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Conic bundle structures on ℚ-Fano threefolds

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Abstract: We obtain a sufficient condition for a Fano threefold with terminal singularities to have a conic bundle structure.

1. Introduction

A *conic bundle* is a proper flat morphism $\pi : X \to Z$ of non-singular varieties such that it is of relative dimension 1 and the anticanonical divisor $-K_X$ is relatively ample. We say that a variety X has a conic bundle structure if there exists a conic bundle $\pi : X' \to Z$ and a birational map $X \dashrightarrow X'$.

Varieties with conic bundle structure play a very important role in the birational classification of algebraic varieties of negative Kodaira dimension. For example, any variety with rational curve fibration has a conic bundle structure [1]. For these varieties there are well-developed techniques to solve rationality problems [1–4].

Another important class of varieties of negative Kodaira dimension is the class of ℚ-*Fano varieties*. Recall that a projective variety X is called ℚ-Fano if it has only terminal ℚ-factorial singularities, the Picard number $\rho(X)$ equals 1, and the anticanonical class $-K_X$ is ample. In fact, these two classes overlap. Moreover, ℚ-Fano varieties with conic bundle structures form a large subclass in the class of all ℚ-Fano varieties. It is very important for birational geometry to investigate and describe those ℚ-Fano varieties that do not have conic bundle structures. This paper is an attempt to do it in dimension three.

To formulate the main result we have to recall some definitions. A normal n-dimensional variety X with only canonical Gorenstein singularities is called a *del Pezzo variety* if there exists an ample Cartier divisor A such that $-K_X = (n-1)A$ (see [5]). Then the intersection number $\mathrm{d}(X) := A^n$ is called

the *degree* of X. We consider only those del Pezzo varieties that have at worst terminal singularities. Del Pezzo varieties of degree ≤ 2 have special names:

(i) a del Pezzo variety X of degree 1 is called *double Veronese cone*;
(ii) a del Pezzo variety of degree 2 is called *quartic double solid*.

Recall that the Fano index $q_{\mathbb{Q}}(X)$ of a $\mathbb{Q}$-Fano variety is the maximal integer that divides the canonical class K_X in the torsion free group $\mathrm{Cl}(X)/\mathrm{Tors}$ (see 2.4). The following fact was proved in [6]:

1.1 Theorem. *Let X be a $\mathbb{Q}$-Fano threefold with $q_{\mathbb{Q}}(X) > 7$. Then X is rational. In particular, X has a lot of conic bundle structures.*

For a normal $\mathbb{Q}$-factorial projective variety X, denote

$$\mathrm{df}(X) := \max\left\{\dim|M| \,\big|\, M \text{ is a Weil divisor such that } -(K_X + M) \text{ is ample}\right\}.$$

Our main result is the following

1.2 Theorem. *Let X be a $\mathbb{Q}$-Fano threefold with $q_{\mathbb{Q}}(X) > 1$. Assume that X has no conic bundle structures. Then we have.*

(i) $\mathrm{df}(X) \leq 3$.
(ii) *If $\mathrm{df}(X) = 3$, then X is a smooth quartic double solid.*
(iii) *If $\mathrm{df}(X) = 2$, then there exists a birational transformation $\Psi : X \dashrightarrow X'$ such that one of the following holds:*

- *X' is a smooth quartic double solid,*
- *X' is a smooth double Veronese cone and Ψ is an isomorphism, or*
- *X' is a double Veronese cone with terminal $\mathbb{Q}$-factorial singularities and X' is singular.*

Note that a *smooth* double Veronese cone has no conic bundle structures according to [7]. Existence of conic bundle structures on smooth quartic double solids and singular double Veronese cones is not known.

2. Preliminaries

2.1. *Notation and terminology*

In this paper we always assume that the ground field $\Bbbk$ is algebraically closed and of characteristic zero. Also, we use the standard notations and definitions of the Minimal Model Program (MMP). When we say that a variety X has terminal (resp. canonical) singularities, this means that the singularities are not worse than that, in particular, X can be smooth.

For a variety X with terminal singularities $\mathbf{B}(X)$ denotes the basket of its singularities (see [8]). Typically, when describing a basket we list only indices of singularities. For example, $\mathbf{B}(X) = (r_1, r_2)$ means that $\mathbf{B}(X)$ contains two points of types $\frac{1}{r_1}(1, -1, b_1)$ and $\frac{1}{r_2}(2, -2, b_2)$. For any normal variety X, $\mathrm{Cl}(X)$ denotes the group of Weil divisors on X modulo linear equivalence. By $\mathrm{Cl}(X)_{\mathrm{T}}$ we denote the torsion subgroup of $\mathrm{Cl}(X)$. If $\mathscr{M}$ is a linear system, then $\mathrm{Bs}(\mathscr{M})$ denotes its base locus.

2.2. Singularities of linear systems

2.2.1 Definition. Let X be a normal variety and let $\mathscr{M}$ be a (not necessarily complete) linear system of Weil divisors. We assume that $\dim(\mathscr{M}) > 0$ and $\mathscr{M}$ has no fixed components. Let μ be non-negative rational number such that $K_X + \mu\mathscr{M}$ is $\mathbb{Q}$-Cartier. For a birational morphism $f : \tilde{X} \to X$, write

$$K_{\tilde{X}} + \mu\tilde{\mathscr{M}} = K_X + \mu\mathscr{M} + \sum e_i E_i,$$

where $\tilde{\mathscr{M}} := f_*^{-1}\mathscr{M}$ is the proper transform of $\mathscr{M}$ on $\tilde{X}$, E_i are prime exceptional divisors, and e_i are rational numbers called discrepancies. Note that in this formula discrepancies are defined with respect to μM, where $M \in \mathscr{M}$ is a general member [9, Sect. 4]. We say that the pair $(X, \mu\mathscr{M})$ *canonical*, resp. *terminal* if $e_i \geq 0$ (resp. $e_i > 0$) for all i and for all birational morphisms $f : \tilde{X} \to X$.

2.2.2 Lemma (see [10], [9], [11]). *Let $(X, \mathscr{M})$ be a pair, where X is a threefold with at worst terminal singularities and $\mathscr{M}$ is a movable linear system on X.*

(i) *If $(X, \mathscr{M})$ is canonical, then a general member $S \in \mathscr{M}$ has only Du Val singularities and in a neighborhood of each point $P \in \mathrm{Bs}(\mathscr{M})$ we have $\mathscr{M} \sim -K_X$,*

(ii) *If $(X, \mathscr{M})$ is terminal, then a general member $S \in \mathscr{M}$ is a smooth surface contained in the smooth locus of X, and $\dim \mathrm{Bs}(\mathscr{M}) \leq 0$.*

2.2.3 Theorem (see [10], [11]). *Let X be a variety with terminal $\mathbb{Q}$-factorial singularities and let $\mathscr{M}$ be a linear system on X without fixed components such that the pair $(X, \mu\mathscr{M})$ is canonical for some $\mu \geq 0$. Let*

$$\varphi : (X, \mu\mathscr{M}) \dashrightarrow (X', \mu\mathscr{M}')$$

be a step of $K_X + \mu\mathscr{M}$-MMP, that is, φ is either a $K_X + \mu\mathscr{M}$-negative extremal divisorial contraction or a $K_X + \mu\mathscr{M}$-flip. Then the variety X' again has terminal $\mathbb{Q}$-factorial singularities and the pair $(X', \mu\mathscr{M}')$ is canonical. If moreover $(X, \mu\mathscr{M})$ is terminal, then so is $(X', \mu\mathscr{M}')$.

2.2.4 Corollary (see [10], [12], [11]). *Let X be a normal variety and let $\mathscr{M}$ be a linear system on X without fixed components such that $K_X + \mu\mathscr{M}$ is $\mathbb{Q}$-Cartier for some $\mu \geq 0$. Then there exists a projective birational morphism*

$$f : (\tilde{X}, \tilde{\mathscr{M}}) \longrightarrow (X, \mathscr{M}),$$

where $\tilde{\mathscr{M}} := f_*^{-1}\mathscr{M}$, such that the pair $(\tilde{X}, \mu\tilde{\mathscr{M}})$ is terminal, $K_{\tilde{X}} + \mu\tilde{\mathscr{M}}$ is f-nef, and $\tilde{X}$ is $\mathbb{Q}$-factorial. Moreover,

$$K_{\tilde{X}} + \mu\tilde{\mathscr{M}} \sim_{\mathbb{Q}} f^*(K_X + \mu\mathscr{M}) - \sum e_i E_i,$$

where E_i are prime exceptional divisors and $e_i \geq 0$ for all i.

Such a pair $(\tilde{X}, \mu\tilde{\mathscr{M}})$ is called *terminal $\mathbb{Q}$-factorial model* of $(X, \mu\mathscr{M})$.

2.2.5 Corollary (see [13], [10], [11]). *Let X be a variety with terminal $\mathbb{Q}$-factorial singularities and let $\mathscr{M}$ be a linear system on X without fixed components such that the pair $(X, \mu\mathscr{M})$ is canonical for some $\mu \geq 0$. Then there exists a projective birational morphism*

$$f : (\tilde{X}, \tilde{\mathscr{M}}) \longrightarrow (X, \mathscr{M}),$$

where $\tilde{\mathcal{M}} := f_*^{-1}\mathcal{M}$, such that $\tilde{X}$ is terminal $\mathbb{Q}$-factorial, the pair $(\tilde{X}, \mu\tilde{\mathcal{M}})$ is canonical,

$$K_{\tilde{X}} + \mu\tilde{\mathcal{M}} = f^*(K_X + \mu\mathcal{M}),$$

and the exceptional locus of f is a prime divisor.

Such a morphism is called an *extremal log crepant blowup* of $(X, \mu\mathcal{M})$.

2.2.6 Corollary (see [12])**.** *Let X be a normal projective variety and let $\mathcal{M}$ be a linear system on X without fixed components such that $K_X + \mu\mathcal{M}$ is canonical for some $\mu \geq 0$. Assume that $K_X + \mu\mathcal{M}$ is not pseudo-effective. Then one can run $K_X + \mu\mathcal{M}$-MMP and end with a Mori fiber space.*

Recall also the following well-known result.

2.2.7 Theorem (see [14])**.** *Let $(X \ni P)$ be a terminal quotient singularity of type $\frac{1}{r}(1, a, r - a)$, let $f: \tilde{X} \to X$ be a divisorial Mori contraction, and let E be the exceptional divisor. Then $f(E) = P$, the contraction f is a weighted blowup with weights $(1, a, r - a)$, and the discrepancy of E equals $a(E, X) = 1/r$.*

2.3. Del Pezzo varieties

2.3.1 Proposition (see [5], [15])**.** *Let X be a three-dimensional del Pezzo variety with terminal singularities. Then*

$$\dim\left|-\tfrac{1}{2}K_X\right| = \mathrm{d}(X) + 1$$

and a general member $M \in \left|-\tfrac{1}{2}K_X\right|$ is a smooth del Pezzo surface of degree $\mathrm{d}(X)$. Moreover, the pair $\left(X, \left|-\tfrac{1}{2}K_X\right|\right)$ is terminal.

2.3.2 Theorem (see [16], [5], [15])**.** *Let X be a three-dimensional del Pezzo variety with terminal singularities.*

(i) *If $\mathrm{d}(X) = 1$ (i.e., X is a double Veronese cone), then X can be realized as a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1^3, 2, 3)$. In this case, the projection*

$$\pi : X \longrightarrow \mathbb{P}(1^3, 2)$$

is a double cover whose branch divisor has degree 6 in $\mathbb{P}(1^3, 2)$.

(ii) *If $\mathrm{d}(X) = 2$ (i.e., X is a quartic double solid), then X can be realized as a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1^4, 2)$. In this case, the projection*

$$\pi : X \longrightarrow \mathbb{P}(1^4) = \mathbb{P}^3$$

is a double cover whose branch divisor is a quartic hypersurface.

2.4. $\mathbb{Q}$-Fano varieties

Now, let X be a $\mathbb{Q}$-Fano variety. It follows from the definition that $\mathrm{Cl}(X)$ is a finitely generated abelian group of rank 1. The numbers

$$\mathrm{q}_{\mathbb{Q}}(X) \quad := \quad \max\{t \in \mathbb{Z} \mid -K_X \sim_{\mathbb{Q}} tA, \ A \text{ is a Weil divisor}\}$$

$$q_w(X) := \max\{t \in \mathbb{Z} \mid -K_X \sim tA, \; A \text{ is a Weil divisor}\}$$

are called the *Fano index* and *Fano-Weil index*, respectively. Clearly, $q_w(X)$ divides $q_{\mathbb{Q}}(X)$ and $q_{\mathbb{Q}}(X) = q_w(X)$ if $\mathrm{Cl}(X)$ is torsion free. For a $\mathbb{Q}$-Fano variety X throughout this paper A will denote a Weil divisor on X such that $-K_X \sim_{\mathbb{Q}} q_{\mathbb{Q}}(X)A$. If $q_{\mathbb{Q}}(X) = q_w(X)$, we take A so that $-K_X \sim q_{\mathbb{Q}}(X)A$. Since $\rho(X) = 1$ and X is $\mathbb{Q}$-factorial, for any Weil divisor M on X we can write $-K_X \sim_{\mathbb{Q}} \lambda M$. In this situation denote

$$\lambda(X, |M|) = \lambda(X, M) := \lambda.$$

Thus $q_{\mathbb{Q}}(X) = \lambda(X, A)$.

2.4.1 Theorem (see [17]). *Let X be a $\mathbb{Q}$-Fano threefold and let M be a smooth surface on X with $\varkappa(M) = -\infty$. Assume that X is not rational. Then one of the following holds:*

 (i) *X is a hypersurface of degree 6 in $\mathbb{P}(1^2, 2^2, 3)$, M is a member of the linear system $|\mathcal{O}_X(2)| = \left| -\frac{2}{3}K_X \right|$, and so $\lambda(X, M) = 3/2$,*
 (ii) *X is a del Pezzo threefold of degree $\mathrm{d}(X) \leq 3$, M is a member of the linear system $\left| -\frac{1}{2}K_X \right|$, and $\lambda(X, M) = 2$.*

Note that in [17] the surface M is supposed to have ample normal bundle. This is automatically satisfied in our case because $\rho(X) = 1$ by our $\mathbb{Q}$-Fano assumption.

2.4.2 Corollary. *Let X be a $\mathbb{Q}$-Fano threefold and let $\mathcal{M}$ be a linear system without fixed components on X such that $\lambda(X, \mathcal{M}) > 1$ and the pair $(X, \mathcal{M})$ is terminal. Assume that X has no conic bundle structures. Then one of the following holds:*

 (i) *$X = X_6 \subset \mathbb{P}(1^3, 2, 3)$ is a double Veronese cone,*
 (ii) *$X = X_4 \subset \mathbb{P}(1^4, 2)$ is a* smooth *quartic double solid.*

Proof. By Lemma 2.2.2(ii) a general member $M \in \mathcal{M}$ is smooth and is contained in the smooth locus of X. By the adjunction formula M is a del Pezzo surface. In particular, M is rational. Thus we can apply Theorem 2.4.1.

Assume that X is such as in 2.4.1(i). Let $\psi(x_1, x'_1, x_2, x'_2, x_3) = 0$ be an equation of $X = X_6 \subset \mathbb{P}(1^2, 2^2, 3)$, where the subscript index of the variables x_i and x'_j is its degree. Since the singularities of X are terminal, ψ contains the term x_3^2. Thus ψ can be written in the form

$$\psi = x_3^2 + \gamma_3(x_2, x'_2) + \phi(x_1, x'_1, x_2, x'_2),$$

where γ_3 is a homogeneous polynomial of degree 3 and ϕ does not contain cubic terms in x_2, x'_2. Clearly, $\gamma_3 \neq 0$ (otherwise X would be singular along the line $\{x_1 = x'_1 = x_3 = 0\}$). Thus after a linear coordinate change we may assume that $\gamma_3 = x_2 x'_2(x_2 + x'_2)$, $x_2 x'^2_2$, or x'^3_2. Then the projection

$$X \dashrightarrow \mathbb{P}(1, 1, 2), \qquad (x_1, x'_1, x_2, x'_2, x_3) \longmapsto (x_1, x'_1, x'_2)$$

is a rational curve fibration because in the affine chart $x'_2 = 1$ the equation ψ becomes quadratic in x_3, x_2. Hence X has a conic bundle structure in this case.

Thus we may assume that X is a del Pezzo threefold of degree $\mathrm{d}(X) \leq 3$ (see 2.4.1(ii)). If $\mathrm{d}(X) = 3$, then $X = X_3 \subset \mathbb{P}^4$ is a cubic with terminal singularities and M is its smooth hyperplane section. In this case the projection $X \dashrightarrow \mathbb{P}^2$ from a line $l \subset X$ is a rational curve fibration.

It remains to consider the case $d(X) = 2$, i.e., the case where $X = X_4 \subset \mathbb{P}(1^4, 2)$ is a quartic double solid. Suppose that X has a singular point, say $P \in X$. Let $\psi(x_0, x_1, x_2, x_3, y) = 0$ be an equation of $X = X_4 \subset \mathbb{P}(1^4, 2)$, where $\deg(x_i) = 1$ and $\deg(y) = 2$. Since the singularities of X are terminal Gorenstein, ψ contains the term y^2. By an obvious coordinate change we may assume that ψ has the form

$$\psi = y^2 + \phi(x_0, x_1, x_2, x_3),$$

where $\deg(\phi) = 4$. Clearly, P is contained in the hyperplane $\{y = 0\}$. Hence, by a linear coordinate change we may assume that $P = (1, 0, 0, 0, 0)$ and so

$$\psi = y^2 + \phi_4(x_1, x_2, x_3) + x_0\phi_3(x_1, x_2, x_3) + x_0^2\phi_2(x_1, x_2, x_3),$$

where $\deg(\phi_i) = i$. As above, the projection

$$X \dashrightarrow \mathbb{P}(1, 1, 1) = \mathbb{P}^2, \qquad (x_0, x_1, x_2, x_3, y) \longmapsto (x_1, x_2, x_3)$$

is a rational curve fibration. Hence X has a conic bundle structure. $\qquad\square$

3. Construction

We recall the construction used in the papers [18–21].

3.1 Assumptions. Let X be a $\mathbb{Q}$-Fano threefold. Consider a linear system $\mathcal{M}$ on X without fixed components. Let $c := \mathrm{ct}(X, \mathcal{M})$ be the canonical threshold of the pair $(X, \mathcal{M})$ [13]. Assume that $\lambda(X, \mathcal{M}) > c$.

3.1.1 Lemma (see [19, Lemma 4.2]). *Let $P \in X$ be a point of index $r > 1$. Assume that $\mathcal{M} \sim -tK_X$ near P, where $0 < t < r$. Then $\mathrm{ct}(X, \mathcal{M}) \le 1/t$ and $\beta \ge t\alpha$.*

Consider an extremal log crepant blowup $f : \tilde{X} \to X$ with respect to $K_X + c\mathcal{M}$ (Corollary 2.2.5). Let E be the exceptional divisor. Recall that $\tilde{X}$ has only terminal $\mathbb{Q}$-factorial singularities. We can write

$$\begin{aligned} K_{\tilde{X}} &\sim_{\mathbb{Q}} f^*K_X + \alpha E, \\ \tilde{\mathcal{M}} &\sim_{\mathbb{Q}} f^*\mathcal{M} - \beta E. \end{aligned} \tag{3.1.2}$$

where $\alpha, \beta \in \mathbb{Q}_{>0}$. Then $c = \alpha/\beta$ and

$$K_{\tilde{X}} + c\tilde{\mathcal{M}} \sim_{\mathbb{Q}} f^*(K_X + c\mathcal{M}).$$

Since $\lambda(X, \mathcal{M}) > c$, the divisor $-(K_{\tilde{X}} + c\tilde{\mathcal{M}})$ is nef. Take $\lambda = \lambda(X, \mathcal{M})$. Then

$$K_{\tilde{X}} + \lambda\tilde{\mathcal{M}} \sim_{\mathbb{Q}} f^*(K_X + \lambda\mathcal{M}) + \alpha(1 - \lambda/c)E \sim_{\mathbb{Q}} \alpha(1 - \lambda/c)E.$$

Put $\delta := \alpha(\lambda/c - 1)$. Then

$$K_{\tilde{X}} + \lambda\tilde{\mathcal{M}} + \delta E \sim_{\mathbb{Q}} 0, \quad \text{where } \delta > 0. \tag{3.1.3}$$

Run the MMP with respect to $K_{\tilde{X}} + c\tilde{\mathcal{M}}$. We obtain the following diagram (Sarkisov link)

$$\tag{3.1.4}$$

Here the varieties $\tilde{X}$ and $\bar{X}$ have only terminal $\mathbb{Q}$-factorial singularities, χ is a composition of $K_{\tilde{X}} + c\tilde{\mathcal{M}}$-log flips, $\rho(\tilde{X}) = \rho(\bar{X}) = 2$, and $\bar{f} : \bar{X} \to \hat{X}$ is an extremal $K_{\bar{X}}$-negative Mori contraction. In particular, $\mathrm{rk}\,\mathrm{Cl}(\hat{X}) = 1$.

In what follows, for a divisor (or a linear system) N on X by $\tilde{N}$ and $\bar{N}$ we denote proper transforms of N on $\tilde{X}$ and $\bar{X}$ respectively. If $\bar{f}$ is birational, then $\hat{N}$ denotes the proper transform of N on $\hat{X}$. Apply χ_* to (3.1.3):

$$K_{\bar{X}} + \lambda\bar{\mathcal{M}} + \delta\bar{E} \sim_{\mathbb{Q}} 0. \tag{3.1.5}$$

3.1.6 Lemma (cf. [6, Lemma 4.7]). *If $\mathcal{M}$ is not composed of a pencil and $\bar{f}$ is not birational, then X has a conic bundle structure.*

Proof. If $\dim(\hat{X}) = 2$, then $\bar{f}$ is a $\mathbb{Q}$-conic bundle and we are done. Assume that $\dim(\hat{X}) = 1$. Then $\bar{f}$ is a del Pezzo fibration and $\hat{X} \simeq \mathbb{P}^1$. By our assumption $\dim(\mathcal{M}) \geq 2$ and a general member of $\mathcal{M}$ is irreducible. Hence $\mathcal{M}$ is not $\bar{f}$-horizontal, i.e., $\mathcal{M}$ is not a pull-back of a linear system on $\hat{X}$. Since the divisor $\bar{E}$ is not movable, it is not contained in fibers and so $\bar{E}$ is $\bar{f}$-ample. For a general fiber F by the adjunction formula and (3.1.5) we have

$$-K_F = -K_{\bar{X}}|_F = \lambda\bar{\mathcal{M}}|_F + \delta\bar{E}|_F,$$

where both $\bar{\mathcal{M}}|_F$ and $\bar{E}|_F$ are integral ample divisors. Since $\lambda > 1$, the surface F is isomorphic either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In this case, $\bar{X}$ is rational and so it has a lot of conic bundle structures. $\square$

3.2. Birational contractions

Assume that the contraction $\bar{f}$ is birational. Then $\hat{X}$ is a $\mathbb{Q}$-Fano threefold. Denote $\hat{q} := \mathrm{q}_{\mathbb{Q}}(\hat{X})$.

3.2.1 Lemma. *If the contraction $\bar{f}$ is birational, then the divisor $\bar{E}$ is not contracted by $\bar{f}$.*

Proof. Indeed, otherwise the map $\bar{f} \circ \chi \circ f^{-1} : X \dashrightarrow \hat{X}$ would be an isomorphism in codimension one. Hence it is an isomorphism. On the other hand, the number of $K_{\hat{X}} + c\mathcal{M}$-crepant divisors on $\hat{X}$ is strictly less than the number of $K_X + c\mathcal{M}$-crepant divisors on X, a contraction. $\square$

It follows from (3.1.5) that

$$K_{\hat{X}} + \lambda\hat{\mathcal{M}} + \delta\hat{E} \sim_{\mathbb{Q}} 0.$$

Therefore,

$$\lambda(\hat{X}, \hat{\mathcal{M}}) > \lambda(X, \mathcal{M}). \tag{3.2.2}$$

3.3. Linear systems

Let $\mathcal{N}_k$ be a non-empty linear system on X such that $q\mathcal{N}_k \sim k(-K_X)$ (here we allow $\mathcal{N}_k$ have fixed components). Let Θ be an ample Weil divisor on $\hat{X}$ generating $\mathrm{Cl}(\hat{X})/\mathrm{Cl}(\hat{X})_{\mathrm{T}}$. We can write

$$\hat{E} \sim_{\mathbb{Q}} e\Theta, \qquad \hat{\mathcal{N}_k} \sim_{\mathbb{Q}} s_k\Theta,$$

where $e \in \mathbb{Z}_{>0}$ by Lemma 3.2.1 and $s_k \in \mathbb{Z}_{\geq 0}$. Thus $s_k = 0$ if and only if $\dim(\mathcal{N}_k) = 0$ and a unique element $\bar{N}$ of $\mathcal{N}_k$ is $\bar{f}$-exceptional. As in (3.1.2), we write

$$\tilde{\mathcal{N}_k} \sim_{\mathbb{Q}} f^*\mathcal{N}_k - \beta_k E. \tag{3.3.1}$$

The relations (3.1.2) and (3.3.1) give us

$$kK_{\tilde{X}} + q\tilde{\mathcal{N}_k} \sim -(q\beta_k - k\alpha)E,$$

where $q\beta_k - k\alpha$ is an integer. From this we obtain

$$k\hat{q} = qs_k + (q\beta_k - k\alpha)e. \tag{3.3.2}$$

3.4. Dimensions of linear systems via computer search

We need certain information on linear systems on $\mathbb{Q}$-Fano threefolds of large Fano index. Most of these facts are contained in the Graded Ring Database [22] or can be obtained by direct computations using algorithms described in [6, 19, 23, 24].

3.4.1 Proposition. *Let X be a $\mathbb{Q}$-Fano threefold and let M be a Weil divisor on X such that $\dim|M| \geq 2$ and $\lambda(X, M) > 1$. Then the linear system $|M|$ is not composed of a pencil.*

Proof. Assume the contrary, that is, $|M| = D + m|L|$, where D is the fixed part of $|M|$ and $|L|$ is a pencil without fixed components. Then $\dim|M| = m \geq 2$. Replacing $|M|$ with $m|L|$ we may assume that $D = 0$. Clearly, $\dim|2L| = 2$ and $q_{\mathbb{Q}}(X) \geq 3$ in our case. Now, running computer search for $\mathbb{Q}$-Fano threefolds with $\dim|L| = 1$ and $\dim|2L| = 2$, we get a contradiction. Interested readers can find a simple PARI/GP code [25] on the author's webpage `https://homepage.mi-ras.ru/~prokhoro/programs/pencil.gp`. □

As above, using computer search one can find that there are 30 possible Hilbert series of $\mathbb{Q}$-Fano threefolds such that $q_{\mathbb{Q}}(X) \geq 3$, $\text{Cl}(X)_{\text{T}} = 0$, and $\text{df}(X) = 3$. As a consequence of this list one obtains.

3.4.2 Proposition. *Let X be a $\mathbb{Q}$-Fano threefold with $q_{\mathbb{Q}}(X) \geq 3$, $\text{Cl}(X) \simeq \mathbb{Z}$ and $\text{df}(X) = 3$. Then the following assertions hold.*

(i) $q_{\mathbb{Q}}(X) \leq 7$, $q_{\mathbb{Q}}(X) \neq 6$, *and the basket $\mathbf{B}(X)$ contains at most one point of index ≥ 8.*
(ii) *If $q_{\mathbb{Q}}(X) = 7$, then $|A| = \varnothing$ and $\dim|2A| = \dim|3A| = 0$.*
(iii) *If $q_{\mathbb{Q}}(X) = 5$, then $\dim|2A| = 0$ and $\dim|3A| = 1$.*
(iv) *If $q_{\mathbb{Q}}(X) = 4$, then $\dim|A| \leq 0$ and $1 \leq \dim|2A| \leq 2$. If moreover, $\dim|2A| = 2$, then $\dim|A| = 0$, $A^3 = 2/11$ and $\mathbf{B}(X) = (11)$.*

Similarly, we have.

3.4.3 Proposition. *Let X be a $\mathbb{Q}$-Fano threefold with $q_{\mathbb{Q}}(X) \geq 3$, $\text{df}(X) = 3$ and $\text{Cl}(X)_{\text{T}} \simeq \mathbb{Z}/n\mathbb{Z}$, where $n > 1$. Let T be a generator of $\text{Cl}(X)_{\text{T}}$. Then the following assertions hold.*

(i) $q_{\mathbb{Q}}(X) \leq 5$,
(ii) *If $q_{\mathbb{Q}}(X) = 5$, then $q_{\mathbb{Q}}(X) = 5$, $n = 2$, $\mathbf{B} = (4^2, 12)$, $A^3 = 1/12$. Moreover, $\dim|A| = 0$, $|A + T| = \varnothing$, $\dim|2A| = \dim|2A + T| = 0$, $\dim|3A| = \dim|3A + T| = 1$.*
(iii) *Assume that $q_{\mathbb{Q}}(X) = 4$ and $\text{df}(X) = 3$. Then $n \in \{2, 5\}$ and for a suitable choice of A we have $|A| = \varnothing$, $\dim|2A| = 1$, and $\dim|3A + kT| = 3$ for any k. Moreover,*

   (a) *if $n = 2$, then $\dim|A + T| = 0$ and $0 \leq \dim|2A + T| \leq 1$;*
   (b) *if $n = 5$, then $\dim|A + kT| = 0$ and $\dim|2A + kT| = 1$ for $k \not\equiv 0 \mod 5$.*

(iv) *Assume that $q_{\mathbb{Q}}(X) = 3$ and $\text{df}(X) = 3$. Then $n \in \{2, 3\}$ and $|A + \Lambda| \neq \varnothing$ for any $\Lambda \in \text{Cl}(X)_{\text{T}}$. Moreover, if $n = 2$ and $\dim|2A| < 3$, then $A^3 = 15/28$ and $\mathbf{B} = (2, 4, 14)$, and $|A| \neq \varnothing$.*

4. Proof of Theorem 1.2

4.1 Assumptions. Let X be a $\mathbb{Q}$-Fano threefold with $q_\mathbb{Q}(X) > 1$ and $df(X) \geq 2$ and let $\mathscr{M}$ be a linear system without fixed components such that $\dim(\mathscr{M}) = df(X)$ and $\lambda(X, \mathscr{M}) > 1$. Assume that X has no conic bundle structures.

4.1.1 Claim. *There exists a sequence of links of the form* (3.1.4)

$$\Phi : X = X^{(1)} \xdashrightarrow{\Psi_1} X^{(2)} \xdashrightarrow{\Psi_2} \cdots \xdashrightarrow{\Psi_{n-1}} X^{(n)} = X' \tag{4.1.2}$$

where each $X^{(i)}$ is a $\mathbb{Q}$-Fano threefold,

$$\lambda\left(X^{(i+1)}, \mathscr{M}^{(i+1)}\right) \geq \lambda\left(X^{(i)}, \mathscr{M}^{(i)}\right).$$

for $\mathscr{M}^{(i+1)} = \Psi_{i*}\mathscr{M}^{(i)}$, and the pair $\left(X', \mathscr{M}' = \mathscr{M}^{(n)}\right)$ is terminal.

Proof. If the pair $(X, \mathscr{M})$ is terminal, we are done. Thus we may assume that $c := ct(X, \mathscr{M}) \leq 1$. By Proposition 3.4.1 the linear system $\mathscr{M}$ is not composed of a pencil. Apply the construction (3.1.4) to $(X, \mathscr{M})$. By Lemma 3.1.6 the contraction $\bar{f}$ is birational and so we obtain a new pair $(\hat{X}, \hat{\mathscr{M}})$, where $\hat{X}$ is a $\mathbb{Q}$-Fano threefold and $\hat{\mathscr{M}} := \Psi_*\mathscr{M}$ is a linear system without fixed components such that $\lambda(\hat{X}, \hat{\mathscr{M}}) > \lambda(X, \mathscr{M})$ (see (3.2.2)). If the pair $(\hat{X}, \hat{\mathscr{M}})$ is terminal, we are done. Otherwise we can repeat the process applying the construction (3.1.4) to $(\hat{X}, \hat{\mathscr{M}})$ and continue. We get a sequence of pairs

$$(X, \mathscr{M}) = (X^{(1)}, \mathscr{M}^{(1)}) \dashrightarrow (X^{(2)}, \mathscr{M}^{(2)}) \dashrightarrow \cdots \dashrightarrow (X^{(n)}, \mathscr{M}^{(n)}) \dashrightarrow \cdots \tag{4.1.3}$$

Since the set of all $\mathbb{Q}$-Fano threefolds is bounded [29], the process terminates with a pair $(X', \mathscr{M}') = (X^{(n)}, \mathscr{M}^{(n)})$ having terminal singularities. $\qquad\square$

By Corollary 2.4.2 X' is a del Pezzo threefold of degree $d(X') \leq 2$ and $\mathscr{M}' = \Psi_*\mathscr{M} \subset |-\frac{1}{2}K_{X'}|$. In particular, $\dim(\mathscr{M}) = \dim(\mathscr{M}') \leq 3$. This proves 1.2(i).

If $X' = X'_6 \subset \mathbb{P}(1^3, 2, 3)$ is a double Veronese cone, then $\dim(\mathscr{M}) = \dim(\mathscr{M}') = 2$. If, furthermore, X' is smooth, then according to [7], any birational model of X' which is a Mori fiber space non-isomorphic to X' is a degree 1 del Pezzo fibration over $\mathbb{P}^1$. In particular, this implies that $X \simeq X'$. If X' is a quartic double solid, then it must be smooth (see Corollary 2.4.2). This proves 1.2(iii).

It remains to prove the assertion of 1.2(ii), i.e., Ψ is an isomorphism in the case $\dim(\mathscr{M}) = 3$. For this purpose we can consider the last step of (4.1.2). We show that $X^{(n)} \simeq X^{(n-1)}$.

4.2 Assumptions. Thus we assume that $\dim(\mathscr{M}) = 3$. We put $X = X^{(n-1)}$ and $\hat{X} = X^{(n)}$. Recall that $\hat{X}$ is a smooth quartic double solid in our case. We are going to use all the notation of Sect. 3. Since $(X, \mathscr{M})$ is not terminal, $c = ct(X, \mathscr{M}) \leq 1$. Recall also the commutative diagram (3.1.4):

$$
\begin{array}{ccc}
& \tilde{X} \xdashrightarrow{\ \chi\ } \bar{X} & \\
{\scriptstyle f}\swarrow & & \searrow{\scriptstyle \bar{f}} \\
X \dashrightarrow\!\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\dashrightarrow & & \hat{X} = \hat{X}_6 \subset \mathbb{P}(1^4, 2)
\end{array} \tag{4.2.1}
$$

Denote by $\bar{F}$ the $\bar{f}$-exceptional divisor and by $\tilde{F} \subset \tilde{X}$ its proper transform. By Lemma 3.2.1 $\bar{F} \neq \bar{E}$. Hence $F := f(\tilde{F})$ is a prime divisor on X. Write

$$\mathscr{M} \sim_{\mathbb{Q}} bA, \qquad F \sim_{\mathbb{Q}} dA, \qquad b, d \in \mathbb{Z}_{>0}.$$

Thus $\lambda := \lambda(X, \mathscr{M}) = q/b > 1$. Clearly,

$$K_{\bar{X}} \sim \bar{f}^* K_{\hat{X}} + a\bar{F},$$

where a is the discrepancy of $\bar{F}$. Since $\hat{X}$ is smooth, a is a positive integer.

4.2.2 Remark. • If $\bar{f}(\bar{F})$ is a point, then by [26] $\bar{f}$ is a weighted blowup of $\bar{f}(\bar{F}) \in \hat{X}$ with weights $(1, w_1, w_2)$, where $\gcd(w_1, w_2) = 1$. In particular, $a = w_1 + w_2 \geq 2$. Moreover, $a = 2$ if and only if $\bar{f}$ is a usual blowup.
 • If $\bar{f}(\bar{F})$ is a curve, then the contraction $\bar{f}$ is the usual blowup at its general point. In particular, $a = 1$.

4.2.3 Claim. $2b = q + ad$. *In particular, $q < 2b$.*

Proof. Since $\mathrm{Bs}(\hat{\mathscr{M}}) = \varnothing$, we have

$$K_{\bar{X}} + 2\bar{\mathscr{M}} = \bar{f}^*(K_{\hat{X}} + 2\hat{\mathscr{M}}) + a\bar{F} \sim a\bar{F}.$$

Taking the pushforward of this relation to X, we get the desired equality. $\qquad\square$

4.2.4 Claim. $\gcd(b, d) = 1$ *and $\mathrm{Cl}(X)_{\mathrm{T}}$ is a cyclic group of order e/d.*

Proof. The group $\mathrm{Cl}(\bar{X})$ is generated by the classes of $\bar{\mathscr{M}}$ and $\bar{E}$:

$$\mathrm{Cl}(\bar{X}) = \mathbb{Z} \cdot [\bar{\mathscr{M}}] \oplus \mathbb{Z} \cdot [\bar{F}].$$

Hence, $\mathrm{Cl}(\tilde{X}) = \mathbb{Z} \cdot [\tilde{\mathscr{M}}] \oplus \mathbb{Z} \cdot [\tilde{F}]$ and we can write

$$E \sim u\tilde{\mathscr{M}} + v\tilde{F}. \tag{4.2.5}$$

Since E is contracted by f, we have $\mathrm{Cl}(X) = \mathrm{Cl}(\tilde{X})/\mathbb{Z} \cdot [E]$ and so $\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, where $n = \gcd(u, v)$. Taking pushforward of (4.2.5) to X and $\hat{X}$ we get $ub + vd = 0$ and $e = u$, respectively. Since the classes of $\mathscr{M}$ and F generate $\mathrm{Cl}(X)$, we have $\gcd(b, d) = 1$. Hence, $v = -nb$ and $e = nd$. $\qquad\square$

4.2.6 Claim. *Assume that $\mathscr{M} \ni tN$, where N is an (effective) Weil divisor and $t \geq 2$. Then $e = d = 1$, $\mathrm{Cl}(X)$ is torsion free, $\mathrm{Supp}(N) = F$, $\bar{f}(\bar{F})$ is a point, and $a \geq 2$.*

Proof. We can write $\tilde{\mathscr{M}} \sim t\tilde{N} + \delta E$, where δ is an non-negative integer. Hence, $\hat{\mathscr{M}} \sim t\bar{f}_*\bar{N} + \delta\hat{E}$. Since the class of $\hat{\mathscr{M}}$ is not divisible in $\mathrm{Cl}(\hat{X})$, we have $\mathrm{Supp}(\bar{N}) = \bar{F}$ and so $\delta = 1$ and $\hat{E} \sim \mathscr{M}$, i.e., $e = 1$. By Claim 4.2.4 $d = 1$ and $\mathrm{Cl}(X)$ is torsion free. Further,

$$\bar{E} \sim \bar{\mathscr{M}} - t\bar{F} \sim \bar{f}^*\hat{E} - t\bar{F},$$

where $t \geq 2$. If $\bar{f}(\bar{F})$ is a curve, then $\hat{E}$ must be singular along $\bar{f}(\bar{F})$. This contradicts [27]. Hence $\bar{f}(\bar{F})$ is a point and $a \geq 2$ by Remark 4.2.2. $\qquad\square$

4.2.7 Claim. *Assume that $\mathscr{M} \ni N_1 + N_2$, where N_1 and N_2 are effective non-zero Weil divisors without common components. If $F \neq \mathrm{Supp}(N_1)$, then $F = \mathrm{Supp}(N_2)$ and N_1 is a prime divisor.*

Proof. As above, write $\tilde{\mathscr{M}} \sim \tilde{N}_1 + \tilde{N}_2 + \delta E$, where $\delta \geq 0$. Hence, $\hat{\mathscr{M}} \sim \bar{f}_* \bar{N}_1 + \bar{f}_* \bar{N}_2 + \delta \hat{E}$. By our assumption $f_* \bar{N}_1 \neq 0$. Since the class of $\hat{\mathscr{M}}$ is not divisible in $\mathrm{Cl}(\hat{X})$, we have $\delta = 0$, $\bar{f}_* \bar{N}_2 = 0$, and $\bar{N}_1$ is a prime divisor. $\qquad\square$

4.2.8 Claim. *If $a > 1$, then $\mathscr{M} \sim N + \delta F$, where δ is a positive integer and N is an effective Weil divisor such that $\dim |N| \geq 2$.*

Proof. By Remark 4.2.2 $\bar{f}(\bar{F})$ is a point, say $\hat{P}$. Let $\hat{\mathscr{M}}_P \subset \hat{\mathscr{M}}$ be the linear subsystem consisting of all divisors from $\hat{\mathscr{M}}$ passing through $\hat{P}$. Then $\dim(\hat{\mathscr{M}}_P) = 2$ and

$$\bar{\mathscr{M}}_P \sim \bar{f}^* \hat{\mathscr{M}}_P - \delta \bar{F} \sim f^* \hat{\mathscr{M}} - \delta \bar{F},$$

where $\delta \geq 1$. Therefore, $\mathscr{M} \sim \mathscr{M}_P + \delta F$, where $\dim(\mathscr{M}_P) = 2$. $\qquad\square$

Now we are in position to prove Theorem 1.2(ii). We show that in our assumptions 4.2 the diagram 4.2.1 does not exist. Consider possibilities for $\hat{X}$ case by case.

4.3. Case $\mathrm{Cl}(X) \simeq \mathbb{Z}$

Then $e = d$ by Claim 4.2.4. Apply Proposition 3.4.2. We obtain the following subcases.

Subcase $q_{\mathbb{Q}}(X) = 7$

Then by 3.4.2(ii) there are prime divisors M_2 and M_3 such that $\mathscr{M} \sim 3M_2 \sim 2M_3$. But then by Claim 4.2.6 we have $M_2 = M_3 = F$, a contradiction.

Subcase $q_{\mathbb{Q}}(X) = 5$

Then $\dim |2A| = 0$ and $\dim |3A| = 1$ by 3.4.2(iii). Hence, $b = 4$ and $\mathscr{M} \ni 2D$, where $D \in |2A|$. By Claim 4.2.6 we have $\mathrm{Supp}(D) = F$, $d = 1$ and $a \geq 2$. Then by Claim 4.2.8 we have $\dim |kA| \geq 2$, where $k \leq 3$, a contradiction.

Subcase $q_{\mathbb{Q}}(X) = 4$.

It follows from Claim 4.2.3 that $b = 3$ and $ad = 2$. Thus $\mathscr{M} \sim 3A$ and $s_3 = 1$. According to Proposition 3.4.2(iv) we have $\dim |A| \leq 0$ and $\dim |2A| \geq 1$. The relation (3.3.2) for $k = 3$ has the form

$$2 = (4\beta_3 - 3\alpha)e,$$

where $\beta_3 \geq 2\alpha$ by Lemma 3.1.1. Hence $\alpha \leq 2/(5e) < 1$ and so $P := f(E)$ is a non-Gorenstein point of X. Let r be its index. Similarly, the relation (3.3.2) for $k = 2$ has the form

$$(\alpha - 2\beta_2)e = 2(s_2 - 1).$$

Since $\dim |2A| > 0$, a general member of $\mathscr{M}_2 = |2A|$ is not contracted on $\hat{X}$, hence $s_2 \geq 1$. Therefore, $\beta_2 \leq \alpha/2$. Since $r\mathscr{M}_2$ is Cartier at P, $r\beta_2$ is an integer. So, $\alpha \geq 2/r$. Hence f cannot be a

Kawamata blowup of $P \in X$ and $P \in X$ is not a cyclic quotient singularity (see Theorem 2.2.7). By Proposition 3.4.2(i) we have $r \geq 7$. Then

$$\frac{1}{5e} \geq \frac{\alpha}{2} \geq \beta_2 \geq \frac{1}{7}, \quad e = 1.$$

By Claim 4.2.4 $d = 1$, $a = 2$, and $\dim |2A| \geq 2$ by Claim 4.2.8. Then by 3.4.2(iv) we have only one possibility: $A^3 = 2/11$ and $\mathbf{B}(X) = (11)$. But this means that $P \in X$ is a cyclic quotient singularity, a contradiction.

Subcase $q_{\mathbb{Q}}(X) = 3$

Then $b = 2$ and $d = a = 1$ by Claim 4.2.3. Hence, $\mathscr{M} \sim 2F$ because $\mathrm{Cl}(X) \simeq \mathbb{Z}$. Then we get a contradiction by Claim 4.2.6.

4.4. Case: $\mathrm{Cl}(X)_{\mathrm{T}} \neq 0$

Let n be the order of $\mathrm{Cl}(X)_{\mathrm{T}}$. Apply Proposition 3.4.3 and consider possibilities for $q_{\mathbb{Q}}(X)$.

Subcase $q_{\mathbb{Q}}(X) = 5$.

Then $n = 2$. By 3.4.3(ii) we have

$$\dim |kA| \leq 1 \quad \text{and} \quad \dim |kA + T| \leq 1 \quad \text{for all } k \leq 3.$$

In particular, $b = 4$. If $a > 1$, then by Claim 4.2.8 we get a contradiction. Thus $a = 1$ and $d = 3$ (see Claim 4.2.3). If $\mathscr{M} = |4A|$, then we get a contradiction by Claim 4.2.6 because $|A| \neq \varnothing$ (see 3.4.3(ii)). Let $\mathscr{M} = |4A + T|$. Then $\mathscr{M} \sim 2N_1 + N_2$, where $N_1 \in |A|$ and $N_2 \in |2A + T|$ are prime divisors. By Claim 4.2.7 we have $F = N_1$ and so $d = 1$, a contradiction.

Subcase $q_{\mathbb{Q}}(X) = 4$

Then $n = 2$ or 5. It follows from Claim 4.2.3 that $b = 3$ and $ad = 2$. If $a > 1$, we get a contradiction by Claim 4.2.8 and 3.4.3(iii). Thus, $a = 1$ and $d = 2$. By 3.4.3(iii) $|A + T| \neq \varnothing$ and $\dim |3A + 3T| = 3$. Take $\mathscr{M} = |3A + 3T|$ and apply the construction (3.1.4). Then by Claim 4.2.6 $\hat{X}$ is not a smooth quartic double solid because $\mathscr{M} \ni 3N$, $N \in |A + T|$. By (3.2.2) we have $\lambda(\hat{X}, \hat{\mathscr{M}}) > \lambda(X, \mathscr{M}) = 4/3$. Therefore, $q_{\mathbb{Q}}(\hat{X}) > 4$. Hence we can proceed with the sequence of Sarkisov links (4.1.2) so that $\lambda(X^{(i)}, \mathscr{M}^{(i)}) > 4/3$ and $q_{\mathbb{Q}}(X^{(i)}) > 4$. By the above considered cases we get a contradiction.

Subcase $q_{\mathbb{Q}}(X) = 3$.

It follows from Claim 4.2.3 that $b = 2$, $a = d = 1$. Hence, $\mathscr{M} \sim_{\mathbb{Q}} 2F$. If $\mathscr{M} \sim 2F$, we get a contradiction by Claim 4.2.6. Therefore, $T := \mathscr{M} - 2F$ is a non-trivial torsion element. We have $K_{\bar{X}} + 2\bar{\mathscr{M}} - \bar{F} \sim 0$. Hence $-K_X \sim 2\mathscr{M} - F \sim 3F + 2T$. Moreover, the group $\mathrm{Cl}(X)$ is generated by the classes of F and T. By Claim 4.2.4 $\hat{E} \nsim \hat{\mathscr{M}}$ because $d = 1$.

Assume that $3T \sim 0$. Then $\mathscr{M} \sim 2(F + 2T)$ and $|F + 2T| = \varnothing$ by Claim 4.2.6. This contradicts 3.4.3(iv). Therefore, $2T \sim 0$. Hence, $-K_X \sim 3F$. If $\dim |2F| = 3$, then we take $\mathscr{M} = |2F|$ and apply the construction (3.1.4). Then by Claim 4.2.6 $\hat{X}$ is not a smooth quartic double solid. By (3.2.2)

we have $\lambda(\hat{X}, \hat{\mathscr{M}}) > \lambda(X, \mathscr{M}) = 3/2$. Therefore, $q_{\mathbb{Q}}(\hat{X}) > 3$. Hence we can proceed with the sequence of Sarkisov links (4.1.2) so that $\lambda(X^{(i)}, \mathscr{M}^{(i)}) > 3/2$ and $q_{\mathbb{Q}}(X^{(i)}) > 3$. By the above considered cases we get a contradiction.

Therefore, $\dim |2F| < 3$. Thus by 3.4.3(iv) we have $\mathbf{B} = (2, 4, 14)$. Then $|F + T| \neq \varnothing$ and for $D \in |F + T|$ we have $-K_X \sim \mathscr{M} + D$,

$$K_{\tilde{X}} + \tilde{\mathscr{M}} + \tilde{D} + \gamma E = f^*(K_X + \mathscr{M} + D) \sim 0.$$

where $\gamma \geq \beta - \alpha = \alpha(c - 1) \geq 0$. Since $D \neq F$, taking pushforward to $\hat{X}$ we obtain $-K_{\hat{X}} \sim \hat{\mathscr{M}} + \hat{D} + \gamma \hat{E}$, where $\hat{D} \neq 0$. This is possible only if $\gamma = 0$. Then the pair $(X, \mathscr{M})$ is canonical. In this case by Lemma 2.2.2(i) near each singular point either D or $\mathscr{M}$ is Cartier. Since $-4K_X \sim 12D \sim 6\mathscr{M}$, the divisor $4K_X$ is Cartier. This contradicts $\mathbf{B} = (2, 4, 14)$.

5. Applications

In this section we consider applications of Theorem 1.2 to the existence of conic bundle structures on rationally connected threefolds.

5.1 Proposition. *Let X be a normal projective threefold and let $\mathscr{M}$ be a non-empty linear system of Weil divisors on X such that*

  (i) *$\mathscr{M}$ is not composed of a pencil,*
 (ii) *$-(K_X + \mathscr{M}) \sim_{\mathbb{Q}} \Theta$, where Θ is a $\mathbb{Q}$-divisor which is $\mathbb{Q}$-Cartier, and*
(iii) *either $\Theta > 0$ or $\Theta = 0$ and the pair $(X, \mathscr{M})$ is not canonical.*

Assume that the variety X has no conic bundle structures. Then there exists a birational transformation $\Psi : X \dashrightarrow X'$, where X' is del Pezzo threefold as in 1.2(iii). *Moreover,*

$$\dim(\mathscr{M}) \leq \mathrm{d}(X') + 1.$$

Proof. We may assume that $\mathscr{M}$ has no fixed components. Let

$$f : (\tilde{X}, \tilde{\mathscr{M}}) \longrightarrow (X, \mathscr{M})$$

be a terminal $\mathbb{Q}$-factorial model of the pair $(X, \mathscr{M})$ (see Corollary 2.2.4). Thus $\tilde{\mathscr{M}} := f_*^{-1}\mathscr{M}$, the pair $(\tilde{X}, \tilde{\mathscr{M}})$ is terminal, $K_{\tilde{X}} + \tilde{\mathscr{M}}$ is f-nef, and the variety $\tilde{X}$ is $\mathbb{Q}$-factorial. We can write

$$K_{\tilde{X}} + \tilde{\mathscr{M}} \sim_{\mathbb{Q}} f^*(K_X + \mathscr{M}) - E',$$

where E' is the exceptional $\mathbb{Q}$-divisor and $E' \geq 0$. Furthermore,

$$f^*\Theta = \tilde{\Theta} + E'',$$

where $\tilde{\Theta}$ is the proper transform of Θ on $\tilde{X}$ and $E'' \geq 0$. Hence,

$$K_{\tilde{X}} + \tilde{\mathscr{M}} + \tilde{\Xi} \sim_{\mathbb{Q}} f^*(K_X + \mathscr{M} + \Theta) \sim_{\mathbb{Q}} 0,$$

where $\tilde{\Xi} := \tilde{\Theta} + E' + E'' \geq 0$. Moreover, $\tilde{\Xi} > 0$ by our assumptions 5.1(iii). In particular, $K_{\tilde{X}} + \tilde{\mathscr{M}}$ is not nef. Run $(K_{\tilde{X}} + \tilde{\mathscr{M}})$-MMP. The divisor $\tilde{\Xi}$ cannot be contracted, so at the end we get a Mori fiber space $(\bar{X}, \bar{\mathscr{M}})/Z$:

$$(\tilde{X}, \tilde{\mathscr{M}}) - - - - - \rightarrow (\bar{X}, \bar{\mathscr{M}})$$
$$\downarrow \varphi$$
$$Z$$

where $\bar{\Xi} \sim_{\mathbb{Q}} -(K_{\bar{X}} + \bar{\mathscr{M}})$ if φ-ample. If $\dim(Z) = 2$, then φ is a $\mathbb{Q}$-conic bundle and then we are done. Assume that $\dim(Z) = 1$. Then φ is a del Pezzo fibration and $Z \simeq \mathbb{P}^1$. Since $\dim(\mathscr{M}) \geq 2$ and a general member of $\mathscr{M}$ is irreducible, $\mathscr{M}$ is not φ-horizontal, i.e., $\mathscr{M}$ is not a pull-back of a linear system on Z. For a general fiber F by the adjunction formula we have

$$K_F = K_{\bar{X}}|_F = \bar{\mathscr{M}}|_F + \bar{\Xi}|_F,$$

where both $\bar{\mathscr{M}}|_F$ and $\bar{\Xi}|_F$ are ample. Since $\bar{\mathscr{M}}|_F$ is an (integral) Cartier divisor, the surface F is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In this case, $\bar{X}$ is rational and so it has a lot of conic bundle structures. Finally, assume that Z is a point. Then $\bar{X}$ is a $\mathbb{Q}$-Fano threefold such that $\mathrm{df}(\bar{X}) \geq 2$ and

$$K_{\bar{X}} + \bar{\mathscr{M}} + \bar{\Xi} \sim_{\mathbb{Q}} 0,$$

where both $\bar{\mathscr{M}}$ and $\bar{\Xi}$ are ample, and $\dim(\bar{\mathscr{M}}) \geq 2$. Thus we can apply Theorem 1.2 to $(\bar{X}, \bar{\mathscr{M}})$. $\square$

Now we consider applications of Theorem 1.2 to $\mathbb{Q}$-Fano threefolds. For simplicity we consider only $\mathbb{Q}$-Fanos whose Weil divisor class group $\mathrm{Cl}(X)$ has no torsions, i.e., $\mathrm{Cl}(X) \simeq \mathbb{Z}$. The collection of invariants $\left(\mathbf{B}(X), q_w(X), A^3\right)$ determines the Hilbert series

$$P_{(X,A)}(t) := \sum_{n=0}^{\infty} \dim H^0(X, \mathscr{O}_X(nA)) \cdot t^n.$$

An abstract collection $(\mathbf{B}, q, \alpha)$, where $\mathbf{B}$ is a basket of terminal singularities, $q \in \mathbb{Z}_{>0}$, and $\alpha \in \mathbb{Q}_{>0}$ is called a *numerical candidate* if there are no numerical obstructions (like orbifold Riemann-Roch theorem [8] and Bogomolov-Miyaoka inequality [14]) for existence of a $\mathbb{Q}$-Fano threefold with corresponding invariants.

Computer search by using algorithm described in [6, 19, 23, 24] (see also [22]) shows that there are at most 472 numerical candidates with $q \geq 3$. Among them Theorem 1.2 is applicable in 313 cases. Similarly, there are at most 1 492 numerical candidates with $q = 2$ and Theorem 1.2 is applicable in 382 cases. We expect that in most of the remaining cases the corresponding $\mathbb{Q}$-Fano threefolds (if they exist) should have conic bundle structure. However proof of this needs a case by case considerations. Let us consider just one example.

5.2 Proposition. *Let X be a $\mathbb{Q}$-Fano threefold with $q_{\mathbb{Q}}(X) = 7$ and $A^3 = 1/78$ (No. 41478 in [22]). Then X is rational. In particular, X has a conic bundle structure.*

Note however that the existence of a $\mathbb{Q}$-Fano threefold X with these invariants is not known.

Proof. In this case the group $\mathrm{Cl}(X)$ is torsion free, $\mathbf{B}(X) = (2, 3, 13)$, $\dim |6A| = 1$, and $\dim |kA| = 0$ for $1 \le k \le 5$. Apply construction (3.1.4) with $\mathcal{M} = |6A|$. Then near the point of index 13 we have $\mathcal{M} \sim 12(-K_X)$. Hence, $\beta_6 \ge 12\alpha$ by Lemma 3.1.1 and so

$$\beta_1 \ge \beta_6/6 \ge 2\alpha.$$

The relation (3.3.2) for $k = 1$ has the form

$$\hat{q} = 7s_1 + (7\beta_1 - \alpha)e \ge 7s_1 + 13\alpha e. \tag{5.2.1}$$

Since $\hat{q} \le 7$, we have $s_1 = 0$ and $\alpha < 1$. Thus f is the Kawamata blowup of a cyclic quotient singularity $P \in X$ of index $r = 2, 3$ or 13. If $r = 2$ or 3, then $A \sim -K_X$ near P. Hence $\beta_1 = \alpha + m_1$, where m_1 is an integer, $m_1 \ge \alpha > 0$. Then (5.2.1) gives us

$$\hat{q} = 6\alpha e + 7m_1 e > 7.$$

This contradicts our assumptions. Thus $r = 13$ and so $A \sim 2(-K_X)$ near P. Hence $\beta_1 = 2\alpha + m_1$, where m_1 is a non-negative integer. Again, from (5.2.1) we obtain

$$\hat{q} = 13\alpha e + 7m_1 e.$$

Thus $m_1 = 0$, $\beta_1 = 2\alpha$, and $\beta_6 = 12\alpha$. By Theorem 2.2.7 $\alpha = 1/13$. Now, the relation (3.3.2) for $k = 6$ has the form:

$$6\hat{q} = 7s_6 + (7\beta_6 - 6\alpha)e = 7s_6 + 6e.$$

Since $\hat{q} < 8$, $s_6 = 0$. Since $\dim |6A| = 1$, this implies that the contraction $\bar{f}$ is not birational, so it is a del Pezzo fibration. Since $s_1 = 0$, we have $\bar{\mathcal{M}} \sim 6\bar{M}_1$ and so $\bar{M}_1$ is a fiber of multiplicity 6. By the main result of [28] the general fiber of $\bar{f}$ is a del Pezzo surface of degree 6. But then $\bar{X}$ must be rational, a contradiction. □

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Conflict of interest

The author declares there is no conflicts of interest.

References

1. V. G. Sarkisov, On conic bundle structures, *Math. USSR, Izv.*, **120** (1982), 355–390. https://doi.org/10.1070/IM1983v020n02ABEH001354

2. V. V. Shokurov, Prym varieties: theory and applications, *Math. USSR Izv.*, **23** (1984), 83–147. https://doi.org/10.1070/IM1984v023n01ABEH001459

3. V. A. Iskovskikh, On the rationality problem for conic bundles, *Duke Math. J.*, **54** (1987), 271–294. https://doi.org/10.1215/S0012-7094-87-05416-0

4. Y. Prokhorov, The rationality problem for conic bundles, *Russian Math. Surv.*, **73** (2018), 375–456. http://dx.doi.org/10.1070/RM9811

5. T. Fujita, *Classification theories of polarized varieties*, vol. 155 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1990.

6. Y. Prokhorov, Rationality of **Q**-Fano threefolds of large Fano index, *ArXiv e-print*, **1903.07105**, URL `https://arxiv.org/abs/1903.07105`, To appear in Recent Developments in Algebraic Geometry: To Miles Reid for his 70th Birthday, edited by Hamid Abban; Gavin Brown; Alexander Kasprzyk; Shigefumi Mori, London Mathematical Society Lecture Note Series Cambridge University Press.

7. M. M. Grinenko, Mori structures on a Fano threefold of index 2 and degree 1, *Proc. Steklov Inst. Math.*, **246** (2004), 103–128.

8. M. Reid, Young person's guide to canonical singularities, in *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, vol. 46 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1987, 345–414. https://doi.org/10.1090/pspum/046.1/927963

9. J. Kollár, Singularities of pairs, in *Algebraic geometry–Santa Cruz 1995*, vol. 62 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1997, 221–287. https://doi.org/10.1090/pspum/062.1/1492525

10. V. Alexeev, General elephants of **Q**-Fano 3-folds, *Compos. Math.*, **91** (1994), 91–116. http://www.numdam.org/item?id=CM_1994__91_1_91_0

11. Y. Prokhorov, Equivariant minimal model program, *Russian Math. Surv.*, **76** (2021), 461–542. https://doi.org/10.1070/rm9990

12. C. Birkar, P. Cascini, C. D. Hacon, J. McKernan, Existence of minimal models for varieties of log general type, *J. Amer. Math. Soc.*, **23** (2010), 405–468. http://dx.doi.org/10.1090/S0894-0347-09-00649-3

13. A. Corti, Factoring birational maps of threefolds after Sarkisov, *J. Algebra. Geom.*, **4** (1995), 223–254.

14. Y. Kawamata, Divisorial contractions to 3-dimensional terminal quotient singularities, in *Higher-dimensional complex varieties (Trento, 1994)*, de Gruyter, Berlin, 1996, 241–246.

15. K.-H. Shin, 3-dimensional Fano varieties with canonical singularities, *Tokyo J. Math.*, **12** (1989), 375–385. https://doi.org/10.3836/tjm/1270133187

16. V. A. Iskovskih, Fano threefolds. I, *Izv. Akad. Nauk SSSR Ser. Mat.*, **41** (1977), 516–562.

17. F. Campana, H. Flenner, Projective threefolds containing a smooth rational surface with ample normal bundle, *J. Reine Angew. Math.*, **440** (1993), 77–98. https://doi.org/10.1515/crll.1993.440.77

18. Y. Prokhorov, The degree of **Q**-Fano threefolds, *Sbornik: Math.*, **198** (2007), 153–174. https://doi.org/10.1070/SM2007v198n11ABEH003901

19. Y. Prokhorov, **Q**-Fano threefolds of large Fano index, I, *Doc. Math.*, **15** (2010), 843–872. https://doi.org/10.3390/mca15050872

20. Y. Prokhorov, Fano threefolds of large Fano index and large degree, *Sbornik: Math.*, **204** (2013), 347–382. https://doi.org/10.1070/SM2013v204n03ABEH004304

21. Y. Prokhorov, M. Reid, On **Q**-Fano threefolds of Fano index 2, in *Minimal Models and Extremal Rays (Kyoto 2011)*, vol. 70 of Adv. Stud. Pure Math., Mathematical Society of Japan, Kinokuniya, Tokyo, 2016, 397–420.

22. G. Brown, A. M. Kasprzyk, Graded Ring Database, 2015. Available from: URL `http://grdb.lboro.ac.uk`.

23. K. Suzuki, On Fano indices of **Q**-Fano 3-folds, *Manuscripta Math.*, **114** (2004), 229–246. https://doi.org/10.1007/s00229-004-0442-4

24. G. Brown, K. Suzuki, Computing certain Fano 3-folds, *Japan J. Indust. Appl. Math.*, **24** (2007), 241–250. https://doi.org/10.1007/BF03167538

25. The PARI Group, Bordeaux, *PARI/GP, version* `2.7.5`, 2008. Available from `http://pari.math.u-bordeaux.fr/`.

26. M. Kawakita, Divisorial contractions in dimension three which contract divisors to smooth points, *Invent. Math.*, **145** (2001), 105–119. https://doi.org/10.1007/s002220100144

27. M. Furushima, M. Tada, Nonnormal Del Pezzo surfaces and Fano threefolds of the first kind, *J. Reine Angew. Math.*, **429** (1992), 183–190. https://doi.org/10.1515/crll.1992.429.183

28. S. Mori, Y. Prokhorov, Multiple fibers of del Pezzo fibrations, *Proc. Steklov Inst. Math.*, **264** (2009), 131–145. https://doi.org/10.1134/S0081543809010167

29. Y. Kawamata, Boundedness of **Q**-Fano threefolds, in *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)*, vol. 131 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1992, 439–445. https://doi.org/10.1090/conm/131.3/1175897

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