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http://www.aimspress.com/journal/era

ERA, 30(3): 1035–1051.
DOI: 10.3934/era.2022054
Received: 18 January 2022
Revised: 15 February 2022
Accepted: 20 February 2022
Published: 08 March 2022

Research article

Existence and nonexistence of global solutions for logarithmic hyperbolic equation

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Abstract: This article is concerned with the initial-boundary value problem for a equation of quasi-hyperbolic type with logarithmic nonlinearity. By applying the Galerkin method and logarithmic Sobolev inequality, we prove the existence of global weak solutions for this problem. In addition, by means of the concavity analysis, we discuss the nonexistence of global solutions in the unstable set and give the lifespan estimation of solutions.

Keywords: logarithmic hyperbolic equation; stable and unstable sets; existence and nonexistence; global solutions; initial-boundary value problem

1. Introduction

In this paper, we study the initial-boundary value problem for logarithmic hyperbolic equation of $p-$Laplacian type

$$u_{tt} + \Delta_p u = |u|^{p-2} u \ln |u|, \quad (x,t) \in \Omega \times R^+, \tag{1.1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times R^+, \tag{1.3}$$

where

$$\Delta_p u = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p-2} \frac{\partial u}{\partial x_i}\Big),\ p > 2. \tag{1.4}$$

$\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, and parameter p fulfills

$$2 < p < +\infty,\ N \le p;\ \ 2 < p \le \frac{Np}{N-p},\ N > p. \tag{1.5}$$

The logarithmic nonlinearity arises in a lot of different areas of physics such as quantum mechanics and inflation cosmology, and is applied to nuclear physics, optics and geophysics [1–8]. Because of these special physical meanings, the research of evolution equations with logarithmic nonlinearity has attracted much attention.

When the logarithmic term in Eq (1.1) is replaced by nonlinear source term $|u|^{r-2}u$, the Eq (1.1) becomes

$$u_{tt} + \Delta_p u = |u|^{r-2}u, \quad (x,t) \in \Omega \times R^+. \tag{1.6}$$

In the case of $p = 2$ and $r > 2$, J. Ball [9] and M. Tsustsumi [10] obtained the blow-up solutions with negative initial energy. By using the concavity method, H. A. Levine and L. E. Payne [11] established the nonexistence of global weak solutions for Eq (1.6) with the conditions (1.2) and (1.3). Y. J. Ye [12] proved the global existence and blow-up result of solutions to the Eq (1.6) with initial-boundary value conditions. S. Ibrahim and A. Lyaghfouri [13] considered the Cauchy problem of (1.6), under appropriate assumptions, they established the finite time blow-up of solutions and, hence, extended a result by V. A. Galaktionov and S. I. Pohozaev [14]. For the Eq (1.6) with dissipative term, Y. J. Ye [15, 16] studied the global solutions by constructing a stable set in $W_0^{1,p}(\Omega)$ and the decay property of solution by using an integral inequality [17].

S. A. Messaoudi and B. S. Said-Houari [18] considered the nonlinear hyperbolic type equation

$$u_{tt} + \Delta_\alpha u + \Delta_\beta u_t - \Delta u_t + a|u_t|^{m-2}u_t = b|u|^{p-2}u,$$

where $a, b > 0, \alpha, \beta, m, p > 2$ and Ω is a bounded domain in $R^N (N \geq 1)$. Under suitable conditions on α, β, m, p, they proved a global nonexistence result of solutions with negative initial energy. For the nonlinear wave equation of p-Laplacian type

$$u_{tt} + \Delta_p u - \Delta u_t + q(x,u) = f(x).$$

C. Chen et al.[19] obtained the global existence and uniqueness of solutions and established the long-time behavior of solutions.

L. C. Nhan and T. X. Le [20] studied the existence and nonexistence of global weak solutions for a class of p-Laplacian evolution equations with logarithmic nonlinearity and gave sufficient conditions for the large time decay and blow-up of solutions. Later, Y. Z. Han et al. [21] also considered this problem. They studied global solutions and blow-up solution for arbitrarily high initial energy. For a mixed pseudo-parabolic p-Laplacian type equation with logarithmic term, under various assumptions about initial values, H. Ding and J. Zhou [22] proved the solution exists globally and blow up in finite time. Moreover, T. Boudjeriou [23] was concerned with the fractional p-Laplacian with logarithmic nonlinearity, by applying the potential well theory and a differential inequality, he proved the existence and decay estimates of global solutions and obtained the blow-up result of solutions.

T. Cazenave and A. Haraux [8] considered the following logarithmic wave equation

$$u_{tt} - \Delta u = u \ln |u|, \tag{1.7}$$

they gave the existence of solutions for the Cauchy problem of Eq (1.7). P. Gorka [4] obtained the global existence of weak solutions for the initial-boundary value problem of Eq (1.7). K. Bartkowski and P. Gorka [7] proved the existence of classical solutions and weak solutions for the corresponding one dimensional Cauchy problem of Eq (1.7). In the case of $0 < \mathcal{E}(0) \leq d$, W. Lian et al. [24]

proved the global existence of solution and obtained the blow-up of solution for the Eq (1.7) with the conditions (1.2) and (1.3).

In this paper, by means of the potential well theory and the concavity analysis method [25–29], we prove the global existence and blow-up of solutions of the problem (1.1)–(1.3).

For simplicity, we denote $L^p(\Omega)$ and $L^2(\Omega)$ norm by $\|\cdot\|_p$ and by $\|\cdot\|$ respectively. The space $W_0^{1,p}(\Omega)$ norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$ is replaced by $\|\nabla\cdot\|_p$.

2. Preliminaries

At first, we define the weak solutions of the problem (1.1)–(1.3) and give a few known lemmas.
Definition 2.1. *[30] If*

$$u \in C([0,T), W_0^{1,p}(\Omega)),\ u_t \in C([0,T), L^2(\Omega)),\ u_{tt} \in C([0,T), W^{-1,p'}(\Omega))$$

and satisfies

$$\int_\Omega u_{tt}\varphi dx + \int_\Omega \Delta_p u\varphi dx = \int_\Omega |u|^{p-2}u\ln|u|\varphi dx,$$

then $u(t)$ is called a weak solution of (1.1)–(1.3) on $[0,T)$, where $\varphi \in W_0^{1,p}(\Omega)$.
Lemma 2.1. *[31] Let q be a real number with $2 \le q < +\infty$ if $2 \le n \le p$ and $2 \le q \le \frac{np}{n-p}$ if $2 < p < n$. Then there exists a positive constant C depending on Ω, p and q such that $\|u\|_q \le C\|\nabla u\|_p$.*
Lemma 2.2. *[30] Let B_0, B, B_1 be Banach spaces with $B_0 \subseteq B \subseteq B_1$ and*

$$X = \{u:\ u \in L^p([0,T]; B_0),\ u_t \in L^q[0,T]; B_1)\},\ 1 \le p,q \le +\infty.$$

Suppose that B_0 is compactly embedded in B and that B is continuously embedded in B_1, then (i) the embedding of X into $L^p(0,T;B)$ is compact if $p < +\infty$. (ii) the embedding of X into $C([0,T];B)$ is compact if $p = +\infty$ and $q > 1$.
Lemma 2.3. *[32] Assume that $u_n(x)$ is a bounded sequence in $L^q(\Omega)$, $1 \le q < +\infty$, $u_n(x) \to u(x)$ a.e.. Then $u(x) \in L^q(\Omega)$ and $u_n(x) \to u(x)$ weakly converges in $L^q(\Omega)$.*
Lemma 2.4. *[33, 34, 35] (L^2-logarithmic Sobolev inequality) If $v \in H_0^1(\Omega)$, then*

$$\int_\Omega |v|^2\ln|v|dx \le \|v\|^2\ln\|v\| + \frac{a^2}{2\pi}\|\nabla v\|^2 - \frac{n}{2}(1+\ln a)\|v\|^2,\quad \forall a > 0. \tag{2.1}$$

In order to deal with the logarithmic term $|u|^{p-2}u\ln|u|$ in Eq (1.1), we introduce the following L^p-logarithmic Sobolev inequality.
Lemma 2.5. *[36] Let $u \in W_0^{1,p}(\Omega)$, then one has the inequality*

$$\int_\Omega |u|^p\ln|u|dx \le \|u\|_p^p\ln\|u\|_p + \frac{(p-2)a^2}{4\pi}\|u\|_p^p + \frac{a^2}{2\pi}\|\nabla u\|_p^p - \frac{n}{p}(1+\ln a)\|u\|_p^p, \tag{2.2}$$

where $a > 0$ is a constant.

For convenience, in the following we are going to give the proof of Lemma 2.5.

Proof. By (2.1) in Lemma 2.4, we have

$$\begin{aligned}\int_\Omega |v|^2 \ln|v|dx &\le \|v\|^2 \ln\|v\| + \frac{a^2}{2\pi}\|\nabla v\|^2 - \frac{n}{2}(1+\ln a)\|v\|^2 \\ &= \int_\Omega |v|^2dx \cdot \ln\Big(\int_\Omega |v|^2 dx\Big)^{\frac{1}{2}} + \frac{a^2}{2\pi}\int_\Omega |\nabla v|^2 dx - \frac{n}{2}(1+\ln a)\int_\Omega |v|^2 dx.\end{aligned} \tag{2.3}$$

Let $v = u^{\frac{p}{2}}$ in (2.3), then we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega |u|^p \ln|u|^p dx &\le \int_\Omega |u|^p dx \cdot \ln\Big(\int_\Omega |u|^p dx\Big)^{\frac{1}{2}} \\ &\quad + \frac{a^2}{2\pi}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx - \frac{n}{2}(1+\ln a)\int_\Omega |u|^p dx \\ &= \frac{1}{2}\|u\|_p^p \ln\|u\|_p^p + \frac{a^2}{2\pi}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx - \frac{n}{2}(1+\ln a)\|u\|_p^p.\end{aligned} \tag{2.4}$$

By direct computation, we get

$$\nabla u^{\frac{p}{2}} = \frac{p}{2}u^{\frac{p}{2}-1}\cdot\nabla u = \frac{p}{2}u^{\frac{p-2}{2}}\cdot\nabla u. \tag{2.5}$$

From (2.5) and Hölder inequality, we receive

$$\begin{aligned}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx &= \frac{p^2}{4}\int_\Omega |u|^{p-2}|\nabla u|^2 dx \\ &\le \frac{p^2}{4}\Big(\int_\Omega |u|^p dx\Big)^{\frac{p-2}{p}}\Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{2}{p}} = \frac{p^2}{4}\|u\|_p^{p-2}\|\nabla u\|_p^2.\end{aligned} \tag{2.6}$$

By Young inequality $XY \le \frac{X^\alpha}{\alpha} + \frac{Y^\beta}{\beta}$ with $\alpha = \frac{p}{p-2}$, $\beta = \frac{p}{2}$, we conclude that

$$\frac{p^2}{4}\|u\|_p^{p-2}\|\nabla u\|_p^2 \le \frac{p^2}{4}\Big(\frac{p-2}{p}\|u\|_p^p + \frac{2}{p}\|\nabla u\|_p^p\Big) = \frac{p(p-2)}{4}\|u\|_p^p + \frac{p}{2}\|\nabla u\|_p^p. \tag{2.7}$$

It follows from (2.4), (2.6) and (2.7) that

$$\frac{1}{2}\int_\Omega |u|^p \ln|u|^p dx \le \frac{1}{2}\|u\|_p^p \ln\|u\|_p^p + \frac{a^2}{2\pi}\Big(\frac{p(p-2)}{4}\|u\|_p^p + \frac{p}{2}\|\nabla u\|_p^p\Big) - \frac{n}{2}(1+\ln a)\|u\|_p^p,$$

which implies

$$\int_\Omega |u|^p \ln|u|dx \le \|u\|_p^p \ln\|u\|_p + \frac{(p-2)a^2}{4\pi}\|u\|_p^p + \frac{a^2}{2\pi}\|\nabla u\|_p^p - \frac{n}{p}(1+\ln a)\|u\|_p^p.$$

This completes the proof of Lemma 2.5.

Next, we define the following functionals

$$\mathcal{J}(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln|u|dx + \frac{1}{p^2}\|u\|_p^p \tag{2.8}$$

and

$$\mathcal{K}(u) = \|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln|u|dx, \tag{2.9}$$

for $u \in W_0^{1,p}(\Omega)$. By (2.8) and (2.9), we have

$$\mathcal{J}(u) = \frac{1}{p^2}\|u\|_p^p + \frac{1}{p}\mathcal{K}(u). \tag{2.10}$$

We denote the energy functional by

$$\mathcal{E}(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln|u|dx + \frac{1}{p^2}\|u\|_p^p = \frac{1}{2}\|u_t\|^2 + \mathcal{J}(u), \tag{2.11}$$

for $u \in W_0^{1,p}(\Omega),\ t \geq 0$.

$$\mathcal{E}(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{p}\|\nabla u_0\|_p^p - \frac{1}{p}\int_\Omega |u_0|^p \ln|u_0|dx + \frac{1}{p^2}\|u_0\|_p^p = \frac{1}{2}\|u_1\|^2 + \mathcal{J}(u_0) \tag{2.12}$$

is the initial total energy.

Moreover, we define the Nehari manifold [37]

$$\mathcal{N} = \{u \in W_0^{1,p}(\Omega)/\{0\} :\ \mathcal{K}(u) = 0,\ \|\nabla u\|_p \neq 0\},$$

the stable set

$$\mathcal{W} = \{u \in W_0^{1,p}(\Omega) :\ \mathcal{K}(u) > 0,\ \mathcal{J}(u) < d\} \cup \{0\}$$

and the unstable set

$$\mathcal{U} = \{u \in W_0^{1,p}(\Omega) :\ \mathcal{K}(u) < 0\ \mathcal{J}(u) < d\},$$

where

$$d = \inf\{\sup_{\theta \geq 0} \mathcal{J}(\theta u) :\ u \in W_0^{1,p}(\Omega),\ \|\nabla u\|_p \neq 0\}. \tag{2.13}$$

It is readily seen that the potential well depth d defined in (2.13) can also be characterized as

$$d = \inf_{u \in \mathcal{N}} \mathcal{J}(u). \tag{2.14}$$

Lemma 2.6. *Let $u \in W_0^{1,p}(\Omega)$ and $\|u\|_p \neq 0$, then we have*

$$(a)\ \ \lim_{\theta \to 0^+} \mathcal{J}(\theta u) = 0,\ \lim_{\theta \to +\infty} \mathcal{J}(\theta u) = -\infty;$$

$$(b)\ \ \mathcal{K}(\theta u) = \theta \mathcal{J}'(\theta u) \begin{cases} > 0, & 0 < \theta < \theta_*, \\ = 0, & \theta = \theta_*, \\ < 0, & \theta_* < \theta < +\infty. \end{cases} \tag{2.15}$$

Proof. (a) For $u \in W_0^{1,p}(\Omega)$,

$$\mathcal{J}(\theta u) = \frac{\theta^p}{p}\|\nabla u\|_p^p - \frac{\theta^p}{p}\int_\Omega |u|^p \ln|u|dx - \frac{\theta^p}{p}\|u\|_p^p \ln\theta + \frac{\theta^p}{p^2}\|u\|_p^p.$$

It is easy to get from $\|u\|_p \neq 0$ that (a) is valid.

(b) An elementary calculation shows that

$$\frac{d}{d\theta}\mathcal{J}(\theta u) = \theta^{p-1}\Big(\|\nabla u\|_p^p - \int_\Omega |u|^p \ln|u|dx - \|u\|_p^p \ln\theta\Big). \tag{2.16}$$

Let $\frac{d}{d\theta}\mathcal{J}(\theta u) = 0$, then we have

$$\theta_* = \exp\Big(\frac{\|\nabla u\|_p^p - \int_\Omega |u|^p \ln|u|dx}{\|u\|_p^p}\Big). \tag{2.17}$$

It follows from (2.9) that

$$\mathcal{K}(\theta u) = \theta^p\Big(\|\nabla u\|_p^p - \int_\Omega |u|^p \ln|u|dx - \|u\|_p^p \ln\theta\Big). \tag{2.18}$$

From (2.16), (2.17) and (2.18), the Eq (2.15) holds.

Lemma 2.7. *Suppose that $u \in W_0^{1,p}(\Omega)$ and $\|\nabla u\|_p \neq 0$. Then $d \geq M$, where $M = \frac{1}{p^2}(2\pi)^{\frac{n}{2}} e^{\frac{2(n+p)-p^2}{2}}$.*

Proof. From Lemma 2.6 and Eq (2.10), one has

$$\sup_{\theta\geq 0}\mathcal{J}(\theta u) = \mathcal{J}(\theta_* u) = \frac{1}{p^2}\|\theta_* u\|_p^p + \frac{1}{p}\mathcal{K}(\theta_* u) = \frac{1}{p^2}\|\theta_* u\|_p^p. \tag{2.19}$$

We get from Lemma 2.5 that

$$\begin{aligned}\mathcal{K}(u) &= \|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln|u|dx \\ &\geq \Big(1 - \frac{a^2}{2\pi}\Big)\|\nabla u\|_p^p + \Big(\frac{n}{p}\ln(ae) - \frac{(p-2)a^2}{4\pi} - \ln\|u\|_p\Big)\|u\|_p^p.\end{aligned}$$

Choosing $a = \sqrt{2\pi}$, we have

$$\begin{aligned}\mathcal{K}(u) &\geq \Big(\frac{n}{p}\ln(\sqrt{2\pi}\, e) - \frac{p-2}{2} - \ln\|u\|_p\Big)\|u\|_p^p \\ &= \Big[\ln\Big((2\pi)^{\frac{n}{2p}} e^{\frac{2(n+p)-p^2}{2p}}\Big) - \ln\|u\|_p\Big]\|u\|_p^p.\end{aligned} \tag{2.20}$$

It follows from $\mathcal{K}(\theta_* u) = 0$ and (2.20) that

$$\ln\Big((2\pi)^{\frac{n}{2}} e^{\frac{2(n+p)-p^2}{2}}\Big) - \ln\|\theta_* u\|_p^p \leq 0,$$

which implies

$$\|\theta_* u\|_p^p \geq (2\pi)^{\frac{n}{2}} e^{\frac{2(n+p)-p^2}{2}}. \tag{2.21}$$

Thus, we obtain from (2.19) and (2.21) that

$$\sup_{\theta\geq 0}\mathcal{J}(\theta u) \geq \frac{1}{p^2}(2\pi)^{\frac{n}{2}} e^{\frac{2(n+p)-p^2}{2}}. \tag{2.22}$$

Thus, by (2.13) and (2.22), we conclude that $d \geq M > 0$.

3. Existence of global solutions

In this section, we state and prove the global existence result for the problem (1.1)–(1.3).

Theorem 3.1. *Assume that p satisfies (1.5). If $u_0 \in W_0^{1,p}(\Omega)$, $u_1 \in L^2(\Omega)$ and $0 < \mathcal{E}(0) < M$, $\mathcal{K}(u_0) \geq 0$, then there is a global weak solution $u(x,t)$ to the problem (1.1)–(1.3) which meets $u(x,t) \in L^\infty([0,+\infty); W_0^{1,p}(\Omega))$, $u_t(x,t) \in L^\infty([0,+\infty); L^2(\Omega))$.*

Proof. Assume that $\{\omega_j\}_{j=1}^\infty$ is a basis of space $W_0^{1,p}(\Omega)$ and that V_k is the subspace of $W_0^{1,p}(\Omega)$ generated by $\{\omega_1, \omega_2, \cdots, \omega_m\}$, $m = 1, 2, \cdots$. We shall look for the approximate solutions $u_m(t) = \sum_{j=1}^{m} g_{jm}(t)\omega_j$ with $g_{jm}(t) \in C^2[0,T]$, $\forall T > 0$. Here the functions $g_{jm}(t)$ fulfil the following system of equations

$$(u_{mtt}, \omega_j) + (\Delta_p u_m, \omega_j) = (|u_m|^{p-2} u_m \ln |u_m|, \omega_j), \ j = 1, 2, \cdots, m \tag{3.1}$$

with initial data

$$u_m(0) = u_{0m}, \ u_{mt}(0) = u_{1m}. \tag{3.2}$$

Because $W_0^{1,p}(\Omega)$ is dense in $L^2(\Omega)$, so there exist α_{jm} and β_{jm} such that

$$u_{0m} = \sum_{j=1}^{m} g_{jm}(0)\omega_j = \sum_{j=1}^{m} \alpha_{jm}\omega_j \to u_0(x) \textit{ strongly in } W_0^{1,p}(\Omega), \ m \to \infty, \tag{3.3}$$

$$u_{1m} = \sum_{j=1}^{m} g'_{jm}(0)\omega_j = \sum_{j=1}^{m} \beta_{jm}\omega_j \to u_1(x) \textit{ strongly in } L^2(\Omega), \ m \to \infty. \tag{3.4}$$

By Picard's iteration method, the solutions $g_{jm}(t)$ for the Cauchy problem (3.1)–(3.2) exist in $t \in [0, t_m)$, $t_m \leq T$. By the uniformly boundedness of functions $g_{jm}(t)$ and the extension theorem, these solutions $g_{jm}(t)$ exists in the whole interval $[0,T]$.

Multiplying both sides of (3.1) by $g'_{jm}(t)$, summing on j from 1 to m and then integrating over $[0,t]$, we obtain

$$\mathcal{E}_m(t) = \frac{1}{2}\|u_{mt}(t)\|^2 + \mathcal{J}(u_m(t)) = \frac{1}{2}\|u_{mt}(0)\|^2 + \mathcal{J}(u_m(0)) = \mathcal{E}_m(0) < M \leq d. \tag{3.5}$$

From (3.5), it is easy to verify

$$u_m(t) \in \mathcal{W}, \ \forall t \in [0,T]. \tag{3.6}$$

Assume that there exists a time $t_1 \in (0,T)$ such that $u_m(t_1) \notin \mathcal{W}$, then, by the continuity of $u_m(t)$ on t, we get $u_m(t_1) \in \partial\mathcal{W}$. Thus, we receive either

$$\mathcal{J}(u_m(t_1)) = d, \tag{3.7}$$

or

$$\mathcal{K}(u_m(t_1)) = 0, \ \|\nabla u_m\|_p \neq 0. \tag{3.8}$$

From (3.5), we have $J(u_m(t_1)) < d$. Thus, the case (3.7) is impossible.

If (3.8) is valid, $u_m(t_1) \in \mathcal{N}$. From (2.13), we obtain $J(u_m(t_1)) \geq d$. This contradicts with (3.5). Therefore, the case (3.8) is also impossible as well.

We deduce from (2.10), (3.5) and (3.6) that

$$M > \mathcal{J}(u_m) = \frac{1}{p^2}\|u_m\|_p^p + \frac{1}{p}\mathcal{K}(u_m) > \frac{1}{p^2}\|u_m\|_p^p, \tag{3.9}$$

which implies that

$$\|u_m\|_p^p < Mp^2. \tag{3.10}$$

Taking $a = \sqrt{\pi}$ in (2.2), we have from Lemma 2.5, Eqs (2.8) and (2.9) that

$$\begin{aligned}
\|\nabla u_m\|_p^p &= 2\mathcal{K}(u_m) + 2\int_\Omega |u_m|^p \ln|u_m|dx - \|\nabla u_m\|_p^p \\
&= 2p\mathcal{J}(u_m) - \frac{2}{p}\|u_m\|_p^p - \|\nabla u_m\|_p^p + 2\int_\Omega |u_m|^p \ln|u_m|dx \\
&\le 2p\mathcal{J}(u_m) - \frac{2}{p}\|u_m\|_p^p + \frac{p-2}{2}\|u_m\|_p^p \\
&-\frac{n}{p}\ln(\pi e^2)\|u_m\|_p^p + 2\|u_m\|_p^p \ln\|u_m\|_p \\
&\le 2p\mathcal{J}(u_m) + \frac{p-2}{2}\|u_m\|_p^p + 2\|u_m\|_p^p \ln\|u_m\|_p < C_M.
\end{aligned} \tag{3.11}$$

Here $C_M = 2pM + \frac{(p-2)p^2}{2}M + 2pM\ln(p^2M)$. From (3.5), we have

$$\|u_{mt}\|^2 < 2M. \tag{3.12}$$

For $u, v \in W_0^{1,p}(\Omega)$, by (1.4), we have $(\Delta_p u, v) = \int_\Omega |\nabla u|^{p-2}|\nabla u|\cdot|\nabla v|dx)$. Hence, from Hölder inequality and (3.11), we obtain

$$\|\Delta_p u\|_{W^{-1,p'}(\Omega)} \le \|\nabla u\|_p^{p-1} < C_M^{\frac{p-1}{p}}. \tag{3.13}$$

It follows from (3.10)–(3.13) that the following limitations are true.

$$u_m \to u \textit{ weakly star in } L^\infty(0,T;W_0^{1,p}(\Omega)), \tag{3.14}$$

$$u_m \to u \textit{ weakly star in } L^\infty(0,T;L^p(\Omega)), \tag{3.15}$$

$$u_{mt} \to u_t \textit{ weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{3.16}$$

$$\Delta_p u_m \to \chi \textit{ weakly star in } L^\infty(0,T;W^{-1,p'}(\Omega)). \tag{3.17}$$

Combining (3.15) and (3.16) with Lemma 2.2 yields

$$u_m \to u \textit{ strongly in } C([0,T];L^2(\Omega)), \tag{3.18}$$

which implies

$$|u_m|^{p-2}u_m\ln|u_m| \to |u|^{p-2}u\ln|u| \textit{ almost everywhere } (x,t)\in\Omega\times(0,T). \tag{3.19}$$

Let $\Omega_1 = \{x \in \Omega : |u_m(x,t) \le 1\}$ and $\Omega_2 = \{x \in \Omega : |u_m(x,t) \ge 1\}$, then by means of direct calculation, we know from Lemma 2.1 and Eq (3.11)

$$\begin{aligned}&\int_\Omega \left| |u_m|^{p-2}u_m \ln |u_m| dx \right|^{p'} \\ &= \int_{\Omega_1} \left| |u_m|^{p-2}u_m \ln |u_m| dx \right|^{p'} + \int_{\Omega_2} \left| |u_m|^{p-2}u_m \ln |u_m| dx \right|^{p'} \\ &\le [(p-1)e]^{-p'}|\Omega| + \left(\frac{n-p}{p(p-1)}\right)^{p'} \int_{\Omega_2} |u_m|^{\frac{np}{n-p}} \\ &\le [(p-1)e]^{-p'}|\Omega| + \left(\frac{n-p}{p(p-1)}\right)^{p'} C^{\frac{np}{n-p}} \|\nabla u_m\|_p^{\frac{np}{n-p}} \le L_M,\end{aligned} \tag{3.20}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $L_M = [(p-1)e]^{-p'}|\Omega| + \left(\frac{n-p}{p(p-1)}\right)^{p'} C^{\frac{np}{n-p}} C_M^{\frac{n}{n-p}}$. From Lemma 2.3, Eqs (3.19) and (3.20), we receive

$$|u_m|^{p-2}u_m \ln |u_m| \to |u|^{p-2}u \ln |u| \ weakly\ in\ L^\infty(0,T; L^{p'}(\Omega)). \tag{3.21}$$

Now, we prove $\chi = \Delta_p u$. For this reason, multiplying both sides of (3.1) by an arbitrary smooth function $\varphi(t) \in C^2[0,T]$ and integrating over $[0,T]$, we have

$$\begin{aligned}&(u_{mt}(T), \varphi(T)\omega_j) + \int_0^T (\Delta_p u_m, \varphi(t)\omega_j)dt = (u_{mt}(0), \varphi(0)\omega_j) \\ &+ \int_0^T (u_{mt}, \varphi'(t)\omega_j)dt + \int_0^T (|u_m|^{p-2}u_m \ln |u_m|, \varphi(t)\omega_j)dt.\end{aligned} \tag{3.22}$$

Taking the limitation of both sides of Eq (3.22) with j fixed and $m \to \infty$, we get

$$\begin{aligned}&(u_t(T), \varphi(T)\omega_j) + \int_0^T (\chi, \varphi(t)\omega_j)dt \\ &= (u_t(0), \varphi(0)\omega_j) + \int_0^T (u_t, \varphi'(t)\omega_j)dt + \int_0^T (|u|^{p-2}u \ln |u|, \varphi(t)\omega_j)dt.\end{aligned} \tag{3.23}$$

By (3.23),we have

$$\begin{aligned}&(u_t(T), \psi(T)) + \int_0^T (\chi, \psi(t))dt \\ &= (u_t(0), \psi(0)) + \int_0^T (u_t, \psi'(t))dt + \int_0^T (|u|^{p-2}u \ln |u|, \psi(t))dt,\end{aligned} \tag{3.24}$$

for every $\psi \in L^2(0,T; W_0^{1,p}(\Omega))$, $\psi' \in L^2(0,T; L^2(\Omega))$. In particular, Setting $\psi = u$ in (3.24) , we obtain

$$\begin{aligned}&(u_t(T), u(T)) + \int_0^T (\chi, u)dt \\ &= (u_t(0), u(0)) + \int_0^T \|u_t(t)\|^2 dt + \int_0^T (|u|^{p-2}u \ln |u|, u)dt.\end{aligned} \tag{3.25}$$

On the other hand, multiplying both sides of (3.1) by $g_{jm}(t)$, summing on j from 1 to m and integrating over $[0,T]$, we get

$$\begin{aligned}(u_{mt}(T),u_m(T))+\int_0^T(\Delta_p u_m,u_m)dt&=(u_{mt}(0),u_m(0)\\+\int_0^T\|u_{mt}\|^2dt+\int_0^T(|u_m|^{p-2}u_m\ln|u_m|,u_m)dt.\end{aligned}\tag{3.26}$$

Taking the inferior limitation on both sides of (3.26) as $m\to\infty$, we have

$$\begin{aligned}&(u_t(T),u(T))+\liminf_{m\to\infty}\int_0^T(\Delta_p u_m,u_m)dt\\&\le(u_t(0),u(0))+\int_0^T\|u_t\|^2dt+\int_0^T(|u|^{p-2}u\ln|u|,u)dt.\end{aligned}\tag{3.27}$$

We conclude from (3.25) and (3.27) that

$$\liminf_{m\to\infty}\int_0^T(\Delta_p u_m,u_m)dt\le\int_0^T(\chi,u)dt.\tag{3.28}$$

By the monotonicity of operator Δ_p, we have

$$\int_0^T(\Delta_p u_m-\Delta_p v,u_m-v)dt\ge0,\ \forall v\in L^\infty(0,T;W_0^{1,p}(\Omega)).\tag{3.29}$$

We get from (3.28) and (3.29) that

$$\begin{aligned}&\liminf_{m\to\infty}\int_0^T(\Delta_p u_m-\Delta_p v,u_m-v)dt\\&\le\int_0^T(\chi,u-v)dt-\int_0^T(\Delta_p v,u-v)dt=\int_0^T(\chi-\Delta_p v,u-v)dt.\end{aligned}\tag{3.30}$$

Combining (3.29) with (3.30) yields that

$$\int_0^T(\chi-\Delta_p v,u-v)dt\ge0.\tag{3.31}$$

Let $v=u-\lambda\omega$, then, by (3.31), we obtain

$$\lambda\int_0^T(\chi-\Delta_p(u-\lambda\omega),\omega)dt\ge0,\tag{3.32}$$

for any $\omega\in L^p(0,T;W_0^{1,p}(\Omega))$ and any real number λ.

As $\lambda>0,\lambda\to0$, from (3.32) and the hemicontinuity of operator Δ_p, we conclude that

$$\int_0^T(\chi-\Delta_p u,\omega)dt\ge0.\tag{3.33}$$

Similarly, when $\lambda < 0, \lambda \to 0$, we have

$$\int_0^T (\chi - \Delta_p u, \omega)dt \le 0. \tag{3.34}$$

Thus, for all $\omega \in L^p(0,T; W_0^{1,p}(\Omega))$, we deduce from (3.33) and (3.34) that

$$\int_0^T (\chi - \Delta_p u, \omega)dt = 0, \tag{3.35}$$

which implies that $\chi = \Delta_p u$.

Next, we prove above solution $u(x,t)$ satisfies (1.2), i.e., $u(x,0) = u_0(x),\ u_t(x,0) = u_1(x)$.

We conclude from Eqs (3.15), (3.16) and Lemma 1.2 that $u(t) : [0,T] \to L^2(\Omega)$ is continuous. Therefore, $u_m(0) \to u(0)$ weakly in $L^2(\Omega)$. According to (3.3), one has $u(0) = u_0$.

To prove $u_t(0) = u_1$, let $\xi(t)$ be a smooth function with $\xi(0) = 1, \xi(T) = 0$. Noting

$$\int_0^T (u_{mtt}, \xi\omega_j)dt = -\int_0^T (u_{mt}, \xi_t\omega_j)dt - (u_{mt}(0), \xi(0)\omega_j).$$

For given j, as $m \to \infty$, we get in the distribution sense

$$\int_0^T (u_{tt}, \xi\omega_j)dt = -\int_0^T (u_t, \xi_t\omega_j)dt - (u_t(0), \xi(0)\omega_j) \tag{3.36}$$

in $\mathcal{D}'([0,T])$. On the other hand,

$$\int_0^T (u_{mtt}, \xi\omega_j)dt = \int_0^T [(-\Delta_p u_m, \xi\omega_j) + (|u_m|^{p-2}u_m \ln|u_m|, \xi\omega_j)]dt$$

converges to

$$\int_0^T [(-\Delta_p u, \xi\omega_j) + (|u|^{p-2}u \ln|u|, \xi\omega_j)]dt = \int_0^T (u_{tt}, \xi\omega_j)dt$$

as $m \to \infty$. Therefore,

$$\int_0^T (u_{tt}, \xi\omega_j)dt = -\int_0^T (u_t, \xi_t\omega_j)dt - (u_1, \xi(0)\omega_j). \tag{3.37}$$

From (3.36) and (3.37), we have $(u_t(0), \omega_j) = (u_1, \omega_j)$. By the density of$\{\omega_j\}_{j=1}^m$ in $L^2(\Omega)$, we get $u_t(0) = u_1$. This completes the proof of Theorem 3.1.

For the case of $\mathcal{K}(u_0) \ge 0$ and $\mathcal{E}(0) = M \le d$, the global existence result of solutions to the problem (1.1)–(1.3) reads as follows:

Theorem 3.2. *Assume that p fulfils (1.5). If $u_0 \in W_0^{1,p}(\Omega),\ u_1 \in L^2(\Omega)$ and $\mathcal{E}(0) = M \le d,\ \mathcal{K}(u_0) \ge 0$, then there exists a global weak solution $u(x,t)$ for the problem (1.1)–(1.3) which satisfies $u(x,t) \in L^\infty([0,+\infty); W_0^{1,p}(\Omega)),\ u_t(x,t) \in L^\infty([0,+\infty); L^2(\Omega))$.*

Proof. For the case $\|\nabla u_0\|_p \ne 0$, let us suppose that $\rho_k = 1 - \frac{1}{k}$ and $u_{0k} = \rho_k u_0,\ k \ge 2$. The problem (1.1)–(1.3) can be written as follows:

$$\begin{cases} u_{tt} + \Delta_p u = |u|^{p-2}u \ln|u|, & (x,t) \in \Omega \times R^+, \\ u(x,0) = u_{0k}(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0,\ (x,t) \in \partial\Omega \times R^+. \end{cases} \tag{3.38}$$

From $\mathcal{K}(u_0) \geq 0$ and Lemma 2.6, we have $\theta^* = \theta^*(u_0) \geq 1$. Accordingly, we get $\mathcal{K}(u_{0k}) > 0$. By (2.3), we obtain

$$0 < \mathcal{J}(u_{0k}) = \frac{1}{p^2}\|u_{0k}\|_p^p + \frac{1}{p}\mathcal{K}(u_{0k}) < \mathcal{J}(u_0). \tag{3.39}$$

Therefore, we receive

$$0 < \mathcal{E}_k(0) = \frac{1}{2}\|u_1\|^2 + \mathcal{J}(u_{0k}) < \frac{1}{2}\|u_1\|^2 + \mathcal{J}(u_0) = \mathcal{E}(0) = M \leq d,$$

which implies that $u_{0k} \in \mathcal{W}$.

For each k, by Theorem 3.1, there exists a global weak solution $u_k(t)$ of the problem (3.38) such that $u_k(t) \in L^\infty([0,+\infty); W_0^{1,p}(\Omega))$, $u_{kt}(t) \in L^\infty([0,+\infty); L^2(\Omega))$ and

$$(u_{kt}, v) + \int_0^t (\Delta_p u_k, v)ds = (u_1, v) + \int_0^t (|u_k|^{p-2}u_k \ln|u_k|, v)ds \tag{3.40}$$

for any $v \in W_0^{1,p}(\Omega)$.

In addition,

$$\mathcal{E}_k(t) = \frac{1}{2}\|u_{kt}\|^2 + \mathcal{J}(u_k) = \frac{1}{2}\|u_1\|^2 + J(u_{0k}) = \mathcal{E}_k(0) < M \leq d. \tag{3.41}$$

By using (3.41) and combining with the same argument as (3.6), we can prove $u_k(t) \in \mathcal{W}$.

For the case $\|\nabla u_0\|_p = 0$, we get $\mathcal{J}(u_0) = 0$ by $\mathcal{K}(u_0) \geq 0$. Thus, we have $\mathcal{E}(0) = \frac{1}{2}\|u_1\|^2 + \mathcal{J}(u_0) = \frac{1}{2}\|u_1\|^2 = M$. Let $\rho_k = 1 - \frac{1}{k}$, $u_{1k} = \rho_k u_1(x)$, $k \geq 2$, we consider the following problem

$$\begin{cases} u_{tt} + \Delta_p u = |u|^{p-2}u \ln|u|, \quad (x,t) \in \Omega \times R^+, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_{1k}(x), \quad x \in \Omega, \\ u(x,t) = 0,\ (x,t) \in \partial\Omega \times R^+. \end{cases} \tag{3.42}$$

Noting

$$0 < \mathcal{E}_k(0) = \frac{1}{2}\|u_{1k}\|^2 + \mathcal{J}(u_0) = \frac{1}{2}\|\rho_k u_1\|^2 < \frac{1}{2}\|u_1\|^2 = M. \tag{3.43}$$

By Eq (3.43) and Theorem 3.1, there is a global weak solution $u_k(t)$ for the problem (3.42) such that $u_k(t) \in L^\infty(0,+\infty; W_0^{1,p}(\Omega))$, $u_{kt}(t) \in L^\infty(0,+\infty; L^2(\Omega))$ and $u_k(t) \in \mathcal{W}$ for each k.

The remainder of the proof for Theorem 3.2 is the same as those of Theorem 3.1. Here, we omit them.

4. Nonexistence of global solutions

Lemma 4.1. *[38, 39] If nonnegative function $\Phi(t) \in C^2$ satisfies*

$$\Phi(t)\Phi''(t) - (1+\rho)\Phi'(t)^2 \geq 0,$$

for $\Phi(0) > 0$, $\Phi'(0) > 0$ and $\rho > 0$, then there exists a time T_* such that $0 < T_* \leq \frac{\Phi(0)}{\rho\Phi'(0)}$ and $\lim\limits_{t\to T_*^-} \Phi(t) = +\infty$.

Lemma 4.2. *Suppose that $u(t)$ is a solution of (1.1)–(1.3). If $u_0 \in \mathcal{U}$ and $\mathcal{E}(0) < d$, then $u(t) \in \mathcal{U}$ and $\mathcal{E}(t) < d,\ \forall t \geq 0$.*
Proof. From the conservation of energy, we obtain $\mathcal{E}(t) = \mathcal{E}(0) < d$. From (2.11), we get

$$\mathcal{J}(u) \leq \mathcal{E}(t) < d. \tag{4.1}$$

Assume that there is $t^* \in [0, +\infty)$ such that $u(t^*) \notin \mathcal{U}$, then by continuity of $\mathcal{K}(u(t))$ on t, we obtain $\mathcal{K}(u(t^*)) = 0$. That means $u(t^*) \in \mathcal{N}$. From (2.14), we have $\mathcal{J}(u(t^*)) \geq d$, which is contradiction with (4.1). Therefore, the conclusion in Lemma 4.2 holds.

Theorem 4.1. *Suppose that $0 < \mathcal{E}(0) < d$ and $\int_\Omega u_0 u_1 dx > 0$, then there is no global weak solution $u(t)$ to the problem (1.1)–(1.3). Namely, there exists a time T_* such that $\lim_{t\to T_*^-} \|u(t)\|^2 = +\infty$, where the lifespan T_* is estimated by $0 < T_* < \frac{4\Psi(0)}{(p-2)\Psi'(0)}$, $\Psi(t)$ is given in (4.19).*
Proof. By $u_0 \in \mathcal{U}$, $\mathcal{E}(0) < d$ and Lemma 4.2, we get $u \in \mathcal{U}$. Thus,

$$\mathcal{K}(u) = \|\nabla u\|_p^p - \int_\Omega |u|^p \ln |u| dx < 0. \tag{4.2}$$

From (2.13) and (2.19), we have

$$d \leq \sup_{\theta \geq 0} \mathcal{J}(\theta u) = \frac{1}{p^2}\|\theta_* u\|_p^p. \tag{4.3}$$

We deduce from (2.17), (4.2) and (4.3) that

$$d \leq \frac{1}{p^2}\|u\|_p^p. \tag{4.4}$$

Let

$$\Psi(t) = \|u(t)\|^2 = \int_\Omega u^2 dx. \tag{4.5}$$

Then there is a real number $\alpha > 0$, which satisfies

$$\Psi(t) \geq \alpha > 0. \tag{4.6}$$

By differentiating on both sides of (4.5), we get

$$\Psi'(t) \ = 2\int_\Omega u u_t dx. \tag{4.7}$$

From (4.7), we obtain

$$\Psi''(t) = 2\|u_t\|^2 + 2\int_\Omega u u_{tt} dx. \tag{4.8}$$

Combining (1.1) with (4.8), we get

$$\Psi''(t) = 2(\|u_t(t)\|^2 + \int_\Omega |u|^p \ln |u| dx - \|\nabla u\|_p^p) = 2[\|u_t(t)\|^2 - \mathcal{K}(u)]. \tag{4.9}$$

By $u \in \mathcal{U}$ and (4.9), we receive $\Psi''(t) > 0$. Combining (4.5), (4.7) and (4.9), we get

$$\begin{aligned}&\Psi(t)\Psi''(t) - \frac{p+2}{4}\Psi'(t)^2\\&= 2\Psi(t)\Big[\|u_t(t)\|^2 + \int_\Omega |u|^p \ln|u|dx - \|\nabla u\|_p^p\Big]\\&-(p+2)\Psi(t)\|u_t(t)\|^2 + (p+2)\Upsilon(t),\end{aligned} \tag{4.10}$$

where

$$\Upsilon(t) = \|u(t)\|^2 \cdot \|u_t(t)\|^2 - \Big(\int_\Omega uu_t dx\Big)^2. \tag{4.11}$$

By Cauchy-Schwarz inequality, we get

$$\Big(\int_\Omega uu_t dx\Big)^2 \le \|u(t)\|^2\|u_t(t)\|^2. \tag{4.12}$$

This inequality (4.12) guarantees $\Upsilon(t) \ge 0$. By (4.10), we have

$$\Psi(t)\Psi''(t) - \frac{p+2}{4}\Psi'(t)^2 \ge \Psi(t)\Pi(t), \tag{4.13}$$

where

$$\Pi(t) = -p\|u_t\|^2 + 2\int_\Omega |u|^p \ln|u|dx - 2\|\nabla u\|_p^p. \tag{4.14}$$

From (2.11) and (4.14), we obtain

$$\Pi(t) = -2p\mathcal{E}(t) + \frac{2}{p}\|u\|_p^p. \tag{4.15}$$

By (4.4), (4.15) and $\mathcal{E}(t) = \mathcal{E}(0) < d$, we get

$$\Pi(t) \ge -2p\mathcal{E}(0) + 2pd = 2p[d - \mathcal{E}(0)] > 0. \tag{4.16}$$

Therefore, there exists $\beta > 0$ such that

$$\Pi(t) \ge \beta > 0. \tag{4.17}$$

Combining (4.6), (4.13) and (4.17), we conclude that

$$\Psi(t)\Psi''(t) - \frac{p+2}{4}\Psi'(t)^2 \ge \alpha\beta > 0,\ \forall t \ge 0. \tag{4.18}$$

Let $\rho = \frac{p-2}{4} > 0$, then, by the differential inequality (4.18) and Lemma 4.1, one has

$$0 < T_* < \frac{4\Psi(0)}{(p-2)\Psi'(0)}, \tag{4.19}$$

and

$$\lim_{t\to T_*^-} \Psi(t) = +\infty. \tag{4.20}$$

From (4.5) and (4.20), we have $\lim_{t\to T_*^-} \|u(t)\|^2 = +\infty$.

This completes the proof of Theorem 4.1.

5. Conclusions

By applying Galerkin method and L^p-Sobolev logarithmic inequality, and combining with the potential well theory, we prove the global existence result of solutions in this paper. Namely, assume that p satisfies (1.5). If $u_0 \in W_0^{1,p}(\Omega)$, $u_1 \in L^2(\Omega)$ and $0 < \mathcal{E}(0) \leq M$, $\mathcal{K}(u_0) \geq 0$, then there is a global weak solution $u(x,t)$ of the problem (1.1)–(1.3). Meanwhile, under the condition of positive initial energy, by using the concavity analysis method, we establish the finite time blow-up result of solutions and give the lifespan estimate of solutions. The result read as follows: If $0 < \mathcal{E}(0) < d$ and $\int_\Omega u_0 u_1 dx > 0$, then the solutions of the problem (1.1)–(1.4) blows up in finite time and the lifespan T_* is estimated by $0 < T_* < \frac{4\Psi(0)}{(p-2)\Psi'(0)}$.

Acknowledgments

The authors would like to thank the reviewers and editors for their help to improve the quality of this article. Moreover, this research was supported by Natural Science Foundation of Zhejiang Province (No. LY17A010009).

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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