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Remarks on the inverse problem of the collinear central configurations in the N-body problem

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Abstract: For a fixed configuration in the collinear N-body problem, the existence of the central configurations is determined by a system of linear equations, which in turn is determined by certain Pfaffians for even or odd N in literatures. In this short note, we prove that the Pfaffians of the associate matrices for all even number collinear configurations are nonzero if and only if the extended Pfaffians of the associate matrices for all odd number collinear configurations are nonzero. Therefore, the inverse problem of the collinear central configurations can be answered and each collinear configuration determines a one-parameter family of masses with a fixed total mass if the Pfaffians of the associate matrix for all collinear even number bodies are nonzero. We also make some remarks on the super central configurations and the number of collinear central configurations under different equivalences, especially a lower bound for the number of collinear central configurations under the geometric equivalence.

Keywords: skew symmetric matrix; Pfaffian; inverse problem of central configurations; geometric equivalence; N-body problem

1. Introduction

In the N-body problem, a central configuration is formed if the position vector of each particle with respect to the center of mass is a common scalar multiple of its acceleration vector. If the bodies are placed in a central configuration and released with zero initial velocity, they will collapse homothetically to a collision at the center of mass. With appropriate velocities, planar central configurations can give rise to the homographic solutions of the N-body problem. Central configurations play an important role in the study of the N-body problem. Although there are many research results on central configurations, understanding central configurations is still far to be completed. Even the problem of the finiteness of the number of central configurations is a challenging question [1]. For an arbitrary given set of masses, the problem of finiteness has been only solved for the collinear N-body problem due to Moulton [2]. As for the planar case, it was solved for $N = 3$ due to Euler [3] and Lagrange [4], for $N = 4$ due to Hampton and Moeckel [5], and partially for $N = 5$ due to Albouy and Kaloshin [6].

A configuration $x = (x_1, x_2, \cdots, x_N)^T$ is *collinear* if all the x_is lie on a straight line. Without loss of generality, we can choose an ordered position vector $x = (x_1, x_2, \cdots, x_N)^T \in \mathbf{R}^N$ with $-\infty < x_1 < x_2 < \cdots < x_N < \infty$ for the collinear N-body problem. A configuration $x = (x_1, x_2, \cdots, x_N)^T$ is called a collinear central configuration for a mass vector $m = (m_1, m_2, \cdots, m_N)^T$, if there exist λ and c such that the following system of algebraic equations holds:

$$Am = -\lambda(x - cL), \tag{1.1}$$

where matrix A is the associated matrix of the configuration x which is denoted by $A \equiv A(x) = A(x_1, x_2, \cdots, x_N) = \left(a_{ij}\right)$, $a_{ij} = \frac{1}{r_{ij}^2}$, $r_{ij} = |x_i - x_j|$ and $a_{ji} = -a_{ij}$ if $1 \le i < j \le N$ and $a_{ii} = 0$. $L = (1, 1, \cdots, 1)^T$. One can easily prove that c is the center of mass from Eq (1.1). Indeed, let

$$C = m_1x_1 + m_2x_2 + \cdots + m_Nx_N, \quad M = m_1 + \cdots + m_N, \quad c = C/M \tag{1.2}$$

be the first moment, total mass and center of mass of the bodies.

Let us introduce two extensions A^x and A^L of the associate matrix A of the configuration x for the odd collinear N-body ($N = 2k - 1$) problem as

$$A^x = A^x(x) = \begin{bmatrix} 0 & x^T \\ -x & A \end{bmatrix}, \qquad A^L = A^L(x) = \begin{bmatrix} 0 & L^T \\ -L & A \end{bmatrix}.$$

A collinear central configuration is also called a *Moulton configuration* after F. R. Moulton [2, 7] who proved that for a fixed mass vector $m = (m_1, m_2, \cdots, m_N)$ with a fixed order of the bodies along the line, there exists a unique collinear central configuration (up to translation and scaling). Moulton also considered the inverse problem: given a collinear configuration, find the mass vectors, if any, for which it is a central configuration. This becomes a system of linear algebraic equations (1.1) and the solutions of mass vector depend on the properties of the Pfaffians of its associated matrix A or its extensions A^x and A^L.

Definition 1.1. By convention, the Pfaffian of a 0×0 matrix is 1. When $N = 2k$ is even, the Pfaffian of the skew-symmetric $2k \times 2k$ matrix A with $k > 0$ can be computed recursively as

$$Pf(A) = \sum_{i=1}^{N-1} (-1)^{i+1} a_{iN} Pf(A_{\hat{i}\hat{N}}), \tag{1.3}$$

where $A_{\hat{i}\hat{N}}$ denotes the sub matrix of A with both the N-th and i-th rows and columns removed.

The associated matrix A is skew-symmetric and its determinant is the square of its Pfaffian, i.e., $\det(A) = (Pf(A))^2$. Moulton's paper [2] assumes that the center of mass is fixed at origin, i.e., $c = 0$ in Eq (1.1) and his results depend on properties of certain Pfaffians. In the even case $N = 2k$, he found that if the center of mass is fixed at $c = 0$ and assuming $Pf(A) \neq 0$ then there is a one parameter family of mass vectors for which the configuration is central. In the odd case $N = 2k - 1$, Moulton found that the inverse problem could not always be solved if the center of mass was fixed at $c = 0$. To have $c = 0$ and Eq (1.1) consistent, $Pf(A^x)$ must be zero and first minor of its associate matrix A must be nonzero.

Buchanan [8] proved that the Pfaffian $Pf(A)$ is nonzero for all even number collinear configurations, and the Pfaffian $Pf(F)$ could be zero for all odd number collinear configurations. But Albouy and Moeckel [9] pointed out that the proof in Buchanan's paper is incorrect and cannot be fixed. They use the tensor notation and define the Pfaffians $K_N(x)$ and $K_N^L(x)$. $K_N(x) \neq 0$ is equivalent to $Pf(A) \neq 0$ for an even configuration. $K_N^L(x) \neq 0$ is equivalent to $Pf(A^L) \neq 0$ for an odd configuration. They conjecture that *(Albouy-Moeckel Conjecture): All the corresponding Pfaffians are actually nonzero for any configurations in the collinear N-body problem.*

The partial results have been obtained in the direction of its complete proof of the conjecture. Albouy and Moeckel proved that the Pfaffian $K_N(x) \neq 0$ analytically for $N = 2, 4$ and numerically for $N = 6$. The masses can be linearly parametrized by λ and c. They also pointed out that the center of mass c is uniquely determined by the configurations in the odd N-body problem. The Pfaffian $K_N^L(x)$ is proved to be nonzero for $N = 3$ analytically. They proved that the corresponding Pfaffian is nonzero for 5-body problem with computer assisted computations. They also found the criteria of general homogeneous potentials to make the corresponding Pfaffians be nonzero.

Xie [10] provided a simple analytical proof of the Pfaffian $Pf(A) \neq 0$ for $N = 6$. Ferrario [11] gives an analytical proof that the conjecture is true for any homogeneity of the potential when $N \leq 6$. It gives a computer-assisted proof for $N \leq 10$ in the Newtonian case. It remarks that an empirical estimate of the time needed to perform the calculation for $N = 12$ with the current algorithm would be of the order of 4 to 5 years on the same computer. An analytical proof is missing for $N > 6$.

In section 2, we prove that the Pfaffians of the associate matrices for all even number collinear configurations are nonzero if and only if the corresponding Pfaffians of all odd number configurations are nonzero. So we only need to study the Pfaffians of the associate matrix for any collinear even number bodies. This is an extension of the results in the paper [10, 11]).

The number of collinear central configurations can be counted differently based on different equivalency. A configuration x is called a super central configuration if x is a central configuration for mass vector m and x is also a central configuration for a distinct permutation of m. In section 3, we make some remarks on the relationship between the existence of super central configuration and the number of collinear central configurations under different equivalences. Especially we give a lower bound for the number of collinear central configurations under the geometric equivalence.

2. Remarks on Pfaffians

Theorem 2.1. *For any ordered even collinear configuration $\xi = (\xi_1, \xi_2, \cdots, \xi_{2k})^T$ with $-\infty < \xi_1 < \xi_2 < \cdots < \xi_{2k} < \infty$, $Pf(A(\xi)) \neq 0$, if and only if, for any ordered odd collinear configuration $x = (x_1, x_2, \cdots, x_{2k-1})^T$ with $-\infty < x_1 < x_2 < \cdots < x_{2k-1} < \infty$, $Pf(A^L(x)) \neq 0$, where $A^L(x) = \begin{bmatrix} 0 & L^T \\ -L & A(x) \end{bmatrix}$ and $L = (1, 1, \cdots, 1)^T$.*

Proof. We first prove the necessary conditions. Assume that $Pf(A^L(x)) \neq 0$ for any odd collinear configuration $x = (x_1, x_2, \cdots, x_{2k-1})^T$.
Let $D = diag\left(1, \frac{1}{|\xi_2-\xi_1|^2}, \frac{1}{|\xi_3-\xi_1|^2}, \cdots, \frac{1}{|\xi_{2k-1}-\xi_1|^2}, \frac{1}{|\xi_{2k}-\xi_1|^2}\right)$. Then $DA(\xi)D = (\tilde{a}_{ij})$ is a skew-symmetric matrix such that the first row of $DA(\xi)D$ equals $(0, 1, 1, \cdots, 1, 1)$, and for $2 \leq i < j \leq 2k$,

$$\tilde{a}_{ij} = \frac{a_{ij}}{a_{1i}a_{1j}} = \frac{(\xi_i - \xi_1)^2(\xi_j - \xi_1)^2}{(\xi_j - \xi_i)^2} = \left(\frac{1}{(\xi_i - \xi_1)} - \frac{1}{(\xi_j - \xi_1)}\right)^{-2}.$$

Let

$$x_{i-1} = \frac{1}{(\xi_1 - \xi_i)} \qquad \text{for } i = 2, 3, \cdots, 2k.$$

Note that $-\infty < x_1 < x_2 < \cdots < x_{2k-1} < 0$ because $-\infty < \xi_1 < \xi_2 < \cdots < \xi_{2k} < \infty$ and $A^L(x) = DA(\xi)D$. Because $\det(A^L(x)) = \det(DA(\xi)D) = (\det(D))^2 \det(A(\xi))$ and $\det(A^L(x)) = (Pf(A^L(x)))^2 \neq 0$, $\det(A(\xi)) \neq 0$. So $Pf(A(\xi)) \neq 0$.

Now we prove the sufficient conditions. Assume that $Pf(A(\xi)) \neq 0$ for any ordered even collinear configuration $\xi = (\xi_1, \xi_2, \cdots, \xi_{2k})^T$.
For any ordered odd collinear configuration $x = (x_1, x_2, \cdots, x_{2k-1})^T$, pick up x_{2k} such that $x_{2k} > x_{2k-1}$. Let $z_i = x_i - x_{2k}$. Then $-\infty < z_1 < z_2 < \cdots < z_{2k-1} < z_{2k} = 0$. Let $\xi_{i+1} = -\frac{1}{z_i}, i = 1, \cdots, 2k-1$ and $\xi_1 = 0$. Then $0 = \xi_1 < \xi_2 < \cdots < \xi_{2k}$. Let $\eta_i = (\xi_1 - \xi_i)^{-1} = -\frac{1}{\xi_i} = z_{i-1}$, $i = 2, \cdots, 2k$.
Let $D = diag\left(1, \frac{1}{|\xi_2-\xi_1|^2}, \frac{1}{|\xi_3-\xi_1|^2}, \cdots, \frac{1}{|\xi_{2k-1}-\xi_1|^2}, \frac{1}{|\xi_{2k}-\xi_1|^2}\right)$. Then $DA(\xi)D = (\tilde{a}_{ij})$ is skew-symmetric such that the first row of $DA(\xi)D$ equals $(0, 1, 1, \ldots, 1)$, and for $2 \le i < j \le 2k$,

$$\tilde{a}_{ij} = \frac{a_{ij}}{a_{1i}a_{1j}} = (\eta_i - \eta_j)^{-2} = (z_{i-1} - z_{j-1})^{-2} = (x_{i-1} - x_{j-1})^{-2} \qquad \text{for } 2 \le i < j \le 2k.$$

So

$$A^L(x) = DA(\xi)D, \qquad \text{and } Pf(A^L(x)) \neq 0$$

because

$$(Pf(A^L(x)))^2 = \det A^L(x) = \det DA(\xi)D = (\det D)^2 (Pf(A(\xi)))^2 \neq 0.$$

This completes the proof. □

With the assumptions on certain Pfaffians such as $K_N(x)$, $K_N^x(x)$, $K_N^L(x)$, $Pf(A)$, $Pf(A^x)$, $Pf(A^L)$ etc, Moulton [2, 7], Albouy and Moeckel [9], Ferrario [11] etc. proved some results which are similar to the following two theorems for the collinear central configurations. But here we only assume that the Pfaffians $Pf(A)$ of the associated matrics of all even number collinear configurations are nonzero. Explicit formulas for masses are also provided for central configurations.

Theorem 2.2. *Let $x = (x_1, x_2, \cdots, x_{2k})^T$ be any even collinear configuration with $x_1 < x_2 < x_3 < \cdots < x_{2k}$. Let A be the associated matrix of x. Assuming that $Pf(A) \neq 0$. Then λ and c can be taken as the two parameters for m,*

$$m = \lambda(-A^{-1}x + cA^{-1}L) \tag{2.1}$$

to make x central. Total mass $M = L^T m = m_1 + m_2 + \cdots + m_{2k}$ is independent of the center of mass c and M is linearly dependent on λ, i.e.,

$$M = -\lambda L^T A^{-1} x. \tag{2.2}$$

Proof. The proof is straight forward by knowing that $L^T A^{-1} L = 0$ because A and A^{-1} are skew-symmetric matrix. We omit the details here. □

Theorem 2.3. *Assume that $Pf(A(\xi)) \neq 0$ for any even collinear configuration $\xi = (\xi_1, \xi_2, \cdots, \xi_{2k})^T$ with $-\infty < \xi_1 < \xi_2 < \cdots < \xi_{2k} < \infty$.*
Let us consider the inverse problem of the collinear $(2k-1)$-body central configurations with $x =$

$(x_1, x_2, \cdots, x_{2k-1})^T$ *and* $-\infty < x_1 < x_2 < \cdots < x_{2k-1} < \infty$. *Let* B_i *be the associated matrix of configuration* $Y_i = (x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{2k-1})^T$, $i = 1, 2, \cdots, 2k-1$, *i.e., configuration* Y_i *is obtained from configuration* x *by deleting* x_i *and the number of bodies in* Y_i *is even.*

1. *The center of mass* c *for any* $(2k-1)$*-body collinear central configuration only depends on the configuration* x *and it is independent of the parameter* λ *and the corresponding mass* m*. More precisely it is given by*
$$c = \frac{x_{2k-1} + v^T B^{-1} y}{1 + v^T B^{-1} L}, \tag{2.3}$$
where $B = B_{2k-1}$, $y = Y_{2k-1} = (x_1, x_2, \cdots, x_{2k-2})^T$, $v = (a_{1(2k-1)}, a_{2(2k-1)}, \cdots, a_{(2k-2)(2k-1)})^T$, *and* $L = (1, 1, \cdots, 1)^T$ *with appropriate dimension. The denominator* $1 + v^T B^{-1} L$ *of* c *in* (2.3) *is always nonzero for any configuration* x.

2. *With the appropriate choice of the center of mass* c *given by* (2.3)*,* m *can be given by a function with two parameters* λ *and* m_{2k-1}
$$m = \begin{bmatrix} -\lambda B^{-1}(y - cL) - m_{2k-1} B^{-1} v \\ m_{2k-1} \end{bmatrix}. \tag{2.4}$$

3. *With the appropriate choice of the center of mass* c *given by* (2.3)*,* m *can be given by a function with two parameters* λ *and* M
$$m = \begin{bmatrix} -\lambda B^{-1}(y - cL + \frac{(L^T B^{-1} y) v}{1 + v^T B^{-1} L}) - \frac{M B^{-1} v}{1 + v^T B^{-1} L} \\ \frac{\lambda L^T B^{-1} y + M}{1 + v^T B^{-1} L} \end{bmatrix}, \tag{2.5}$$
where M *is the total mass.*

Proof. From the $(2k-1)$-body central configuration equations (1.1), we can rewrite the first $(2k-2)$ equations as
$$B(m_1, m_2, \cdots, m_{2k-2})^T = -\lambda(y - cL) - m_{2k-1} v.$$
Because B is even, B^{-1} exists by assumption.
$$(m_1, m_2, \cdots, m_{2k-2})^T = -\lambda B^{-1}(y - cL) - m_{2k-1} B^{-1} v.$$
Substituting this into the last equation $-v^T(m_1, m_2, \cdots, m_{2k-2})^T = -\lambda(x_{2k-1} - c)$ in (1.1), we have
$$-\lambda v^T B^{-1}(y - cL) - m_{2k-1} v^T B^{-1} v = \lambda(x_{2k-1} - c),$$
which implies
$$c(1 + v^T B^{-1} L) = x_{2k-1} + v^T B^{-1} y$$
because $v^T B^{-1} v = 0$. By Proposition 2.3 in [10],
$$(1 + v^T B^{-1} L) = \frac{1}{Pf(B)}(Pf(B) - Pf(B_1) + Pf(B_2) - \cdots + Pf(B_{2k-2}))$$
$$= \frac{1}{Pf(B)} Pf(A^L(x)) \neq 0.$$
So c is well defined by Eq (2.3). The proof of the rest of the theorem would be standard and the detail is omitted. □

Remark 2.4. 1. In the papers [2, 8], they assume the center of mass is at the origin. From the proof of the above theorems, the duality between the even and odd cases is due to the equivalence of the corresponding Pfaffians. So it is not appropriate to fix the center of mass at origin for the odd case because the center of mass for the odd central configuration is determined by the positions only.

2. For the odd case, the Pfaffian $Pf(A^x)$, which is $Pf(F)$ in Moulton's paper [2], must be zero because the center of mass c is fixed at origin. This is equivalent to $x_{2k-1} + v^T B^{-1} y = 0$ in Eq (2.3). In fact, $x_{2k-1} + v^T B^{-1} y = \frac{1}{Pf(B)} Pf(A^x)$ and $c = \frac{Pf(A^x)}{Pf(A^L)}$. $Pf(A^L)$ is also equivalent to $K_n^L(x)$ in the odd case in [9]. The Pfaffian $K_N^x(x) \neq 0$ in [9] is equivalent to $Pf(A^x) \neq 0$. When $N = 3$, $K_N^x(x)$ is not zero since the configuration is chosen to be $x = (0, 1, 1 + r)$ and the center of mass is not at origin.
3. With a fixed total mass M and a fixed collinear configuration, it is easy to know that each collinear configuration determines a one-parameter family of masses from Eqs (2.1), (2.2) or (2.5). This confirms that any collinear configuration can be a central configuration for certain masses (negative masses are allowed here). For the inverse problem with positive masses, Ouyang and Xie [12] proved that the central configurations do not exist for some configurations in the collinear Newtonian four-body problem. Ferrario [11] showed that for any homogeneity and $N \geq 4$ there are explicit regions of the configuration space without solutions of the inverse problem.
4. Albouy and Moeckel's conjecture can be modified only for even case.
   Variant of Albouy-Moeckel's Conjecture: For any $k \geq 1$ and any collinear configuration $x = (x_1, x_2, \cdots, x_{2k})$ with $-\infty < x_1 < x_2 < \cdots < x_{2k} < \infty$, the Pfaffian of its associate matrix $A(x)$ is nonzero, i.e., $Pf(A) \neq 0$.
   The conjecture has been proved analytically for $k = 1, 2, 3$ but it is still missing an analytical proof for $k \geq 4$.

3. Number of central configurations under different equivalences

Historically, there were three different ways to define the equivalent classes in the collinear N-body problem. They are called *permutation equivalence, geometric equivalence* and *mass equivalence*. Before we give some remarks on the relationship among the three equivalences, we recall some definitions and properties.

We borrow the notations in [13–15]). For any $N \in \mathbf{N}$ (the set of integers), we denote by $P(N)$ the set of all permutations of $\{1, 2, \cdots, N\}$. For any element $\tau \in P(N)$, we use $\tau = (\tau(1), \tau(2), \cdots, \tau(N))$ to denote the permutation τ. We also denote a permutation of $m = (m_1, m_2, \cdots, m_N)$ by $m(\tau) = (m_{\tau(1)}, m_{\tau(2)}, \cdots, m_{\tau(N)})$ for $\tau \in P(N)$. We define the converse permutation of τ by $con(\tau) = (\tau(N), \cdots, \tau(1))$ and denote by $^{\#}B$ the number of elements in a set B.

To define the equivalence of collinear central configurations, we fix the order of the positions as

$$W(N) = \left\{ x = (x_1, x_2, \cdots, x_N) \in \mathbf{R}^N | -\infty < x_1 < x_2 < \cdots < x_N < \infty \right\}.$$

Note that when we say by $x = (x_1, \cdots, x_N) \in W(N)$ is a collinear CC for $m(\alpha) \equiv (m_{\alpha(1)}, \cdots, m_{\alpha(N)})$ with some $\alpha \in P(N)$, we always mean that $m_{\alpha(i)}$ is located at x_i for all $i = 1, \cdots, N$.

Definition 3.1. Geometric Equivalence.
Fix $m \in (\mathbf{R}^+)^N$ and let $x = (x_1, \cdots, x_N)$ and $y = (y_1, \cdots, y_N) \in W(N)$ be two collinear CCs for $m(\zeta)$ and $m(\eta)$ with $\zeta \in P(N)$ and $\eta \in P(N)$. Then $(m(\zeta), x)$ and $(m(\eta), y)$ are *geometrically equivalent*, denoted

by $x \sim_G y$, if either $x = a(y-b)$ or $x = a(con(y)-b)$ for some $a \in \mathbf{R}\setminus\{0\}$ and $b = (b_0, b_0, \cdots, b_0) \in \mathbf{R}^N$. We denote by $L_G(N,m)$ the set of all geometric equivalence classes of the N-body collinear central configurations for any given mass vector $m \in (\mathbf{R}^+)^N$.

Definition 3.2. Permutation Equivalence.
Fix $m \in (\mathbf{R}^+)^N$ and let $x = (x_1, \cdots, x_N)$ and $y = (y_1, \cdots, y_N) \in W(N)$ be two collinear CCs for $m(\zeta)$ and $m(\eta)$ with $\zeta \in P(N)$ and $\eta \in P(N)$. Then $(m(\zeta), x)$ and $(m(\eta), y)$ are *permutation equivalent*, denoted by $x \sim_P y$, if $x \sim_G y$ and either $\zeta = \eta$ or $\zeta = con(\eta)$. We denote by $L_P(N,m)$ the set of all permutation equivalence classes of the N-body collinear central configurations for any given mass vector $m \in (\mathbf{R}^+)^N$.

Definition 3.3. Mass Equivalence.
Fix $m \in (\mathbf{R}^+)^N$ and let $x = (x_1, \cdots, x_N)$ and $y = (y_1, \cdots, y_N) \in W(N)$ be two collinear CCs for $m(\zeta)$ and $m(\eta)$ with $\zeta \in P(N)$ and $\eta \in P(N)$. Then $(m(\zeta), x)$ and $(m(\eta), y)$ are *mass equivalent*, denoted by $x \sim_M y$, if $x \sim_G y$ and either $m(\zeta) = m(\eta)$ or $m(\zeta) = m(con(\eta))$. We denote by $L_M(N,m)$ the set of all mass equivalence classes of the N-body collinear central configurations for any given mass vector $m \in (\mathbf{R}^+)^N$.

Remark 3.4. (1) The permutation equivalence in Definition 3.2 is used in most papers and books. Generally speaking, permutation of bodies makes difference in permutation equivalence. This is why it is called permutation equivalence. Moulton [2] showed there is a unique collinear central configuration for an ordering mass vector which implies $L_P(m,N) = N!/2$ for any positive masses.
(2) Directly from the definitions, we can derive that ${}^\#L_G(N,m) \leq^\# L_M(N,m) \leq^\# L_P(N,m)$. Wintner [7] studied $L_M(3,m)$. Long-Sun [13, 14] systematically studied these three equivalence for collinear three-body problem. Especially, they found that there is a singular algebraic hypersurface in the mass space on which ${}^\#L_G(3,m) = 2$ while ${}^\#L_P(3,m) =^\# L_M(3,m) = 3$. They also gave an explicit expression for ${}^\#L_M(N,m)$ for any m. Xie [16] reinvestigated the collinear three-body central configurations under geometric equivalence and he gave a direct parametric expression for the singular algebraic hypersurface in the mass space and his proof involved the concept of super central configurations [17]. In [15], Ouyang-Xie studied the case of collinear four body problem and they showed that ${}^\#L_G(4,m) =^\# L_M(4,m)$ if the four masses are not mutually distinct. If the four masses $m = (m_1, m_2, m_3, m_4)$ are mutually distinct and m in a singular hyper surface, then ${}^\#L_G(4,m) = 11$ while ${}^\#L_P(4,m) =^\# L_M(4,m) = 12$.
(3) The decreasing phenomena of the central configurations under geometric equivalence is related to the existence of super central configurations as shown for the collinear three-body case [17] and the collinear four-body case [15].

Definition 3.5. Super Central Configurations.
For a given central configuration $x \in W(N)$ with mass vector $m = (m_1, m_2, \cdots, m_N)$, let $S_m(x)$ be the permutational admissible set about m by

$$S_m(x) = \{m(\alpha) | x \text{ is a central configuration for } m(\alpha), \alpha \in P(N) \text{ and } m(\alpha) \neq m\}.$$

The configuration x is called a super central configuration for mass vector m if the set $S_m(x)$ is nonempty, i.e., ${}^\#S_m(x) > 0$.

Here we estimate the lower bounds for ${}^\#L_G(N,m)$. First we show a property about the set of super central configurations.

Theorem 3.6. *Assume that the Pfaffian $Pf(A) \neq 0$ of the associate matrix for any even collinear configuration. Let a fixed $x \in W(N)$ be a collinear central configuration for $m \in (\mathbf{R}^+)^N$. We have $^\# S_m(x) \leq 1$.*

Proof. We prove this result by contradiction. Assume that $^\# S_m(x) \geq 2$. Then there exist $\tau \in P(N)$ and $\alpha \in P(N)$ such that $m(\tau) \in S_m(x)$ and $m(\alpha) \in S_m(x)$ and $m(\tau) \neq m(\alpha) \neq m$. So x is a central configuration for mass m, $m(\alpha)$, and $m(\tau)$.

Case one: N is odd.

By theorem 2.3 and Eq (2.5), there exist λ, λ_τ, and λ_α such that

$$m_i = \lambda f_i + M g_i, \qquad m_{\tau(i)} = \lambda_\tau f_i + M g_i, \qquad m_{\alpha(i)} = \lambda_\alpha f_i + M g_i \tag{3.1}$$

for $i = 1, 2, \cdots, N$, where M is the total mass which is a constant, f_i and g_i depend on the configuration x only. Since $m(\tau) \neq m(\alpha) \neq m$, we have $\lambda_\tau \neq \lambda_\alpha \neq \lambda$. Without loss of generality, we can assume $0 < \lambda < \lambda_\tau < \lambda_\alpha$. Let m_i be one of the largest masses in $\{m_1, m_2, \cdots, m_N\}$.

(A) If $f_i > 0$, then $m_{\tau(i)} = \lambda_\tau f_i + M g_i > \lambda f_i + M g_i = m_i$, which is contradiction to that m_i is the largest value.

(B) If $f_i < 0$, then $m_i > m_{\tau(i)} > m_{\alpha(i)}$. Since $m(\tau)$ is a permutation of m, there exists a $j \neq i$ such that $m_{\tau(j)} = m_i$ and $m_i > m_j$, which means $m_{\tau(j)}$ has the largest value. This can be chosen because m_i has the largest value while $m_{\tau(i)}$ does not have. This implies that $f_j > 0$ and $\lambda < \lambda_\tau < \lambda_\alpha$. Therefore $m_{\tau(j)} < m_{\alpha(j)}$ which is contradiction to the fact that $m_{\tau(j)}$ has the largest value.

(C) If $f_i = 0$, then $m_i = m_{\tau(i)} = m_{\alpha(i)}$. If m_i is not the only largest mass, for example m_k is also one of the largest masses, by argument (A) and (B), $f_k = 0$ and $m_k = m_{\tau(k)} = m_{\alpha(k)}$. This implies that all the largest masses in m are also the largest masses in $m(\tau)$ and in $m(\alpha)$ at the same position on the line. Then we can consider the second largest masses in m which become the largest value among the rest of bodies in m and in $m(\tau)$ and $m(\alpha)$. Similar arguments as above can show that all the second largest masses in m are also the second largest masses in $m(\tau)$ and in $m(\alpha)$ at the same position on the line. By induction, $m = m(\tau) = m(\alpha)$ which is a contradiction.

So the assumption $^\# S_m(x) \geq 2$ is wrong. The inequality $^\# S_m(x) \leq 1$ for odd number of bodies holds.

Case two: N is even.

By theorem 2.2 and Eq (2.1) and Eq (2.2), M and λ don't vary in central configurations for m, $m(\alpha)$, and $m(\tau)$. Then there exist c, c_τ, and c_α such that

$$m_i = \lambda s_i + c t_i, \qquad m_{\tau(i)} = \lambda s_i + c_\tau t_i, \qquad m_{\alpha(i)} = \lambda s_i + c_\alpha t_i \tag{3.2}$$

for $i = 1, 2, \cdots, N$, where λ is determined by the total mass M and the configuation x from Eq (2.2), s_i and t_i depend on the configuration x only. Now we can use the same arguments to prove that $^\# S_m(x) \leq 1$ for even number of bodies.

This completes the proof for $^\# S_m(x) \leq 1$. □

Remark 3.7. From the definition of geometric equivalence, two geometric equivalent central configurations have similar geometric shape. If $^\# S_m(x) = 1$ and $m(\alpha) \in P(N)$, then the two central configurations (m, x) and $(m(\alpha), x)$ are geometric equivalent. But they are not mass equivalent and permutation equivalent if $m(\alpha) \neq m$. The decreasing number between $^\# L_M(m, N)$ and $^\# L_G(m, N)$ is determined by the number of super central configurations for $m(\alpha), \alpha \in P(N)$.

Theorem 3.8. *Assume that the Pfaffian $Pf(A) \neq 0$ of the associate matrix for any even collinear configuration. Let $m = (m_1, m_2, \cdots, m_N) \in (\mathbf{R}^+)^N$.*

1. $\frac{{}^{\#}L_M(N,m)}{2} \leq^{\#} L_G(N,m) \leq^{\#} L_M(N,m)$.
2. *If $m_1, m_2, \cdots, m_N$ are distinct to each other, then $N!/4 \leq^{\#} L_G(N,m) \leq N!/2$.*

Proof. 1. If $m(\alpha) = m$ for $\alpha \in P(N)$, then the central configurations for m and $m(\alpha)$ are the same. Since ${}^{\#}S_m(x) \leq 1$, x can be a central configuration for at most two order of mass vector m and $m(\alpha)$ with $m \neq m(\alpha)$. Such x may be counted as two different central configurations under mass equivalence, but it is counted as one central configuration under geometric equivalence. The possible smallest number of ${}^{\#}L_G(N,m)$ occurs when every central configuration for $m(\alpha)$ is a super central configuration. This gives the inequality in this theorem.

2. If $m_1, m_2, \cdots, m_N$ are distinct to each other, then ${}^{\#}L_M(N,m) =^{\#} L_P(N,m) = N!/2$ by Moulton. As a result of above, we have the bounds for $N!/4 \leq^{\#} L_G(N,m) \leq n!/2$.

□

The exact number of central configurations under permutation equivalence ${}^{\#}L_P(N,m)$ and under mass equivalence ${}^{\#}L_M(N,m)$ is known for any collinear N-body problem. But it is unknown for the number of central configurations under geometric equivalence. Even ${}^{\#}L_G(N,m)$ is still unknown for $N = 5$. The classification of super central configurations in the collinear 5-body problem is recently studied in [18]. Some properties are proved there, which help to determine the value of ${}^{\#}L_G(N,m)$.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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