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Research article

Infinitely many periodic solutions for ordinary $p(t)$-Laplacian differential systems

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Abstract: In this paper, we consider the existence of infinitely many periodic solutions for some ordinary $p(t)$-Laplacian differential systems by minimax methods in critical point theory.

Keywords: periodic solutions; $p(t)$-Laplacian system; critical points; variational method

1. Introduction and main results

Consider periodic solution of the ordinary $p(t)$-Laplacian differential system

$$\begin{cases} \frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) + \nabla F(t, u(t)) = 0, & u \in \mathbb{R}^N, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $p(t) > 1$ is a T-periodic continuous function, $F(t, x)$ is T-periodic in t for any $x \in \mathbb{R}^N$. Here we always assume that $F(t, x)$ satisfies the following condition.

(A) $F(t, x)$ is measurable in t for any $x \in \mathbb{R}^N$ and continuously differentiable in x for $a.e.$ $t \in [0, T]$, and there exist two functions $a \in C(\mathbb{R}^+; \mathbb{R}^+), b \in L^1([0, T]; \mathbb{R}^+)$ such that

$$|F(t, x)| \le a(|x|)b(t), \ |\nabla F(t, x)| \le a(|x|)b(t), \quad \forall\, x \in \mathbb{R}^N \textit{ and a.e. } t \in [0, T],$$

where $\nabla F(t, x)$ denotes the gradient of $F(t, x)$ in x.

In the sequel of this paper, we set $p^- := \min\limits_{0\le t\le T} p(t)$, $p^+ := \max\limits_{0\le t\le T} p(t)$ and $q^- > 1$ such that $\frac{1}{p^-} + \frac{1}{q^-} = 1$.

We define the generalized Lebesgue space as

$$L_T^{p(t)} = L^{p(t)}(S_T; \mathbb{R}^N) := \left\{u \in L^1(S_T; \mathbb{R}^N) : \int_0^T |u|^{p(t)} dt < \infty\right\}, \quad S_T = \mathbb{R}/T\mathbb{Z}$$

with the norm

$$\|u\|_{L^{p(t)}} = \inf\left\{\lambda > 0 : \int_0^T \left|\frac{u}{\lambda}\right|^{p(t)} dt \le 1\right\}.$$

$L_T^{p(t)}$ is a kind of generalized Orlicz space. We also define the generalized Sobolev space by

$$W_T^{1,p(t)} = W^{1,p(t)}(S_T;\mathbb{R}^N) := \left\{u \in L^{p(t)}(S_T;\mathbb{R}^N) : \dot{u} \in L^{p(t)}(S_T;\mathbb{R}^N)\right\}$$

with the norm

$$\|u\|_{W_T^{1,p(t)}} = \|u\| = \|u\|_{L^{p(t)}} + \|\dot{u}\|_{L^{p(t)}}.$$

$W_T^{1,p(t)}$ is a kind of generalized Orlicz-Sobolev space. In our case with the condition $p(t) > 1$, the two spaces $L_T^{p(t)}$ and $W_T^{1,p(t)}$ are both reflective Banach spaces with the norms defined above. It is known that there is a compact embedding $W_T^{1,p(t)} \hookrightarrow C([0,T];\mathbb{R}^N)$. One can refer [1] for details.

The corresponding functional φ of (1.1) on $W_T^{1,p(t)}$ is given by

$$\varphi(u) = \int_0^T \frac{|\dot{u}(t)|^{p(t)}}{p(t)}dt - \int_0^T F(t,u(t))dt,\ u \in W_T^{1,p(t)}. \tag{1.2}$$

Wang and Yuan in [2] verified that the functional φ denoted in (1.2) is continuously differentiable, *i.e.*, there holds

$$\langle \varphi'(u), v\rangle = \int_0^T [(|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{v}(t)) - (\nabla F(t,u(t)), v(t))]dt$$

for any $u, v \in W_T^{1,p(t)}$. So the critical points of φ correspond to the solutions of problem (1.1). Then Zhang et al. in [3] proved that the functional φ defined in (1.2) is weakly lower semicontinuous on $W_T^{1,p(t)}$.

When $p(t) = 2$, problem (1.1) reduces to the following Hamiltonian system

$$\begin{cases} \ddot{u}(t) + \nabla F(t,u(t)) = 0 \quad a.e.\ t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.3}$$

which has been extensively investigated in many literatures (see, *e.g.*, [4–8] and the references therein). Particularly, under the following conditions:

(M_1) there exists $g \in L^1([0,T];\mathbb{R}^+)$ such that $|\nabla F(t,x)| \le g(t)$ for $\forall x \in \mathbb{R}^N$ and *a.e.* $t \in [0,T]$,

(M_2) $\int_0^T F(t,x)dt \to \pm\infty$ as $|x| \to +\infty$,

Mawhin and Willem in [6] established the existence of solutions for problem (1.3).

When $N = 1$, $T = 2\pi$, Habets, Manasevich and Zanolin in [4] established the existence of two sequences of different periodic solutions for problem (1.3) when the potential $F(t,x)$ satisfies the following oscillating conditions:

$$\limsup_{a\to\pm\infty} \int_0^{2\pi} F(t,a)dt = +\infty, \quad \liminf_{b\to\pm\infty} \int_0^{2\pi} F(t,b)dt = -\infty.$$

When the nonlinearity $\nabla F(t,x)$ is sublinear, *i.e.*, there exist $f, g \in L^1([0,T];\mathbb{R}^+)$ and $\alpha \in [0,1)$ such that

$$|\nabla F(t,x)| \le f(t)|x|^\alpha + g(t), \qquad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0,T],$$

and $F(t,x)$ satisfies oscillating conditions, *i.e.*, there holds

$$\limsup_{r\to+\infty} \inf_{x\in\mathbb{R}^N,|x|=r} |x|^{-2\alpha} \int_0^T F(t,x)dt = +\infty,$$

$$\liminf_{R\to+\infty}\sup_{x\in\mathbb{R}^N,|x|=R}|x|^{-2\alpha}\int_0^T F(t,x)dt=-\infty,$$

Zhang and Tang in [8] obtained the existence of two sequences of different periodic solutions for problem (1.3).

When $p(t) = p$ is a constant for general $p > 1$, problem (1.1) reduces to the following ordinary p-Laplacian problem

$$\begin{cases} \frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) + \nabla F(t,u(t)) = 0, \quad a.e.\, t\in[0,T], \\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0. \end{cases} \tag{1.4}$$

In recent years, the existence results for problem (1.4) was considered via the variational methods (see, *e.g.*, [9–11] and references therein). Specially, L*ü*, O'Regan and Agarwal in [10] generalized the results of [4] to the p-Laplacian problem (1.4) and established the existence of two sequences of different periodic solutions for problem (1.4). Li, Agarwal and Tang in [9] also established the existence of two sequences of different periodic solutions for problem (1.4) for the case with the potential $F(t,x)$ possessing the mixed nonlinearity, that is, $F(t,x) = F_1(t,x) + F_2(t,x)$ with $F_1(t,x)$ being (λ,μ)-subconvex in x, *i.e.*, function $F_1(t,\cdot):\mathbb{R}^N\to\mathbb{R}$ satisfies

$$F_1(t,\lambda(x+y)) \le \mu(F_1(t,x)+F_1(t,y)) \tag{1.5}$$

for some $\lambda,\mu > 0$ and each $x,y\in\mathbb{R}^N$ (see [7] for details) and $\nabla F_2(t,x)$ being p-sublinear, namely, there exist $f,g\in L^1([0,T];\mathbb{R}^+)$ and $\alpha\in[0,p-1)$ such that

$$|\nabla F_2(t,x)| \le f(t)|x|^\alpha + g(t), \quad \forall\, x\in\mathbb{R}^N \textit{ and a.e. } t\in[0,T].$$

Then they also considered the other case with the nonlinearity $\nabla F(t,x)$ being p-linear, *i.e.*, there exist $f,g\in L^1([0,T];\mathbb{R}^+)$ such that

$$|\nabla F(t,x)| \le f(t)|x|^{p-1} + g(t), \quad \forall\, x\in\mathbb{R}^N \textit{ and a.e. } t\in[0,T].$$

Many scholars are interested in the nonlinear problems including the $p(t)$-Laplacian problem, this kind of problems can be used to model the dynamical phenomena arising from the field of elastic mechanics. One can refer [12, 13] and the references therein for more application of the $p(t)$-Laplacian problem to various practical issues. The $p(t)$-Laplacian problem has more complicated nonlinearity than that of the p-Laplacian problem, for example, the second order differential operator is not homogeneous, this may causes some troubles such as some classical theories and methods, for instance, the Sobolev spaces theory are not applicable.

In recent years, the study of variational problems and elliptic partial differential equations with nonstandard growth conditions has been an interesting topic (see, *e.g.*, [13, 14] and references therein). In 2001, Fan et al. in [14, 15] established some basic theory of space $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. In 2003, Fan et al. in [1] first studied the ordinary $p(t)$-Laplacian problem (1.1). Subsequently, Wang and Ruan in [2] established the existence and multiplicity of periodic solutions for the ordinary $p(t)$-Laplacian problem (1.1) under the generalized Ambrosetti-Rabinowitz conditions. Recently, many scholars considered the existence of periodic solutions of the ordinary $p(t)$-Laplacian problem (1.1). Some existence theorems of the ordinary $p(t)$-Laplacian problem (1.1) are established by using the least action principle and minimax methods in critical point theory (see, e.g., [1, 3, 16] and the references therein).

In this paper, motivated by the results of [4, 5, 8–10], we first aim to consider the ordinary $p(t)$-Laplacian problem (1.1) with a mixed nonlinearity, that is, $F(t, x) = F_1(t, x) + F_2(t, x)$ with $F_1(t, x)$ being (λ, μ)-subconvex and $\nabla F_2(t, x)$ being p^--sublinear. Then we consider the other case with the nonlinearity $\nabla F(t, x)$ being p^--linear. Following the idea of paper [8,9], we obtain the existence of two sequences of different periodic solutions for problem (1.1). The main results of this paper are stated as follows.

Theorem 1.1 *Assume that $F(t, x) = F_1(t, x) + F_2(t, x)$ where $F_1(t, x)$ and $F_2(t, x)$ satisfy condition (A) and the following conditions.*

(H_1) *$F_1(t, \cdot)$ is (λ, μ)-subconvex defined in (1.5) with $\lambda > \frac{1}{2}$ and $1 \le 2\mu < (2\lambda)^{p^-}$ for a.e. $t \in [0, T]$, and there exist $\alpha \in [0, p^- - 1)$, $f, g \in L^1([0, T]; \mathbb{R}^+)$ such that*

$$|\nabla F_2(t, x)| \le f(t)|x|^{\alpha} + g(t), \quad for\ \forall\, x \in \mathbb{R}^N \ and\ a.e.\ t \in [0, T].$$

(H_2) *There exists a positive sequence $\{R_n\}$ with $\lim_{n\to+\infty} R_n = +\infty$, such that*

$$\lim_{n\to+\infty} \inf_{x\in\mathbb{R}^N, |x|=R_n} \int_0^T F(t, x)dt = +\infty$$

and there exists a positive sequence $\{r_m\}$ with $\lim_{m\to+\infty} r_m = +\infty$, such that

$$\lim_{m\to+\infty} \sup_{x\in\mathbb{R}^N, |x|=r_m} \frac{1}{r_m^{q^- \alpha}} \Big(\mu \int_0^T F_1(t, \frac{x}{\lambda})dt + \int_0^T F_2(t, x)dt\Big) = -\infty.$$

Then problem (1.1) possesses two sequences of solutions $\{u_n^{\pm}\}$ with $\varphi(u_n^{\pm}) \to \pm\infty$ as $n \to +\infty$.

Remark 1.2 Theorem 1.1 above generalizes Theorem 1.1 of [10], Theorem 1.1 of [8], Theorem 1 of [5] and Theorem 1.1 of [9]. There are functions $F(t, x)$ satisfying the conditions in Theorem 1.1 but not satisfying the conditions in [5, 8–10]. For example, for $p(t) = 4 + cos\omega t$ with $\omega = \frac{2\pi}{T}$, we take

$$F_1(t, x) = 5 + \sin |x|^6, \quad F_2(t, x) = |x|^{7/4} \sin(\ln(|x|^2 + 1)) + (e(t), x),$$

where $e \in L^1([0, T]; \mathbb{R}^N)$ and $\int_0^T e(t)dt = 0$. Then $F(t, x) = F_1(t, x) + F_2(t, x)$ is such a function.

Theorem 1.3 *Assume that $F(t, x)$ satisfies condition (A) and the following conditions.*

(H_3) *There exist $f, g \in L^1([0, T]; \mathbb{R}^+)$ with $\|f\|_{L^1} < \alpha(p)$ such that*

$$|\nabla F(t, x)| \le f(t)|x|^{p^- -1} + g(t), \quad \forall\, x \in \mathbb{R}^N \ and\ a.e.\ t \in [0, T],$$

where $\alpha(p)$ is a positive constant depending on the function $p(t)$, and so on the constants $p^{\pm}$.

(H_4) *There exists a positive sequence $\{R_n\}$ with $\lim_{n\to+\infty} R_n = +\infty$, such that*

$$\lim_{n\to+\infty} \inf_{x\in\mathbb{R}^N, |x|=R_n} \int_0^T F(t, x)dt = +\infty$$

and there exists a positive sequence $\{r_m\}$ with $\lim_{m\to+\infty} r_m = +\infty$, such that

$$\lim_{m\to+\infty} \sup_{x\in\mathbb{R}^N, |x|=r_m} \frac{1}{r_m^{p^-}} \int_0^T F(t, x)dt < -\beta(p),$$

where $\beta(p)$ is a positive constant depending on the function $p(t)$, and so on the constants $p^{\pm}$. Then problem (1.1) possesses two sequences of solutions $\{u_n^{\pm}\}$ such that $\varphi(u_n^{\pm}) \to \pm\infty$ as $n \to +\infty$.

Remark 1.4 In the process of the proof of Theorem 1.3, the constants $\alpha(p)$ and $\beta(p)$ will be defined clearly (see (3.20) and (3.26)). Theorem 1.3 above generalizes Theorem 1.3 of [9] since here the range of the function $p(t)$ is wider than theirs. There exist functions satisfying the conditions of Theorem 1.3. For example, for $p(t) = \frac{5}{4} + \frac{1}{8}cos\omega t$, one can take

$$F(t,x) = \frac{1}{T}|x|^{\frac{17}{8}} \sin(\frac{7}{16}\ln(|x|^2+1)).$$

Then $F(t,x)$ satisfies the conditions of Theorem 1.3 with $p^- = \frac{9}{8}, p^+ = \frac{11}{8}, q^- = 9, \theta_{p^-} = 1, f(t) = \frac{3}{T}$.

2. Preliminaries

For the convenience of readers, we first introduce two important properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(t)}$ and $W_T^{1,p(t)}$.

We define

$$\widetilde{W}_T^{1,p(t)} = \Big\{u \in W_T^{1,p(t)} : \int_0^T u(s)ds = 0\Big\}.$$

For $u \in W_T^{1,p(t)}$, let

$$\bar{u} = \frac{1}{T}\int_0^T u(s)ds,$$

then $\tilde{u}(t) = u(t) - \bar{u} \in \widetilde{W}_T^{1,p(t)}$. So we get

$$W_T^{1,p(t)} = \mathbb{R}^N \oplus \widetilde{W}_T^{1,p(t)}.$$

Lemma 2.1 [16] *For all $u \in \widetilde{W}_T^{1,p(t)}$, there exist constants C_0', C_0 such that*

$$\begin{aligned} &\|u\|_\infty \le C_0'\|\dot{u}\|_{L^{p(t)}}, \\ &\|u\|_\infty \le 2C_0\Big[\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{1}{p^-}} + T^{\frac{1}{p^-}}\Big], \end{aligned}$$

where C_0', $C_0 > 0$.

Lemma 2.2 [1] *Let $u = \bar{u} + \tilde{u} \in W_T^{1,p(t)}$ with $\bar{u} \in \mathbb{R}^N$ and $\tilde{u} \in \widetilde{W}_T^{1,p(t)}$, then the norm $\|\dot{\tilde{u}}\|_{L^{p(t)}}$ is an equivalent norm on $\widetilde{W}_T^{1,p(t)}$ and $|\bar{u}| + \|\dot{u}\|_{L^{p(t)}}$ is an equivalent norm on $W_T^{1,p(t)}$. Therefore, for $u \in W_T^{1,p(t)}$,*

$$\|u\| \to +\infty \Rightarrow |\bar{u}| + \int_0^T |\dot{u}(t)|^{p(t)}dt \to +\infty.$$

Next we give two important properties of minimax methods as follows.

Lemma 2.3 [6] *Let K be a compact metric space, X a Banach space and $K_0 \subset K$ is a closed subset, $\chi \in C(K_0; X)$. We define a complete metric space M as*

$$M = \{\gamma \in C(K;X) : \gamma(s) = \chi(s),\ \forall\, s \in K_0\}$$

with the usual norm and metric. Let $\varphi \in C^1(X;R)$ *and*

$$c = \inf_{\gamma\in M}\max_{s\in K}\varphi(\gamma(s)),\ \ c_1 = \max_{\chi(K_0)}\varphi.$$

If $c > c_1$*, then for each* $\varepsilon > 0$ *and each* $\gamma \in M$ *such that*

$$\max_{s\in K}\varphi(\gamma(s)) \le c+\varepsilon,$$

there exists $v \in X$ *such that*

$$c-\varepsilon \le \varphi(v) \le \max_{s\in K}\varphi(\gamma(s)),\ \ dist(v,\gamma(K)) \le \varepsilon^{1/2},\ \ |\varphi'(v)| \le \varepsilon^{1/2}.$$

Corollary 2.4 [6] *Under the conditions of Lemma 2.3, for each sequence* (γ_k) *in* M *such that*

$$\max_K \varphi(\gamma_k) \to c,$$

there exists a sequence (v_k) *in* X *such that*

$$\varphi(v_k) \to c,\ \ dist(v_k,\gamma_k(K)) \to 0,\ \ |\varphi'(v_k)| \to 0 \quad as\ k \to +\infty.$$

3. Proofs of theorems

Proof of Theorem 1.1. The proof is divided into four steps.
Step 1. We demonstrate that $\varphi(u) \to +\infty$ as $\|u\| \to \infty$ for $u \in \widetilde{W}_T^{1,p(t)}$.

Let $\beta = \log_{2\lambda}(2\mu)$, then one gets $0 \le \beta < p^-$. For $|x| > 1$, there exists a positive integer n such that

$$n-1 < \log_{2\lambda}|x| \le n.$$

Therefore one has $|x|^\beta > (2\lambda)^{(n-1)\beta} = (2\mu)^{n-1}$ and $|x| \le (2\lambda)^n$. Combining the conditions (A) and (H_1), one obtains

$$F_1(t,x) \le 2\mu F_1(t,\frac{x}{2\lambda}) \le \cdots \le (2\mu)^n F_1(t,\frac{x}{(2\lambda)^n}) \le 2\mu|x|^\beta a_0 b(t)$$

for any $|x| > 1$ and *a.e.* $t \in [0,T]$, where $a_0 = \max_{0\le s\le 1} a(s)$. Therefore one gets

$$F_1(t,x) \le (2\mu|x|^\beta + 1)a_0 b(t) \tag{3.1}$$

for any $|x| > 1$ and *a.e.* $t \in [0,T]$. In the sequel, we denote by C a suitable positive constant which may take different value in various estimations.

Combining Lemma 2.1, (H_1) and (3.1), for the functional φ defined in (1.2), we obtain

$$\begin{aligned}\varphi(u) =& \int_0^T \frac{|\dot u(t)|^{p(t)}}{p(t)}dt - \int_0^T F(t,u(t))dt\\ \ge& \frac{1}{p^+}\int_0^T |\dot u(t)|^{p(t)}dt - \int_0^T F_1(t,u(t))dt - \int_0^T F_2(t,u(t))dt\end{aligned}$$

$$
\begin{aligned}
\geq &\frac{1}{p^+}\int_0^T |\dot{u}(t)|^{p(t)}dt - \int_0^T (2\mu|u(t)|^{\beta}+1)a_0 b(t)dt \\
&- \frac{1}{\alpha+1}\int_0^T f(t)|u(t)|^{\alpha+1}dt - \int_0^T g(t)|u(t)|dt \\
\geq &\frac{1}{p^+}\int_0^T |\dot{u}(t)|^{p(t)}dt - 2\mu a_0\|u\|_\infty^{\beta}\int_0^T b(t)dt \\
&- \int_0^T a_0 b(t)dt - \|u\|_\infty^{\alpha+1}\frac{\|f\|_{L^1}}{\alpha+1} - \|u\|_\infty\|g\|_{L^1} \\
\geq &\frac{1}{p^+}\int_0^T |\dot{u}(t)|^{p(t)}dt - C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{\beta}{p^-}} - C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{\alpha+1}{p^-}} \\
&- C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{1}{p^-}} - C
\end{aligned} \tag{3.2}
$$

for any $u \in \widetilde{W}_T^{1,p(t)}$. From Lemma 2.2, estimate (3.2) implies that

$$\varphi(u) \to +\infty \quad as\ \|u\| \to +\infty \quad for\ u \in \widetilde{W}_T^{1,p(t)}.$$

Step 2. We verify that for the positive sequence $\{r_m\}$ defined in (H_2), there holds

$$\lim_{m\to+\infty}\inf_{u\in W_{r_m}} \varphi(u) = +\infty, \tag{3.3}$$

where $W_{r_m} = \{u \in \mathbb{R}^N \| u| = r_m\} \oplus \widetilde{W}_T^{1,p(t)}$.

By Lemma 2.1, (H_1) and Young inequality, we obtain

$$
\begin{aligned}
&\Big|\int_0^T (F_2(t,u(t)) - F_2(t,\bar{u}))dt\Big| \\
=&\Big|\int_0^T\int_0^1 (\nabla F_2(t,\bar{u}+s\tilde{u}(t)),\tilde{u}(t))dsdt\Big| \\
\leq &\int_0^T\int_0^1 f(t)|\bar{u}+s\tilde{u}(t)|^{\alpha}|\tilde{u}(t)|dsdt + \int_0^T\int_0^1 g(t)|\tilde{u}(t)|dsdt \\
\leq &2^{p^- -1}(|\bar{u}|^{\alpha} + \|\tilde{u}\|_\infty^{\alpha})\|\tilde{u}\|_\infty\int_0^T f(t)dt + \|\tilde{u}\|_\infty\int_0^T g(t)dt \\
=&\Big((\frac{p^-}{2p^+})^{1/p^-}\frac{\|\tilde{u}\|_\infty}{2C_0}\Big)\Big((\frac{p^-}{2p^+})^{-1/p^-}2^{p^-}C_0\int_0^T f(t)dt\Big)|\bar{u}|^{\alpha} \\
&+ 2^{p^- -1}\|\tilde{u}\|_\infty^{\alpha+1}\int_0^T f(t)dt + \|\tilde{u}\|_\infty\int_0^T g(t)dt \\
\leq &\frac{1}{2p^+}\int_0^T |\dot{u}(t)|^{p(t)}dt + C|\bar{u}|^{q^-\alpha} + C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{\alpha+1}{p^-}} \\
&+ C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{1}{p^-}} + C
\end{aligned} \tag{3.4}
$$

for any $u \in W_T^{1,p(t)}$, where C_0 is the constant defined in Lemma 2.1.

Hence from (H_1), (3.1) and (3.4), for the functional φ defined in (1.2), we have

$$
\begin{aligned}
\varphi(u) =& \int_0^T \frac{|\dot{u}(t)|^{p(t)}}{p(t)} dt - \int_0^T F(t, u(t)) dt \\
\geq & \frac{1}{p^+} \int_0^T |\dot{u}(t)|^{p(t)} dt - \int_0^T F_1(t, u(t)) dt - \int_0^T F_2(t, u(t)) dt \\
\geq & \frac{1}{p^+} \int_0^T |\dot{u}(t)|^{p(t)} dt - \mu \int_0^T F_1(t, \frac{\bar{u}}{\lambda}) dt - \mu \int_0^T F_1(t, \frac{\tilde{u}(t)}{\lambda}) dt \\
& - \int_0^T (F_2(t, u(t)) - F_2(t, \bar{u})) dt - \int_0^T F_2(t, \bar{u}) dt \\
\geq & \frac{1}{2p^+} \int_0^T |\dot{u}(t)|^{p(t)} dt - \mu \int_0^T F_1(t, \frac{\bar{u}}{\lambda}) dt - \mu(2\mu\lambda^{-\beta} \|\tilde{u}\|_\infty^\beta + 1) \int_0^T a_0 b(t) dt - C|\bar{u}|^{q^- \alpha} \\
& - \int_0^T F_2(t, \bar{u}) dt - C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt\Big)^{\frac{\alpha+1}{p^-}} - C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt\Big)^{\frac{1}{p^-}} - C \\
\geq & \frac{1}{2p^+} \int_0^T |\dot{u}(t)|^{p(t)} dt - C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt\Big)^{\frac{\alpha+1}{p^-}} \\
& - C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt\Big)^{\frac{1}{p^-}} - C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt\Big)^{\frac{\beta}{p^-}} \\
& - |\bar{u}|^{q^- \alpha}\Big(\frac{1}{|\bar{u}|^{q^- \alpha}}\Big(\mu \int_0^T F_1(t, \frac{\bar{u}}{\lambda}) dt + \int_0^T F_2(t, \bar{u}) dt + C\Big)\Big) - C.
\end{aligned}
\tag{3.5}
$$

One can check that

$$
\begin{aligned}
B = \inf_{\tilde{u} \in \widetilde{W}_T^{1,p(t)}} \Big(& \frac{1}{2p^+} \int_0^T |\dot{u}(t)|^{p(t)} dt - C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt\Big)^{\frac{\alpha+1}{p^-}} \\
& - C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt\Big)^{\frac{1}{p^-}} - C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt\Big)^{\frac{\beta}{p^-}}\Big) > -\infty
\end{aligned}
$$

and

$$
\inf_{u \in W_{r_m}} \varphi(u) \geq B - C - \sup_{\bar{u} \in \mathbb{R}^N, |\bar{u}| = r_m} |r_m|^{q^- \alpha}\Big(\frac{1}{|r_m|^{q^- \alpha}}\Big(\mu \int_0^T F_1(t, \frac{\bar{u}}{\lambda}) dt + \int_0^T F_2(t, \bar{u}) dt\Big)\Big).
$$

Thus the above inequality and (H_2) imply (3.3).

Step 3. Prove (i) of Theorem 1.1. For the positive sequence $\{R_n\}$ defined in the condition (H_2), we define

$$
\Gamma_n = \{\gamma \in C(B_{R_n}; W_T^{1,p(t)}) \,|\, \gamma|\partial B_{R_n} = id|\partial B_{R_n}\}
$$

and

$$
c_n = \inf_{\gamma \in \Gamma_n} \max_{x \in B_{R_n}} \varphi(\gamma(x)), \tag{3.6}
$$

where

$$
B_\delta = \{x \in \mathbb{R}^N \,||x| \leq \delta\}.
$$

One can verify that $\gamma(B_{R_n}) \cap \widetilde{W}_T^{1,p(t)} \neq \emptyset$ (for more details see [10]).

From Step 1, we know that φ is coercive on $\widetilde{W}_T^{1,p(t)}$. Since $W_T^{1,p(t)}$ is a reflective Banach space, hence there is a constant M such that

$$\max_{x\in B_{R_n}} \varphi(\gamma(x)) \geq \inf_{u\in\widetilde{W}_T^{1,p(t)}} \varphi(u) = M. \tag{3.7}$$

Combining (3.6) and (3.7), one obtains that $c_n \geq M$.

As in (3.3), we can easily verify that

$$\lim_{n\to+\infty} \sup_{u\in\mathbb{R}^N,|u|=R_n} \varphi(u) = -\infty. \tag{3.8}$$

From (3.8), for large enough n, we obtain

$$c_n > \sup_{u\in\mathbb{R}^N,|u|=R_n} \varphi(u).$$

Now in Lemma 2.3 and Corollary 2.4, we take the compact set $K = B_{R_n}$ and $K_0 = \partial B_{R_n}$, then for any sequence $\{\gamma_k\} \subset \Gamma_n$ such that

$$\max_{x\in B_{R_n}} \varphi(\gamma_k(x)) \to c_n, \tag{3.9}$$

there exists a sequence $\{v_k\} \subset W_T^{1,p(t)}$ satisfying

$$\varphi(v_k) \to c_n, \quad dist(v_k, \gamma_k(B_{R_n})) \to 0, \quad |\varphi'(v_k)| \to 0, \text{ as } k \to \infty. \tag{3.10}$$

Combining (3.6), (3.9) and (3.10), for k large enough, we have

$$c_n \leq \max_{x\in B_{R_n}} \varphi(\gamma_k(x)) \leq c_n + 1 \tag{3.11}$$

Define $\pi : W_T^{1,p(t)} \to \mathbb{R}^N$ as

$$\pi(u) = \frac{1}{T}\int_0^T u(t)dt, \ for\ u \in W_T^{1,p(t)}.$$

For fixed n, we obtain

$$|\pi(\gamma_k(B_{R_n}))| \leq K_n.$$

for some positive constant K_n, here $|S| := \sup_{x\in S} \|x\|$ for a set $S \subset \mathbb{R}^N$.

We now take $\omega_k \in \gamma_k(B_{R_n})$ such that

$$\|v_k - \omega_k\| \leq 1. \tag{3.12}$$

We write $\omega_k = \bar{\omega}_k + \tilde{\omega}_k$ with $\bar{\omega}_k \in \mathbb{R}^N$ and $\tilde{\omega}_k \in \widetilde{W}_T^{1,p(t)}$. So there holds $|\bar{\omega}_k| \leq K_n$, which implies that

$$\mu\int_0^T F_1(t, \frac{\bar{\omega}_k}{\lambda})dt + \int_0^T F_2(t, \bar{\omega}_k)dt$$

is bounded. Combining (H_1), (3.1), (3.4), (3.11) and Lemma 2.1, we obtain

$$
\begin{aligned}
c_n + 1 \geq & \varphi(\omega_k) \\
= & \int_0^T \frac{|\dot{\omega}_k(t)|^{p(t)}}{p(t)} dt - \int_0^T F(t, \omega_k(t)) dt \\
\geq & \frac{1}{p^+} \int_0^T |\dot{\omega}_k(t)|^{p(t)} dt - \int_0^T F_1(t, \omega_k(t)) dt - \int_0^T F_2(t, \omega_k(t)) dt \\
\geq & \frac{1}{p^+} \int_0^T |\dot{\omega}_k(t)|^{p(t)} dt - \mu \int_0^T F_1(t, \frac{\bar{\omega}_k}{\lambda}) dt - \mu \int_0^T F_1(t, \frac{\tilde{\omega}_k(t)}{\lambda}) dt \\
& - \int_0^T (F_2(t, \omega_k(t)) - F_2(t, \bar{\omega}_k)) dt - \int_0^T F_2(t, \bar{\omega}_k) dt \\
\geq & \frac{1}{2p^+} \int_0^T |\dot{\omega}_k(t)|^{p(t)} dt - \mu \int_0^T F_1(t, \frac{\bar{\omega}_k}{\lambda}) dt - \mu(2\mu\lambda^{-\beta} \|\tilde{\omega}_k\|_\infty^\beta + 1) \int_0^T a_0 b(t) dt - C \\
& - \int_0^T F_2(t, \bar{\omega}_k) dt - C|\bar{\omega}_k|^{q^- \alpha} - C\Big(\int_0^T |\dot{\omega}_k(t)|^{p(t)} dt \Big)^{\frac{\alpha+1}{p^-}} - C\Big(\int_0^T |\dot{\omega}_k(t)|^{p(t)} dt \Big)^{\frac{1}{p^-}} \\
\geq & \frac{1}{2p^+} \int_0^T |\dot{\omega}_k(t)|^{p(t)} dt - C\Big(\int_0^T |\dot{\omega}_k(t)|^{p(t)} dt \Big)^{\frac{\alpha+1}{p^-}} \\
& - C\Big(\int_0^T |\dot{\omega}_k(t)|^{p(t)} dt \Big)^{\frac{1}{p^-}} - C\Big(\int_0^T |\dot{\omega}_k(t)|^{p(t)} dt \Big)^{\frac{\beta}{p^-}} - C.
\end{aligned} \tag{3.13}
$$

Therefore one can check that $\{\tilde{\omega}_k\}$ and $\{\omega_k\}$ is bounded from Lemma 2.2 and (3.13). By (3.12), we obtain that $\{v_k\}$ is bounded. By a standard argument, one can easy to verify that $\{v_k\}$ possesses a convergent subsequence. It is still denoted by $\{v_k\}$. One can refer [2] for more details about the proof of (PS) condition.

Let

$$u_n^+ = \lim_{k \to +\infty} v_k.$$

Then from (3.10), we obtain

$$\varphi(u_n^+) = \lim_{k \to +\infty} \varphi(v_k) = c_n, \quad \varphi'(u_n^+) = \lim_{k \to +\infty} \varphi'(v_k) = 0.$$

This means u_n^+ is a solution of problem (1.1) with $\varphi(u_n^+) = c_n$.

For all $\gamma \in \Gamma_n$, by definition, there holds $\gamma(B_{R_n}) \cap W_{r_m} \neq \emptyset$. Then we get

$$\max_{x \in B_{R_n}} \varphi(\gamma(x)) \geq \inf_{u \in W_{r_m}} \varphi(u). \tag{3.14}$$

Now combining (3.3), (3.6) and (3.14), we obtain

$$\lim_{n \to +\infty} c_n = +\infty.$$

Hence there exists a sequence of periodic solutions $\{u_n^+\}$ which are minimax-type critical points of the functional φ and $\varphi(u_n^+) \to +\infty$ as $n \to +\infty$. So (i) of Theorem 1.1 is proved.

Step 4. Prove (ii) of Theorem 1.1. For the sequence $\{r_n\}$ defined in condition (H_2) and as the same from (3.3), we define P_n as

$$P_n = \{u \in W_T^{1,p(t)} | u = \bar{u} + \tilde{u}, \bar{u} \in \mathbb{R}^N, |\bar{u}| \le r_n, \tilde{u} \in \widetilde{W}_T^{1,p(t)}\}.$$

For $u \in P_n$, it is easy to see that

$$\mu \int_0^T F_1(t, \frac{\bar{u}}{\lambda})dt + \int_0^T F_2(t, \bar{u})dt$$

is bounded.

Combining (H_1), (3.1) and (3.4), for $u \in P_n$, by the same process as in (3.13), we get

$$\begin{aligned}\varphi(u) =& \int_0^T \frac{|\dot{u}(t)|^{p(t)}}{p(t)}dt - \int_0^T F(t, u(t))dt \\ \ge& \frac{1}{2p^+} \int_0^T |\dot{u}(t)|^{p(t)}dt - C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{\alpha+1}{p^-}} - C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{1}{p^-}} \\ &- C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{\beta}{p^-}} - C. \end{aligned} \tag{3.15}$$

Thus the functional φ is bounded below in P_n.

Set

$$\rho_n = \inf_{u \in P_n} \varphi(u) \tag{3.16}$$

and let $\{u_k\}$ be a sequence on P_n such that

$$\varphi(u_k) \to \rho_n \; as \, k \to \infty.$$

We write

$$u_k = \bar{u}_k + \tilde{u}_k,\ \bar{u}_k \in \mathbb{R}^N, |\bar{u}_k| \le r_n,\ \tilde{u}_k \in \widetilde{W}_T^{1,p(t)}.$$

From (3.15) and (3.16), it is easy to see that $\{u_k\}$ is bounded sequence in $W_T^{1,p(t)}$. Then $\{u_k\}$ possesses a subsequence, which is still denoted by $\{u_k\}$ such that

$$u_k \rightharpoonup u_n^- \;\; in \;\; W_T^{1,p(t)}.$$

As P_n is a closed convex subset of $W_T^{1,p(t)}$, one has $u_n^- \in P_n$.

Since the functional φ is weakly lower semicontinuous, we obtain

$$\rho_n = \lim_{k \to \infty} \varphi(u_k) \ge \varphi(u_n^-).$$

Therefore we get

$$\rho_n = \varphi(u_n^-). \tag{3.17}$$

From (3.3), for large n, we get that $u_n^- \notin \{\bar{u} + \tilde{u}, \bar{u} \in \mathbb{R}^N, |\bar{u}| = r_n\}$, and hence $u_n^- \in IntP_n = \{u \in W_T^{1,p(t)} | u = \bar{u} + \tilde{u}, |\bar{u}| < r_n\}$. Thus combining (3.16) and (3.17), one obtains

$$\varphi'(u_n^-) = 0$$

and u_n^- is a solution of problem (1.1).

Since $r_n \to +\infty$ as $n \to +\infty$, we can choose the sequence $\{R_n\}$ in the condition (H_2) and as the same in (3.8) satisfying $0 < R_n < r_n$. By (3.16), we obtain

$$\varphi(u_n^-) \leq \sup_{x\in\mathbb{R}^N, |x|=R_n} \varphi(x). \tag{3.18}$$

From (3.8) and (3.18), we have

$$\varphi(u_n^-) \to -\infty \quad as\, n \to \infty.$$

So (ii) of Theorem 1.1 is true. The proof of Theorem 1.1 is complete.

Proof of Theorem 1.3. The proof is divided into two steps.

Step 1. We show that $\varphi(u) \to +\infty$ as $\|u\| \to \infty$ for $u \in \widetilde{W}_T^{1,p(t)}$.

Combining Lemma 2.1 and (H_3), we obtain

$$\begin{aligned}
\varphi(u) &= \int_0^T \frac{|\dot{u}(t)|^{p(t)}}{p(t)} dt - \int_0^T F(t, u(t))dt \\
&\geq \frac{1}{p^+} \int_0^T |\dot{u}(t)|^{p(t)} dt - \frac{1}{p^-} \int_0^T f(t)|u(t)|^{p^-} dt - \int_0^T g(t)|u(t)|dt \\
&\geq \frac{1}{p^+} \int_0^T |\dot{u}(t)|^{p(t)} dt - \frac{\|f\|_{L^1}}{p^-} \|u\|_\infty^{p^-} - \|u\|_\infty \|g\|_{L^1} \\
&\geq \Big(\frac{1}{p^+} - \frac{(2C_0)^{p^-}}{p^-} \|f\|_{L^1}\Big) \int_0^T |\dot{u}(t)|^{p(t)} dt - C\Big(\int_0^T |\dot{u}(t)|^{p(t)} dt\Big)^{\frac{1}{p^-}} - C,
\end{aligned} \tag{3.19}$$

for any $u \in \widetilde{W}_T^{1,p(t)}$, where C_0 is the positive constant defined in Lemma 2.1. We now define the constant $\alpha(p)$ in Theorem 1.3 as

$$\alpha(p) = \frac{p^-}{2p^+ \theta_{p^-} (2C_0)^{p^-}}, \tag{3.20}$$

where θ_{p^-} is a positive constant defined by $\theta_{p^-} = 1$ for $p^- \in (1, 2]$, and $\theta_{p^-} = 2^{p^- - 2}$ for $p^- > 2$. Now if $\|f\|_{L^1} < \alpha(p)$, in view of Lemma 2.2, the estimate (3.19) implies that

$$\varphi(u) \to +\infty \quad as\ \|u\| \to +\infty \quad for\ u \in \widetilde{W}_T^{1,p(t)}.$$

Step 2. We verify that for the positive sequence $\{r_m\}$ defined in condition (H_4), there holds

$$\lim_{m\to+\infty} \inf_{u\in W_{r_m}} \varphi(u) = +\infty, \tag{3.21}$$

where $W_{r_m} = \{u \in \mathbb{R}^N \| u| = r_m\} \oplus \widetilde{W}_T^{1,p(t)}$.

For $u \in W_T^{1,p(t)}$ with $u = \bar{u} + \tilde{u}$ where $|\bar{u}| = r_m$ and $\tilde{u} \in \widetilde{W}_T^{1,p(t)}$. Here we recall that r_m is defined in condition (H_4), namely we have

$$\lim_{m\to+\infty} \sup_{x\in\mathbb{R}^N, |x|=r_m} \frac{1}{|r_m|^{p^-}} \int_0^T F(t, x)dt < -\beta(p). \tag{3.22}$$

Combining (H_3), we obtain

$$
\begin{aligned}
&\Big|\int_0^T (F(t,u(t)) - F(t,\bar{u}))dt\Big| \\
=&\Big|\int_0^T \int_0^1 (\nabla F(t,\bar{u}+s\tilde{u}(t)),\tilde{u}(t))dsdt\Big| \\
\le&\int_0^T \int_0^1 f(t)|\bar{u}+s\tilde{u}(t)|^{p^- -1}|\tilde{u}(t)|dsdt + \int_0^T \int_0^1 g(t)|\tilde{u}(t)|dsdt \\
\le&\theta_{p^-}\Big(|\bar{u}|^{p^- -1}\|\tilde{u}\|_\infty + \frac{1}{p^-}\|\tilde{u}\|_\infty^{p^-}\Big)\|f\|_{L^1} + \|\tilde{u}\|_\infty\|g\|_{L^1} \\
\le&\frac{2\theta_{p^-}}{p^-}\|f\|_{L^1}\|\tilde{u}\|_\infty^{p^-} + \frac{\alpha(p)\theta_{p^-}}{q^-}|\bar{u}|^{p^-} + C\|\tilde{u}\|_\infty,
\end{aligned}
\tag{3.23}
$$

for any $u \in W_T^{1,p(t)}$, where $\frac{1}{p^-} + \frac{1}{q^-} = 1$. Where in (3.23), in the second estimate, we have used the inequality $(a+b)^c \le a^c + b^c$ for $a, b \ge 0$, $c \in (0,1]$, $(a+b)^c \le 2^{c-1}(a^c + b^c)$ for $a, b \ge 0$, $c \in (1,+\infty)$, and in the last estimate, we have used the Young inequality $ab \le \frac{a^{p^-}}{p^-} + \frac{b^{q^-}}{q^-}$, $a, b \ge 0$. In the Lemma 2.1 of [11], we know that for a fixed $\rho > 1$ and any $\varepsilon > 0$, there is a number $B(\varepsilon) > 0$ such that

$$
(a+b)^\rho \le (1+\varepsilon)a^\rho + B(\varepsilon)b^\rho, \ \forall a, b > 0.
$$

Now we take $\varepsilon > 0$ such that $\alpha(p) \ge (1+\varepsilon)\|f\|_{L^1} + \varepsilon$, and $\rho = p^-$, from Lemma 2.1, there holds

$$
\|f\|_{L^1}\|\tilde{u}\|_\infty^{p^-} \le (\alpha(p)-\varepsilon)(2C_0)^{p^-}\int_0^T |\dot{\tilde{u}}|^{p(t)}dt + C.
$$

Thus we get

$$
\begin{aligned}
&\Big|\int_0^T (F(t,u(t)) - F(t,\bar{u}))dt\Big| \\
\le&\frac{2\theta_{p^-}(2C_0)^{p^-}(\alpha(p)-\varepsilon)}{p^-}\int_0^T |\dot{\tilde{u}}|^{p(t)}dt + \frac{\theta_{p^-}\alpha(p)}{q^-}|\bar{u}|^{p^-} + C\left(\int_0^T |\dot{\tilde{u}}|^{p(t)}dt\right)^{\frac{1}{p^-}} + C.
\end{aligned}
\tag{3.24}
$$

Therefore, from (H_3) and (3.24), we have

$$
\begin{aligned}
\varphi(u) =& \int_0^T \frac{|\dot{u}(t)|^{p(t)}}{p(t)}dt - \int_0^T F(t,u(t))dt \\
\ge&\frac{1}{p^+}\int_0^T |\dot{u}(t)|^{p(t)}dt - \int_0^T (F(t,u(t)) - F(t,\bar{u}))dt - \int_0^T F(t,\bar{u})dt \\
\ge&\left(\frac{1}{p^+} - \frac{2\theta_{p^-}(2C_0)^{p^-}(\alpha(p)-\varepsilon)}{p^-}\right)\int_0^T |\dot{u}(t)|^{p(t)}dt - C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{1}{p^-}} \\
&- |\bar{u}|^{p^-}\Big(\frac{1}{|\bar{u}|^{p^-}}\int_0^T F(t,\bar{u})dt + \frac{\theta_{p^-}\alpha(p)}{q^-}\Big) - C
\end{aligned}
\tag{3.25}
$$

for all $u \in W_T^{1,p(t)}$. From (3.20), we can check that

$$
\frac{1}{p^+} - \frac{2\theta_{p^-}(2C_0)^{p^-}(\alpha(p)-\varepsilon)}{p^-} > 0.
$$

Hence we obtain

$$
\begin{aligned}
D = \inf_{\tilde{u}\in \widetilde{W}_T^{1,p(t)}} &\Big(\frac{1}{p^+} - \frac{2\theta_{p^-}(2C_0)^{p^-}(\alpha(p)-\varepsilon)}{p^-}\Big)\int_0^T |\dot{u}(t)|^{p(t)}dt \\
&- C\Big(\int_0^T |\dot{u}(t)|^{p(t)}dt\Big)^{\frac{1}{p^-}}\Big) > -\infty.
\end{aligned}
$$

Now define the constant $\beta(p)$ in Theorem 1.3 as

$$\beta(p) = \frac{p^-}{2p^+q^-(2C_0)^{p^-}}. \tag{3.26}$$

Therefore combining the above expression and (3.22), we obtain (3.21).

The remainder is similar to that as in the proof of Theorem 1.1, we omit the detail here.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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