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Research article

Decision self-information based on parameterized fuzzy β neighborhood and its application in three-way multi-attribute group decision-making

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Abstract: As a special kind of entropy, decision self-information effectively considers the uncertainty information of both the lower and upper approximations. However, it is limited to rough binary relations, which limits its application to complex problems. In addition, parameterized fuzzy β covering, as an extension of the covering-based rough set model, can effectively characterize the similarity between samples. We combine decision self-information with a parameterized fuzzy β neighborhood to propose decision self-information in fuzzy environments, and we study its important properties. On this basis, a three-way multi-attribute group decision-making algorithm is established, and a practical problem is solved. The effectiveness of the proposed method is verified by experimental analysis.

Keywords: fuzzy β covering; self-information; three-way decision; multi-attribute decision-making

1. Introduction

Zakowski [1] first proposed the covering-based rough set model [2], which is a natural extension of the classical rough set model and an effective tool to deal with uncertain information. However, like traditional rough sets, covering-based rough sets deal with discrete attributes that belong or do not belong in a dataset, which limits their application in complex environments. To this end, Dubois and Prade [3] introduced the concept of fuzzy rough sets and extended rough set theory to the fuzzy environment, and scholars have proposed various improved fuzzy rough set models. Ma [4] defined two pairs of fuzzy approximation operators in the covering-based fuzzy approximation space, which show the properties and topological importance of the complementary neighborhood. D'eer et al. [5] discussed the relationship between various fuzzy covering-based fuzzy rough set models. Ma [6] proposed the concept of fuzzy β covering and fuzzy β neighborhoods. Zhan et al. [7] improved the fuzzy β neighborhood, proposed a covering-based variable-precision fuzzy rough set model, and applied it to multi-attribute decision-making. Zhang et al. [8] explained the fuzzy binary relation in fuzzy β approximation space from the perspective of pessimism and optimism, which makes up for the defect that the fuzzy β neighborhood operator cannot obtain the fuzzy binary relation between objects. However, the fuzzy β covering-based model proposed by Ma cannot guarantee that the lower approximation is included in the upper approximation. Subsequently, Zhang et al. [9] and Huang et al. [10] proposed a parameterized fuzzy β covering-based model that guarantees that the lower approximation is included in the upper approximation while reducing the influence of noisy data. Dai et al. [11,12] constructed four kinds of fuzzy β neighborhood operators with reflexivity by using fuzzy logic operators and used fuzzy β covering relations to describe the similarity between samples.

Information entropy [13] is another important and effective method to characterize information uncertainty, which is widely used in the fields of artificial intelligence, multi-attribute decision-making, attribute reduction, and information security. In recent years, information entropy has been combined with rough set theory in various types of entropy models [14,15]. Liao et al. [16] considering the scale diversity between different attributes, proposed a new uncertainty measure, which provides effective support for some decision-making constrained by test cost. Li et al [17]. proposed an uncertainty measurement method for fuzzy relational information systems, and gave an axiomatic definition of granularity measurement. Wang et al. [18–20] constructed various types of entropy according to different binary relations, among them a special form of entropy, decision self-information [21], which takes into account uncertainty information in both the lower and upper approximations. However, decision self-information is limited to rough binary relations, which limits its application to complex problems. We combine decision self-information with parameterized fuzzy β covering to enable its application in fuzzy environments.

In an increasingly complex social environment, multi-attribute decision-making problems are part of daily life. Traditional decision-making methods [22,23] are insufficient to solve complex uncertainty problems in real life, and many methods have been proposed [24–27]. Zhang et al. [28] constructed a reflexive fuzzy α neighborhood operator, proposed a fuzzy α rough set model based on the fuzzy neighborhood operator, and applied it to multi-attribute decision-making. Wang and Miao [29] proposed exponential hesitant fuzzy entropy and gave a hesitant fuzzy multi-attribute decision-making model based on the entropy weight method. Yao [30] proposed three ideas to solve complex and uncertain multi-attribute decision-making problems. In recent years, the three-way decision model has been successfully applied in various fields [31–34]. Zhang et al. [35] proposed a classification and ranking decision method based on three-way decision theory and the TOPSIS model. Ye et al. [36] established a three-way multi-attribute decision-making model in an incomplete environment. Zhang et al. [37] proposed a three-way decision-making model based on a utility function, and Zhan et al. [38] proposed a relative utility function and established a three-way multi-attribute decision-making model based on utility theory in incomplete fuzzy information systems. Decision research using behavioral theory is a hot topic recently, applying regret theory to multi-attribute decision making can reflect the risk attitude and psychological behavior of decision makers and improve the scientificity of decision making [39–41]. The above models have one thing in common: they involve only one decision-maker or multiple decision-makers that agree. However, due to different backgrounds, decision-making experience, and subjective preferences, the opinions of decision-makers may diverge and cannot be compromised. We select one of multiple decision-makers who is most suitable to make a decision.

We combine parameterized fuzzy β covering and decision self-information, propose decision self-information based on a parameterized fuzzy β neighborhood to determine the most suitable decision-maker, and propose a three-way multi-attribute group decision-making model based on a parameterized fuzzy β neighborhood. The classification and ranking results of all alternatives can be obtained. The effectiveness of the proposed method is experimentally verified.

2. Basic knowledge

2.1. Parameterized fuzzy β neighborhood

The parameterized fuzzy β covering [10], as an extension of the covering-based rough set model, can effectively characterize the similarity between samples.

Let $\mathbb{C} = \{C_1, C_2, \cdots, C_m\}$ be the fuzzy β covering group of $U, \beta \in [0,1]$, and let $(U, \mathbb{C})$ be a fuzzy β covering information list. If $\mathcal{P} \subseteq \mathbb{C}$, then for all $x \in U$, the fuzzy β neighborhood of x with regard to $\mathcal{P}$ is

$$\mathcal{N}_{\mathcal{P}}^{\beta}(x) = \cap\{K | K \in C, C \in \mathcal{P}, K(x) \geqslant \beta\}.$$

Given real numbers $\lambda \in [0,1]$and $x \in U$, the parameterized fuzzy β neighborhood is defined as

$$\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}(x)(y) = \begin{cases} 0, & \mathcal{N}_{\mathcal{P}}^{\beta}(x)(y) < \lambda; \\ \mathcal{N}_{\mathcal{P}}^{\beta}(x)(y), & \mathcal{N}_{\mathcal{P}}^{\beta}(x)(y) \geqslant \lambda; \end{cases}$$

where λ is the fuzzy β neighborhood radius.

Let $(U, \mathbb{C})$ be a fuzzy β covering information list, $\lambda \in [0,1]$, and $\mathcal{P} \subseteq \mathbb{C}$. Then for all $X \in \mathcal{F}(U)$, the lower and upper approximations of X are respectively

$$\underline{C}_{\mathcal{P}}^{\beta,\lambda}(X)(x) = \begin{cases} \bigwedge_{y \in U} \left\{ \left(1 - \mathcal{N}_{\mathcal{P}}^{\beta,\lambda}(x)(y)\right) \vee X(y) \right\}, & X(x) \geqslant 1 - \beta; \\ 0, & X(x) < 1 - \beta; \end{cases}$$

$$\bar{C}_{\mathcal{P}}^{\beta,\lambda}(X)(x) = \begin{cases} \bigvee_{y \in U} \left\{ \mathcal{N}_{\mathcal{P}}^{\beta,\lambda}(x)(y) \wedge X(y) \right\}, & X(x) \leqslant \beta; \\ 1, & X(x) > \beta. \end{cases}$$

2.2. Three-way decision-related theories

Based on the three-way decision model [30], Zhang [37] and Zhan et al. [38] proposed a three-way decision model using utility theory to improve classification accuracy.

Suppose the state set $\Omega = \{T, \neg T\}$ indicates that an object belongs to states T and $\neg T$. $\mathfrak{A} = \{a_P, a_B, a_N\}$ is an action set, where a_P, a_B, a_N represent acceptance, delay, and rejection, respectively. Table 1 gives the corresponding utility and relative utility of alternatives x_i in the two states of the three actions. u_{PP}, u_{BP}, and u_{NP} denote the utility of alternative x_i in taking actions a_P, a_B, and a_N, respectively, in T. Similarly, u_{PN}, u_{BN}, and u_{NN} denote the utility of alternative x_i in adopting a_P, a_B, and a_N, respectively, in $\neg T$.

The relative utility function can be understood as follows. When the utility of action a_P is used as the criterion and $x_i \in T$, $\tilde{u}_{PP}, \tilde{u}_{BP}$, and 0 are the relative utility functions of a_P, a_B, and a_N, respectively; similarly, when the utility of action a_N is used as the criterion and $x_i \in \neg T$, 0, $\tilde{u}_{BN}$, and $\tilde{u}_{NN}$ are the relative utility functions of a_P, a_B, and a_N, respectively; where$\tilde{u}_{PP} = u_{PP} - u_{NP}$, $\tilde{u}_{BP} = u_{BP} - u_{NP}$, $\tilde{u}_{BN} = u_{BN} - u_{PN}$, and $\tilde{u}_{NN} = u_{NN} - u_{PN}$.

Table 1. Two types of utility functions.

	Primitive utility function		Relative utility function	
	$T(P)$	$\neg T(N)$	$T(P)$	$\neg T(N)$
a_P	u_{PP}	u_{PN}	$\tilde{u}_{PP}$	0
a_B	u_{BP}	u_{BN}	$\tilde{u}_{BP}$	$\tilde{u}_{BN}$
a_N	u_{NP}	u_{NN}	0	$\tilde{u}_{NN}$

Suppose $[x]_R$ is a class of objects with respect to x induced by the binary relation R, and x is a conditional probability of T such that $Pr(T|[x]_R)$. Then, based on the relative utility function, the expected utility $\mathcal{U}(a_\diamond|x)(\diamond{=}P,B,N)$ of x can be calculated as

$$\mathcal{U}(a_\diamond|x) = \tilde{u}_{\diamond P} Pr(T|[x]_R) + \tilde{u}_{\diamond N} Pr(\neg T|[x]_R).$$

According to the Bayesian decision rule, the action with the greatest utility value should be selected, which leads to the following rule:

(P) if $\mathcal{U}(a_P|x) \geqslant \mathcal{U}(a_B|x)$ and $\mathcal{U}(a_P|x) \geqslant \mathcal{U}(a_N|x)$, then $x \in Pos(T)$;

(B) if $\mathcal{U}(a_B|x) \geqslant \mathcal{U}(a_P|x)$ and $\mathcal{U}(a_B|x) \geqslant \mathcal{U}(a_N|x)$, then $x \in Bnd(T)$;

(N) if $\mathcal{U}(a_N|x) \geqslant \mathcal{U}(a_P|x)$ and $\mathcal{U}(a_N|x) \geqslant \mathcal{U}(a_B|x)$, then $x \in Neg(T)$,

where $Pos(T)$, $Bnd(T)$, and $Neg(T)$, respectively, indicate the accepted domain, delayed domain, and rejected domain.

Based on $Pr(T|[x]_R) + Pr(\neg T|[x]_R) = 1$, $(P) - (N)$gives the following equivalent rules:

(P') if $Pr(T|x) \geqslant \alpha$ and $Pr(T|x) \geqslant \gamma$, then $x \in Pos(T)$;

(B') if $Pr(T|x) \leqslant \alpha$ and $Pr(T|x) \geqslant \beta$, then $x \in Bnd(T)$; and

(N') if $Pr(T|x) \leqslant \gamma$ and $Pr(T|x) \leqslant \beta$, then $x \in Neg(T)$,

where thresholds α, β, and γ can be expressed as:

$$\alpha = \frac{\tilde{u}_{BN}}{\tilde{u}_{BN} + (\tilde{u}_{PP} - \tilde{u}_{BP})},\ \beta = \frac{(\tilde{u}_{NN} - \tilde{u}_{BN})}{(\tilde{u}_{NN} - \tilde{u}_{BN}) + \tilde{u}_{BP}},\ \gamma = \frac{\tilde{u}_{NN}}{\tilde{u}_{NN} + \tilde{u}_{PP}}.$$

3. Decision self-information based on parameterized fuzzy β neighborhood

We improve decision self-information so that it can be applied in fuzzy environments based on the idea of decision self-information studied by Wang et al [21]. We use the upper and lower approximations based on parameterized fuzzy β neighborhoods to construct three decision indicators with different meanings to calculate the decision accuracy and roughness. Then four uncertainty measures are constructed, i.e., decision self-information based on parameterized fuzzy β neighborhoods, and we study their important properties.

Definition 1. Let $(U,\mathbb{C},D)$ be a fuzzy β covering decision information list. $\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}(x)$ is the fuzzy β neighborhood induced by $\mathcal{P}$ on U, $\lambda \in [0,1]$, $\mathcal{P} \subseteq \mathbb{C}$, and $T$is the target set obtained by the decision attribute. Then, for decision index $dec(T)$ of fuzzy set T, the decision index $cert_{\mathcal{P}}(T)$ is determined, and the possible decision index $poss_{\mathcal{P}}(T)$ is defined as:

$$dec(T) = |T|, cert_{\mathcal{P}}(T) = \left|\underline{C}_{\mathcal{P}}^{\beta,\lambda}(T)\right|, poss_{\mathcal{P}}(T) = \left|\bar{C}_{\mathcal{P}}^{\beta,\lambda}(T)\right|,$$

where $|.|$ represents the cardinality of the fuzzy set, $\underline{C}_{\mathcal{P}}^{\beta,\lambda}$ and $\bar{C}_{\mathcal{P}}^{\beta,\lambda}$ are the lower and upper approximations, respectively, constructed by $\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}(x)$. According to neighborhood fuzzy rough set theory, the definite decision index $cert_{\mathcal{P}}(T)$ of T is used as the lower approximation cardinality, indicating the degree of membership that the object definitely belongs to T; the possible decision index $poss_{\mathcal{P}}(T)$ of T is used as the upper approximation cardinality, indicating that the object may belong to T degrees of affiliation.

Proposition 3.1. $cert_{\mathcal{P}}(T) \leqslant dec(T) \leqslant poss_{\mathcal{P}}(T)$.

Proof. From $\underline{C}_{\mathcal{P}}^{\beta,\lambda}(T) \subseteq T \subseteq \bar{C}_{\mathcal{P}}^{\beta,\lambda}(T)$, we obtain $cert_{\mathcal{P}}(T) \leqslant dec(T) \leqslant poss_{\mathcal{P}}(T)$.

Proposition 3.2. *If* $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathbb{C}$, *then:*

(1) $cert_{\mathcal{P}_1}(T) \leqslant cert_{\mathcal{P}_2}(T)$;

(2) $poss_{\mathcal{P}_1}(T) \geqslant poss_{\mathcal{P}_2}(T)$.

Proof. (1) Since $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathbb{C}$, then $\mathcal{N}_{\mathcal{P}_1}^{\beta,\lambda}(x) \supseteq \mathcal{N}_{\mathcal{P}_2}^{\beta,\lambda}(x)$, and $1 - \mathcal{N}_{\mathcal{P}_1}^{\beta,\lambda}(x) \subseteq 1 - \mathcal{N}_{\mathcal{P}_2}^{\beta,\lambda}(x)$. From the structure of $\underline{C}_{\mathcal{P}}^{\beta,\lambda}(T)$ we obtain $\underline{C}_{\mathcal{P}_1}^{\beta,\lambda}(T) \subseteq \underline{C}_{\mathcal{P}_2}^{\beta,\lambda}(T)$ or $\left|\underline{C}_{\mathcal{P}_1}^{\beta,\lambda}(T)\right| \leqslant \left|\underline{C}_{\mathcal{P}_2}^{\beta,\lambda}(T)\right|$, and $cert_{\mathcal{P}_1}(T) \leqslant cert_{\mathcal{P}_2}(T)$.

(2) Since $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathbb{C}$, we obtain $[x]_{\mathcal{P}_1}^{\beta,\lambda} \supseteq [x]_{\mathcal{P}_2}^{\beta,\lambda}$. Furthermore, from the structure of $\bar{C}_{\mathcal{P}}^{\beta,\lambda}(T)$ we obtain $\bar{C}_{\mathcal{P}_1}^{\beta,\lambda}(T) \supseteq \bar{C}_{\mathcal{P}_2}^{\beta,\lambda}(T)$ or $\left|\bar{C}_{\mathcal{P}_1}^{\beta,\lambda}(T)\right| \geqslant \left|\bar{C}_{\mathcal{P}_2}^{\beta,\lambda}(T)\right|$, so we also obtain $poss_{\mathcal{P}_1}(T) \geqslant poss_{\mathcal{P}_2}(T)$.

Proposition 3.2 shows that both the definite and possible decision indexes are monotonic. As the number of attributes increases, the decision-making index increases, as does the decision-making consistency. As the number of attributes increases, the possible decision indicators decrease, and the decision uncertainty decreases.

Definition 2. Let $\mathcal{P} \subseteq \mathbb{C}$ and T be the target set obtained from the decision attribute. Then the accuracy $\alpha_{\mathcal{P}}^{1}(T)$ and roughness $\rho_{\mathcal{P}}^{1}(T)$ of the decision index are determined as

$$\alpha_{\mathcal{P}}^{1}(T) = \frac{cert_{\mathcal{P}}(T)}{dec(T)}, \rho_{\mathcal{P}}^{1}(T) = 1 - \frac{cert_{\mathcal{P}}(T)}{dec(T)}.$$

It is clear that by Proposition 3.1, $0 \leqslant \alpha_{\mathcal{P}}^{1}(T), \rho_{\mathcal{P}}^{1}(T) \leqslant 1$.

Proposition 3.3. *Let* $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathbb{C}$. *Then:*

(1) $\alpha_{\mathcal{P}_1}^{1}(T) \leqslant \alpha_{\mathcal{P}_2}^{1}(T)$,

(2) $\rho^1_{\mathcal{P}_1}(T) \geqslant \rho^1_{\mathcal{P}_2}(T)$.

Proof. (1) By Proposition 3.2, we know that $cert_{\mathcal{P}_1}(T) \leqslant cert_{\mathcal{P}_2}(T)$. Hence $\frac{cert_{\mathcal{P}_1}(T)}{dec(T)} \leqslant \frac{cert_{\mathcal{P}_2}(T)}{dec(T)}$, and therefore $\alpha^1_{\mathcal{P}_1}(T) \leqslant \alpha^1_{\mathcal{P}_2}(T)$.

(2) The proof is similar to that of (1).

Proposition 3.3 shows that the accuracy and roughness of the definite decision index are monotonic.

Definition 3. Let $(U,\mathbb{C},D)$ be a fuzzy β covering decision information list, $\mathcal{P} \subseteq \mathbb{C}$, and T the target set obtained from the decision attribute. Then the definite decision self-information definition of $(U,\mathbb{C},D)$ is

$$I^1_{\mathcal{P}}(T) = -\rho^1_{\mathcal{P}}(T)\ln\alpha^1_{\mathcal{P}}(T).$$

Proposition 3.4. *Let* $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathbb{C}$. *Then* $I^1_{\mathcal{P}_1}(T) \geqslant I^1_{\mathcal{P}_2}(T)$.

Proof. By Proposition 3.3, we know that $\alpha^1_{\mathcal{P}_1}(T) \leqslant \alpha^1_{\mathcal{P}_2}(T)$ and $\rho^1_{\mathcal{P}_1}(T) \geqslant \rho^1_{\mathcal{P}_2}(T)$. Therefore, $I^1_{\mathcal{P}_1}(T) \geqslant I^1_{\mathcal{P}_2}(T)$.

Definition 4. Let $\mathcal{P} \subseteq \mathbb{C}$ and T be the target set obtained from the decision attribute. Then the accuracy $\alpha^2_{\mathcal{P}}(T)$ and roughness $\rho^2_{\mathcal{P}}(T)$ of the possible decision index are

$$\alpha^2_{\mathcal{P}}(T) = \frac{dec(T)}{poss_{\mathcal{P}}(T)}, \rho^2_{\mathcal{P}}(T) = 1 - \frac{dec(T)}{poss_{\mathcal{P}}(T)}.$$

It is clear that by Proposition 3.1, $0 \leqslant \alpha^2_{\mathcal{P}}(T), \rho^2_{\mathcal{P}}(T) \leqslant 1$.

Proposition 3.5. *Let* $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathbb{C}$. *Then:*

(1) $\alpha^2_{\mathcal{P}_1}(T) \leqslant \alpha^2_{\mathcal{P}_2}(T)$;

(2) $\rho^2_{\mathcal{P}_1}(T) \geqslant \rho^2_{\mathcal{P}_2}(T)$.

Proof. The proof is similar to that of Proposition 3.3.

Proposition 3.5 shows that the accuracy and roughness of the possible decision index are monotonic.

Definition 5. Let $(U,\mathbb{C},D)$ be a fuzzy β covering decision information list, $\mathcal{P} \subseteq \mathbb{C}$, and T the target set obtained from the decision attribute. Then the possible decision self-information definition of $(U,\mathbb{C},D)$ is

$$I^2_{\mathcal{P}}(T) = -\rho^2_{\mathcal{P}}(T)\ln\alpha^2_{\mathcal{P}}(T).$$

Proposition 3.6. *Let* $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathbb{C}$. *Then* $I^2_{\mathcal{P}_1}(T) \geqslant I^2_{\mathcal{P}_2}(T)$.

Proof. By Proposition 3.5, we know that $\alpha^2_{\mathcal{P}_1}(T) \leqslant \alpha^2_{\mathcal{P}_2}(T)$ and $\rho^2_{\mathcal{P}_1}(T) \geqslant \rho^2_{\mathcal{P}_2}(T)$. Therefore, $I^2_{\mathcal{P}_1}(T) \geqslant I^2_{\mathcal{P}_2}(T)$.

Next, we propose another two types of decision self-information to characterize the uncertainty of fuzzy information, and we consider using both upper and lower approximation information to measure the uncertainty of the target concept.

Definition 6. Let $\mathcal{P} \subseteq \mathbb{C}$, and let T be the target set obtained from the decision attribute. Then the corresponding accuracy $\alpha^3_{\mathcal{P}}(T)$ and roughness $\rho^3_{\mathcal{P}}(T)$ of the decision index are:

$$\alpha^3_{\mathcal{P}}(T) = \frac{cert_{\mathcal{P}}(T)}{poss_{\mathcal{P}}(T)}, \rho^3_{\mathcal{P}}(T) = 1 - \frac{cert_{\mathcal{P}}(T)}{poss_{\mathcal{P}}(T)}.$$

It is clear that by Proposition 3.1, $0 \leqslant \alpha^3_{\mathcal{P}}(T), \rho^3_{\mathcal{P}}(T) \leqslant 1$.

Proposition 3.7. *Let* $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathbb{C}$. *Then:*

(1) $\alpha^3_{\mathcal{P}_1}(T) \leqslant \alpha^3_{\mathcal{P}_2}(T)$;

(2) $\rho^3_{\mathcal{P}_1}(T) \geqslant \rho^3_{\mathcal{P}_2}(T)$.

Proof. The proof is similar to that of Proposition 3.3.

Proposition 3.7 shows that the precision and roughness of relative decision indicators are monotonic.

Definition 7. Let $(U,\mathbb{C},D)$ be a fuzzy β covering decision information list, $\mathcal{P} \subseteq \mathbb{C}$, and T the target set obtained by the decision attribute. Then the relative decision self-information definition of $(U,\mathbb{C},D)$ is $I^3_{\mathcal{P}}(T) = -\rho^3_{\mathcal{P}}(T)\ln\alpha^3_{\mathcal{P}}(T)$.

Proposition 3.8. *Let* $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathbb{C}$. *Then* $I^3_{\mathcal{P}_1}(T) \geqslant I^3_{\mathcal{P}_2}(T)$.

Proof. By Proposition 3.7, we know that $\alpha^3_{\mathcal{P}_1}(T) \leqslant \alpha^3_{\mathcal{P}_2}(T)$ and $\rho^3_{\mathcal{P}_1}(T) \geqslant \rho^3_{\mathcal{P}_2}(T)$. Therefore, $I^3_{\mathcal{P}_1}(T) \geqslant I^3_{\mathcal{P}_2}(T)$.

Example 1. Suppose there is a fuzzy β covering information list $(U,\mathbb{C},D)$, where $U = \{x_1,x_2,x_3,x_4,x_5\}$, $\mathbb{C} = \{C_1,C_2,C_3,C_4\}$, $D = \{T_1,T_2\}$, with data as shown in Table 2. Let $\beta = 0.6$, $\lambda = 0.3$. According to the fuzzy β covering decision information list $(U,\mathbb{C},D)$, the parameterized fuzzy β domain is obtained, as shown in Table 3.

Table 2. Fuzzy β covering decision information table $(U,\mathbb{C},D)$.

	C_1	C_2	C_3	C_4	T_1	T_2
x_1	0.6	0.6	0.55	0.51	0.47	0.46
x_2	0.5	0.5	0.6	0.61	0.62	0.65
x_3	0.63	0.6	0.58	0.73	0.64	0.33
x_4	0.52	0.8	0.8	0.55	0.69	0.53
x_5	0.56	0.4	0.43	0.6	0.39	0.48

Table 3. Parameterized fuzzy β neighborhood.

$\mathcal{N}^{\beta,\lambda}_{\mathbb{C}}(x_i)/U$	x_1	x_2	x_3	x_4	x_5
$\mathcal{N}^{\beta,\lambda}_{\mathbb{C}}(x_1)$	0.6	0.51	0.51	0.55	0.51
$\mathcal{N}^{\beta,\lambda}_{\mathbb{C}}(x_2)$	0.5	0.6	0.5	0.5	0.61
$\mathcal{N}^{\beta,\lambda}_{\mathbb{C}}(x_3)$	0.6	0.58	0.6	0.58	0.73
$\mathcal{N}^{\beta,\lambda}_{\mathbb{C}}(x_4)$	0.52	0.55	0.52	0.8	0.55
$\mathcal{N}^{\beta,\lambda}_{\mathbb{C}}(x_5)$	0.4	0.43	0.4	0.4	0.6

Furthermore, we can obtain the upper and lower approximations of the parameterized fuzzy β neighborhood:

$$\underline{C}^{\beta,\lambda}_{\mathbb{C}}(T_1) = \frac{0.47}{x_1} + \frac{0.62}{x_2} + \frac{0.64}{x_3} + \frac{0.69}{x_4} + \frac{0}{x_5},$$

$$\bar{C}^{\beta,\lambda}_{\mathbb{C}}(T_1) = \frac{0.47}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{0.39}{x_5},$$

$$\underline{C}^{\beta,\lambda}_{\mathbb{C}}(T_2) = \frac{0.46}{x_1} + \frac{0.65}{x_2} + \frac{0}{x_3} + \frac{0.53}{x_4} + \frac{0.48}{x_5},$$

$$\bar{C}_{\mathbb{C}}^{\beta,\lambda}(T_2) = \frac{0.46}{x_1} + \frac{1}{x_2} + \frac{0.33}{x_3} + \frac{0.53}{x_4} + \frac{0.48}{x_5}.$$

Therefore, we can obtain the decision index $dec(T_i)$ of $T_i (i=1,2)$, and determine the decision index $cert_{\mathbb{C}}(T_i)$ and possible decision index $poss_{\mathbb{C}}(T_i)$. The results are shown in Table 4. Furthermore, we can obtain the values of three kinds of decision self-information of T_i, as shown in Table 5.

Table 4. Values of each decision index.

	T_1	T_2
$dec(T_i)$	2.81	2.45
$cert_{\mathbb{C}}(T_i)$	2.42	2.12
$poss_{\mathbb{C}}(T_i)$	3.86	2.8

Table 5. Three decision self-information of T_i.

	T_1	T_2
$I_{\mathbb{C}}^1(T_i)$	0.0207	0.0195
$I_{\mathbb{C}}^2(T_i)$	0.0864	0.0167
$I_{\mathbb{C}}^3(T_i)$	0.1742	0.0676

4. Decision self-information based on parameterized fuzzy β neighborhood

4.1. Parameterized fuzzy β neighborhood class

Next, we construct the parameterized fuzzy β neighborhood class and convert it to the classic set.

Definition 8. Let $(U,\mathbb{C})$ be a fuzzy β covering information list, $\mathcal{P} \subseteq \mathbb{C}$, and let $\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}(x)$ be parameterized fuzzy β neighborhoods. Then the parameterized fuzzy β neighborhood class is

$$[x]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}} = \left\{y \in U | \mathcal{N}_{\mathcal{P}}^{\beta,\lambda}(x)(y) \geqslant \beta\right\}.$$

Proposition 4.1. *Let $(U,\mathbb{C})$ be a fuzzy β covering information list, $\mathcal{P} \subseteq \mathbb{C}$, and let $[x]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}$ be parameterized fuzzy β neighborhood classes. If $\lambda \leqslant \beta$, then*

(1) for any $x \in U$, $x \in [x]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}$;

(2) $\cup_{x \in U} [x]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}} = U$.

Proof. (1) From the definition of the fuzzy β neighborhood $\mathcal{N}_{\mathcal{P}}^{\beta}(x)$, we know that $\mathcal{N}_{\mathcal{P}}^{\beta}(x)(x) \geqslant \beta$. When $\lambda \leqslant \beta$ and $\mathcal{N}_{\mathcal{P}}^{\beta}(x)(x) \geqslant \beta \geqslant \lambda$, then $\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}(x)(x) = \mathcal{N}_{\mathcal{P}}^{\beta}(x)(x) \geqslant \beta$, and therefore $x \in [x]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}$.

(2) It is clear from (1).

Proposition 4.1 shows that the parameterized fuzzy β neighborhood class $[x]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}$ is reflexive when $\lambda \leqslant \beta$, and the union of the parameterized fuzzy β neighborhood classes for all objects can cover the domain of discourse U.

4.2. Construction of three-way decision-making model

Next, we construct conditional probabilities based on parameterized fuzzy β neighborhood classes, and establish a three-way decision model.

Definition 9. Let $(U,\mathbb{C},D)$ be a fuzzy β covering decision information list, $\mathcal{P} \subseteq \mathbb{C}$, let $[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}$ be parameterized fuzzy β neighborhood classes, and let $T(x_i)$ be a decision attribute value of target $x_i \in U$. Then the conditional probability of target x_i is

$$Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) = \frac{\sum_{x_j \in [x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}} T(x_j)}{\left|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right|}.$$

Conditional probability $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right)$ shows that target x_i in $[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}$ belongs to the probability of the target set T.

Proposition 4.2. *Let* $\neg T$ *be the complement of fuzzy set* T *. Then* $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) + Pr\left(\neg T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) = 1$.

Proof. Since $\neg T$ is the complement of the fuzzy set T, then $\forall x_i \in U$, $\neg T(x_i) = 1 - T(x_i)$. Therefore,

$$Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) + Pr\left(\neg T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) = \frac{\sum_{x_j \in [x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}} T(x_j)}{\left|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right|} + \frac{\sum_{x_j \in [x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}} \neg T(x_j)}{\left|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right|}$$

$$= \frac{\sum_{x_j \in [x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}} \left(T(x_j) + \neg T(x_j)\right)}{\left|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right|} = \frac{\left|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right|}{\left|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right|} = 1.$$

Example 2. (continued from Example 1). The parameterized fuzzy β neighborhood class can be obtained from the parameterized fuzzy β neighborhood in Table 3:

$$[x_1]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_1\}, [x_2]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_2,x_5\}, [x_3]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_1,x_3,x_5\}, [x_4]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_4\}, [x_5]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_5\}.$$

Furthermore, as an example, we can calculate the conditional probability of T_2:

$$Pr\left(T_2|[x_1]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}}\right) = 0.46, Pr\left(T_2|[x_2]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}}\right) = 0.57, Pr\left(T_2|[x_3]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}}\right) = 0.42, Pr\left(T_2|[x_4]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}}\right) = 0.53, Pr\left(T_2|[x_5]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}}\right) = 0.48.$$

According to the relative utility function studied by Zhan et al. [38], the standard deviation of the utility values of all alternatives given by the decision-maker is used to measure the dispersion of the decision-maker's preference:

$$\phi = \sqrt{(\sum_{i=1}^{n}(T(x_i) - \bar{T})^2)/(n-1)},$$

where $\bar{T} = (\sum_{i=1}^{n} T(x_i))/n$ is the average of the utility values of all the alternatives below state T. The larger the value of ϕ, the better the decision-maker's ability to distinguish all alternatives, i.e., the greater the priority difference. According to the utility value $T(x_i)$, taking into account the priority difference, we calculate the relative utility function of taking action a_P in state T,

$$\hat{u}_{PP}^{i} = \begin{cases} (T(x_i))^{1-\phi}, & T(x_i) \geqslant \bar{T}; \\ (T(x_i))^{1/(1-\phi)}, & T(x_i) < \bar{T}. \end{cases}$$

Similarly, we calculate the relative utility function of taking action a_N in state $\neg T$:

$$\hat{u}_{NN}^{i} = \begin{cases} (1-T(x_i))^{1-\phi}, & T(x_i) \leqslant \bar{T}; \\ (1-T(x_i))^{1/(1-\phi)}, & T(x_i) > \bar{T}. \end{cases}$$

A risk coefficient, $\sigma \in (0.5,1]$, is introduced to calculate the relative utility function of adopted behavior a_B under different states, i.e., $\hat{u}_{BP}^{i} = \sigma\hat{u}_{PP}^{i}$ and $\hat{u}_{BN}^{i} = \sigma\hat{u}_{NN}^{i}$.

According to the relative utility function [38] and our constructed conditional probability, the expected utility values of three behaviors of all objects x_i are calculated as:

$$\mathcal{U}(a_P|x_i) = \hat{u}_{PP}^{i} Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) + \hat{u}_{PN}^{i} Pr\left(\neg T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right),$$

$$\mathcal{U}(a_B|x_i) = \hat{u}_{BP}^{i} Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) + \hat{u}_{BN}^{i} Pr\left(\neg T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right),$$

$$\mathcal{U}(a_N|x_i) = \hat{u}_{NP}^{i} Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) + \hat{u}_{NN}^{i} Pr\left(\neg T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right).$$

Since $\hat{u}_{PN}^{i} = 0$ and $\hat{u}_{NP}^{i} = 0$, we can simplify these to:

$$\mathcal{U}(a_P|x_i) = \hat{u}_{PP}^{i} Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right),$$

$$\mathcal{U}(a_B|x_i) = \hat{u}_{BP}^{i} Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) + \hat{u}_{BN}^{i} Pr\left(\neg T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right),$$

$$\mathcal{U}(a_N|x_i) = \hat{u}_{NN}^{i} Pr\left(\neg T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right).$$

According to the Bayesian decision rule, the action with the greatest utility value should be selected, so the following three decision rules can be obtained:

(P) if $\mathcal{U}(a_P|x_i) \geqslant \mathcal{U}(a_B|x_i)$ and $\mathcal{U}(a_P|x_i) \geqslant \mathcal{U}(a_N|x_i)$, then $x_i \in Pos(T)$;

(B) if $\mathcal{U}(a_P|x_i) \geqslant \mathcal{U}(a_B|x_i)$ and $\mathcal{U}(a_B|x_i) \geqslant \mathcal{U}(a_N|x_i)$, then $x_i \in Bnd(T)$;

(N) if $\mathcal{U}(a_N|x_i) \geqslant \mathcal{U}(a_P|x_i)$ and $\mathcal{U}(a_N|x_i) \geqslant \mathcal{U}(a_B|x_i)$,then $x_i \in Neg(T)$,

where $Pos(T)$, $Bnd(T)$, and $Neg(T)$ indicate the accepted, delayed, and rejected domain, respectively.

According to Proposition 4.2, $(P) - (N)$ is equivalent to the following rule:

$(P1)$ if $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) \geqslant \hat{\alpha}_i$ and $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) \geqslant \hat{\gamma}_i$, then $x_i \in Pos(T)$;

$(B1)$ if $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) \leqslant \hat{\alpha}_i$ and $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) \geqslant \hat{\beta}_i$, then $x_i \in Bnd(T)$;

$(N1)$ if $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) \leqslant \hat{\gamma}_i$ and $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) \leqslant \hat{\beta}_i$,then $x_i \in Neg(T)$,

where thresholds $\hat{\alpha}_i$, $\hat{\beta}_i$, and $\hat{\gamma}_i$ can be calculated as

$$\hat{\alpha}_i = \frac{\hat{u}_{BN}^i}{\hat{u}_{BN}^i + \left(\hat{u}_{PP}^i - \hat{u}_{BP}^i\right)}, \hat{\beta}_i = \frac{\hat{u}_{NN}^i - \hat{u}_{BN}^i}{\left(\hat{u}_{NN}^i - \tilde{u}_{BN}^i\right) + \hat{u}_{BP}^i}, \hat{\gamma}_i = \frac{\hat{u}_{NN}^i}{\hat{u}_{NN}^i + \hat{u}_{PP}^i}.$$

If $\sigma \in (0.5,1]$, then $\tilde{\beta}_i < \tilde{\gamma}_i < \tilde{\alpha}_i$, and $(P1)-(N1)$ can be simplified as follows:

$(P2)$ if $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) \geqslant \hat{\alpha}_i$, then $x_i \in Pos(T)$;

$(B2)$ if $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) \leqslant \hat{\alpha}_i$ and $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) \geqslant \hat{\beta}_i$, then $x_i \in Bnd(T)$;

$(N2)$ if $Pr\left(T|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) \leqslant \hat{\beta}_i$, then $x_i \in Neg(T)$.

Example 3. (continued from Example 2). Letting $\sigma = 0.6$, we calculate all relative utility function values, as shown in Table 6. It is further possible to calculate thresholds $\hat{\alpha}_i$ and $\hat{\beta}_i$. Some important results can be seen in Table 7.

Table 6. Relative utility functions of all objects.

	$\hat{u}_{PP}^i$	$\hat{u}_{BP}^i$	$\hat{u}_{NP}^i$	$\hat{u}_{PN}^i$	$\hat{u}_{BN}^i$	$\hat{u}_{NN}^i$
x_1	0.4154	0.2493	0	0	0.3480	0.5800
x_2	0.6833	0.4100	0	0	0.1830	0.3050
x_3	0.2853	0.1712	0	0	0.4211	0.7019
x_4	0.5705	0.3423	0	0	0.2554	0.4257
x_5	0.4359	0.2616	0	0	0.3366	0.5610

Table 7. Conditional probability and two thresholds.

	$Pr\left(T_2|[x_i]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}}\right)$	$\hat{\alpha}_i$	$\hat{\beta}_i$
x_1	0.46	0.6768	0.4821
x_2	0.57	0.4010	0.2293
x_3	0.42	0.7868	0.6212
x_4	0.53	0.5281	0.3322
x_5	0.48	0.6587	0.4618

According to the decision rule, $(P2)-(N2)$ lets us attain all of the final decision behaviors of the targets:

$$Pos(T_2) = \{x_2, x_4\}, Bnd(T_2) = \{x_5\}, Neg(T_2) = \{x_1, x_3\}.$$

5. Three-way multi-attribute group decision-making model based on parameterized fuzzy β neighborhood

We consider that the relative decision self-information $I_{\mathcal{P}}^3(T)$ contains both upper and lower approximation information. Hence, we build a three-way multi-attribute group decision model based on relative decision self-information to solve real-life problems.

5.1. Problem statement

In the real world, the uncertainty and complexity of the social environment bring certain difficulties to decision-makers, and an important decision can require multiple decision-makers, whose evaluations can differ due to their knowledge, experience, and subjective factors. When they cannot reach an agreement, we need to choose the most suitable decision-maker. The parameterized fuzzy β covering, as an extension of the covering-based rough set model, provides an effective method to deal with uncertain information. We establish a three-way multi-attribute decision-making model based on parameterized fuzzy β neighborhoods to solve the uncertain multi-attribute decision-making problem in the real world when multiple decision-makers disagree.

The parameterized fuzzy β neighborhoods of all alternatives are obtained based on fuzzy β covering decision information list $(U,\mathbb{C},D)$, and the upper and lower approximations of all decision-makers are further obtained. We use relative decision self-information to measure the uncertainty of all decision-makers and select the one with the smallest entropy value. We construct conditional probabilities using parameterized fuzzy β neighborhoods and use this to further revise the decision-maker's decision preference. We calculate the relative utility function values of all the alternatives. Using classification rule $(P2)-(N2)$ and comparing the magnitude between the conditional probability and thresholds $\hat{\alpha}_i$ and $\hat{\beta}_i$, we determine the final decision action for each alternative.

Finally, we can calculate the expected utility value of all the alternatives to take the final decision action,

$$EU(x_i)=\begin{cases}\mathcal{U}(a_P|x_i), & x_i\in Pos(T);\\ \mathcal{U}(a_B|x_i), & x_i\in Bnd(T);\\ \mathcal{U}(a_N|x_i), & x_i\in Neg(T).\end{cases}$$

All alternatives can be sorted according to the expected utility values and priorities of the three domains. We sort according to the expected utility value of each domain, $x_i,x_j\in Pos(T)$ and $\mathcal{U}(a_P|x_i)>\mathcal{U}(a_P|x_j)$; then $x_i\succ x_j$. Then we consider the priority of each domain as
$Pos(T)\succ Bnd(T)\succ Neg(T)$.

3.2. Three-way multi-attribute group decision-making algorithm based on parameterized fuzzy neighborhood

According to the above properties and decision rules, we can obtain a three-way multi-attribute group decision-making algorithm based on parameterized fuzzy neighborhood.

Input: Fuzzy β covering decision information list $(U,\mathbb{C},D)$, evaluation of all alternatives by l decision-makers $D=\{T_1,T_2,\cdots,T_l\}$ and λ

Output: The most suitable decision-maker, and the classification and ranking of each alternative

Step 1 The decision information list $(U,\mathbb{C},D)$ is covered by fuzzy β, and the parameterized fuzzy β neighborhoods $\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_i)$ of all alternatives are calculated;

Step 2 Calculate the lower approximation $\underline{C}_{\mathbb{C}}^{\beta,\lambda}(T_j)$ and upper approximation $\bar{C}_{\mathbb{C}}^{\beta,\lambda}(T_j)$ based on the neighborhood of parameterized fuzzy β for all decision-makers, where $(j=1,2,\cdots,l)$;

Step 3 Calculate the decision self-information $I_{\mathbb{C}}^{3}(T_j)$ of all decision-makers;

Step 4 Find the smallest value of the decision self-information $I_{\mathbb{C}}^{3}(T_k)=\min\{I_{\mathbb{C}}^{3}(T_1),I_{\mathbb{C}}^{3}(T_2),\cdots,I_{\mathbb{C}}^{3}(T_l)\}$, and then the most suitable decision-maker is T_k;

Step 5 According to Definitions 8 and 9, calculate the conditional probabilities of each alternative, $Pr\left(T_k|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right)$;

Step 6 Calculate the relative utility function values and thresholds $\hat{\alpha}_i$ and $\hat{\beta}_i$ for all alternatives from the relative utility function in Section 3.2;

Step 7 According to the decision rule, $(P2)-(N2)$ obtains the domain corresponding to the final decision behavior of all alternatives;

Step 8 Calculate expected utility value $EU(x_i)$ of all alternatives;

Step 9 Compare the priorities of $Pos(T_k) \succ Bnd(T_k) \succ Neg(T_k)$ and the expected utility values of the alternatives in each domain to rank all the alternatives.

The pseudo-code program is as follows:

Algorithm 1: Three-way multi-attribute group decision-making algorithm based on parameterized fuzzy neighborhood

Input: $\lambda,\beta,(U,\mathbb{C},D),D=\{T_1,T_2,\cdots,T_l\}$

Output: The most suitable decision-maker, and the classification and ranking of each alternative

$n \leftarrow |U|; l \leftarrow |T|$

for $i = 1 \rightarrow n$

  for $k = 1 \rightarrow n$

    if $\mathcal{N}_{\mathbb{C}}^{\beta}(x_i)(x_k) \geqslant \lambda$ then $\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_i)(x_k) = \mathcal{N}_{\mathbb{C}}^{\beta}(x_i)(x_k)$

else $\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_i)(x_k) = 0$

Cycle calculate

$\underline{C}_{\mathbb{C}}^{\beta,\lambda}(T_j)(x_i) and \bar{C}_{\mathbb{C}}^{\beta,\lambda}(T_j)(x_i)$

Calculate $I_{\mathbb{C}}^{3}(T_j)$

$T_k \leftarrow \min_i\{I_{\mathbb{C}}^{3}(T_i)\}$

Cycle calculate

  $Pr\left(T_j|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right) and Pr\left(\neg T_j|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right)$

  $\mathcal{U}(a_P|x_i), \mathcal{U}(a_B|x_i) and \mathcal{U}(a_N|x_i)$

  calculate the threshold $\hat{\alpha}_i and \hat{\beta}_i$

Determine $x_i \in Pos(T_j), x_i \in Bnd(T_j)$ or $x_i \in Neg(T_j)$

Calculate $EU(x_i)$

Compare $Pos(T_j) \succ Bnd(T_j) \succ Neg(T_j)$ and the expected utility values of the alternatives in each domain to rank all the alternatives.

Return

The time complexity of calculating the neighborhoods $\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_i)$ of all alternatives is $O(n^2)$, the time complexity of calculating the lower approximation $\underline{C}_{\mathbb{C}}^{\beta,\lambda}(T_j)$ and upper approximation $\bar{C}_{\mathbb{C}}^{\beta,\lambda}(T_j)$ is $O(n^2 \times l)$, the time complexity of calculating the decision self-information $I_{\mathbb{C}}^{3}(T_j)$ of all decision-makers is $O(n \times l)$, the time complexity of finding the smallest value of the decision self-information $I_{\mathbb{C}}^{3}(T_k)$ is $O(n \times l)$, the time complexity of calculating the conditional probabilities of each alternative $Pr\left(T_k|[x_i]_{\mathcal{N}_{\mathcal{P}}^{\beta,\lambda}}\right)$ and the relative utility function values and thresholds $\hat{\alpha}_i$ and $\hat{\beta}_i$ for all alternatives is $O(n^2 \times l)$, the time complexity of calculating the domain corresponding to the final decision behavior of all alternatives and expected utility value $EU(x_i)$ of all alternatives is $O(n \times l)$, So the total time complexity of Algorithm 1 is $O(n^2 \times l)$.

5.3. Numerical example

We use examples from the literature [25] to verify the effectiveness of the proposed method.

Example 4. An investment company intends to select some projects for investment, and decision-makers make choices based on the benefits that each project can bring. There are eight investment projects $U = \{x_1, x_2, \cdots, x_8\}$, which the company considers from five aspects $\mathbb{C} = \{C_1, C_2, C_3, C_4, C_5\}$, which represent expected benefits, environmental factors, market saturation, social benefits, and energy conservation. C_2 and C_3 are cost attributes, and the rest are benefit attributes. The attribute weight $W = \{0.3, 0.1, 0.3, 0.2, 0.1\}$ is transformed to the evaluation result of the benefit standard, as shown in Table 8. Three experts are evaluating these eight projects, with results as shown in Table 9.

Table 8. Attribute evaluation table of each investment project.

	C_1	C_2	C_3	C_4	C_5
x_1	0.8	0.6	0.7	0.8	0.9
x_2	0.9	0.5	0.5	0.7	0.6
x_3	0.3	0.6	0.4	0.4	0.3
x_4	0.5	0.8	0.8	0.7	0.6
x_5	0.7	0.4	0.4	0.5	0.8
x_6	0.4	0.2	0.3	0.7	0.3
x_7	0.9	0.5	0.9	0.8	0.7
x_8	0.6	0.2	0.2	0.3	0.4

Table 9. Assessment of eight projects by three experts.

	T_1	T_2	T_3
x_1	0.76	0.48	0.6
x_2	0.67	0.45	0.65
x_3	0.38	0.10	0.33
x_4	0.67	0.42	0.78
x_5	0.55	0.32	0.48
x_6	0.4	0.23	0.38
x_7	0.82	0.68	0.64
x_8	0.36	0.19	0.34

We obtain the parameterized fuzzy β neighborhoods of all alternatives based on the fuzzy β coverage decision information list $(U, \mathbb{C}, D)$. Let $\beta = 0.6, \lambda = 0.3$, as shown in Table 10.

Then we can obtain the lower and upper approximations of the three experts based on the parameterized fuzzy β neighborhood:

$$\underline{C}_{\mathbb{C}}^{\beta,\lambda}(T_1) = \frac{0.76}{x_1} + \frac{0.67}{x_2} + \frac{0}{x_3} + \frac{0.67}{x_4} + \frac{0.55}{x_5} + \frac{0.4}{x_6} + \frac{0.82}{x_7} + \frac{0}{x_8},$$

$$\bar{C}_{\mathbb{C}}^{\beta,\lambda}(T_1) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{0.38}{x_3} + \frac{1}{x_4} + \frac{0.55}{x_5} + \frac{0.4}{x_6} + \frac{1}{x_7} + \frac{0.36}{x_8},$$

$$\underline{C}_{\mathbb{C}}^{\beta,\lambda}(T_2) = \frac{0.48}{x_1} + \frac{0.45}{x_2} + \frac{0}{x_3} + \frac{0.42}{x_4} + \frac{0}{x_5} + \frac{0}{x_6} + \frac{0.68}{x_7} + \frac{0}{x_8},$$

$$\bar{C}_{\mathbb{C}}^{\beta,\lambda}(T_2) = \frac{0.48}{x_1} + \frac{0.45}{x_2} + \frac{0.1}{x_3} + \frac{0.42}{x_4} + \frac{0.32}{x_5} + \frac{0.23}{x_6} + \frac{1}{x_7} + \frac{0.19}{x_8},$$

$$\underline{C}_{\mathbb{C}}^{\beta,\lambda}(T_3) = \frac{0.6}{x_1} + \frac{0.65}{x_2} + \frac{0}{x_3} + \frac{0.78}{x_4} + \frac{0.48}{x_5} + \frac{0}{x_6} + \frac{0.64}{x_7} + \frac{0}{x_8},$$

$$\bar{C}_{\mathbb{C}}^{\beta,\lambda}(T_3) = \frac{0.6}{x_1} + \frac{1}{x_2} + \frac{0.33}{x_3} + \frac{1}{x_4} + \frac{0.48}{x_5} + \frac{0.38}{x_6} + \frac{1}{x_7} + \frac{0.34}{x_8}.$$

The relative decision self-information of the three experts is calculated as:

$$I_{\mathbb{C}}^3(T_1) = 0.1233, I_{\mathbb{C}}^3(T_2) = 0.1644, I_{\mathbb{C}}^3(T_3) = 0.1882.$$

From this, we obtain $I_{\mathbb{C}}^3(T_1) < I_{\mathbb{C}}^3(T_2) < I_{\mathbb{C}}^3(T_3)$, from which we see that the most suitable expert is T_1. We can get the parameterized fuzzy β neighborhood class from the table as:

$$[x_1]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_1\}, [x_2]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_1,x_2,x_7\}, [x_3]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_1,x_2,x_3,x_4\}, [x_4]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_1,x_2,x_4\},$$

$$[x_5]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_1,x_2,x_5,x_7\}, [x_6]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_1,x_2,x_4,x_6,x_7\}, [x_7]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_1,x_7\},$$

$$[x_8]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}} = \{x_1,x_2,x_5,x_7,x_8\}.$$

Table 10. Parameterized fuzzy β neighborhood of all investment projects.

$\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_i)/U$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_1)$	0.6	0.8	0.6	0.6	0.8	0.8	0.7	0.8
$\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_2)$	0.5	0.6	0.5	0.5	0.6	0.7	0.5	0.9
$\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_3)$	0.3	0.3	0.6	0.3	0.3	0.4	0.3	0.3
$\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_4)$	0.5	0.5	0.8	0.6	0.5	0.7	0.5	0.5
$\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_5)$	0.4	0.5	0.4	0.4	0.7	0.5	0.4	0.7
$\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_6)$	0	0.3	0	0	0.3	0.7	0.3	0.4
$\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_7)$	0.5	0.7	0.5	0.5	0.7	0.8	0.7	0.9
$\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}(x_8)$	0	0.3	0	0	0.4	0.3	0	0.6

From the relative utility function and the decision preference of T_1, two thresholds and conditional probabilities can be obtained, as shown in Table 11.

To more intuitively show the relationship between the conditional probability and the threshold, we show a comparison chart between them, as shown in Figure 1. From the decision rule $(P2) - (N2)$, the final decision classification result of expert T_1 can be obtained as:

$$Pos(T_1) = \{x_1,x_2,x_4,x_5,x_7\}, Bnd(T_1) = \{x_3,x_6,x_8\}, Neg(T_1) = \emptyset.$$

The expected utility of all investment projects can then be calculated, as shown in Figure 2, from which a complete ranking can be obtained: $x_7 \succ x_1 \succ x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_8$.

The company can make decisions on which projects to invest in based on the final decision classification and ranking results of expert T_1.

Table 11. Conditional probabilities and thresholds of each project.

	$Pr\left(T_1 \mid [x_i]_{\mathcal{N}_{\mathbb{C}}^{\beta,\lambda}}\right)$	$\hat{\alpha}_i$	$\hat{\beta}_i$
x_1	0.76	0.2477	0.1277
x_2	0.75	0.3500	0.1931
x_3	0.62	0.7675	0.5947
x_4	0.7	0.3500	0.1931
x_5	0.7	0.6178	0.4181
x_6	0.66	0.7511	0.5729
x_7	0.79	0.1789	0.0883
x_8	0.63	0.7835	0.6166

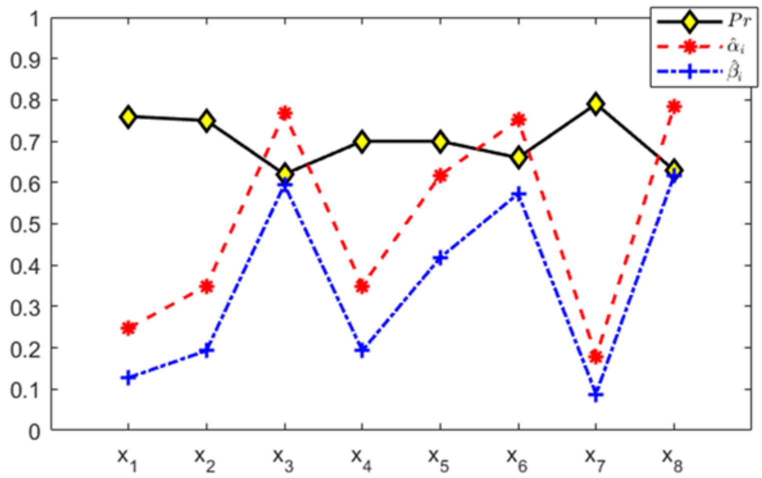

Figure 1. Comparison of conditional probabilities with two thresholds.

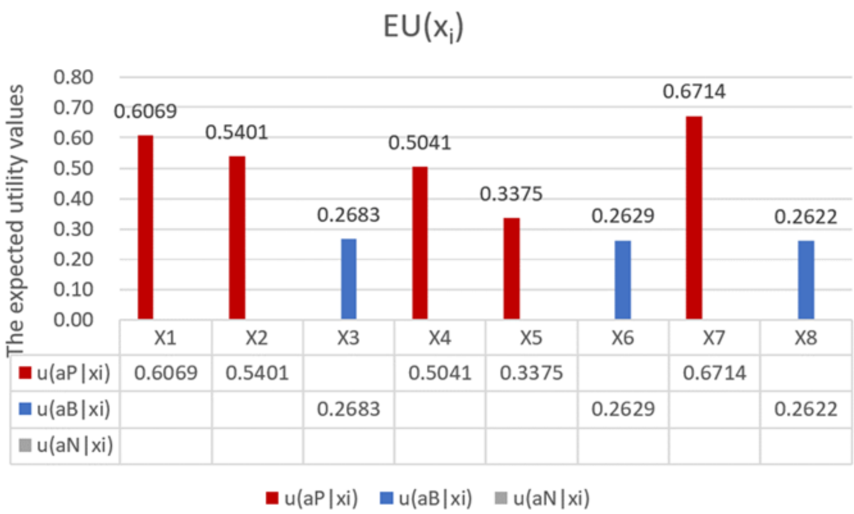

Figure 2. Expected utility values of all investment projects.

5.4. Experiment analysis

To illustrate the effectiveness of our method, we compare it with state-of-the-art and traditional decision-making methods, i.e., the methods of Zhan et al. [38], Ye et al. [34], and Zhang et al. [35], the TOPSIS method [23], and the WAA operator method [22]. The classifications and ranking results of different methods are shown in Tables 12 and 13.

Table 12. Classification results of different methods.

	$Pos(T)$	$Bnd(T)$	$Neg(T)$
Our method	$\{x_1,x_2,x_4,x_5,x_7\}$	$\{x_3,x_6,x_8\}$	$\emptyset$
Zhan et al.'s method	$\{x_1,x_2,x_4,x_7\}$	$\{x_3,x_5,x_6\}$	$\{x_8\}$
Ye et al.'s method	$\{x_1,x_2,x_4,x_5,x_7\}$	$\{x_3,x_6,x_8\}$	$\emptyset$
Zhang et al.'s method	$\{x_7\}$	$\{x_1,x_2,x_4\}$	$\{x_3,x_5,x_6,x_8\}$

Table 13. Ranking results of different methods.

		Ranking	Optimal
Our method	T_1	$x_7 \succ x_1 \succ x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_8$	x_7
	T_2	$x_7 \succ x_4 \succ x_2 \succ x_1 \succ x_3 \succ x_8 \succ x_6 \succ x_5$	x_7
	T_3	$x_4 \succ x_2 \succ x_7 \succ x_1 \succ x_5 \succ x_6 \succ x_8 \succ x_3$	x_4
Zhan et al.'s method		$x_7 \succ x_1 \succ x_2 \succ x_4 \succ x_3 \succ x_6 \succ x_5 \succ x_8$	x_7
Ye et al.'s method		$x_7 \succ x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3 \succ x_8 \succ x_6$	x_7
Zhang et al.'s method		$x_7 \succ x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_6 \succ x_8 \succ x_3$	x_7
TOPSIS method		$x_7 \succ x_1 \succ x_2 \succ x_4 \succ x_5 \succ x_6 \succ x_8 \succ x_3$	x_7
WAA operator method		$x_7 \succ x_1 \succ x_2 \approx x_4 \succ x_5 \succ x_6 \succ x_3 \succ x_8$	x_7

Table 13 includes the ranking results of experts $T_i(i=1,2,3)$. It can be found that the results of expert T_1 are most similar to those of other methods, and the optimal objects are all x_7, while the optimal results of expert T_3 are x_4, indicating that experts T_3 and T_1 are different. By the method in this paper, expert T_1 can be selected from the three experts $T_i(i=1,2,3)$ for decision-making, with results basically consistent with those of other methods, which shows that the proposed method is effective. To observe the difference between the ranking results of our and other methods, we compare the ranking results of different methods in Figure 3.

To further illustrate the effectiveness of the proposed method, $SRCC$ is used to analyze the correlation between the ranking results of different methods, as shown in Table 14.

A ratio greater than 0.8 between the ranking results of two methods indicates that the correlation between them is significant. It can be seen from the table that the differences between the proposed method and the other methods are greater than 0.8, indicating the effectiveness of the method.

From the above analysis, we can find that the results obtained by using the decision information of T_1 expert is the most reasonable. Due to the lack of decision-making experience of T_3 expert on this issue, the results obtained by using the decision-making information of T_3 expert is not ideal. Therefore, the proposed model can effectively improve the scientificity of decision-making while comparing the decision-making information of many experts and avoiding incorporating the lack of experience expert information.

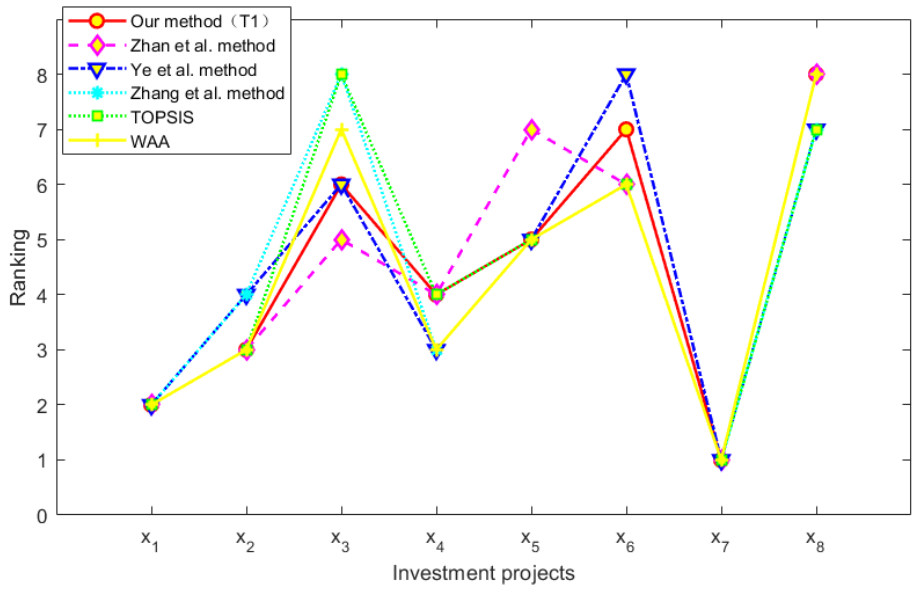

Figure 3. Comparison of ranking results of different methods.

Table 14. $SRCCs$ between different methods.

	Our method	Zhan et al.'s method	Ye et al.'s method	Zhang et al.'s method	TOPSIS method	WAA operator method
Our method	1	0.9286	0.9524	0.9048	0.9286	0.9643
Zhan et al.'s method		1	0.8571	0.8095	0.8333	0.8929
Ye et al.'s method			1	0.9048	0.8810	0.9167
Zhang et al.'s method				1	0.9762	0.9643
TOPSIS method					1	0.9643
WAA operator method						1

6. Conclusions

Decision self-information is a special kind of entropy and is an effective tool to characterize uncertain information. In this paper, the parameterized fuzzy β neighborhood was combined with decision self-information to extend it to the fuzzy environment and apply it to multi-attribute group decision-making. We defined three kinds of decision-making self-information, studied their important properties, and defined the parameterized fuzzy β neighborhood class and the corresponding conditional probability to establish a three-way decision-making model. We applied relative decision self-information, including both upper and lower approximation information, to three-way multi-attribute group decision-making, solving the problem of disagreement among multiple decision-makers in the real world. A three-way multi-attribute group decision-making algorithm based on a parameterized fuzzy β neighborhood was proposed and was used to solve a practical example. An experimental analysis showed the effectiveness of the proposed method. The main contributions of this paper are listed as follows:

(1) In this paper, the advantages of parametric fuzzy β neighborhood satisfying reflexivity and effectively reducing the influence of noise data are used to construct decision self-information based on parametric fuzzy β neighborhood. This measure can effectively describe the target concept in fuzzy environment.

(2) In order to avoid incorporating inexperienced expert information in the process of group decision-making, we construct a three-way multi-attribute group decision-making algorithm based on parametric fuzzy β neighborhood to measure multiple experts and select the most suitable experts for decision-making. The advantage of doing so is that the process can both compare the decision-making information of multiple experts and avoid fusing the information of inexperienced experts.

In solving multi-attribute decision-making problems, we will consider the impact of risk aversion or benefit maximization of psychological behavior on decision-making, which is a direction worthy of further study. In addition, group consensus decision-making based on regret theory will be one of our future research directions.

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Conflict of interest

The authors declare there is no conflict of interest.

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