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http://www.aimspress.com/journal/era

ERA, 30(7): 2600–2617.
DOI: 10.3934/era.2022133
Received: 31 December 2020
Revised: 28 February 2022
Accepted: 10 March 2022
Published: 16 May 2022

Research article

Chaotic oscillations of 1D wave equation with or without energy-injections

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Abstract: It is interesting and challenging to study chaotic phenomena in partial differential equations. In this paper, we mainly study the problems for oscillations governed by 1D wave equation with general nonlinear feedback control law and energy-conserving or energy-injecting effects at the boundaries. We show that i) energy-injecting effect at the boundary is the necessary condition for the onset of chaos when the nonlinear feedback law is an odd function; ii) chaos never occurs if the nonlinear feedback law is an even function; iii) when one of the two ends is fixed, only the effect of self-regulation at the other end can still cause the onset of chaos; whereas if one of the two ends is free, there will never be chaos for any feedback control law at the other end. In addition, we give a sufficient condition about the general feedback law at one of two ends to ensure the occurrence of chaos. Numerical simulations are provided to demonstrate the effectiveness of the theoretical outcomes.

Keywords: chaotic vibrations; wave equation; wave propagation; generalized boundary condition

1. Introduction

First, I take this opportunity to express my great admiration toward Professor Goong Chen by dedicating this paper to him on the occasion of his 70th birthday.

In this paper, we mainly consider the problem of vibrations governed by 1D wave equation $w_{tt} - c^2 w_{xx} = 0$, where c denotes wave propagation speed, associated with a generalized boundary condition. Recall the definition of chaos for this kind of system, which is firstly introduced in [1], as below:

Definition 1.1. Consider an initial-boundary problem (S) governed by 1D wave equation $w_{tt} - c^2 w_{xx} = 0$ defined on a segment I, where $c > 0$ denotes the propagation speed of the wave. The system is said to be chaotic if there exists a large class of initial data (w_0, w_1) such that

(i) $|w_x| + |w_t|$ is uniformly bounded,

(ii) $V(t) \overset{def}{=} V_I(w_x(\cdot, t)) + V_I(w_t(\cdot, t)) < +\infty$ for all $t \geq 0$,

(iii) $\liminf_{t\to+\infty} \frac{\ln V(t)}{t} > 0$.

Remark 1.1. When chaos occurs, the function $x \mapsto (|w_x| + |w_t|)(x,t)$ is uniformly bounded, whereas the length of the curve $\{(x, \ (|w_x| + |w_t|)(x,t)), \ x \in [0,1]\}$ grows exponentially w.r.t. time t. Therefore, the system (S) must undergo extremely complex oscillations as time t increasing.

Remark 1.2. There are lots of work about chaos studies, see e.g., [2–4] and references therein. As we know, there is no a common mathematical definition for chaos, which is actually a challenge to give. However, Li-Yorke chaos is probably one of the most popular and acceptable notions of chaos. We will study the relationship between Li-Yorke chaos and chaos in the sense of Definition 1.1 in our future work. Particularly, if chaos happens and the solution (w_x, w_t) can be represented by two interval maps, denoted by K_1 and K_2, respectively, then K_1 and K_2 have positive entropy, which implies K_1 and K_2 are chaotic in the sense of Li-Yorke.

Let us take a classical model to introduce the research background and our motivation, as below:

$$\begin{cases} w_{tt} - w_{xx} = 0, \ \ x \in (0,1), \ t > 0, \\ w_x(0,t) = -\eta w_t(0,t), \ \eta \neq 1, \ \ t > 0, \\ w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t), \ \ 0 < \alpha < 1, \ \beta \geq 0, \ t > 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad 0 \leq x \leq 1, \end{cases} \tag{1.1}$$

where α,β, and η are given constants. If $\eta = 1$, the system (1.1) is not well-posed. Thus throughout this paper we assume $\eta \neq 1$. The wave equation itself is linear and represents the infinite-dimensional harmonic oscillator. Let

$$E(t) = \frac{1}{2}\int_0^1 |\nabla w(x,t)|^2 dx = \frac{1}{2}\int_0^1 \left[w_x^2(x,t) + w_t^2(x,t)\right] dx,$$

be the energy function of this system. And assume that (1.1) admits a C^2 solution, then the boundary conditions show

$$\frac{d}{dt}E(t) = \eta w_t^2(0,t) + w_t^2(1,t)[\alpha - \beta w_t^2(1,t)].$$

The right-handed side boundary condition (at $x = 1$) is nonlinear when $\beta \neq 0$, which is usually called a *van der Pol* type boundary condition (see, e.g., [1, 5–9]). The left-handed side boundary condition (at $x = 0$) is linear, where $\eta > 0$ indicates that energy is being injected into the system at $x = 0$. Thus if $\eta > 0$, the system (1.1) has a self-excited mechanism that supplies energy to the system itself, which induces irregular vibrations [1, 6]. In particular, when $\eta = 0$, the free end at $x = 0$ has no effect to the energy, which is also called an energy-conserving boundary condition.

The existence and uniqueness of the classical solution of (1.1) can be found in [5, 6]. Furthermore, the system (1.1) has a smooth solution $w \in C^2$ if the initial data satisfy

$$w_0 \in C_0^2([0,1]), \quad w_1 \in C_0^1([0,1]), \tag{1.2}$$

where

$$C_0^k([0,1]) = \{f \in C^k([0,1]) \mid f^{(i)}(0) = f^{(i)}(1) = 0, \ 0 \leq i \leq k\}, \ k \in \mathbb{N}, \tag{1.3}$$

see Theorem 6.1 in [6]. The weak solution as well as its numerical approximation are discussed in [9].

The PDE system (1.1) has received considerable attention since it exhibits many interesting and complicated dynamical phenomena, such as limit cycles and chaotic behavior of (w_t, w_x) when the parameters α, β and η assume certain values [1, 6]. Different from dynamics of a system of ODEs, this is a simple and useful infinite-dimensional model for the study of spatiotemporal behaviors as time evolutes. For instance, the propagation of acoustic waves in a pipe satisfies the linear wave equation: $w_{tt} - w_{xx} = 0$. As we know, the solution of 1D wave equation describes a superposition of two traveling wave with arbitrary profiles, one propagating with unit speed to the left, the other with unit speed to the right. The boundary conditions appeared in (1.1) can create irregularly acoustical vibrations ([1, 6, 7]). This type of vibrations, for example, can be generated by noise signals radiated from underwater vehicles, and there are intensive research for the properties of acoustical vibrations in current literature (see e.g., [10] and references therein). Hence the study of this type of vibration is not only important but also may lead to a better understanding of the dynamics of acoustic systems.

In this paper, we mainly consider the oscillation problems described by the following models:

$$\begin{cases} w_{tt}(x,t) - c^2 w_{xx}(x,t) = 0, \quad x \in (0,1), \ t > 0, \\ w_x(0,t) = -\eta w_t(0,t), \ \eta \neq c^{-1}, \quad t > 0, \\ w_x(1,t) = h(w_t(1,t)), \ t > 0, \\ w_0(x) = w(x,0), \quad w_1(x) = w_t(x,0), \quad 0 \le x \le 1, \end{cases} \tag{1.4}$$

$$\begin{cases} z_{tt}(x,t) - c^2 z_{xx}(x,t) = 0, \quad x \in (0,1), \ t > 0, \\ z(0,t) = 0, \quad t > 0, \\ z_x(1,t) = h(z_t(1,t)), \ t > 0, \\ z_0(x) = z(x,0), \quad z_1(x) = z_t(x,0), \quad 0 \le x \le 1, \end{cases} \tag{1.5}$$

where the function $h \in C^0(\mathbb{R})$ satisfies

(A1) $\phi : t \mapsto \frac{1}{2}\left(h(t) - \frac{1}{c}t\right)$ strictly monotonically decreases on $\mathbb{R}$,

(A2) $\phi(\mathbb{R}) = \mathbb{R}$.

It is clear that $\phi^{-1} : \mathbb{R} \to \mathbb{R}$ is well-defined and strictly decreases on $\mathbb{R}$. In fact, the model (1.4) is a generalized case of the van del Pol type boundary condition, $h(x) = \alpha x - \beta x^3$, say. Denote the total energy of system (1.4) as

$$E(t) = \frac{1}{2c^2} \int_0^1 \left[c^2 w_x^2(x,t) + w_t^2(x,t) \right] dx,$$

then the boundary conditions show

$$\frac{d}{dt} E(t) = \eta w_t^2(0,t) + w_t^2(1,t) \cdot \frac{h(w_t(1,t))}{w_t(1,t)}.$$

Hence $\eta > 0$ can cause the energy of system to increase. Moreover, the system (1.1) has a generalized self-excited mechanism if we assume that, roughly speaking, $\frac{h(y)}{y} > 0$ if $|y|$ is small and $\frac{h(y)}{y} < 0$ if $|y|$ is large. Thus if $\eta > 0$, the system (1.4) has a self-excited mechanism that supplies energy to the system itself, which could induce irregular oscillations. The interesting part is that the system (1.5) may have chaos even though there is no energy supplier at the boundary.

We can treat the initial-boundary problems (1.4) and (1.5) by using *wave propagation method*. Let's take (1.4) as a start. It is well-known that w, a solution of 1D wave equation, has the following form:

$$w(x,t) = L(x+ct) + R(x-ct), \tag{1.6}$$

where L and R are two C^1 functions. Then the gradient of w can be represented as follows:

$$w_t(x,t) = cL'(x+ct) - cR'(x-ct), \quad w_x(x,t) = L'(x+ct) + R'(x-ct), \tag{1.7}$$

for $x \in [0,1], t \geq 0$. Introduce two new variables

$$u(x,t) = L'(x+ct),\ v(x,t) = R'(x-ct), \tag{1.8}$$

which are called the *Riemann invariants*. It is evident that u and v keeps constants along the lines $x+ct = const.$ and $x-ct = const.$, respectively, which are referred to as *characteristics*.

When $t > 0$ and $x = 0$, from the boundary condition at the left end $x = 0$,

$$\begin{aligned} v(0,t) &= \frac{cw_x(0,t) - w_t(0,t)}{2c} \\ &= \frac{-c\eta w_t(0,t) - w_t(0,t)}{2c} = -\frac{1+c\eta}{2c} w_t(0,t) \\ &= -\frac{1+c\eta}{2}(u(0,t) - v(0,t)), \end{aligned}$$

that is

$$v(0,t) = \frac{c\eta+1}{c\eta-1} u(0,t) \stackrel{def}{=} \gamma(\eta)\cdot u(0,t). \tag{1.9}$$

Without confusion, we also take $\gamma(\eta)$ as a linear map.

When $t > 0$ and $x = 1$, from the boundary condition at the right end $x = 1$, it follows:

$$h\left(c(u(1,t) - v(1,t))\right) - \frac{1}{c}\cdot c\left(u(1,t) - v(1,t)\right) - 2v(1,t) = 0,$$

that is

$$v(1,t) = \phi\left(c(u(1,t) - v(1,t))\right),$$

which determines a reflection relationship between $u(1,t)$ and $v(1,t)$ as follow

$$u(1,t) = \frac{1}{c}\cdot\phi^{-1}\left(v(1,t)\right) + v(1,t) \stackrel{def}{=} \varphi\left(v(1,t)\right), \tag{1.10}$$

where

$$\varphi(x) = \frac{1}{c}\cdot\phi^{-1}(x) + x. \tag{1.11}$$

When $t = 0$, from the initial conditions,

$$u_0(x) \stackrel{def}{=} u(x,0) = \frac{cw_0'(x) + w_1(x)}{2c}, \quad v_0(x) \stackrel{def}{=} v(x,0) = \frac{cw_0'(x) - w_1(x)}{2c}, \tag{1.12}$$

which are referred to as the initial date of (u, v).

For $x \in [0,1]$ and $\tau \geq 0$, it follows from the boundary reflections (1.9) and (1.10) that

$$\begin{aligned} u\left(x, \tau + \frac{2}{c}\right) &= L'(x + c\tau + 2) = u\left(1, \frac{1}{c}x + \tau + \frac{1}{c}\right) = \varphi\left(v\left(1, \frac{1}{c}x + \tau + \frac{1}{c}\right)\right) \\ &= \varphi\left(R'(-x - c\tau)\right) = \varphi\left(v\left(0, \frac{1}{c}x + \tau\right)\right) \\ &= \varphi\left(\gamma(\eta) \cdot u(x, \tau)\right), \end{aligned}$$

and

$$v\left(x, \tau + \frac{2}{c}\right) = \gamma(\eta) \circ \varphi\left(v(x, \tau)\right);$$

inductively,

$$u\left(x, \tau + \frac{2n}{c}\right) = \left(\varphi \circ \gamma(\eta)\right)^n \left(u(x, \tau)\right), \quad v\left(x, \tau + \frac{2n}{c}\right) = \left(\gamma(\eta) \circ \varphi\right)^n \left(v(x, \tau)\right), \tag{1.13}$$

where the superscript $n \in \mathbb{N}$ denotes the nth iteration of a function. Analogously, for $x \in [0,1]$ and $\tau \in [0, \frac{2}{c})$,

$$u(x, \tau) = \begin{cases} u_0(x + c\tau), & 0 \leq c\tau \leq 1 - x, \\ \varphi(v_0(2 - x - c\tau)), & 1 - x < c\tau \leq 2 - x, \\ \varphi \circ \gamma(\eta)(u_0(x + c\tau - 2)), & 2 - x < c\tau < 2, \end{cases} \tag{1.14}$$

and

$$v(x, \tau) = \begin{cases} v_0(x - c\tau), & 0 \leq c\tau \leq x, \\ \gamma(\eta)(u_0(c\tau - x)), & x < c\tau \leq x + 1, \\ \gamma(\eta) \circ \varphi(v_0(2 + x - c\tau)), & x + 1 < c\tau < 2. \end{cases} \tag{1.15}$$

Equations (1.13)–(1.15) show that the system (1.4) is solvable and the dynamics of the solution (w, w_x, w_t) to the equation (1.4) can be uniquely determined by the initial data and the following two functions:

$$\psi_\eta = \gamma(\eta) \circ \varphi, \;\; g = \varphi \circ \gamma(\eta). \tag{1.16}$$

Note that $g = \gamma^{-1}(\eta) \circ \psi_\eta \circ \gamma(\eta)$, that is to say there is topological conjugacy between ψ_η and g. Therefore one only needs to consider one of them, say ψ_η.

The paper will be organized as follows. In the next two sections, we will present the necessary and sufficient conditions to cause the onset of chaos, respectively. Section 4 shows, as a special case, in which the wave equation has a fixed end, that chaos can occur with the effects of self-regulations and energy-conservation. In Section 5, there are some applications of the theoretical outcomes. In the last section, it is the numerical simulations.

2. Necessary conditions for the onset of chaos

In this section, we firstly give a necessary condition for the onset of chaos in the following system:

$$\begin{cases} w_{tt}(x, t) - c^2 w_{xx}(x, t) = 0, \;\; x \in (0, 1), \; t > 0, \\ w_x(0, t) = -\eta w_t(0, t), \; \eta \neq c^{-1}, \;\; t > 0, \\ w_x(1, t) = h(w_t(1, t)), \; t > 0, \\ w_0(x) = w(x, 0), \quad w_1(x) = w_t(x, 0), \quad 0 \leq x \leq 1, \end{cases} \tag{2.1}$$

where h satisfies hypotheses $(A1)$ and $(A2)$. In addition, assume the function $t \mapsto \left(h(t) + \frac{1}{c}t\right)$ is piece-wise monotone.

Theorem 2.1. *Suppose the system* (2.1) *is chaotic in the sense of Definition 1.1. Then h is not even and $\eta \neq 0$.*

Proof. It is equivalent to prove that there is no chaos in the system (2.1) if $\eta = 0$ or h is an even function.

Firstly, assume h is an even function. Let $\eta \in \mathbb{R}$ and $\eta \neq c^{-1}$. Recall the function ψ_η given by (1.16):

$$\psi_\eta = \gamma(\eta) \circ \varphi, \ \varphi(x) = \frac{1}{c} \cdot \phi^{-1}(x) + x.$$

Introduce a new map Q from $\mathbb{R}$ to $\mathbb{R}$ as follow:

$$Q(y) = \frac{1}{2}\left(h(y) + \frac{1}{c}y\right). \tag{2.2}$$

For $x \in \mathbb{R}$, let $y = \phi^{-1}(x)$, then

$$\begin{aligned}\psi_\eta(x) &= \gamma(\eta)\left(\frac{1}{c}y + \phi(y)\right) = \frac{1}{2}\gamma(\eta)\left(\frac{1}{c}y + h(y)\right)\\ &= \gamma(\eta) \cdot Q \circ \phi^{-1}(x).\end{aligned}$$

Since ϕ^{-1} is strictly decreasing and $\gamma(\eta) \in \mathbb{R}^*$, ψ is monotonic if and only if Q is monotonic. Let $y_1, \ y_2 \in \mathbb{R}$ with $y_1 < y_2$. From the hypothesis h being an even function, it follows

$$\begin{aligned}Q(y_2) - Q(y_1) &= \frac{1}{2}\left(h(-y_2) - \frac{1}{c}(-y_2)\right) - \frac{1}{2}\left(h(-y_1) - \frac{1}{c}(-y_1)\right)\\ &= \phi(-y_2) - \phi(-y_1)\\ &> 0,\end{aligned}$$

which implies Q strictly monotonically increases. Therefore ψ strictly monotonically increases (de-creases) if $\gamma(\eta) < 0(\gamma(\eta) > 0)$. It is easily seen that

$$\begin{aligned}w_t(x,t) &= c\left[\gamma^{-1}(\eta) \circ \psi^n\left(\gamma(\eta)u(x,\tau)\right) - \psi^n\left(v(x,\tau)\right)\right],\\ w_x(x,t) &= \gamma^{-1}(\eta) \circ \psi^n\left(\gamma(\eta)u(x,\tau)\right) + \psi^n\left(v(x,\tau)\right).\end{aligned}$$

Hence the dynamics of (w_x, w_t) is simple, and the chaos doesn't occur in the system.

Next, assume $\eta = 0$, in other words, the system is free at the left end. In this case,

$$\psi_0(x) = -x - \phi^{-1}(x).$$

We will show that there is no period point of ψ_0 with period 2. Let $x_0 \in \mathbb{R}$ satisfy $x_0 < \psi_0(x_0)$. Define two functions as follows:

$$q : x \mapsto -x + \psi_0(x_0) + x_0, \qquad k = \psi_0 - q.$$

Since $\phi^{-1}(\cdot)$ strictly monotonically decreases and

$$k(x) = -x - \phi^{-1}(x) - q(x) = \phi^{-1}(x_0) - \phi^{-1}(x),$$

k strictly monotonically increases in $[x_0,\ \psi_0(x_0)]$. Hence

$$k(\psi_0(x_0)) = \psi_0^2(x_0) - x_0 > k(x_0) = 0.$$

That implies there are no period points of ψ_0 with period 2. By *Sharkovsky's Theorem*, there are no periods of ψ_0 with period lager than 2. By virtue of the Main Theorem 6 in [12], there is still no chaos in the system when $\eta = 0$. □

Next, we consider the following system governed by 1D wave equation with a fixed end:

$$\begin{cases} z_{tt}(x,t) - c^2 z_{xx}(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ z(0,t) = 0, \quad t > 0, \\ z_x(1,t) = h(z_t(1,t)),\ t > 0, \\ z_0(x) = z(x,0), \quad z_1(x) = z_t(x,0), \quad 0 \le x \le 1, \end{cases} \tag{2.3}$$

where h satisfies hypotheses $(A1)$ and $(A2)$.

Theorem 2.2. *Suppose the system* (2.3) *is chaotic in the sense of Definition 1.1. Then h is neither an even function nor an odd function.*

Proof. Put

$$\psi_\infty(x) = x + \phi^{-1}(x).$$

By the analysis in Section 1, it is clear that (z_x, z_t) can be represented by iterations of ψ_∞ and initial data. Hence, the dynamics of (z_x, z_t) is completely determined by ψ_∞.

If h is even, ψ_∞ is monotonically monotone. If h is odd, $-\psi_\infty$ has no periodic points of period larger than 2. Therefore, chaos never occurs in the system 2.3 if h is either an even function or an odd function. □

Remark 2.1. Chaos can definitely happen in the system 2.3 for a special kind of h. One can find more details about that in the later section.

3. Chaotic oscillations of 1D wave equation with a general boundary feedback control law

In this section, we mainly try to determine some sufficient conditions to ensure the onset of chaotic oscillations in the following system:

$$\begin{cases} w_{tt}(x,t) - c^2 w_{xx}(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ w_x(0,t) = -\eta w_t(0,t),\ \eta \neq c^{-1}, \quad t > 0, \\ w_x(1,t) = h(w_t(1,t)),\ t > 0, \\ w_0(x) = w(x,0), \quad w_1(x) = w_t(x,0), \quad 0 \le x \le 1, \end{cases} \tag{3.1}$$

where $h \in C^0(\mathbb{R})$ satisfies $h(0) = 0$, hypotheses $(A1)$ and $(A2)$. When $w_t(1,t) \equiv 0$, there should be no signals feedback to $w_x(1,t)$, that's to say the right end should be free. Therefore it is reasonable to let $h(0) = 0$.

According to the analysis in the first section, we have known that the function ψ_η given by (1.16) plays a vital role in studying the dynamics of system (3.1). We firstly give two lemmas that are useful in analyzing dynamics of ψ_η, as follow:

Lemma 3.1. *Let I be a non-degenerate closed interval, $J \subseteq \mathbb{R}$ and $F : I \to J$, $G : J \to \mathbb{R}$. Assume that $V_J G < +\infty$ and F is piecewise monotone. Then*

$$V_{F(I)}G \leq V_I G \circ F < +\infty.$$

Proof. Let $I_1 = [a, b]$ be a monotone interval of F. Without loss of generality, let F monotonically increase in I_1. Let

$$P : a = t_0 \leq t_1 \leq \cdots \leq t_n = b,$$

be a partition of I_1. Then

$$V_{I_1}[G \circ F, P] = V_{F(I_1)}[G, P'] \leq V_{F(I_1)}G,$$

where

$$P' : F(a) = F(t_0) \leq F(t_1) \leq \cdots \leq F(t_n) = F(b),$$

is a partition of $F(I_1)$. Consequently, $V_{I_1}G \circ F \leq V_{F(I_1)}G$. Conversely, if

$$P' : F(a) = q_0 \leq q_1 \leq \cdots \leq q_n = F(b),$$

is a partition of of $F(I_1)$, then

$$P : a = \tilde{F}^{-1}(q_0) \leq \tilde{F}^{-1}(q_1) \leq \cdots \leq \tilde{F}^{-1}(q_n) \leq q_{n+1} = b,$$

where $\tilde{F}^{-1}(q) = \min\{x \in I_1, F(x) = q\}$, is a partition of I_1. That implies

$$V_{F(I_1)}[G, P'] = V_{I_1}[G \circ F, P] \leq V_{I_1}G \circ F.$$

Therefore, $V_{I_1}G \circ F = V_{F(I_1)}G$.

Let $(I_j)_{1 \leq j \leq n}$ be a finite sequence consisting of monotone intervals of F. Assume $\sqcup_{1 \leq j \leq n} I_j = I$ and $\#(I_i \cap I_j) \leq 1$ provided $i \neq j$. Then,

$$\begin{aligned} V_I G \circ F &= \sum_{j=1}^{n} V_{I_j} G \circ F = \sum_{j=1}^{n} V_{F(I_j)} G \\ &\geq V_{\sqcup_{1 \leq j \leq n} F(I_j)} G = V_{F(I)} G. \end{aligned}$$

□

Lemma 3.2. *Let I be a closed interval and $F \in C^0(I, I)$ be piecewise monotone. If there exist non-degenerate subintervals $I_1, I_2 \subseteq A$ with $Card(I_1 \cap I_2) \leq 1$ such that $I_2 \subseteq F(I_1)$ and $I_1 \cup I_2 \subseteq F(I_2)$, then*

$$\liminf_{n \to +\infty} \frac{\ln V_{I_1 \cup I_2} F^n}{n} \geq \ln \frac{1 + \sqrt{5}}{2}. \tag{3.2}$$

Proof. Let two subintervals $I_1, I_2 \subseteq A$ satisfy the hypothesis. Let $n \in \mathbb{N}$. Put

$$x_n = V_{I_2} F^n, \quad y_n = V_{I_1 \cup I_2} F^n,$$

where $F^0 = Id_{\mathbb{R}}$ if $n = 0$. Note that $F(\cdot)$ is continuous and piecewise monotone, it follows from Lemma 3.1 that

$$x_{n+1} = V_{I_2} F^n \circ F \geq V_{F(I_2)} F^n \geq V_{I_1 \cup I_2} F^n = y_n,$$

and

$$\begin{aligned} y_{n+2} &= V_{I_1} F^{n+1} \circ F + V_{I_2} F^{n+1} \circ F \\ &\geq V_{F(I_1)} F^{n+1} + V_{F(I_2)} F^{n+1} \\ &\geq V_{I_2} F^{n+1} + V_{I_1 \cup I_2} F^{n+1} = x_{n+1} + y_{n+1} \\ &\geq y_n + y_{n+1}. \end{aligned}$$

Let $(z_n)_{n \in \mathbb{N}}$ be *Fibonacci sequence*, i.e., $z_{n+2} = z_{n+1} + z_n$, with the initial data $z_0 = 0, z_1 = 1$. It is well known that

$$z_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right], \quad n \in \mathbb{N}.$$

It is clear that $y_1 > 0$ and $\forall n \in \mathbb{N}, y_n \geq y_1 z_n$. Therefore,

$$\liminf_{n \to +\infty} \frac{\ln V_{I_1 \cup I_2} F^n}{n} = \liminf_{n \to +\infty} \frac{\ln y_n}{n} \geq \lim_{n \to +\infty} \frac{\ln y_1 z_n}{n} = \ln \frac{1+\sqrt{5}}{2}.$$

This competes the proof. □

Proposition 3.1. *Let $h \in C^0(\mathbb{R})$ satisfy the hypothesises $(A1) - (A2)$ and ψ_η be given by (1.16). In addition, assume $h(0) = 0$ and*

(i) non-constant function $Q : t \mapsto \frac{1}{2}\left(h(t) + \frac{1}{c}t\right)$ is piecewise monotone,

(ii) there is at least one solution to the equation $h(y) + c^{-1}y = 0$ in $\mathbb{R}^*$.

Then there exist $A > 0$ and a non-degenerate interval $I \subseteq (0, +\infty) \setminus \{c^{-1}\}$ such that for any $\eta \in I$, $[0, A]$ is an invariant set of ψ_η and

$$\liminf_{n \to +\infty} \frac{\ln V_{[0,A]} \psi_\eta^n}{n} \geq \ln \frac{1+\sqrt{5}}{2}. \tag{3.3}$$

Proof. We need to finish the following two steps:

(1) Determine an invariant interval $[0, A]$ of ψ_η.

(2) To make sure the hypothesises of Lemma 3.2 holds.

It has been known that

$$\psi_\eta = \gamma(\eta) \cdot \varphi = \gamma(\eta) \cdot Q \circ \phi^{-1}.$$

Put

$$S^- = \{y < 0 \mid Q(y) = 0 \text{ and } \exists\, t \in (y, 0), Q(t) \neq 0\}$$

and

$$S^+ = \{y > 0 \mid Q(y) = 0 \text{ and } \exists\, t \in (0, y), Q(t) \neq 0\}.$$

By the hypothesises $(i) - (ii)$, $S^- \neq \emptyset$ or $S^+ \neq \emptyset$. Without loss of generality, assume $S^- \neq \emptyset$. We take

$$\bar{y} = \max S^-, \quad A = -c^{-1}\bar{y}, \tag{3.4}$$

then

$$\bar{y} < 0,\ A = \phi(\bar{y}) > 0,\ \psi_\eta(A) = 0. \tag{3.5}$$

Moreover, let

$$M = \max\{Q(t) \mid t \in [\bar{y}, 0]\}, \quad m = \min\{Q(t) \mid t \in [\bar{y}, 0]\}.$$

By the definition of S^-, it is evident that $M = 0$ or $m = 0$.

Case 1: Assume $M = 0$. That implies $m < 0$. Put

$$y_0 = \max\{t \in [\bar{y}, 0] \mid Q(t) = m\}, \quad x_0 = \phi(y_0).$$

For any $\eta \in [0, c^{-1})$, it is easily seen that

$$\psi_\eta(x_0) = \max\{\psi_\eta(x) \mid x \in [0, A]\} = m \cdot \gamma(\eta).$$

and $\psi_\eta(x) < \psi_\eta(x_0)$ provided that $x \in [0, x_0)$. We need to prove that $m \cdot \gamma(0) < A$. Proceed the proof by contradiction. Assume $m \cdot \gamma(0) \geq A$, which implies

$$\psi_0\left([0, x_0]\right) \cap \psi_0\left([x_0, A]\right) \supseteq [0, \psi_0(x_0)] \supseteq [0, A] = [0, x_0] \cup [x_0, A].$$

Thus ψ_0 is turbulent. According to Lemma 3 in [11], ψ_0 has periodic points of all periods. But from the proof of Theorem 2.1, ψ_0 has no periodic points whose period is larger than 2. We have thus reached a contradiction. Since that $m \cdot \gamma(\cdot)$ is strictly increasing in $[0, c^{-1})$ and $m \cdot \gamma(\eta) \to +\infty$ as $\eta \to (c^{-1})^-$, it is reasonable to put

$$\bar{\eta} = \gamma^{-1}\left(\frac{m}{A}\right) \in (0, c^{-1}). \tag{3.6}$$

Then for $\eta \in [0, \bar{\eta}]$, we have

$$\forall x \in [0, A], \quad 0 \leq \psi_\eta(x) \leq A,$$

that is to say $[0, A]$ is an invariant set of ψ_η provided $\eta \in [0, \bar{\eta}]$. Put

$$\underline{\eta} = \begin{cases} \gamma^{-1}\left(\frac{m}{x_0}\right), \text{ if } \frac{m}{x_0} < -1, \\ 0, \text{ else}, \end{cases} \tag{3.7}$$

then $\underline{\eta} < \bar{\eta}$ and $\psi(x_0) > x_0$ if $\eta > \underline{\eta}$. For $\eta \in [\underline{\eta}, c^{-1})$, consider the following set:

$$S(\eta, x_0) = \{x \in (0, x_0) \mid \gamma(\eta) \cdot \varphi(x) = x_0\}.$$

It is clear that $S(\eta, x_0)$ is closed. Since x_0 is uniquely determined by h and h is independent of η, the following function is well-defined:

$$\alpha(\eta) = \max S(\eta, x_0), \quad \eta \in [\underline{\eta}, c^{-1}). \tag{3.8}$$

We first prove that $\alpha(\cdot)$ is strictly monotonically decreasing in $[\underline{\eta}, c^{-1})$. Let $t_1, t_2 \in (\underline{\eta}, c^{-1})$ with $t_1 < t_2$. From the definition of $\alpha(\cdot)$, it follows

$$\begin{aligned} &\gamma(t_1)\cdot\varphi(\alpha(t_1)) = x_0, \quad \gamma(t_1)\cdot\varphi(x) > x_0, \quad x \in (\alpha(t_1), x_0],\\ &\gamma(t_2)\cdot\varphi(\alpha(t_2)) = x_0, \quad \gamma(t_2)\cdot\varphi(x) > x_0, \quad x \in (\alpha(t_2), x_0]. \end{aligned} \tag{3.9}$$

Since that

$$\gamma(t_1)\cdot\varphi(\alpha(t_2)) < \gamma(t_2)\cdot\varphi(\alpha(t_2)) = x_0,\ \gamma(t_1)\cdot\varphi(x_0) > x_0,$$

by applying the continuity of $\gamma(t_1)\cdot\varphi(\cdot)$ there exists $x \in (\alpha(t_2), x_0)$ such that $\gamma(t_1)\cdot\varphi(x) = x_0$. According to the definition of $\alpha(t_1)$, we have

$$\alpha(t_2) < x \le \alpha(t_1).$$

Therefore, $\alpha(\cdot)$ strictly monotonically decreases in $[\underline{\eta}, c^{-1})$.

Next, prove that $\alpha(\cdot)$ is continuous from the left. Let $t_0 \in (\underline{\eta}, x_0)$ be fixed. Since α is monotone, we have

$$\alpha(t_0) \le L \overset{def}{=} \lim_{t\to t_0^-} \alpha(t) < +\infty. \tag{3.10}$$

By the continuity of $\eta \mapsto \gamma(\eta)$ and $x \mapsto \varphi(x)$, we obtain

$$\lim_{\varepsilon\to 0^+} \gamma(t_0-\varepsilon) = \gamma(t_0), \quad \lim_{\varepsilon\to 0^+} \varphi(\alpha(t_0-\varepsilon)) = \varphi(L), \tag{3.11}$$

which gives $\gamma(t_0)\cdot\varphi(L) = \gamma(t_0)\cdot\varphi(\alpha(t_0)) = x_0$. From the definition of $\alpha(\cdot)$ and (3.10), it follows

$$L \le \alpha(t_0) \le L = \lim_{t\to t_0^-} \alpha(t).$$

Therefore $\alpha(\cdot)$ is continuous from the left.

In particular, $\psi_{\bar\eta}(\alpha(\underline{\eta})) = \gamma(\underline{\eta})\cdot\varphi(\alpha(\bar\eta)) = x_0$ and $\psi_{\bar\eta}(x_0) = \gamma(\underline{\eta})\cdot\varphi(x_0) = A$. Define a function as follow:

$$K(s) = \alpha(\underline{\eta} - s) - \psi^2_{\bar\eta - s}(x_0). \tag{3.12}$$

It is easily seen that $K(\cdot)$ is continuous from right at $s = 0$. Note that

$$K(0) = \alpha(\underline{\eta}) - \psi_{\bar\eta}(\psi_{\bar\eta}(x_0)) = \alpha(\underline{\eta}) - \psi_{\bar\eta}(A) = \alpha(\underline{\eta}) > 0,$$

hence there exists $\rho_0 \in (0, \bar\eta - \underline{\eta})$ such that

$$\forall s \in [0, \rho_0], K(s) = \alpha(\underline{\eta} - s) - \psi^2_{\bar\eta - s}(x_0) > 0. \tag{3.13}$$

Put

$$I = [\bar\eta - \rho_0, \bar\eta] \tag{3.14}$$

and

$$J_1 = [\alpha(\eta), x_0],\ J_2 = [x_0, \psi_\eta(x_0)],\ \eta \in I. \tag{3.15}$$

Let $\eta \in I$ be fixed. By (3.13),

$$\begin{aligned} &\psi_\eta(J_1) \supseteq \left[\psi_\eta(\alpha(\eta)), \psi_\eta(x_0)\right] = J_2,\\ &\psi_\eta(J_2) \supseteq \left[\psi^2_\eta(x_0), \psi_\eta(x_0)\right] \supseteq [\alpha(\eta), \psi_\eta(x_0)] \supseteq J_1 \cup J_2, \end{aligned}$$

which is to say hypotheses of Lemma 3.2 holds. Therefore for any $\eta \in I$, by applying Lemma 3.2 we have

$$\liminf_{n\to+\infty} \frac{\ln V_{[0,A]}\psi_\eta^n}{n} \geq \liminf_{n\to+\infty} \frac{\ln V_{J_1\cup J_2}\psi_\eta^n}{n} \geq \ln \frac{1+\sqrt{5}}{2}. \tag{3.16}$$

For the case $m = 0$, one just needs to consider $\eta > c^{-1}$. The proof is similar, we omit it. □

Theorem 3.1. *Consider the system* (3.1) *with the hypotheses* $(A1) - (A2)$. *Suppose* $h(0) = 0$ *and*

(i) non-constant function $Q : t \mapsto \frac{1}{2}\left(h(t) + \frac{1}{c}t\right)$ *is piecewise monotone,*

(ii) there is at least one solution to the equation $h(y) + c^{-1}y = 0$ *in* $\mathbb{R}^*$.

Then there exists a non-degenerate interval I such that for all $\eta \in I$ the system (3.1) *is chaotic in the sense of Definition 1.1.*

Proof. Without loss of generality, assume the equation $h(y) + c^{-1}y = 0$ has at least one solution in $(-\infty, 0)$. We still use the symbols given in the proof of Proposition 3.1, such as A, x_0, $\alpha(\eta)$ and so on. Let I and J_1, J_2 be given by (3.14) and (3.15), respectively. Take $\eta \in I$.

Let (w_0, w_1) be the initial data of system (3.1) and (w, w_x, w_t) be a solution of system (3.1). Recall the *Riemann invariants* as follow

$$u(x,t) = \frac{cw_x(x,t) + w_t(x,t)}{2c},\ v(x,t) = \frac{cw_x(x,t) - w_t(x,t)}{2c},\ x \in [0,1],\ t \geq 0,$$

and $u_0(\cdot) = u(\cdot, 0)$ and $v_0(\cdot) = v(\cdot, 0)$ are the initial data. Let $t > 2c^{-1}$, $n = [\frac{ct}{2}]$ and $\tau = t - 2nc^{-1}$ be fixed. It is clear that $0 \leq c\tau < 2$. By using (1.13), (1.14) and (1.15), we obtain

$$u(x,t) = \begin{cases} \gamma^{-1}(\eta) \circ \psi_\eta^n\left(\gamma(\eta) \cdot u_0(x + c\tau)\right), & 0 \leq c\tau \leq 1 - x, \\ \gamma^{-1}(\eta) \circ \psi_\eta^{n+1}\left(v_0(2 - x - c\tau)\right), & 1 - x < c\tau \leq 2 - x, \\ \gamma^{-1}(\eta) \circ \psi_\eta^n\left(\gamma(\eta) \cdot u_0(x + c\tau - 2)\right), & 2 - x < c\tau < 2, \end{cases} \tag{3.17}$$

and

$$v(x,t) = \begin{cases} \psi_\eta^n\left(v_0(x - c\tau)\right), & 0 \leq c\tau \leq x, \\ \psi_\eta^n\left(\gamma(\eta) \cdot u_0(c\tau - x)\right), & x < c\tau \leq x + 1, \\ \psi_\eta^{n+1}\left(v_0(2 + x - c\tau)\right), & x + 1 < c\tau < 2. \end{cases} \tag{3.18}$$

Since $[0, A]$ is an invariant set of ψ_η, $|u| + |v|$ is uniformly bounded if

$$Range(\gamma(\eta) \cdot u_0) \cup Range(v_0) \subseteq [0, A]. \tag{3.19}$$

In addition, assume that u_0 and v_0 are piecewise monotone and

$$[0, \alpha(\eta)] \subseteq Range(\gamma(\eta) \cdot u_0) \cap Range(v_0). \tag{3.20}$$

Consider the total variations of $u(\cdot, t)$ and $v(\cdot, t)$ on $[0, 1]$. When $0 \leq c\tau < 1$, from (1.13)-(1.15) and Lemma 3.1, it follows

$$\begin{aligned} V_{[0,1]}u(\cdot, t) &= V_{[0,1-c\tau]}u(\cdot, t) + V_{[1-c\tau,1]}u(\cdot, t) \\ &= V_{[c\tau,1]}\gamma^{-1}(\eta) \circ \psi_\eta^n \circ \gamma(\eta) \circ u_0 + V_{[1-c\tau,1]}\gamma^{-1}(\eta) \circ \psi_\eta^{n+1} \circ v_0 \\ &= \left|\gamma^{-1}(\eta)\right| \left(V_{[c\tau,1]}\psi_\eta^n \circ \gamma(\eta) \circ u_0 + V_{[1-c\tau,1]}\psi_\eta^{n+1} \circ v_0\right), \end{aligned}$$

$$\begin{aligned} V_{[0,1]}v(\cdot,t) &= V_{[0,c\tau]}v(\cdot,t) + V_{[c\tau,1]}v(\cdot,t) \\ &= V_{[0,c\tau]}\psi_\eta^n \circ \gamma(\eta) \circ u_0 + V_{[0,1-c\tau]}\psi_\eta^n \circ v_0. \end{aligned}$$

Note that $0 < \left|\gamma^{-1}(\eta)\right| < 1$, by Lemma 3.1 and (3.20) we obtain

$$\begin{aligned} V_{[0,1]}u(\cdot,t) + V_{[0,1]}v(\cdot,t) &\geq \left|\gamma^{-1}(\eta)\right| V_{\gamma(\eta)\cdot u_0([0,1])}\psi_\eta^n \\ &\geq \left|\gamma^{-1}(\eta)\right| V_{[0,\alpha(\eta)]}\psi_\eta^n \geq \left|\gamma^{-1}(\eta)\right| V_{\psi_\eta^2[0,\alpha(\eta)]}\psi_\eta^{n-2} \\ &\geq \left|\gamma^{-1}(\eta)\right| V_{J_1\cup J_2}\psi_\eta^{n-2}. \end{aligned}$$

Then Proposition 3.1 and (3.16) show that

$$\begin{aligned} \frac{\ln\left(V_{[0,1]}u(\cdot,t) + V_{[0,1]}v(\cdot,t)\right)}{t} &\geq \frac{\ln\left|\gamma^{-1}(\eta)\right|}{t} + \frac{\ln\left(V_{J_1\cup J_2}\psi_\eta^{n-2}\right)}{t} \\ &\geq \frac{c\ln\left(V_{J_1\cup J_2}\psi_\eta^{n-2}\right)}{2n+2} + \frac{\ln\left|\gamma^{-1}(\eta)\right|}{t} \\ &\geq \frac{c}{2}\ln\frac{1+\sqrt{5}}{2}, \text{ as } t \to +\infty. \end{aligned} \tag{3.21}$$

When $1 \leq c\tau < 2$, in the same way we can obtain

$$\begin{aligned} \frac{\ln\left(V_{[0,1]}u(\cdot,t) + V_{[0,1]}v(\cdot,t)\right)}{t} &\geq \frac{c\ln\left(V_{J_1\cup J_2}\psi_\eta^{n-1}\right)}{2n+2} + \frac{\ln\left|\gamma^{-1}(\eta)\right|}{t} \\ &\geq \frac{c}{2}\ln\frac{1+\sqrt{5}}{2}, \text{ as } t \to +\infty. \end{aligned} \tag{3.22}$$

It is evident that

$$V_{[0,1]}u(\cdot,t) + V_{[0,1]}v(\cdot,t) \leq \max\{1, c^{-1}\}\left(V_{[0,1]}w_x(\cdot,t) + V_{[0,1]}w_t(\cdot,t)\right). \tag{3.23}$$

Hence

$$\liminf_{t\to+\infty} \frac{\ln\left(V_{[0,1]}w_x(\cdot,t) + V_{[0,1]}w_t(\cdot,t)\right)}{t} \geq \frac{c}{2}\ln\frac{1+\sqrt{5}}{2} > 0.$$

Therefore, the system (3.1) is chaotic in the sense of Definition 1.1. □

4. Chaotic oscillations of 1D wave equation with a fixed end

In this section, we mainly consider the oscillation problem governed by 1D wave equation with a fixed end. We will show that only the effect of self-regulation effect at one of the two ends can cause the onset of chaos. But if the fixed end is replaced by a free end, the system never has chaos. As we know, both a fixed end and a free end are called the energy-conserving boundary condition. Even though either a fixed end or a free end has the same effect to the energy of the system, the systems may show completely different dynamics: one with chaos, the other without chaos. Thus the relationship between chaos and energy of the system is much more complicated.

Theorem 4.1. *Consider an initial-boundary problem governed by 1D wave equation with a fixed end as follows:*

$$\begin{cases} z_{tt}(x,t) - c^2 z_{xx}(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ z(0,t) = 0, \quad t > 0, \\ z_x(1,t) = h(z_t(1,t)),\ t > 0, \\ z_0(x) = z(x,0), \quad z_1(x) = z_t(x,0), \quad 0 \le x \le 1. \end{cases} \tag{4.1}$$

There exists a piecewise linear function h so that the system is chaotic in the sense of Definition 1.1.

Proof. Without loss of generality, we take $c = 1$. Put

$$h_\theta(x) = \begin{cases} -2x + 2(\theta^2 - 3), \quad x \in [2(\theta^2 - 3), -2), \\ (1 - \theta^2)x, \quad x \in [-2, 0], \\ (\theta - 1)x, \quad x \in (0, 1], \\ -2x + (\theta + 1),\ x \in (1, \theta + 1], \\ -x, \quad \text{else}, \end{cases} \tag{4.2}$$

where $\theta \in (0,1)$ is a fixed parameter. It is clear that $h_\theta \in C^0(\mathbb{R})$ and

$$\phi_\theta : x \mapsto \frac{1}{2}(h_\theta(x) - x)$$

is strictly monotonically decreasing and $\phi_\theta(\mathbb{R}) = \mathbb{R}$. Then ϕ_θ^{-1} exists. Let

$$Q_\theta(y) = \frac{1}{2}(h_\theta(y) + y),$$

then

$$\psi_\theta(x) = x + \phi_\theta^{-1}(x) = Q_\theta \circ \phi_\theta^{-1}(x).$$

Let $u = \frac{z_x + z_t}{2}$ and $u = \frac{z_x - z_t}{2}$ be the *Riemann invariants* of system 4.1. For some initial date (u_0, v_0), suppose (u, v) is solved for $t \le 2$. Then for $t = 2n + \tau$ with $\tau \in [0,2)$ and $n \in \mathbb{N}$, we have

$$u(x,t) = \psi_\theta^n(u(x,\tau)), \quad v(x,t) = \psi_\theta^n(v(x,\tau)),\ x \in [0,1]. \tag{4.3}$$

Put

$$I_1 = [-\theta - 1, \frac{1}{2}\theta - 1],\ I_2 = [\frac{1}{2}\theta - 1, 0],\ I_3 = [0, \theta^2].$$

They are monotone intervals of ψ_θ, respectively. A simple calculation shows that

$$\psi_\theta(-\theta - 1) = 0,\ \psi_\theta(\frac{1}{2}\theta - 1) = \frac{1}{2}\theta,\ \psi_\theta(0) = 0,\ \psi_\theta(\theta^2) = \theta^2 - 2.$$

Let $\theta \in (0,1)$ be sufficiently small such that $\frac{1}{2}\theta \ge \theta^2$ and $\theta^2 - 2 \le -\theta - 1$, which implies

$$I_3 \subseteq \psi_\theta(I_1) \cap \psi_\theta(I_2),\ I_1 \cup I_2 \subseteq \psi_\theta(I_3).$$

Let $n \in \mathbb{N}$. Put

$$x_n = V_{I_1}\psi_\theta^n,\ y_n = V_{I_2}\psi_\theta^n,\ z_n = V_{I_3}\psi_\theta^n.$$

Lemma 3.2 shows that

$$x_{n+1} \geq V_{\psi_\theta(I_1)}\psi_\theta^n \geq V_{I_3}\psi_\theta^n = z_n,\ y_{n+1} \geq z_n,\ z_{n+2} \geq x_{n+1} + y_{n+1},$$

which implies

$$z_{n+2} \geq 2z_n.$$

Inductively, we obtain

$$z_{2n+2} \geq 2^n z_2 > 0,\ z_{2n+1} \geq 2^n z_1 > 0.$$

Therefore,

$$\liminf_{n\to+\infty} \frac{\ln z_n}{n} \geq \frac{1}{2}\ln 2 > 0. \tag{4.4}$$

A simple calculation shows that

$$\psi_\theta(x) = 0,\ x \in (-\infty, -\theta - 1] \cup [2(3 - \theta^2), +\infty].$$

By (4.3), $|u| + |v|$ is always uniformly bounded for any initial date. Analogous to the proof of Theorem 3.1, there exists a large class of initial data such that

$$\liminf_{t\to+\infty} \frac{\ln\left(V_{[0,1]}z_x(\cdot,t) + V_{[0,1]}z_t(\cdot,t)\right)}{t} \geq \frac{1}{4}\ln 2 > 0.$$

Therefore, the proof completes. □

5. Applications

We first consider a problem about perturbations at boundaries, described by the following model:

$$\begin{cases} w_{tt}(x,t) - c^2 w_{xx}(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ w_x(0,t) = -\eta w_t(0,t),\ \eta \neq c^{-1}, \quad t > 0, \\ w_x(1,t) = -c^{-1}w_t(1,t) + \varepsilon\sin(w_t(1,t)),\ t > 0, \\ w_0(x) = w(x,0), \quad w_1(x) = w_t(x,0), \quad 0 \leq x \leq 1. \end{cases} \tag{5.1}$$

If $\varepsilon = 0$, for any initial data (w_0, w_1) and $\eta \neq c^{-1}$, by using wave propagation method we can obtain

$$\forall t > 2, x \in [0,1],\ w_x(x,t) = w_t(x,t) = 0.$$

That shows system (5.1) is of global asymptotical stability.

Assume $\varepsilon \neq 0$ and $|\varepsilon|$ is sufficiently small. Put

$$h_\varepsilon(y) = -c^{-1}y + \varepsilon\sin(y)$$

It is evident that *i*) $y \mapsto h_\varepsilon(y) + c^{-1}y$ is piecewise monotone, *ii*) $y \mapsto h_\varepsilon(y) - c^{-1}y$ strictly monotonically decreasing, *iii*) there are infinite solutions to the equation $h_\varepsilon(y) + c^{-1}y = 0$ in $\mathbb{R}^*$. That shows the hypotheses of Theorem 3.1 hold. Therefore, system (5.1) is chaotic in the sense of Definition 1.1.

The above analysis shows that this kind of system is no longer stable under small perturbations at one of the boundaries. That is to say this kind of system is not of structural stability.

As we have stated in the previous sections, if the feedback control law is even chaos never happen. In particular, consider the system described by the following model:

$$\begin{cases} z_{tt}(x,t) - z_{xx}(x,t) = 0, \ \ x \in (0,1), \ t > 0, \\ z_x(0,t) = -\eta z_t(0,t), \ \eta \neq 1, \ \text{or}\ z(0,t) = 0, \ t > 0, \\ z_z(1,t) = \cos(z_t(1,t)) - 1, \ t > 0, \\ z_0(x) = z(x,0), \quad z_1(x) = z_t(x,0), \quad 0 \le x \le 1. \end{cases} \tag{5.2}$$

The feedback control law at the right end is periodic and even. Therefore there is no chaos in the system (5.2). Moreover, there should exists a compact attractor in the system 5.2.

6. Numerical simulations

In this section, we will give some numerical simulations to validate the theoretical results of this paper. We first consider the system as follow:

$$\begin{cases} z_{tt}(x,t) - z_{xx}(x,t) = 0, \ \ x \in (0,1), \ t > 0, \\ z(0,t) = 0, \ \ t > 0, \\ z_x(1,t) = h(z_t(1,t)), \ t > 0, \\ z_0(x) = z(x,0), \quad z_1(x) = z_t(x,0), \quad 0 \le x \le 1, \end{cases} \tag{6.1}$$

where

$$h(x) = \begin{cases} -2x - \frac{11}{2}, \ \ x \in [-\frac{11}{2}, -2), \\ \frac{3}{4}x, \quad x \in [-2, 0], \\ -\frac{1}{2}x, \quad x \in (0, 1], \\ -2x + \frac{3}{2}, \ x \in (1, \frac{3}{2}], \\ -x, \quad \text{else}. \end{cases} \tag{6.2}$$

Choose

$$w_0 = 0, \ w_1(x) = \frac{1}{4}\sin^4(2\pi x), \ x \in [0,1],$$

as the initial data of system (6.1). Put

$$\psi_\infty(x) = \begin{cases} \frac{1}{3}x - \frac{11}{6}, \ \ x \in (\frac{1}{4}, \frac{11}{2}], \\ -7x, \quad x \in (0, \frac{1}{4}], \\ -\frac{1}{3}x, \quad x \in (-\frac{3}{4}, 0], \\ \frac{1}{3}x + \frac{1}{2}, \ x \in [-\frac{3}{2}, -\frac{3}{4}], \\ 0, \quad \text{else}. \end{cases} \tag{6.3}$$

As we show in Section 4, the solution (z, z_x, z_t) can be represented by the iterations of this ψ_∞ and initial data (z_0, z_1). Note that h and ψ_∞ are piecewise linear, thus the feedback control at the boundaries is easy to implement and the solutions of system are much simpler to be solved and represented.

We present the graphics in some detail, for z_x, z_t for $98 \le t \le 100$. Figure 1 shows that z_x, z_t are extremely oscillatory in every direction of space and time.

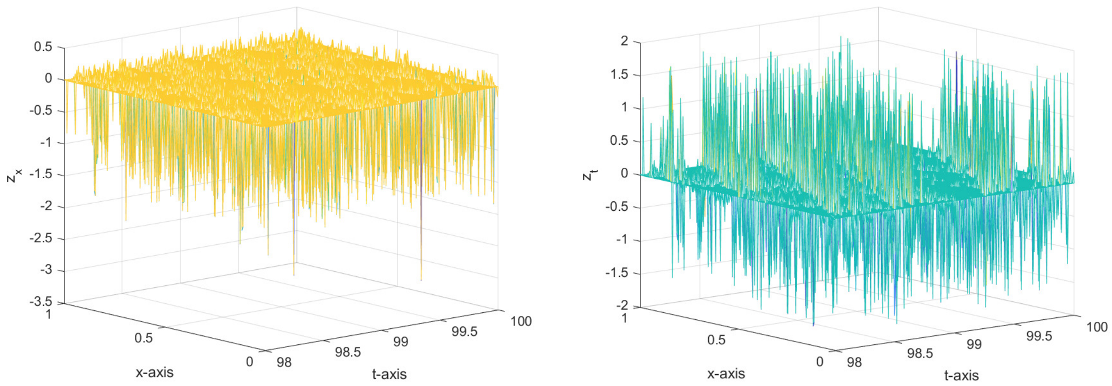

Figure 1. The profile of z_x (left) z_t(right) for $t \in [98, 100]$.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant 11671410 and the Natural Science Foundation of Guangdong Province, China (No.2022A1515012153).

Conflict of interest

The authors declare there is no conflicts of interest.

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