0\\ -1 & 4 & -1 & 0 & -1 & 0\\ 0 & -1 & 4 & 0 & 0 & -1\\ -1 & 0 & 0 & 4 & -1 & 0\\ 0 & -1 & 0 & -1 & 4 & -1\\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix},$$

then A is positive-definite and the eigenvalues of A are

$$\lambda_1 = 3-\sqrt{2},\quad \lambda_2 = 3,\quad \lambda_3 = 5-\sqrt{2},\quad \lambda_4 = 3+\sqrt{2},\quad \lambda_5 = 5,\quad \lambda_6 = 5+\sqrt{2}.$$

Let $m = 2$, $k = 3$. Then, $5-\sqrt{2} \le \liminf_{|u|\to\infty} \frac{f((i,j),u)}{u} = \limsup_{|u|\to\infty} \frac{f((i,j),u)}{u} = 4 \le 3+\sqrt{2}$, which means that $(\mathbf{f_1})$ is satisfied. Further, there exists $\delta > 0$ such that, when $|u(i,j)| \le \delta$, the following holds for any $(i,j) \in \mathbb{Z}(1,3)\times\mathbb{Z}(1,2)$:

$$2F((i,j),u) - 3u^2 = 2\ln(2-u^2) + u^2 \ge 0.$$

In fact, for any $(i,j) \in \Omega$, we can choose $\delta = 1 > 0$, and then $0 < [u(i,j)]^2 \le 1$ for $0 < |u(i,j)| \le 1$. This means that $\ln(2-u^2) \ge 0$ and $2F((i,j),u) - 3u^2 \ge 0$. Thus, $(\mathbf{F_0^+})$ holds and all conditions of Theorem 3.2 are satisfied. Consequently, Theorem 3.2 ensures that (4.2)–(1.2) admits at least four nontrivial solutions.

More clearly, using Matlab, we find that problem (4.2)–(1.2) has 36 nontrivial solutions. Some examples of these solutions are as follows: $(0.5061, 0.6548, 0.5061, 0.5061, 0.6548, 0.5061)$, $(-0.5061, -0.6548, -0.5061, -0.5061, -0.6548, -0.5061)$, $(-1.9858, 2.1015, 0.6084, 1.9858, -2.1015, -0.6084)$, and $(1.2137, -4.7492, 1.2137, 13.9612, -1.5032, 13.9612)$.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. J. S. Yu, J. Li, Discrete-time models for interactive wild and sterile mosquitoes with general time steps, *Math. Biosci.*, **346** (2022), 108797. https://doi.org/10.1016/j.mbs.2022.108797
2. Y. H. Long, L. Wang, Global dynamics of a delayed two-patch discrete SIR disease model, *Commun. Nonlinear Sci. Numer. Simul.*, **83** (2020), 105117. https://doi.org/10.1016/j.cnsns.2019.105117
3. Y. H. Long, Existence of multiple and sign-changing solutions for a second-order nonlinear functional difference equation with periodic coefficients, *J. Differ. Equation Appl.*, **26** (2020), 966–986. https://doi.org/10.1080/10236198.2020.1804557
4. Y. H. Long, Existence of two homoclinic solutions for a nonperiodic difference equation with a perturbation, *AIMS Math.*, **6** (2021), 4786–4802. https://doi.org/10.3934/math.2021281
5. Y. H. Long, J. L. Chen, Existence of multiple solutions to second-order discrete Neumann boundary value problems, *Appl. Math. Lett.*, **83** (2018), 7–14. https://doi.org/10.1016/j.aml.2018.03.006
6. J. S. Yu, Z. M. Guo, X. F. Zou, Periodic solutions of second order self-adjoint difference equations, *J. Lond. Math. Soc.*, **71** (2005), 146–160. https://doi.org/10.1112/S0024610704005939
7. Y. H. Long, S. H. Wang, J. L. Chen, Multiple solutions of fourth-order difference equations with different boundary conditions, *Bound. Value Probl.*, **2019** (2019), 152. https://doi.org/10.1186/s13661-019-1265-2
8. Y. H. Long, S. H. Wang, Multiple solutions for nonlinear functional difference equations by the invariant sets of descending flow, *J. Differ. Equation Appl.*, **25** (2019), 1768–1789. https://doi.org/10.1080/10236198.2019.1694014
9. Y. H. Long, Q. Q. Zhang, Sign-changing solutions of a discrete fourth-order Lidstone problem with three parameters, *J. Appl. Anal. Comput.*, 2022.
10. S. Heidarkhani, F. Gharehgazlouei, M. Imbesi, Existence and multiplicity of homoclinic solutions for a difference equation, *Electron. J. Differ. Equations*, **115** (2020), 1–12. https://www.webofscience.com/wos/alldb/full-record/WOS:000591718300001
11. F. Gharehgazlouei, S. Heidarkhani, New existence criterion of infinitely many solutions for partial discrete Dirichlet problems, *Tbilisi Math. J.*, **13** (2020), 43–51. http://dx.doi.org/10.32513/tbilisi/1601344897

12. M. Bohner, G. Caristi, S. Heidarkhani, S. Moradi, Existence of at least one homoclinic solution for a nonlinear second-order difference equation, *Int. J. Nonlinear Sci. Numer. Simul.*, **20** (2019), 433–439. https://doi.org/10.1515/ijnsns-2018-0223

13. S. Heidarkhani, G. A. Afrouzi, S. Moradi, An existence result for discrete anisotropic equations, *Taiwanese J. Math.*, **22** (2018), 725–739. http://dx.doi.org/10.11650/tjm/170801

14. S. Heidarkhani, G. A. Afrouzi, S. Moradi, G. Caristi, Existence of multiple solutions for a perturbed discrete anisotropic equation, *J. Differ. Equation Appl.*, **23** (2017), 1491–1507. https://doi.org/10.1080/10236198.2017.1337108

15. S. Heidarkhani, M. Imbesi, Multiple solutions for partial discrete Dirichlet problems depending on a real parameter, *J. Differ. Equation Appl.*, **21** (2015), 96–110. https://doi.org/10.1080/10236198.2014.988619

16. Z. Zhou, J. X. Ling, Infinitely many positive solutions for a discrete two point nonlinear boundary value problem with ϕ_c-Laplacian, *Appl. Math. Lett.*, **91** (2019), 28–34. https://doi.org/10.1016/j.aml.2018.11.016

17. Y. H. Long, X. Q. Deng, Existence and multiplicity solutions for discrete Kirchhoff type problems, *Appl. Math. Lett.*, **126** (2022), 107817. https://doi.org/10.1016/j.aml.2021.107817

18. Y. H. Long, Multiple results on nontrivial solutions of discrete Kirchhoff type problems, *J. Appl. Math. Comput.*, 2022. https://doi.org/10.1007/s12190-022-01731-0.

19. Y. H. Long, Nontrivial solutions of discrete Kirchhoff type problems via Morse theory, *Adv. Nonlinear Anal.*, **11** (2022), 1352–1364.

20. Y. H. Long, H. Zhang, Three nontrivial solutions for second-order partial difference equation via Morse theory, *J. Funct. Spaces*, 2022. https://doi.org/10.1155/2022/1564961

21. S. J. Du, Z. Zhou, On the existence of multiple solutions for a partial discrete Dirichlet boundary value problem with mean curvature operator, *Adv. Nonlinear Anal.*, **11** (2022), 198–211. https://doi.org/10.1515/anona-2020-0195

22. S. H. Wang, Z. Zhou, Three solutions for a partial discrete Dirichlet boundary value problem with p-Laplacian, *Bound. Value Probl.*, **2021** (2021), 39. https://doi.org/10.1186/s13661-021-01514-9

23. S. S. Cheng, Partial difference equations, *Taylor Francis*, 2003. https://doi.org/10.1201/9780367801052

24. H. S. Tang, W. Luo, X. Li, M. J. Ma, Nontrivial solutions of discrete elliptic boundary value problems, *Comput. Math. Appl.*, **55** (2008), 1854–1860. https://doi.org/10.1016/j.camwa.2007.08.030

25. G. Zhang, Existence of nontrivial solutions for discrete elliptic boundary value problems, *Numer. Methods Partial Differ. Equations*, **22** (2006), 1479–1488. https://doi.org/10.1002/num.20164

26. J. B. Su, L. G. Zhao, An elliptic resonance problem with multiple solutions, *J. Math. Anal. Appl.*, **319** (2006), 604–616. https://doi.org/10.1016/j.jmaa.2005.10.059

27. Q. Wang, W. J. Liu, M. Wang, Nontrivial periodic solutions for second-order differential delay equations, *J. Appl. Math. Comput.*, **7** (2017), 931–941. http://dx.doi.org/10.11948/2017058

28. K. C. Chang, Infinite dimensional Morse theory and multiple solutions problem, *Birkhäuser Boston*, Boston, 1993. http://dx.doi.org/10.1007/978-1-4612-0385-8

29. K. C. Chang, Solutions of asymptotically linear operator via Morse theory, *Comm. Pure Appl. Math.*, **34** (1981), 693–712. https://doi.org/10.1002/cpa.3160340503

30. Y. H. Long, H. P. Shi, X. Q. Peng, Nontrivial periodic solutions to delay difference equations via Morse theory, *Open Math.*, **16** (2018), 885–896. https://doi.org/10.1515/math-2018-0077

31. M. Imbesi, G. M. Bisci, Discrete elliptic Dirichlet problems and nonlinear algebraic systems, *Mediterr. J. Math.*, **13** (2016), 263–278. http://dx.doi.org/10.1007/s00009-014-0490-2

32. P. Bartolo, V. Benci, D. Fortunato, Abstract critical point theorems and applications to non-linear problems with strong resonance at infinity, *Nonlinear Anal.*, **7** (1983), 981–1012. https://doi.org/10.1016/0362-546X(83)90115-3

33. J. Mawhin, M. Willem, Critical point theory and Hamiltonian systems, *Springer-Verlag*, Berlin, 1989. http://dx.doi.org/10.1007/978-1-4757-2061-7

34. Z. P. Liang, J. B. Su, Multiple solutions for semilinear elliptic boundary value problems with double resonance, *J. Math. Anal. Appl.*, **354** (2009), 147–158. https://doi.org/10.1016/j.jmaa.2008.12.053

35. J. B. Su, Multiplicity results for asymptotically linear elliptic problems at resonance, *J. Math. Anal. Appl.*, **278** (2003), 397–408. https://doi.org/10.1016/S0022-247X(02)00707-2.