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Research article

A stochastic linear-quadratic differential game with time-inconsistency

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Abstract: We consider a general stochastic linear-quadratic differential game with time-inconsistency. The time-inconsistency arises from the presence of quadratic terms of the expected state as well as state-dependent term in the objective functionals. We define an equilibrium strategy, which is different from the classical one, and derive a sufficient condition for equilibrium strategies via a system of forward-backward stochastic differential equation. When the state is one-dimensional and the coefficients are all deterministic, we find an explicit equilibrium strategy. The uniqueness of such equilibrium strategy is also given.

Keywords: time-inconsistency; stochastic linear-quadratic differential game; equilibrium strategy; BSDE; forward-backward stochastic differential equation

1. Introduction

Time inconsistency in dynamic decision making is often observed in social systems and daily life. Motivated by practical applications, especially in mathematical economics and finance, time-inconsistency control problems have recently attracted considerable research interest and efforts attempting to seek equilibrium, instead of optimal controls. At a conceptual level, the idea is that a decision made by the controller at every instant of time is considered as a game against all the decisions made by the future incarnations of the controller. An "equilibrium" control is therefore one such that any deviation from it at any time instant will be worse off. The study on time inconsistency by economists can be dated back to Stroz [1] and Phelps [2,3] in models with discrete time (see [4] and [5] for further developments), and adapted by Karp [6,7], and by Ekeland and Lazrak [8–13] to the case of continuous time. In the LQ control problems, Yong [14] studied a time-inconsistent deterministic model and derived equilibrium controls via some integral equations.

It is natural to study time inconsistency in the stochastic models. Ekeland and Pirvu [15] studied the non-exponential discounting which leads to time inconsistency in an agent's investment-consumption policies in a Merton model. Grenadier and Wang [16] also studied the hyperbolic discounting problem in an optimal stopping model. In a Markovian systems, Björk and Murgoci [17] proposed a definition of a general stochastic control problem with time inconsistent terms, and proposed some sufficient condition for a control to be solution by a system of integro-differential equations. They constructed some solutions for some examples including an LQ one, but it looks very hard to find not-to-harsh condition on parameters to ensure the existence of a solution. Björk, Murgoci and Zhou [18] also constructed an equilibrium for a mean-variance portfolio selection with state-dependent risk aversion. Basak and Chabakauri [19] studied the mean-variance portfolio selection problem and got more details on the constructed solution. Hu, Jin and Zhou [20, 21] studied the general LQ control problem with time inconsistent terms in a non-Markovian system and constructed an unique equilibrium for quite general LQ control problem, including a non-Markovian system.

To the best of our knowledge, most of the time-inconsistent problems are associated with the control problems though we use the game formulation to define its equilibrium. In the problems of game theory, the literatures about time inconsistency is little [22, 23]. However, the definitions of equilibrium strategies in the above two papers are based on some corresponding control problems like before. In this paper, we formulate a general stochastic LQ differential game, where the objective functional of each player include both a quadratic term of the expected state and a state-dependent term. These non-standard terms each introduces time inconsistency into the problem in somewhat different ways. We define our equilibrium via open-loop controls. Then we derive a general sufficient condition for equilibrium strategies through a system of forward-backward stochastic differential equations (FBSDEs). An intriguing feature of these FBSDEs is that a time parameter is involved; so these form a flow of FBSDEs. When the state process is scalar valued and all the coefficients are deterministic functions of time, we are able to reduce this flow of FBSDEs into several Riccati-like ODEs. Comparing to the ODEs in [20], though the state process is scalar valued, the unknowns are matrix-valued because of two players. Therefore, such ODEs are harder to solve than those of [20]. Under some more stronger conditions, we obtain explicitly an equilibrium strategy, which turns out to be a linear feedback. We also prove that the equilibrium strategy we obtained is unique.

The rest of the paper is organized as follows. The next section is devoted to the formulation of our problem and the definition of equilibrium strategy. In Section 3, we apply the spike variation technique to derive a flow of FBSEDs and a sufficient condition of equilibrium strategies. Based on this general results, we solve in Section 4 the case when the state is one dimensional and all the coefficients are deterministic. The uniqueness of such equilibrium strategy is also proved in this section.

2. Problem setting

Let $T > 0$ be the end of a finite time horizon, and let $(W_t)_{0\le t\le T} = (W_t^1, ..., W_t^d)_{0\le t\le T}$ be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $(\mathcal{F}_t)$ the augmented filtration generated by (W_t).

Let $\mathbb{S}^n$ be the set of symmetric $n \times n$ real matrices; $L^2_{\mathcal{F}}(\Omega, \mathbb{R}^l)$ be the set of square-integrable random variables; $L^2_{\mathcal{F}}(t, T; \mathbb{R}^n)$ be the set of $\{\mathcal{F}_s\}_{s\in[t,T]}$-adapted square-integrable processes; and $L^2_{\mathcal{F}}(\Omega; C(t, T; \mathbb{R}^n))$ be the set of continuous $\{\mathcal{F}_s\}_{s\in[t,T]}$-adapted square-integrable processes.

We consider a continuous-time, n-dimensional nonhomogeneous linear controlled system:

$$dX_s = [A_sX_s + B'_{1,s}u_{1,s} + B'_{2,s}u_{2,s} + b_s]ds + \sum_{j=1}^{d}[C^j_sX_s + D^j_{1,s}u_{1,s} + D^j_{2,s}u_{2,s} + \sigma^j_s]dW^j_s, \quad X_0 = x_0. \quad (2.1)$$

Here A is a bounded deterministic function on $[0,T]$ with value in $\mathbb{R}^{n\times n}$. The other parameters B_1, B_2, C, D_1, D_2 are all essentially bounded adapted processes on $[0,T]$ with values in $\mathbb{R}^{l\times n}, \mathbb{R}^{l\times n}, \mathbb{R}^{n\times n}, \mathbb{R}^{n\times l}, \mathbb{R}^{n\times l}$, respectively; b and σ^j are stochastic processes in $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$. The processes $u_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$, $i = 1,2$ are the controls, and X is the state process valued in $\mathbb{R}^n$. Finally, $x_0 \in \mathbb{R}^n$ is the initial state. It is obvious that for any controls $u_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$, $i = 1,2$, there exists a unique solution $X \in L^2_{\mathcal{F}}(\Omega, C(0,T;\mathbb{R}^n))$.

As time evolves, we need to consider the controlled system starting from time $t \in [0,T]$ and state $x_t \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$:

$$dX_s = [A_sX_s + B'_{1,s}u_{1,s} + B'_{2,s}u_{2,s} + b_s]ds + \sum_{j=1}^{d}[C^j_sX_s + D^j_{1,s}u_{1,s} + D^j_{2,s}u_{2,s} + \sigma^j_s]dW^j_s, \quad X_t = x_t. \quad (2.2)$$

For any controls $u_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$, $i = 1,2$, there exists a unique solution $X^{t,x_t,u_1,u_2} \in L^2_{\mathcal{F}}(\Omega, C(0,T;\mathbb{R}^n))$.

We consider a two-person differential game problem. At any time t with the system state $X_t = x_t$, the i-th ($i = 1,2$) person's aim is to minimize her cost (if maximize, we can times the following function by -1):

$$\begin{aligned} J_i(t,x_t;u_1,u_2) &= \frac{1}{2}\mathbb{E}_t\int_t^T[\langle Q_{i,s}X_s, X_s\rangle + \langle R_{i,s}u_{i,s}, u_{i,s}\rangle]ds + \frac{1}{2}\mathbb{E}_t[\langle G_iX_T, X_T\rangle] \\ &\quad -\frac{1}{2}\langle h_i\mathbb{E}_t[X_T], \mathbb{E}_t[X_T]\rangle - \langle \lambda_i x_t + \mu_i, \mathbb{E}_t[X_T]\rangle \end{aligned} \quad (2.3)$$

over $u_1, u_2 \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^l)$, where $X = X^{t,x_t,u_1,u_2}$, and $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$. Here, for $i = 1,2$, Q_i and R_i are both given essentially bounded adapted process on $[0,T]$ with values in $\mathbb{S}^n$ and $\mathbb{S}^l$, respectively, $G_i, h_i, \lambda_i, \mu_i$ are all constants in $\mathbb{S}^n$, $\mathbb{S}^n$, $\mathbb{R}^{n\times n}$ and $\mathbb{R}^n$, respectively. Furthermore, we assume that Q_i, R_i are non-negative definite almost surely and G_i are non-negative definite.

Given a control pair (u_1^*, u_2^*). For any $t \in [0,T)$, $\epsilon > 0$, and $v_1, v_2 \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^l)$, define

$$u^{t,\epsilon,v_i}_{i,s} = u^*_{i,s} + v_i\mathbf{1}_{s\in[t,t+\epsilon)}, \qquad s \in [t,T],\ i = 1,2. \quad (2.4)$$

Because each person at time $t > 0$ wants to minimize his/her cost as we claimed before, we have

Definition 2.1. *Let $(u_1^*, u_2^*) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$ be a given strategy pair, and let X^* be the state process corresponding to (u_1^*, u_2^*). The strategy pair (u_1^*, u_2^*) is called an equilibrium if*

$$\lim_{\epsilon\downarrow 0}\frac{J_1(t,X^*_t;u_1^{t,\epsilon,v_1},u_2^*) - J_1(t,X^*_t;u_1^*,u_2^*)}{\epsilon} \geq 0, \quad (2.5)$$

$$\lim_{\epsilon\downarrow 0}\frac{J_2(t,X^*_t;u_1^*,u_2^{t,\epsilon,v_2}) - J_2(t,X^*_t;u_1^*,u_2^*)}{\epsilon} \geq 0, \quad (2.6)$$

where u_i^{t,ϵ,v_i}, $i = 1,2$ are defined by (2.4), for any $t \in [0,T)$ and $v_1, v_2 \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^l)$.

Remark. The above definition means that, in each time t, the equilibrium is a static Nash equilibrium in a corresponding game.

3. Main result

Let (u_1^*, u_2^*) be a fixed strategy pair, and let X^* be the corresponding state process. For any $t \in [0, T)$, as a similar arguments of Theorem 5.1 in pp. 309 of [24], defined in the time interval $[t, T]$, there exist adapted processes $(p_i(\cdot;t), (k_i^j(\cdot;t)_{j=1,2,\ldots,d})) \in L^2_{\mathcal{F}}(t, T; \mathbb{R}^n) \times (L^2_{\mathcal{F}}(t, T; \mathbb{R}^n))^d$ and $(P_i(\cdot;t), (K_i^j(\cdot;t)_{j=1,2,\ldots,d})) \in L^2_{\mathcal{F}}(t, T; \mathbb{S}^n) \times (L^2_{\mathcal{F}}(t, T; \mathbb{S}^n))^d$ for $i = 1, 2$ satisfying the following equations:

$$\begin{cases} dp_i(s;t) = -[A_s' p_i(s;t) + \sum_{j=1}^d (C_s^j)' k_i^j(s;t) + Q_{i,s} X_s^*]ds + \sum_{j=1}^d k_i^j(s;t) dW_s^j, \quad s \in [t, T], \\ p_i(T;t) = G_i X_T^* - h_i \mathbb{E}_t[X_T^*] - \lambda_i X_t^* - \mu_i, \end{cases} \tag{3.1}$$

$$\begin{cases} dP_i(s;t) = -\Big\{A_s' P_i(s;t) + P_i(s;t) A_s + Q_{i,s} + \sum_{j=1}^d [(C_s^j)' P_i(s;t) C_s^j + (C_s^j)' K_i^j(s;t) + K_i^j(s;t) C_s^j]\Big\} ds \\ \qquad\qquad + \sum_{j=1}^d K_i^j(s;t) dW_i^j, \quad s \in [t, T], \\ P_i(T;t) = G_i, \end{cases} \tag{3.2}$$

for $i = 1, 2$. From the assumption that Q_i and G_i are non-negative definite, it follows that $P_i(s;t)$ are non-negative definite for $i = 1, 2$.

Proposition 1. *For any $t \in [0, T)$, $\epsilon > 0$, and $v_1, v_2 \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^l)$, define $u_i^{t,\epsilon,v_i}, i = 1, 2$ by (2.4). Then*

$$J_1(t, X_t^*; u_1^{t,\epsilon,v_1}, u_2^*) - J_1(t, X_t^*; u_1^*, u_2^*) = \mathbb{E}_t \int_t^{t+\epsilon} \Big\{\langle \Lambda_1(s;t), v_1 \rangle + \frac{1}{2} \langle H_1(s;t) v_1, v_1 \rangle \Big\} ds + o(\epsilon), \tag{3.3}$$

$$J_2(t, X_t^*; u_1^*, u_2^{t,\epsilon,v_2}) - J_2(t, X_t^*; u_1^*, u_2^*) = \mathbb{E}_t \int_t^{t+\epsilon} \Big\{\langle \Lambda_2(s;t), v_2 \rangle + \frac{1}{2} \langle H_2(s;t) v_2, v_2 \rangle \Big\} ds + o(\epsilon), \tag{3.4}$$

where $\Lambda_i(s;t) = B_{i,s} p_i(s;t) + \sum_{j=1}^d (D_{i,s}^j)' k_i^j(s;t) + R_{i,s} u_{i,s}^*$ and $H_i(s;t) = R_{i,s} + \sum_{j=1}^d (D_{i,s}^j)' P_i(s;t) D_{i,s}^j$ for $i = 1, 2$.

Proof. Let X^{t,ϵ,v_1,v_2} be the state process corresponding to $u_i^{t,\epsilon,v_i}, i = 1, 2$. Then by standard perturbation approach (cf. [20, 25] or pp. 126-128 of [24]), we have

$$X_s^{t,\epsilon,v_1,v_2} = X_s^* + Y_s^{t,\epsilon,v_1,v_2} + Z_s^{t,\epsilon,v_1,v_2}, \quad s \in [t, T], \tag{3.5}$$

where $Y \equiv Y^{t,\epsilon,v_1,v_2}$ and $Z \equiv Z^{t,\epsilon,v_1,v_2}$ satisfy

$$\begin{cases} dY_s = A_s Y_s ds + \sum_{j=1}^d [C_s^j Y_s + D_{1,s}^j v_1 \mathbf{1}_{s \in [t,t+\epsilon)} + D_{2,s}^j v_2 \mathbf{1}_{s \in [t,t+\epsilon)}] dW_s^j, \quad s \in [t, T], \\ Y_t = 0, \end{cases} \tag{3.6}$$

$$\begin{cases} dZ_s = [A_s Z_s + B_{1,s}' v_1 \mathbf{1}_{s \in [t,t+\epsilon)} + B_{2,s}' v_2 \mathbf{1}_{s \in [t,t+\epsilon)}] ds + \sum_{j=1}^d C_s^j Z_s dW_s^j, \quad s \in [t, T], \\ Z_t = 0. \end{cases} \tag{3.7}$$

Moreover, by Theorem 4.4 in [24], we have

$$\mathbb{E}_t \Big[\sup_{s \in [t,T)} |Y_s|^2 \Big] = O(\epsilon), \quad \mathbb{E}_t \Big[\sup_{s \in [t,T)} |Z_s|^2 \Big] = O(\epsilon^2). \tag{3.8}$$

With A being deterministic, it follows from the dynamics of Y that, for any $s \in [t, T]$, we have

$$\mathbb{E}_t[Y_s] = \int_t^s \mathbb{E}_t[A_s Y_\tau] d\tau = \int_t^s A_s \mathbb{E}_t[Y_\tau] d\tau. \tag{3.9}$$

Hence we conclude that

$$\mathbb{E}_t[Y_s] = 0 \quad s \in [t, T]. \tag{3.10}$$

By these estimates, we can calculate

$$\begin{aligned}
& J_i(t, X_t^*; u_1^{t,\epsilon,v_1}, u_2^{t,\epsilon,v_2}) - J_i(t, X_t^*; u_1^*, u_2^*) \\
&= \frac{1}{2}\mathbb{E}_t \int_t^T [\langle Q_{i,s}(2X_s^* + Y_s + Z_s), Y_s + Z_s\rangle + \langle R_{i,s}(2u_i^* + v_i), v_i\rangle \mathbf{1}_{s\in[t,t+\epsilon)}]ds \\
&\quad + \mathbb{E}_t[\langle G_i X_T^*, Y_T + Z_T\rangle] + \frac{1}{2}\mathbb{E}_t[\langle G_i(Y_T + Z_T), Y_T + Z_T\rangle] \\
&\quad - \langle h_i \mathbb{E}_t[X_T^*] + \lambda_i X_t^* + \mu_i, \mathbb{E}_t[Y_T + Z_T]\rangle - \frac{1}{2}\langle h_i \mathbb{E}_t[Y_T + Z_T], \mathbb{E}_t[Y_T + Z_T]\rangle \\
&= \frac{1}{2}\mathbb{E}_t \int_t^T [\langle Q_{i,s}(2X_s^* + Y_s + Z_s), Y_s + Z_s\rangle + \langle R_{i,s}(2u_i^* + v_i), v_i\rangle \mathbf{1}_{s\in[t,t+\epsilon)}]ds \\
&\quad + \mathbb{E}_t[\langle G_i X_T^* - h_i \mathbb{E}_t[X_T^*] - \lambda_i X_t^* - \mu_i, Y_T + Z_T\rangle + \frac{1}{2}\langle G_i(Y_T + Z_T), Y_T + Z_T\rangle] + o(\epsilon).
\end{aligned} \tag{3.11}$$

Recalling that $(p_i(\cdot; t), k_i(\cdot; t))$ and $(P_i(\cdot; t), K_i(\cdot; t))$ solve, respectively, BSDEs (3.1) and (3.2) for $i = 1, 2$, we have

$$\begin{aligned}
& \mathbb{E}_t[\langle G_i X_T^* - h_i \mathbb{E}_t[X_T^*] - \lambda_i X_t^* - \mu_i, Y_T + Z_T\rangle] \\
&= \mathbb{E}_t[\langle p_i(T; t), Y_T + Z_T\rangle] \\
&= \mathbb{E}_t\Big[\int_t^T d\langle p_i(s; t), Y_s + Z_s\rangle\Big] \\
&= \mathbb{E}_t \int_t^T \Big[\langle p_i(s; t), A_s(Y_s + Z_s) + B_{1,s}' v_1 \mathbf{1}_{s\in[t,t+\epsilon)} + B_{2,s}' v_2 \mathbf{1}_{s\in[t,t+\epsilon)}\rangle \\
&\quad - \langle A_s' p_i(s; t) + \sum_{j=1}^d (C_s^j)' k_i^j(s; t) + Q_{i,s} X_s^*, Y_s + Z_s\rangle \\
&\quad + \sum_{j=1}^d \langle k_i^j(s; t), C_s^j(Y_s + Z_s) + D_{1,s}^j v_1 \mathbf{1}_{s\in[t,t+\epsilon)} + D_{2,s}^j v_2 \mathbf{1}_{s\in[t,t+\epsilon)}\rangle\Big]ds \\
&= \mathbb{E}_t \int_t^T \Big[\langle -Q_{i,s} X_s^*\rangle + \Big\langle B_{1,s} p_i(s; t) + \sum_{j=1}^d (D_{1,s}^j)' k_i^j(s; t), v_1 \mathbf{1}_{s\in[t,t+\epsilon)}\Big\rangle \\
&\quad + \Big\langle B_{2,s} p_i(s; t) + \sum_{j=1}^d (D_{2,s}^j)' k_i^j(s; t), v_2 \mathbf{1}_{s\in[t,t+\epsilon)}\Big\rangle\Big]ds
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
& \mathbb{E}_t[\frac{1}{2}\langle G_i(Y_T + Z_T), Y_T + Z_T\rangle] \\
&= \mathbb{E}_t[\frac{1}{2}\langle P_i(T; t)(Y_T + Z_T), Y_T + Z_T\rangle] \\
&= \mathbb{E}_t\Big[\int_t^T d\langle P_i(s; t)(Y_s + Z_s), Y_s + Z_s\rangle\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_t \int_t^T \Big\{ \langle P_i(s;t)(Y_s+Z_s), A_s(Y_s+Z_s) + B'_{1,s}v_1\mathbf{1}_{s\in[t,t+\epsilon)} + B'_{2,s}v_2\mathbf{1}_{s\in[t,t+\epsilon)} \rangle \\
&\quad + \langle P_i(s;t)[A_s(Y_s+Z_s) + B'_{1,s}v_1\mathbf{1}_{s\in[t,t+\epsilon)} + B'_{2,s}v_2\mathbf{1}_{s\in[t,t+\epsilon)}], Y_s+Z_s \rangle \\
&\quad - \langle [A'_s P_i(s;t) + P_i(s;t)A_s + Q_{i,s} \\
&\qquad + \sum_{j=1}^d ((C_s^j)' P_i(s;t) C_s^j + (C_s^j)' K_i^j(s;t) + K_i^j(s;t) C_s^j)](Y_s+Z_s), Y_s+Z_s \rangle \\
&\quad + \sum_{j=1}^d \langle K_i^j(s;t)(Y_s+Z_s), C_s^j(Y_s+Z_s) + D_{1,s}^j v_1\mathbf{1}_{s\in[t,t+\epsilon)} + D_{2,s}^j v_2\mathbf{1}_{s\in[t,t+\epsilon)} \rangle \\
&\quad + \sum_{j=1}^d \langle K_i^j(s;t)[C_s^j(Y_s+Z_s) + D_{1,s}^j v_1\mathbf{1}_{s\in[t,t+\epsilon)} + D_{2,s}^j v_2\mathbf{1}_{s\in[t,t+\epsilon)}], Y_s+Z_s \rangle \\
&\quad + \sum_{j=1}^d \langle P_i(s;t)[C_s^j(Y_s+Z_s) + D_{1,s}^j v_1\mathbf{1}_{s\in[t,t+\epsilon)} + D_{2,s}^j v_2\mathbf{1}_{s\in[t,t+\epsilon)}], \\
&\qquad\qquad C_s^j(Y_s+Z_s) + D_{1,s}^j v_1\mathbf{1}_{s\in[t,t+\epsilon)} + D_{2,s}^j v_2\mathbf{1}_{s\in[t,t+\epsilon)} \rangle \Big\} ds \\
&= \mathbb{E}_t \int_t^T \Big[- \langle Q_{i,s}(Y_s+Z_s), Y_s+Z_s \rangle \\
&\quad + \sum_{j=1}^d \langle P_i(s;t)[D_{1,s}^j v_1 + D_{2,s}^j v_2], D_{1,s}^j v_1 + D_{2,s}^j v_2 \rangle \mathbf{1}_{s\in[t,t+\epsilon)} \Big] ds + o(\epsilon) \qquad (3.13)
\end{aligned}
$$

Combining (3.11)-(3.13), we have

$$
\begin{aligned}
&J_i(t, X_t^*; u_1^{t,\epsilon,v_1}, u_2^{t,\epsilon,v_2}) - J_i(t, X_t^*; u_1^*, u_2^*) \\
&= \mathbb{E}_t \int_t^T \Big[\frac{1}{2} \langle R_{i,s}(2u_i^* + v_i), v_i \rangle \mathbf{1}_{s\in[t,t+\epsilon)} + \Big\langle B_{1,s} p_i(s;t) + \sum_{j=1}^d (D_{1,s}^j)' k_i^j(s;t), v_1\mathbf{1}_{s\in[t,t+\epsilon)} \Big\rangle \\
&\quad + \Big\langle B_{2,s} p_i(s;t) + \sum_{j=1}^d (D_{2,s}^j)' k_i^j(s;t), v_2\mathbf{1}_{s\in[t,t+\epsilon)} \Big\rangle \\
&\quad + \frac{1}{2} \sum_{j=1}^d \langle P_i(s;t)[D_{1,s}^j v_1 + D_{2,s}^j v_2], D_{1,s}^j v_1 + D_{2,s}^j v_2 \rangle \mathbf{1}_{s\in[t,t+\epsilon)} \Big] ds + o(\epsilon). \qquad (3.14)
\end{aligned}
$$

Take $i = 1$, we let $v_2 = 0$, then $u_2^{t,\epsilon,v_2} = u_2^*$, from (3.14), we obtain

$$
\begin{aligned}
&J_1(t, X_t^*; u_1^{t,\epsilon,v_1}, u_2^*) - J_1(t, X_t^*; u_1^*, u_2^*) \\
&= \mathbb{E}_t \int_t^T \Big\{ \Big\langle R_{1,s}u_1^* + B_{1,s} p_1(s;t) + \sum_{j=1}^d (D_{1,s}^j)' k_1^j(s;t), v_1\mathbf{1}_{s\in[t,t+\epsilon)} \Big\rangle \\
&\quad + \frac{1}{2} \Big\langle \Big[R_{1,s} + \sum_{j=1}^d (D_{1,s}^j)' P_1(s;t) D_{1,s}^j \Big] v_1, v_1 \Big\rangle \Big\} ds
\end{aligned}
$$

$$= \mathbb{E}_t \int_t^{t+\epsilon} \Big\{ \langle \Lambda_1(s;t), v_1 \rangle + \frac{1}{2} \langle H_1(s;t) v_1, v_1 \rangle \Big\} ds + o(\epsilon). \tag{3.15}$$

This proves (3.3), and similarly, we obtain (3.4). □

Because of $R_{i,s}$ and $P_i(s;t), i = 1,2$ are non-negative definite, $H_i(s;t),\ i = 1,2$ are also non-negative definite. In view of (3.3)-(3.4), a sufficient condition for an equilibrium is

$$\mathbb{E}_t \int_t^T |\Lambda_i(s;t)| ds < +\infty, \quad \lim_{s \downarrow t} \mathbb{E}_t[\Lambda_i(s;t)] = 0 \ a.s. \ \forall t \in [0,T], \quad i = 1,2. \tag{3.16}$$

By an arguments similar to the proof of Proposition 3.3 in [21], we have the following lemma:

Lemma 3.1. *For any triple of state and control processes* (X^*, u_1^*, u_2^*)*, the solution to BSDE (3.1) in* $L^2(0,T;\mathbb{R}^n) \times (L^2(0,T;\mathbb{R}^n))^d$ *satisfies* $k_i(s;t_1) = k_i(s;t_2)$ *for a.e.* $s \geq \max\{t_1,t_2\},\ i = 1,2$*. Furthermore, there exist* $\rho_i \in L^2(0,T;\mathbb{R}^l)$*,*$\delta_i \in L^2(0,T;\mathbb{R}^{l\times n})$ *and* $\xi_i \in L^2(\Omega; C(0,T;\mathbb{R}^n))$*, such that*

$$\Lambda_i(s;t) = \rho_i(s) + \delta_i(s)\xi_i(t), \qquad i = 1,2. \tag{3.17}$$

Therefore, we have another characterization for equilibrium strategies:

Proposition 2. *Given a strategy pair* $(u_1^*, u_2^*) \in L^2(0,T;\mathbb{R}^l) \times L^2(0,T;\mathbb{R}^l)$*. Denote* X^* *as the state process, and* $(p_i(\cdot;t), (k_i^j(\cdot;t)_{j=1,2,\ldots,d})) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^n) \times (L^2_{\mathcal{F}}(t,T;\mathbb{R}^n))^d$ *as the unique solution for the BSDE (3.1), with* $k_i(s) = k_i(s;t)$ *according to Lemma 3.1 for* $i = 1,2$ *respectively. For* $i = 1,2$*, letting*

$$\Lambda_i(s,t) = B_{i,s} p_i(s;t) + \sum_{j=1}^d (D_{j,s})' k(s;t)^j + R_{i,s} u_{i,s}^*, \quad s \in [t,T], \tag{3.18}$$

then u^* *is an equilibrium strategy if and only if*

$$\Lambda_i(t,t) = 0, \ a.s., \ a.e. \ t \in [0,T], \quad i = 1,2. \tag{3.19}$$

Proof. From (3.17), we have $\Lambda_1(s;t) = \rho_1(s) + \delta_1(s)\xi_1(t)$. Since δ_1 is essentially bounded and ξ_1 is continuous, we have

$$\lim_{\epsilon \downarrow 0} \mathbb{E}_t \left[\frac{1}{\epsilon} \int_t^{t+\epsilon} |\delta_1(s)(\xi_1(s) - \xi_1(t))| ds \right] \leq c \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \mathbb{E}_t[|\xi_1(s) - \xi_1(t)|] ds = 0,$$

and hence

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \mathbb{E}_t[\Lambda_1(s;t)] ds = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \mathbb{E}_t[\Lambda_1(s;s)] ds.$$

Therefore, if (3.19) holds, we have

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \mathbb{E}_t[\Lambda_1(s;t)] ds = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \mathbb{E}_t[\Lambda_1(s;s)] ds = 0.$$

When $i = 2$, we can prove (3.19) similarly.

Conversely, if (3.16) holds, then $\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \mathbb{E}_t[\Lambda_i(s;s)] ds = 0, i = 1,2$ leading to (3.19) by virtue of Lemma 3.4 of [21]. □

The following is the main general result for the stochastic LQ differential game with time-inconsistency.

Theorem 3.2. *A strategy pair* $(u_1^*, u_2^*) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$ *is an equilibrium strategy pair if the following two conditions hold for any time t:*

(i) The system of SDEs

$$\begin{cases} dX_s^* = [A_s X_s^* + B'_{1,s}u^*_{1,s} + B'_{2,s}u^*_{2,s} + b_s]ds + \sum_{j=1}^d [C_s^j X_s^* + D^j_{1,s}u^*_{1,s} + D^j_{2,s}u^*_{2,s} + \sigma_s^j]dW_s^j, \\ X_0^* = x_0, \\ dp_1(s;t) = -[A'_s p_1(s;t) + \sum_{j=1}^d (C_s^j)' k_1^j(s;t) + Q_{1,s}X_s^*]ds + \sum_{j=1}^d k_1^j(s;t)dW_s^j, \quad s \in [t,T], \\ p_1(T;t) = G_1 X_T^* - h_1 \mathbb{E}_t[X_T^*] - \lambda_1 X_t^* - \mu_1, \\ dp_2(s;t) = -[A'_s p_2(s;t) + \sum_{j=1}^d (C_s^j)' k_2^j(s;t) + Q_{2,s}X_s^*]ds + \sum_{j=1}^d k_2^j(s;t)dW_s^j, \quad s \in [t,T], \\ p_2(T;t) = G_2 X_T^* - h_2 \mathbb{E}_t[X_T^*] - \lambda_2 X_t^* - \mu_2, \end{cases} \tag{3.20}$$

admits a solution $(X^*, p_1, k_1, p_2, k_2)$*;*

(ii) $\Lambda_i(s;t) = R_{i,s}u^*_{i,s} + B_{i,s}p_i(s;t) + \sum_{j=1}^d (D^j_{i,s})' k_i^j(s;t), i = 1,2$ *satisfy condition (3.19).*

Proof. Given a strategy pair $(u_1^*, u_2^*) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$ satisfying (i) and (ii), then for any $v_1, v_2 \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^l)$, define $\Lambda_i, H_i, i = 1,2$ as in Proposition 1. We have

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} \frac{J_1(t, X_t^*; u_1^{t,\epsilon,v_1}, u_2^*) - J_1(t, X_t^*; u_1^*, u_2^*)}{\epsilon} \\ &= \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}_t \int_t^{t+\epsilon} \Big\{ \langle \Lambda_1(s;t), v_1 \rangle + \frac{1}{2} \langle H_1(s;t) v_1, v_1 \rangle \Big\} ds}{\epsilon} \\ &\geq \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}_t \int_t^{t+\epsilon} \langle \Lambda_1(s;t), v_1 \rangle ds}{\epsilon} \\ &= 0, \end{aligned} \tag{3.21}$$

proving the first condition of Definition 2.1, and the proof of the second condition is similar. □

Theorem 3.2 involve the existence of solutions to a flow of FBSDEs along with other conditions. The system (3.20) is more complicated than system (3.6) in [20]. As declared in [20], "proving the general existence for this type of FBSEs remains an outstanding open problem", it is also true for our system (3.20).

In the rest of this paper, we will focus on the case when $n = 1$. When $n = 1$, the state process X is a scalar-valued rocess evolving by the dynamics

$$dX_s = [A_s X_s + B'_{1,s}u_{1,s} + B'_{2,s}u_{2,s} + b_s]ds + [C_s X_s + D_{1,s}u_{1,s} + D_{2,s}u_{2,s} + \sigma_s]' dW_s, \quad X_0 = x_0, \tag{3.22}$$

where A is a bounded deterministic scalar function on $[0,T]$. The other parameters B, C, D are all essentially bounded and $\mathcal{F}_t$-adapted processes on $[0,T]$ with values in $\mathbb{R}^l, \mathbb{R}^d, \mathbb{R}^{d\times l}$, respectively. Moreover, $b \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and $\sigma \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.

In this case, the adjoint equations for the equilibrium strategy become

$$\begin{cases} dp_i(s;t) = -[A'_s p_i(s;t) + (C_s)' k_i(s;t) + Q_{i,s}X_s^*]ds + k_i(s;t)' dW_s, \quad s \in [t,T], \\ p_i(T;t) = G_i X_T^* - h_i \mathbb{E}_t[X_T^*] - \lambda_i X_t^* - \mu_i, \end{cases} \tag{3.23}$$

$$\begin{cases} dP_i(s;t) = -[(2A_s + |C_s|^2)P_i(s;t) + 2C_s'K(s;t) + Q_{i,s}]ds + K_i(s;t)'dW_s, \quad s \in [t,T], \\ P_i(T;t) = G_i, \end{cases} \tag{3.24}$$

for $i = 1,2$. For convenience, we also state here the $n = 1$ version of Theorem 3.2:

Theorem 3.3. *A strategy pair* $(u_1^*, u_2^*) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$ *is an equilibrium strategy pair if, for any time* $t \in [0,T)$,

(i) The system of SDEs

$$\begin{cases} dX_s^* = [A_sX_s^* + B_{1,s}'u_{1,s}^* + B_{2,s}'u_{2,s}^* + b_s]ds + [C_sX_s^* + D_{1,s}u_{1,s}^* + D_{2,s}u_{2,s}^* + \sigma_s]'dW_s, \\ X_0^* = x_0, \\ dp_1(s;t) = -[A_sp_1(s;t) + (C_s)'k_1(s;t) + Q_{1,s}X_s^*]ds + k_1(s;t)'dW_s, \quad s \in [t,T], \\ p_1(T;t) = G_1X_T^* - h_1\mathbb{E}_t[X_T^*] - \lambda_1X_t^* - \mu_1, \\ dp_2(s;t) = -[A_sp_2(s;t) + (C_s)'k_2(s;t) + Q_{2,s}X_s^*]ds + k_2(s;t)'dW_s, \quad s \in [t,T], \\ p_2(T;t) = G_2X_T^* - h_2\mathbb{E}_t[X_T^*] - \lambda_2X_t^* - \mu_2, \end{cases} \tag{3.25}$$

admits a solution $(X^*, p_1, k_1, p_2, k_2)$*;*

(ii) $\Lambda_i(s;t) = R_{i,s}u_{i,s}^* + B_{i,s}p_i(s;t) + (D_{i,s})'k_i(s;t), i = 1,2$ *satisfy condition (3.19).*

4. The deterministic coefficients case

The unique solvability of (3.25) remains a challenging open problem even for the case $n = 1$. However, we are able to solve this problem when the parameters $A, B_1, B_2, C, D_1, D_2, b, \sigma, Q_1, Q_2, R_1$ and R_2 are all deterministic functions.

Throughout this section we assume all the parameters are deterministic functions of t. In this case, since G_1, G_2 have been also assumed to be deterministic, the BSDEs (3.24) turns out to be ODEs with solutions $K_i \equiv 0$ and $P_i(s;t) = G_ie^{\int_s^T(2A_u+|C_u|^2)du} + \int_s^T e^{\int_s^T(2A_u+|C_u|^2)du}Q_{i,v}dv$ for $i = 1,2$. Hence, the equilibrium strategy will be characterized through a system of coupled Riccati-type equations.

4.1. The uniqueness of the equilibrium strategy

As in classical LQ control, we attempt to look for a linear feedback equilibrium strategy pair. For such purpose, motivated by [20], given any $t \in [0,T]$, we consider the following process:

$$p_i(s;t) = M_{i,s}X_s^* - N_{i,s}\mathbb{E}_t[X_s^*] - \Gamma_{i,s}X_t^* + \Phi_{i,s}, \ 0 \le t \le s \le T, \quad i = 1,2, \tag{4.1}$$

where $M_i, N_i, \Gamma_i, \Phi_i$ are deterministic differentiable functions with $\dot{M}_i = m_i, \dot{N}_i = n_i, \dot{\Gamma}_i = \gamma_i$ and $\dot{\Phi}_i = \phi_i$ for $i = 1,2$. The advantage of this process is to separate the variables $X_s^*, \mathbb{E}_t[X_s^*]$ and X_t^* in the solutions $p_i(s;t), i = 1,2$, thereby reducing the complicated FBSDEs to some ODEs.

For any fixed t, applying Ito's formula to (4.1) in the time variable s, we obtain, for $i = 1,2$,

$$\begin{aligned} dp_i(s;t) \ = \ & \{M_{i,s}(A_sX_s^* + B_{1,s}'u_{1,s}^* + B_{2,s}'u_{2,s}^* + b_s) + m_{i,s}X_s^* - N_{i,s}\mathbb{E}_t[A_sX_s^* + B_{1,s}'u_{1,s}^* + B_{2,s}'u_{2,s}^* + b_s] \\ & -n_{i,s}\mathbb{E}_t[X_s^*] - \gamma_{i,s}X_t^* + \phi_{i,s}\}ds + M_{i,s}(C_sX_s^* + D_{1,s}u_{1,s}^* + D_{2,s}u_{2,s}^* + \sigma_s)'dW_s. \end{aligned} \tag{4.2}$$

Comparing the dW_s term of $dp_i(s;t)$ in (3.25) and (4.2), we have

$$k_i(s;t) = M_{i,s}[C_sX_s^* + D_{1,s}u_{1,s}^* + D_{2,s}u_{2,s}^* + \sigma_s], \ s \in [t,T], \quad i = 1,2. \tag{4.3}$$

Notice that $k(s;t)$ turns out to be independent of t.

Putting the above expressions (4.1) and (4.3) of $p_i(s;t)$ and $k_i(s;t), i = 1,2$ into (3.19), we have

$$R_{i,s}u^*_{i,s} + B_{i,s}[(M_{i,s} - N_{i,s} - \Gamma_{i,s})X^*_s + \Phi_{i,s}] + D'_{i,s}M_{i,s}[C_sX^*_s + D_{1,s}u^*_{1,s} + D_{2,s}u^*_{2,s} + \sigma_s] = 0, \ s \in [0,T], \quad (4.4)$$

for $i = 1,2$. Then we can formally deduce

$$u^*_{i,s} = \alpha_{i,s}X^*_s + \beta_{i,s}, \quad i = 1,2. \quad (4.5)$$

Let $M_s = \mathrm{diag}(M_{1,s}I_l, M_{2,s}I_l), N_s = \mathrm{diag}(N_{1,s}I_l, N_{2,s}I_l), \Gamma_s = \mathrm{diag}(\Gamma_{1,s}I_l, \Gamma_{2,s}I_l), \Phi_s = \mathrm{diag}(\Phi_{1,s}I_l, \Phi_{2,s}I_l)$, $R_s = \mathrm{diag}(R_{1,s}, R_{2,s})$, $B_s = \begin{pmatrix} B_{1,s} \\ B_{2,s} \end{pmatrix}$, $D_s = \begin{pmatrix} D_{1,s}, & D_{2,s} \end{pmatrix}$, $u^*_s = \begin{pmatrix} u^*_{1,s} \\ u^*_{2,s} \end{pmatrix}$, $\alpha_s = \begin{pmatrix} \alpha_{1,s} \\ \alpha_{2,s} \end{pmatrix}$ and $\beta_s = \begin{pmatrix} \beta_{1,s} \\ \beta_{2,s} \end{pmatrix}$. Then from (4.4), we have

$$R_su^*_s + [(M_s - N_s - \Gamma_s)X^*_s + \Phi_s]B_s + M_sD'_s[C_sX^*_s + D_s(\alpha_sX^*_s + \beta_s) + \sigma_s] = 0, \ s \in [0,T] \quad (4.6)$$

and hence

$$\begin{aligned} \alpha_s &= -(R_s + M_sD'_sD_s)^{-1}[(M_s - N_s - \Gamma_s)B_s + M_sD'_sC_s], & (4.7) \\ \beta_s &= -(R_s + M_sD'_sD_s)^{-1}(\Phi_sB_s + M_sD'_s\sigma_s). & (4.8) \end{aligned}$$

Next, comparing the ds term of $dp_i(s;t)$ in (3.25) and (4.2) (we supress the argument s here), we have

$$\begin{aligned} & M_i[AX^* + B'(\alpha X^* + \beta) + b] + m_iX^* - N_i\{A\mathbb{E}_t[X^*] + B'\mathbb{E}_t[\alpha X^* + \beta] + b\} - n_i\mathbb{E}_t[X^*] - \gamma_iX^*_t + \phi_i \\ & = -[A(M_iX^* - N_i\mathbb{E}_t[X^*] - \Gamma_iX^*_t + \Phi_i) + M_iC'(CX^* + D(\alpha X^* + \beta) + \sigma)]. \quad (4.9) \end{aligned}$$

Notice in the above that $X^* = X^*_s$ and $\mathbb{E}_t[X^*] = \mathbb{E}_t[X^*_s]$ due to the omission of s. This leads to the following equations for $M_i, N_i, \Gamma_i, \Phi_i$:

$$\begin{cases} \dot{M}_i = -(2A + |C|^2)M_i - Q_i + M_i(B' + C'D)(R + MD'D)^{-1}[(M - N - \Gamma)B + MD'C], \ s \in [0,T], \\ M_{i,T} = G_i; \end{cases} \quad (4.10)$$

$$\begin{cases} \dot{N}_i = -2AN_i + N_iB'(R + MD'D)^{-1}[(M - N - \Gamma)B + MD'C], \ s \in [0,T], \\ N_{i,T} = h_i; \end{cases} \quad (4.11)$$

$$\begin{cases} \dot{\Gamma}_i = -A\Gamma_i, \ s \in [0,T], \\ \Gamma_{i,T} = \lambda_i; \end{cases} \quad (4.12)$$

$$\begin{cases} \dot{\Phi}_i = -\{A - [B'(M - N) + C'DM](R + MD'D)^{-1}B\}\Phi_i - (M_i - N_i)b - M_iC'\sigma \\ \qquad -[(M_i - N_i)B' + M_iC'D](R + MD'D)^{-1}MD'\sigma, \ s \in [0,T], \\ \Phi_{i,T} = -\mu_i. \end{cases} \quad (4.13)$$

Though $M_i, N_i, \Gamma_i, \Phi_i, i = 1,2$ are scalars, M, N, Γ, Φ are now matrices because of two players. Therefore, the above equations are more complicated than the similar equations (4.5)–(4.8) in [20]. Before we solve the equations (4.10)–(4.13), we first prove that, if exist, the equilibrium constructed above is the unique equilibrium. Indeed, we have

Theorem 4.1. *Let*

$$\mathcal{L}_1 = \left\{X(\cdot;\cdot) : X(\cdot;t) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}),\ \sup_{t\in[0,T]} \mathbb{E}\left[\sup_{s\geq t} |X(s;t)|^2\right] < +\infty\right\} \quad (4.14)$$

and

$$\mathcal{L}_2 = \left\{Y(\cdot;\cdot) : Y(\cdot;t) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^d),\ \sup_{t\in[0,T]} \mathbb{E}\left[\int_t^T |X(s;t)|^2 ds\right] < +\infty\right\}. \quad (4.15)$$

Suppose all the parameters $A, B_1, B_2, C, D_1, D_2, b, \sigma, Q_1, Q_2, R_1$ and R_2 are all deterministic. When $(M_i, N_i, \Gamma_i, \Phi_i), i = 1,2$ exist, and for $i = 1,2$, $(p_i(s;t), k_i(s;t)) \in \mathcal{L}_1 \times \mathcal{L}_2$, the equilibrium strategy is unique.

Proof. Suppose there is another equilibrium (X, u_1, u_2), then the BSDE (3.1), with X^* replaced by X, admits a solution $(p_i(s;t), k_i(s), u_{i,s})$ for $i = 1,2$, which satisfies $B_{i,s}p_i(s;s) + D'_{i,s}k_i(s) + R_{i,s}u_{i,s} = 0$ for a.e. $s \in [0,T]$. For $i = 1,2$, define

$$\begin{aligned} \bar{p}_i(s;t) &\triangleq p_i(s;t) - [M_{i,s}X_s - N_{i,s}\mathbb{E}_t[X_s] - \Gamma_{i,s} + \Phi_{i,s}], & (4.16)\\ \bar{k}_i(s;t) &\triangleq k_i(s) - M_{i,s}(C_sX_s + D_{1,s}u_{1,s} + D_{2,s}u_{2,s} + \sigma_s), & (4.17)\end{aligned}$$

where $k_i(s) = k_i(s;t)$ by Lemma 3.1.

We define $p(s;t) = \mathrm{diag}(p_1(s;t)I_l, p_2(s;t)I_l)$, $\bar{p}(s;t) = \mathrm{diag}(\bar{p}_1(s;t)I_l, \bar{p}_2(s;t)I_l)$, and $u = \begin{pmatrix} u_{1,s} \\ u_{2,s}\end{pmatrix}$. By the equilibrium condition (3.19), we have

$$\begin{aligned} 0 &= \begin{pmatrix} B_{1,s}p_1(s;s) + D'_{1,s}k_1(s) + R_{1,s}u_{1,s} \\ B_{2,s}p_2(s;s) + D'_{2,s}k_2(s) + R_{2,s}u_{2,s}\end{pmatrix} \\ &= p(s;s)B_s + \begin{pmatrix} D'_{1,s}k_1(s) \\ D'_{2,s}k_2(s)\end{pmatrix} + R_su_s \\ &= [\bar{p}(s;s) + X_s(M_s - N_s - \Gamma_s) + \Phi_s]B_s + \begin{pmatrix} D'_{1,s}\bar{k}_1(s) \\ D'_{2,s}\bar{k}_2(s)\end{pmatrix} + M_sD'_s(C_sX_s + D_su_s + \sigma_s) + R_su_s \\ &= \bar{p}(s;s)B_s + \begin{pmatrix} D'_{1,s}\bar{k}_1(s) \\ D'_{2,s}\bar{k}_2(s)\end{pmatrix} + X_s[(M_s - N_s - \Gamma_s)B_s + M_sD'_sC_s] + \Phi_sB_s + M_sD'_s\sigma_s \\ &\quad +(R_s + M_sD'_sD_s)u_s. \end{aligned} \quad (4.18)$$

Since $R_s + M_sD'_sD_s$ is invertible, we have

$$\begin{aligned} u_s = -(R_s + M_sD'_sD_s)^{-1}\Bigg\{&\bar{p}(s;s)B_s + \begin{pmatrix} D'_{1,s}\bar{k}_1(s) \\ D'_{2,s}\bar{k}_2(s)\end{pmatrix} \\ &+X_s[(M_s - N_s - \Gamma_s)B_s + M_sD'_sC_s] + \Phi_sB_s + M_sD'_s\sigma_s\Bigg\}, \end{aligned} \quad (4.19)$$

and hence for $i = 1,2$,

$$\begin{aligned} d\bar{p}_i(s;t) &= dp_i(s;t) - d[M_{i,s}X_s - N_{i,s}\mathbb{E}_t[X_s] - \Gamma_{i,s} + \Phi_{i,s}] \\ &= -[A_sp_i(s;t) + C'_sk_i(s) + Q_{i,s}X_s]ds + k'_i(s)dW_s - d[M_{i,s}X_s - N_{i,s}\mathbb{E}_t[X_s] - \Gamma_{i,s}X_t + \Phi_{i,s}] \end{aligned}$$

$$
\begin{aligned}
= & -\Big\{A_s\bar{p}_i(s;t) + C_s'\bar{k}_i(s) + A_s(M_{i,s}X_s - N_{i,s}\mathbb{E}_t[X_s] - \Gamma_{i,s}X_t + \Phi_{i,s}) \\
& \quad + C_s'M_{i,s}(C_sX_s + D_{1,s}u_{1,s} + D_{2,s}u_{2,s} + \sigma_s)\Big\}ds \\
& \quad + [\bar{k}_i(s) - M_{i,s}(C_sX_s + D_{1,s}u_{1,s} + D_{2,s}u_{2,s} + \sigma_s)]'dW_s \\
& -\Big\{M_{i,s}[A_sX_s + B_s'u_s + b_s] + m_{i,s}X_s - N_{i,s}(A_s\mathbb{E}_t[X_s] + B_s'\mathbb{E}_t[u_s] + b_s) \\
& \quad - n_{i,s}\mathbb{E}_t[X_s] - \gamma_{i,s}X_t + \phi_{i,s}\Big\}ds \\
& - M_{i,s}[C_sX_s + D_su_s + \sigma_s]'dW_s \\
= & -\Big\{A_s\bar{p}_i(s;t) + C_s'\bar{k}_i(s) - M_{i,s}(B_s' + C_s'D_s)(R_s + M_sD_s'D_s)^{-1}\left[B_s\bar{p}(s;s) + \begin{pmatrix} D_{1,s}'\bar{k}_1(s) \\ D_{2,s}'\bar{k}_2(s)\end{pmatrix}\right] \\
& \quad N_{i,s}B_s'(R_s + M_sD_s'D_s)^{-1}\mathbb{E}_t\left[B_s\bar{p}(s;s) + \begin{pmatrix} D_{1,s}'\bar{k}_1(s) \\ D_{2,s}'\bar{k}_2(s)\end{pmatrix}\right]\Big\}ds + \bar{k}_i(s)'dW_s,
\end{aligned} \tag{4.20}
$$

where we suppress the subscript s for the parameters, and we have used the equations (4.10)–(4.13) for $M_i, N_i, \Gamma_i, \Phi_i$ in the last equality. From (4.16) and (4.17), we have $(\bar{p}_i, \bar{k}_i) \in \mathcal{L}_1 \times \mathcal{L}_2$. Therefore, by Theorem 4.2 of [21], we obtain $\bar{p}(s;t) \equiv 0$ and $\bar{k}(s) \equiv 0$.

Finally, plugging $\bar{p} \equiv \bar{k} \equiv 0$ into u of (4.19), we get u being the same form of feedback strategy as in (4.5), and hence (X, u_1, u_2) is the same as (X^*, u_1^*, u_2^*) which defined by (4.5) and (3.25). □

4.2. Existence of the equilibrium strategies

The solutions to (4.12) is

$$
\Gamma_{i,s} = \lambda_i e^{\int_s^T A_t dt}, \tag{4.21}
$$

for $i = 1, 2$. Let $\tilde{N} = N_1/N_2$, from (4.11), we have $\dot{\tilde{N}} = 0$, and hence

$$
\tilde{N} \equiv \frac{h_1}{h_2}, \quad N_2 \equiv \frac{h_2}{h_1}N_1. \tag{4.22}
$$

Equations (4.10) and (4.11) form a system of coupled Riccati-type equations for (M_1, M_2, N_1):

$$
\begin{cases}
\dot{M}_1 = -[2A + |C|^2 + B'\Gamma(R + MD'D)^{-1}(B + D'C)]M_1 - Q_1 \\
\qquad\quad + (B + D'C)'(R + MD'D)^{-1}M(B + D'C)M_1 - B'N(R + MD'D)^{-1}(B + D'C)M_1, \\
M_{1,T} = G_1; \\
\dot{M}_2 = -[2A + |C|^2 + B'\Gamma(R + MD'D)^{-1}(B + D'C)]M_2 - Q_2 \\
\qquad\quad + (B + D'C)'(R + MD'D)^{-1}M(B + D'C)M_2 - B'N(R + MD'D)^{-1}(B + D'C)M_2, \\
M_{2,T} = G_2; \\
\dot{N}_1 = -2AN_i + N_iB'(R + MD'D)^{-1}[(M - N - \Gamma)B + MD'C], \\
N_{1,T} = h_1.
\end{cases} \tag{4.23}
$$

Finally, once we get the solution for (M_1, M_2, N_1), (4.13) is a simple ODE. Therefore, it is crucial to solve (4.23).

Formally, we define $\tilde{M} = \frac{M_1}{M_2}$ and $J_1 = \frac{M_1}{N_1}$ and study the following equation for $(M_1, \tilde{M}, J_1)$:

$$\begin{cases} \dot{M}_1 = -[2A + |C|^2 + B'\Gamma(R + MD'D)^{-1}(B + D'C)]M_1 - Q_1 \\ \qquad +(B + D'C)'(R + MD'D)^{-1}M(B + D'C)M_1 - B'N(R + MD'D)^{-1}(B + D'C)M_1, \\ M_{1,T} = G_1; \\ \dot{\tilde{M}} = -(\frac{Q_1}{M_1} - \frac{Q_2}{M_1}\tilde{M})\tilde{M}, \\ \tilde{M}_T = \frac{G_1}{G_2}; \\ \dot{J}_1 = -[|C|^2 - C'D(R + MD'D)^{-1}M(B + D'C) + B'\Gamma(R + MD'D)^{-1}D'C + \frac{Q_1}{M_1}]J_1 \\ \qquad -C'D(R + MD'D)^{-1}M \operatorname{diag}(I_l, \frac{h_2}{h_1}\tilde{M}I_l)B, \\ J_{1,T} = \frac{G_1}{h_1}, \end{cases} \tag{4.24}$$

where $M = \operatorname{diag}(M_1 I_l, \frac{M_1}{\tilde{M}} I_l), N = \operatorname{diag}(\frac{M_1}{J_1} I_l, \frac{h_2}{h_1}\frac{M_1}{J_1} I_l)$ and $\Gamma = \operatorname{diag}(\lambda_1 e^{\int_s^T A_t dt} I_l, \lambda_2 e^{\int_s^T A_t dt} I_l)$.

By a direct calculation, we have

Proposition 3. *If the system (4.24) admits a positive solution $(M_1, \tilde{M}, J_1)$, then the system (4.23) admits a solution (M_1, M_2, N_1).*

In the following, we will use the truncation method to study the system (4.24). For convenienc, we use the following notations:

$$a \vee b = \max\{a, b\}, \qquad \forall a, b \in \mathbb{R}, \tag{4.25}$$

$$a \wedge b = \min\{a, b\}, \qquad \forall a, b \in \mathbb{R}. \tag{4.26}$$

Moreover, for a matrix $M \in \mathbb{R}^{m\times n}$ and a real number c, we define

$$(M \vee c)_{i,j} = M_{i,j} \vee c, \qquad \forall 1 \le i \le m, 1 \le j \le n, \tag{4.27}$$

$$(M \wedge c)_{i,j} = M_{i,j} \wedge c, \qquad \forall 1 \le i \le m, 1 \le j \le n. \tag{4.28}$$

We first consider the standard case where $R - \delta I \ge 0$ for some $\delta > 0$. We have

Theorem 4.2. *Assume that $R - \delta I \ge 0$ for some $\delta > 0$ and $G \ge h > 0$. Then (4.24), and hence (4.23) admit unique solution if*
(i) there exists a constant $\lambda \ge 0$ such that $B = \lambda D'C$;
(ii) $\frac{|C|^2}{2l} D'D - (\lambda + 1)D'CC'D \ge 0$.

Proof. For fixed $c > 0$ and $K > 0$, consider the following truncated system of (4.24):

$$\begin{cases} \dot{M}_1 = -[2A + |C|^2 + B'\Gamma(R + M_c^+ D'D)^{-1}(B + D'C)]M_1 - Q_1 \\ \qquad +(B + D'C)'(R + M_c^+ D'D)^{-1}(M_c^+ \wedge K)(B + D'C)M_1 \\ \qquad -B'(N_c^+ \wedge K)(R + M_c^+ D'D)^{-1}(B + D'C)M_1, \\ M_{1,T} = G_1; \\ \dot{\tilde{M}} = -(\frac{Q_1}{M_1 \vee c} - \frac{Q_2}{M_1 \vee c}\tilde{M} \wedge K)\tilde{M}, \\ \tilde{M}_T = \frac{G_1}{G_2}; \\ \dot{J}_1 = -\lambda^{(1)} J_1 - C'D(R + M_c^+ D'D)^{-1}(M_c^+ \wedge K)\operatorname{diag}(I_l, \frac{h_2}{h_1}(\tilde{M} \wedge K)I_l)B, \\ J_{1,T} = \frac{G_1}{h_1}, \end{cases} \tag{4.29}$$

where $M_c^+ = \mathrm{diag}((M_1 \vee 0)I_l, \frac{M_1\vee 0}{\tilde{M}\vee c}I_l)$, $N_c^+ = \mathrm{diag}(\frac{M_1\vee 0}{J_1\vee c}I_l, \frac{h_2}{h_1}\frac{M_1\vee 0}{J_1\vee c}I_l)$ and

$$\lambda^{(1)} = |C|^2 - C'D(R + M_c^+D'D)^{-1}(M_c^+ \wedge K)(B + D'C) + B'\Gamma(R + M_c^+D'D)^{-1}D'C + \frac{Q_1}{M_1 \vee c}. \quad (4.30)$$

Since $R-\delta I \geq 0$, the above system (4.29) is locally Lipschitz with linear growth, and hence it admits a unique solution $(M_1^{c,K}, \tilde{M}^{c,K}, J_1^{c,K})$. We will omit the superscript (c, K) when there is no confusion.

We are going to prove that $J_1 \geq 1$ and that $M_1, \tilde{M} \in [L_1, L_2]$ for some $L_1, L_2 > 0$ independent of c and K appearing in the truncation functions. We denote

$$\begin{aligned}\lambda^{(2)} \;=\; & (2A + |C|^2 + B'\Gamma(R + M_c^+D'D)^{-1}(B + D'C)) \\ & -(B + D'C)'(R + M_c^+D'D)^{-1}(M_c^+ \wedge K)(B + D'C) \\ & -B'(N_c^+ \wedge K)(R + M_c^+D'D)^{-1}(B + D'C). \quad (4.31)\end{aligned}$$

Then $\lambda^{(2)}$ is bounded, and M_1 satisfies

$$\dot{M}_1 + \lambda^{(2)}M_1 + Q_1 = 0, \quad M_{1,T} = G_1. \quad (4.32)$$

Hence $M_1 > 0$. Similarly, we have $\tilde{M} > 0$.

The equation for $\tilde{M}$ is

$$\begin{cases} -\dot{\tilde{M}} = (\frac{Q_1}{M_1\vee c}\tilde{M} - \frac{Q_2}{M_1\vee c}(\tilde{M} \wedge K)\tilde{M}, \\ \tilde{M}_T = \frac{G_1}{G_2}; \end{cases} \quad (4.33)$$

hence $\tilde{M}$ admits an upper bound L_2 independent of c and K. Choosing $K = L_2$ and examining again (4.33), we deduce that there exists $L_1 > 0$ independent of c and K such that $\tilde{M} \geq L_1$. Indeed, we can choose $L_1 = \min_{0\leq t\leq T} \frac{Q_{1,t}}{Q_{2,t}} \wedge \frac{G1}{G_2}$ and $L_2 = \max_{0\leq t\leq T} \frac{Q_{1,t}}{Q_{2,t}} \vee \frac{G1}{G_2}$. As a result, choosing $c < L_1$, the terms M_c^+ can be replaced by $M = \mathrm{diag}(M_1I_l, \frac{M_1}{\tilde{M}}I_l)$, respectively, in (4.29) without changing their values.

Now we prove $J \geq 1$. Denote $\tilde{J} = J_1 - 1$, then $\tilde{J}$ satisfies the ODE:

$$\dot{\tilde{J}} = -\lambda^{(1)}\tilde{J} - [\lambda^{(1)} + C'D(R + MD'D)^{-1}(M \wedge K)\mathrm{diag}(I_l, \frac{h_2}{h_1}\tilde{M}I_l)B] = -\lambda^{(1)}\tilde{J} - a^{(1)}, \quad (4.34)$$

where

$$\begin{aligned} a^{(1)} \;=\; & \lambda^{(1)} + C'D(R + MD'D)^{-1}(M \wedge K)\mathrm{diag}(I_l, \frac{h_2}{h_1}\tilde{M}I_l)B \\ =\; & |C|^2 - (\lambda + 1)C'D(R + MD'D)^{-1}(M \wedge K)D'C + C'D\Gamma(R + MD'D)^{-1}(M \wedge K)D'C + +\frac{Q_1}{M_1 \vee c} \\ & +C'D(R + MD'D)^{-1}(M \wedge K)\mathrm{diag}(I_l, \frac{h_2}{h_1}\tilde{M}I_l)D'C \\ & \geq |C|^2 - (\lambda + 1)C'D(R + MD'D)^{-1}MD'C + C'D\Gamma(R + MD'D)^{-1}(M \wedge K)D'C + +\frac{Q_1}{M_1 \vee c} \\ & = tr\left\{(R + MD'D)^{-1}\frac{|C|^2 + Q_1/(M_1 \vee c)}{2l}(R + MD'D)\right\} - (\lambda + 1)tr\{(R + MD'D)^{-1}D'CC'DM\} \\ & = tr\left\{(R + MD'D)^{-1}H\right\} \quad (4.35)\end{aligned}$$

with $H = \frac{|C|^2 + Q_1/(M_1 \vee c)}{2l}(R + D'DM) - (\lambda + 1)D'CC'DM$.

When c is small enough such that $R - cD'D \geq 0$, we have

$$\frac{Q_1}{M_1 \vee c}(R + MD'D) \geq \frac{Q_1}{L_2} D'D. \tag{4.36}$$

Hence,

$$H \geq (\frac{|C|^2}{2l} D'D - (\lambda + 1)D'CC'D)M \geq 0, \tag{4.37}$$

and consequently $a^{(1)} \geq tr\{(R + MD'D)^{-1}H\} \geq 0$. We then deduce that $\tilde{J} \geq 0$, and hence $J_1 \geq 1$. The boundness of M_1 can be proved by a similar argument in the proof of Theorem 4.2 in [20]. □

Similarly, for the singular case $R \equiv 0$, we have

Theorem 4.3. *Given $G_1 \geq h_1 \geq 1, R \equiv 0$, if $B = \lambda D'C$ and $|C|^2 - (\lambda + 1)C'D(D'D)^{-1}D'C \geq 0$, then (4.24) and (4.23) admit a unique positive solution.*

Concluding the above two theorems, we can present our main results of this section:

Theorem 4.4. *Given $G_1 \geq h_1 \geq 1$ and $B = \lambda D'C$. The (4.23) admits a unique positive solution (M_1, M_2, N_1) in the following two cases:*

(i) $R - \delta I \geq 0$ for some $\delta > 0$, $\frac{|C|^2}{2l} D'D - (\lambda + 1)D'CC'D \geq 0$;

(ii) $R \equiv 0$, $|C|^2 - (\lambda + 1)C'D(D'D)^{-1}D'C \geq 0$.

Proof. Define $p_i(s;t)$ and $k_i(s;t)$ by (4.1) and (4.3), respectively. It is straightforward to check that $(u_1^*, u_2^*, X^*, p_1, p_2, k_1, k_2)$ satisfies the system of SDEs (3.25). Moreover, in the both cases, we can check that $\alpha_{i,s}$ and $\beta_{i,s}$ in (4.5) are all uniformly bounded, and hence $u_i^* \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$ and $X^* \in L^2(\Omega; C(0, T; \mathbb{R}))$.

Finally, denote $\Lambda_i(s;t) = R_{i,s}u^*_{i,s} + p_i(s;t)B_{i,s} + (D_{i,s})'k_i(s;t), i = 1, 2$. Plugging p_i, k_i, u_i^* define in (4.1),(4.3) and (4.5) into Λ_i, we have

$$\Lambda_i(s;t) = R_{i,s}u^*_{i,s} + (M_{i,s}X^*_s - N_{i,s}\mathbb{E}_t[X^*_s] - \Gamma_{i,s}X^*_t + \Phi_{i,s})B_{i,s} + M_{i,s}D'_{i,s}[C_sX^*_s + D_{1,s}u^*_{1,s} + D_{2,s}u^*_{2,s} + \sigma_s] \tag{4.38}$$

and hence,

$$\begin{aligned}
\Lambda(t;t) &\triangleq \begin{pmatrix} \Lambda_1(t;t) \\ \Lambda_2(t;t) \end{pmatrix} \\
&= (R_t + M_tD'_tD_t)u^*_t + M_t(B_t + D'_tC_t)X^*_t - N_tB_t\mathbb{E}_t[X^*_t] - \Gamma_tB_tX^*_t + (\Phi_tB_t + M_tD'_t\sigma_t) \\
&= -[(M_t - N_t - \Gamma_t)B_t + M_tD'_tC_t]X^*_t - (\Phi_tB_t + M_tD'_t\sigma_t) \\
&\quad + M_t(B_t + D'_tC_t)X^*_t - N_tB_tX^*_t - \Gamma_tB_tX^*_t + (\Phi_tB_t + M_tD'_t\sigma_t) \\
&= 0.
\end{aligned} \tag{4.39}$$

Therefore, Λ_i satisfies the seond condition in (3.19). □

5. Conclusions

We investigate a general stochastic linear-quadratic differential game, where the objective functional of each player include both a quadratic term of the expected state and a state-dependent term. As discussed in detail in Björk and Murgoci [17] and [18], the last two terms in each objective functional, respectively, introduce two sources of time inconsistency into the differential game problem. That is to say, the usual equilibrium aspect is not a proper way when the players at 0 cannot commit the players at all intermediate times to implement the decisions they have planed. With the time-inconsistency, the notion "equilibrium" needs to be extended in an appropriate way. We turn to adopt the concept of equilibrium strategy between the players at all different times, which is at any time, an equilibrium "infinitesimally" via spike variation. By applying the spike variation technique, We derive a sufficient condition for equilibrium strategies via a system of forward-backward stochastic differential equation. The unique solvability of such FBSDEs remains a challenging open problem.

For a special case, when the state is one-dimensional and the coefficients are all deterministic, the equilibrium strategy will be characterized through a system of coupled Riccati-type equations. At last, we find an explicit equilibrium strategy, which is also proved be the unique equilibrium strategy.

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Conflict of interest

The authors declare there is no conflicts of interest.

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