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Research article

Multiple positive solutions for a bi-nonlocal Kirchhoff-Schrödinger-Poisson system with critical growth

Guaiqi Tian[1], **Hongmin Suo**[1,*] **and Yucheng An**[2]

[1] School of Data Sciences and Information Engineering, Guizhou Minzu University, Guiyang 550025, China

[2] School of Sciences, Guizhou University of Engineering Science, Bijie 551700, China

* **Correspondence:** Email: gzmysxx88@sina.com.

Abstract: In this article, we study the following bi-nonlocal Kirchhoff-Schrödinger-Poisson system with critical growth:

$$\begin{cases} -\left(\int_\Omega |\nabla u|^2 dx\right)^r \Delta u + \phi u = u^5 + \lambda\left(\int_\Omega F(x,u)dx\right)^s f(x,u), & \text{in } \Omega, \\ -\Delta\phi = u^2, u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $\lambda > 0$, $0 \leq r < 1$, $0 < s < \frac{1-r}{3(r+1)}$ and $f(x,u)$ satisfies some suitable assumptions. By using the concentration compactness principle, the multiplicity of positive solutions for the above system is established.

Keywords: Kirchhoff-Schrödinger-Poisson systems; positive solutions; critical growth; concentration compactness principle; multiplicity

1. Introduction and main result

This paper is concerned with the following Kirchhoff-Schrödinger-Poisson system:

$$\begin{cases} -\left(\int_\Omega |\nabla u|^2 dx\right)^r \Delta u + \phi u = u^5 + \lambda\left(\int_\Omega F(x,u)dx\right)^s f(x,u), & \text{in } \Omega, \\ -\Delta\phi = u^2, u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $\lambda > 0$, $0 \leq r < 1$, $0 < s < \frac{1-r}{3(r+1)}$ and $F(x,u) = \displaystyle\int_0^u f(x,\xi)d\xi$. We assume that $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exist constants $a_1, a_2 > 0$ and $\frac{6(r+1)}{r+2} < q < \frac{4}{s+1}, 0 < s < \frac{1-r}{3(r+1)}$, such that

$$a_1 t^{q-1} \leq f(x,t) \leq a_2 t^{q-1} \quad \text{for any } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{1.2}$$

When $\left(\displaystyle\int_\Omega |\nabla u|^2 dx\right)^r = 1$ and $s = 0$, the system (1.1) reduces to the boundary value problem

$$\begin{cases} -\Delta u + Vu + \phi u = f(x,u), & \text{in } \Omega, \\ -\Delta \phi = u^2, u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Problem (1.3) has been extensively studied, by using variational methods and critical point theory under suitable assumptions on V, f; see [1–7] and the references therein.

On the other hand, considering just the first equation in (1.1) with the potential equal to zero, we have the problem

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $a, b > 0$, which was proposed by Kirchhoff in [8] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The appearance of the nonlocal term $\int_\Omega |\nabla u|^2 dx$ in the equations makes them important in many physical applications. We have to point out that such nonlocal problems appear in other fields like biological systems, such as population density, where u describes a process which depends on the average of itself (see [9]). The Kirchhoff type problem (1.4) with critical growth began to call the attention of researchers; we can see [10–18] and the references therein.

In particular, Che et al. in [19] considered the following Kirchhoff-Schrödinger-Poisson system with critical growth:

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u + \phi u = \lambda g(x)|u|^{q-1} + h(x)u^5, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, u > 0, & \text{in } \mathbb{R}^3, \end{cases}$$

where $a > 0, b \geq 0, q \in [4,6)$, and $\lambda > 0$ is a parameter. Under some suitable conditions on $V(x)$, $g(x)$ and $h(x)$, by using the Nehari manifold technique and Ljusternik-Schnirelmann category theory, they established the number of positive solutions with the topology of the global maximum set of h when λ is small enough. Furthermore, with the aid of the mountain pass theorem, they obtained an existence result for λ sufficiently large.

In [20], Chabrowski investigated the bi-nonlocal problem for the nonlinear elliptic equation of the form

$$\begin{cases} -\left(\displaystyle\int_\Omega |\nabla u|^2 dx\right)^s \Delta u = Q(x)|u|^{p-2}u + \left(\displaystyle\int_\Omega |u|^q dx\right)^r |u|^{q-2}u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $2 < p \leq 2^*, 2 < q < 2^*, s > 0, r > 0$, and $2^* = \frac{2N}{N-2}(N \geq 3)$ denotes the critical Sobolev exponent. The existence of solutions in critical and subcritical cases is obtained by using the variational method. A similar problem with Dirichlet boundary conditions has been considered in [21]. Motivated by the above references, we study the existence of multiple positive solutions for system (1.1). Our main difficulties are as follows: The critical growth of system (1.1) leads to the lack of compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, and it is difficult to prove the energy functional belongs to the range where the (PS) condition holds. We overcome this difficulty by using the concentration compactness principle.

Throughout this paper, we make use of the following notations:

- The space $H_0^1(\Omega)$ is equipped with the norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$, and the norm in $L^p(\Omega)$ is denoted by $|u|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$;
- $C, C_1, C_2, \ldots$ denote various positive constants, which may vary from line to line;
- We denote by S_ρ (respectively, B_ρ) the sphere (respectively, the closed ball) of center zero and radius ρ, i.e., $S_\rho = \{u \in H_0^1(\Omega) : \|u\| = \rho\}$, $B_\rho = \{u \in H_0^1(\Omega) : \|u\| \leq \rho\}$;
- $\rightarrow$ (respectively, $\rightharpoonup$) denotes strong (respectively, weak) convergence;
- Let S be the best Sobolev constant, namely,

$$S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^6 dx\right)^{1/3}}. \tag{1.5}$$

Our main result is the following:

Theorem 1.1. *Assume that* $q(s+1) < 4 \leq 2(r+1) < 6$, $\lambda > 0$, *and* $0 < s < \frac{1-r}{3(r+1)}$. *Then, there exists* $\Lambda_* > 0$ *such that for any* $\lambda \in (0, \Lambda_*)$, *system (1.1) has at least two positive solutions.*

Remark 1.1. *As we shall see, in the system (1.1), when* $f(x,u) = |u|^{q-2}u$, $F(x,u) = |u|^q$, *Chabrowski established the existence of solutions. In this paper, due to the nonlocal term* ϕu *in (1.1), new treatments are needed for our problem. Therefore, in this article, we extend the relevant results of [20].*

2. Proof of Theorem 1.1

First, by using the Lax-Milgram theorem, for each $u \in H_0^1(\Omega)$, there exists a unique solution $\phi_u \in H_0^1(\Omega)$ which satisfies the second equation of system (1.1). We substitute ϕ_u into the first equation of system (1.1), and then system (1.1) is transformed into the following problem:

$$\begin{cases} -\left(\int_\Omega |\nabla u|^2 dx\right)^r \Delta u + \phi_u u = u^5 + \lambda \left(\int_\Omega F(x,u) dx\right)^s f(x,u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We define the energy functional corresponding to problem (2.1) by

$$I_\lambda(u) = \frac{1}{2(r+1)} \left(\int_\Omega |\nabla u|^2 dx\right)^{r+1} + \frac{1}{4} \int_\Omega \phi_u u^2 dx - \frac{1}{6} \int_\Omega |u|^6 dx - \frac{\lambda}{s+1} \left(\int_\Omega F(x,u) dx\right)^{s+1}.$$

We say that a function $u \in H_0^1(\Omega)$ is called a weak solution of problem (2.1) if for every $\varphi \in H_0^1(\Omega)$, there holds

$$\left(\int_\Omega |\nabla u|^2 dx\right)^r \int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \phi_u u \varphi dx - \int_\Omega |u|^4 u \varphi dx - \lambda \left(\int_\Omega F(x,u)dx\right)^s \int_\Omega f(x,u)\varphi dx = 0.$$

Before proving Theorem 1.1, we give the following important Lemma.

Lemma 2.1. *(See [22, 23]) For all $u \in H_0^1(\Omega)$, there exists a unique solution $\phi_u \in H_0^1(\Omega)$ of*

$$\begin{cases} -\Delta\phi = u^2, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

and
(1) $\|\phi_u\|^2 = \int_\Omega \phi_u u^2 dx$;
(2) $\phi_u \geq 0$. *Moreover,* $\phi_u > 0$ *when* $u \neq 0$;
(3) *For each* $t \neq 0$, $\phi_{tu} = t^2\phi_u$;
(4)

$$\int_\Omega |\nabla \phi_u|^2 dx = \int_\Omega \phi_u u^2 dx \leq S^{-1}|u|_{12/5}^4 \leq C\|u\|^4;$$

(5) *If* $u_n \rightharpoonup u$ *in* $H_0^1(\Omega)$, *then* $\phi_{u_n} \to \phi_u$ *in* $H_0^1(\Omega)$, *and*

$$\int_\Omega \phi_{u_n} u_n v dx \to \int_\Omega \phi_u u v dx, \quad \forall\, v \in H_0^1(\Omega).$$

Lemma 2.2. *There exist constants $\delta, \rho, \Lambda_0 > 0$, for all $\lambda \in (0, \Lambda_0)$ such that the functional I_λ satisfies the following conditions:*
(i) $I_\lambda|_{u\in S_\rho} \geq \delta > 0$, $\inf_{u\in B_\rho} I_\lambda(u) < 0$;
(ii) *There exists $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.*

Proof. (i) According to Hölder's inequality and (1.5), one has

$$\begin{aligned} \left(\int_\Omega |u|^q dx\right)^{s+1} &\leq \left[\left(\int_\Omega |u|^6 dx\right)^{\frac{q}{6}} \left(\int_\Omega 1^{\frac{6}{6-q}} dx\right)^{\frac{6-q}{6}}\right]^{s+1} \\ &\leq |\Omega|^{\frac{(6-q)(s+1)}{6}} S^{-\frac{q(s+1)}{2}} \|u\|^{q(s+1)}. \end{aligned} \tag{2.2}$$

By (1.2), we have

$$a_1|t|^q \leq f(x,t)t \leq a_2|t|^q \quad \text{for any } (x,t) \in \overline{\Omega}\times\mathbb{R}, \tag{2.3}$$

and

$$\frac{a_1}{q}|t|^q \leq F(x,t) \leq \frac{a_2}{q}|t|^q \quad \text{for any } (x,t) \in \overline{\Omega}\times\mathbb{R}. \tag{2.4}$$

Therefore, it follows from (1.5), (2.2) and (2.4) that

$$\begin{aligned}
I_\lambda(u) &\geq \frac{1}{2(r+1)}\|u\|^{2(r+1)} - \frac{1}{6}S^{-3}\|u\|^6 - \frac{\lambda}{s+1}\left(\frac{a_2}{q}\right)^{s+1}\left(\int_\Omega |u|^q dx\right)^{s+1} \\
&\geq \frac{1}{2(r+1)}\|u\|^{2(r+1)} - \frac{1}{6}S^{-3}\|u\|^6 - \frac{\lambda}{s+1}\left(\frac{a_2}{q}|\Omega|^{\frac{6-q}{6}}S^{-\frac{q}{2}}\|u\|^q\right)^{s+1} \\
&= \|u\|^{q(s+1)}\left[\frac{\|u\|^{2(r+1)-q(s+1)}}{2(r+1)} - \frac{S^{-3}\|u\|^{6-q(s+1)}}{6} - \frac{\lambda}{s+1}\left(\frac{a_2}{q}|\Omega|^{\frac{6-q}{6}}S^{-\frac{q}{2}}\right)^{s+1}\right].
\end{aligned}$$

Let $H(t) = \frac{1}{2(r+1)}t^{2(r+1)-q(s+1)} - \frac{1}{6}S^{-3}t^{6-q(s+1)}$ for $t > 0$, and then there exists

$$\rho = \left[\frac{3S^3[2(r+1) - q(s+1)]}{(r+1)[6 - q(s+1)]}\right]^{\frac{1}{6-2(r+1)}} > 0,$$

such that $\max_{t>0} H(t) = H(\rho) > 0$. Setting

$$\Lambda_0 = \frac{(s+1)q^{s+1}S^{\frac{q(s+1)}{2}}}{a_2^{s+1}|\Omega|^{\frac{(6-q)(s+1)}{6}}}H(\rho),$$

there exists a constant $\delta > 0$, such that $I_\lambda|_{u\in S_\rho} \geq \delta$ for each $\lambda \in (0, \Lambda_0)$. Moreover, for every $u \in H_0^1(\Omega)\backslash\{0\}$, we get

$$\begin{aligned}
\lim_{\tau\to 0^+}\frac{I_\lambda(\tau u)}{\tau^{q(s+1)}} &= \lim_{\tau\to 0^+} -\frac{\lambda}{s+1}\left(\int_\Omega F(x,\tau u)dx\right)^{s+1} \\
&\leq -\frac{\lambda}{s+1}\left(\frac{a_1}{q}\right)^{s+1}\left(\int_\Omega |u|^q dx\right)^{s+1} < 0.
\end{aligned}$$

So, we obtain $I_\lambda(\tau u) < 0$ for all $u \neq 0$ and τ small enough. Hence, for $\|u\|$ small enough, we have

$$m = \inf_{u\in B_\rho} I_\lambda(u) < 0.$$

(ii) Set $u \in H_0^1(\Omega)$, and we get

$$\begin{aligned}
I_\lambda(\tau u) \leq &\frac{\tau^{2(r+1)}}{2(r+1)}\left(\int_\Omega |\nabla u|^2 dx\right)^{r+1} + \frac{\tau^4}{4}\int_\Omega \phi_u u^2 dx - \frac{\tau^6}{6}\int_\Omega |u|^6 dx \\
&- \lambda\frac{\tau^{q(s+1)}}{s+1}\left(\frac{a_1}{q}\right)^{s+1}\left(\int_\Omega |u|^q dx\right)^{s+1} \to -\infty,
\end{aligned}$$

as $\tau \to +\infty$, which implies that $I_\lambda(\tau u) < 0$ for $\tau > 0$ large enough. Consequently, we can find $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$. The proof is complete. □

Lemma 2.3. *Assume that $\lambda > 0$, $q(s+1) < 4 \leq 2(r+1) < 6$, and $0 < s < \frac{1-r}{3(r+1)}$. Then, the functional I_λ satisfies the $(PS)_{c_\lambda}$ condition for each*

$$c_\lambda < c_* = \frac{2-r}{6(r+1)}S^{\frac{3(r+1)}{2-r}} - D\lambda^{\frac{2(r+1)}{2(r+1)-q(s+1)}},$$

where

$$D = \frac{(2-r)[2(r+1) - q(s+1)]}{6q(r+1)(s+1)}\left[\frac{6a_2^{s+1} - q(s+1)a_1^{s+1}}{2q^s(2-r)}|\Omega|^{\frac{(6-q)(s+1)}{6}}S^{-\frac{q(s+1)}{2}}\right]^{\frac{2(r+1)}{2(r+1)-q(s+1)}}.$$

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a (PS) sequence for I_λ at the level c_λ, that is,

$$I_\lambda(u_n) \to c_\lambda, \text{ and } I'_\lambda(u_n) \to 0 \text{ as } n \to \infty. \tag{2.5}$$

Combining with (2.2)–(2.4), we get

$$\begin{aligned}
c_\lambda + 1 + o(\|u_n\|) &\geq I_\lambda(u_n) - \frac{1}{6}\langle I'_\lambda(u_n), u_n\rangle \\
&= \left(\frac{1}{2(r+1)} - \frac{1}{6}\right)\|u_n\|^{2(r+1)} + \left(\frac{1}{4} - \frac{1}{6}\right)\int_\Omega \phi_{u_n} u_n^2 dx \\
&\quad - \lambda \frac{1}{s+1}\left(\int_\Omega F(x,u_n)dx\right)^{s+1} + \frac{\lambda}{6}\left(\int_\Omega F(x,u_n)dx\right)^{s}\int_\Omega f(x,u_n)u_n dx \\
&\geq \left(\frac{1}{2(r+1)} - \frac{1}{6}\right)\|u_n\|^{2(r+1)} + \lambda\left[\frac{q}{6}\left(\frac{a_1}{q}\right)^{s+1} - \frac{1}{s+1}\left(\frac{a_2}{q}\right)^{s+1}\right]\left(\int_\Omega |u_n|^q dx\right)^{s+1} \\
&\geq \left(\frac{1}{2(r+1)} - \frac{1}{6}\right)\|u_n\|^{2(r+1)} + \lambda\left[\frac{q}{6}\left(\frac{a_1}{q}\right)^{s+1} - \frac{1}{s+1}\left(\frac{a_2}{q}\right)^{s+1}\right]|\Omega|^{\frac{(6-q)(s+1)}{6}} S^{-\frac{q(s+1)}{2}}\|u_n\|^{q(s+1)}.
\end{aligned}$$

Then, this implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ for all $q(s+1) < 4 \leq 2(r+1) < 6$. Thus, we may assume up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } H_0^1(\Omega), \\ u_n \to u, & \text{strongly in } L^p(\Omega)\ (1 \leq p < 6), \\ u_n(x) \to u(x), & \text{a.e. in } \Omega, \end{cases} \tag{2.6}$$

as $n \to \infty$. Next, we prove that $u_n \to u$ strongly in $H_0^1(\Omega)$. By using the concentration compactness principle (see [24]), there exist an at most countable set J, a family of points $\{x_j\}_{j\in J} \subset \bar{\Omega}$, and positive numbers $\{\nu_j\}_{j\in J}$, $\{\mu_j\}_{j\in J}$ such that

$$|u_n|^6 \rightharpoonup d\nu = |u|^6 + \sum_{j\in J}\nu_j\delta_{x_j},$$

$$|\nabla u_n|^2 \rightharpoonup d\mu \geq |\nabla u|^2 + \sum_{j\in J}\mu_j\delta_{x_j}.$$

Moreover, we have

$$\mu_j, \nu_j \geq 0,\ \mu_j \geq S\nu_j^{\frac{1}{3}}. \tag{2.7}$$

Let $\varphi_{\varepsilon,j}(x)$ be a smooth cut-off function centered at x_j such that $0 \leq \varphi_{\varepsilon,j} \leq 1$, $|\nabla\varphi_{\varepsilon,j}| \leq \frac{2}{\varepsilon}$, $\varepsilon > 0$, and

$$\varphi_{\varepsilon,j}(x) = \begin{cases} 1, & \text{in } B(x_j, \frac{\varepsilon}{2}), \\ 0, & \text{in } \Omega \setminus B(x_j, \varepsilon). \end{cases} \tag{2.8}$$

Noting that $\{u_n\varphi_{\varepsilon,j}\}$ is bounded in $H_0^1(\Omega)$ uniformly for n, combining with (2.6) and (2.8), we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^q\varphi_{\varepsilon,j}dx \leq \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B(x_j,\varepsilon)}|u_n|^q dx = 0. \tag{2.9}$$

Similarly, we can obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \phi_{u_n}u_n^2\varphi_{\varepsilon,j}dx \le \lim_{\varepsilon\to 0}\int_{B(x_j,\varepsilon)}\phi_u u^2\varphi_{\varepsilon,j}dx = 0. \tag{2.10}$$

By using the Hölder inequality and $|\nabla\varphi_{\varepsilon,j}| \le \frac{2}{\varepsilon}$, there exists $C_2 > 0$, and we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \langle\nabla u_n, \nabla\varphi_{\varepsilon,j}\rangle u_n dx\\
\le& \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_\Omega |\nabla u_n|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega |u_n|^2|\nabla\varphi_{\varepsilon,j}|^2dx\right)^{\frac{1}{2}}\\
\le& C_1\lim_{\varepsilon\to 0}\left(\int_{B(x_j,\varepsilon)}|u|^6dx\right)^{\frac{1}{6}}\left(\int_{B(x_j,\varepsilon)}|\nabla\varphi_{\varepsilon,j}|^3dx\right)^{\frac{1}{3}}\\
\le& C_1\lim_{\varepsilon\to 0}\left(\int_{B(x_j,\varepsilon)}|u|^6dx\right)^{\frac{1}{6}}\left[\int_{B(x_j,\varepsilon)}\left(\frac{2}{\varepsilon}\right)^3dx\right]^{\frac{1}{3}}\\
\le& C_2\lim_{\varepsilon\to 0}\left(\int_{B(x_j,\varepsilon)}|u|^6dx\right)^{\frac{1}{6}}\\
=& 0.
\end{aligned} \tag{2.11}$$

We also derive that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |\nabla u_n|^2\varphi_{\varepsilon,j}dx \ge \lim_{\varepsilon\to 0}\int_\Omega |\nabla u|^2\varphi_{\varepsilon,j}dx + \mu_j = \mu_j, \tag{2.12}$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^6\varphi_{\varepsilon,j}dx = \lim_{\varepsilon\to 0}\int_\Omega |u|^6\varphi_{\varepsilon,j}dx + \nu_j = \nu_j. \tag{2.13}$$

By (2.5) and (2.9)–(2.13), we get

$$\begin{aligned}
0 =& \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle I_\lambda'(u_n), u_n\varphi_{\varepsilon,j}\rangle\\
=& \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{\left(\int_\Omega |\nabla u_n|^2dx\right)^r\int_\Omega\left\langle\nabla u_n, \nabla(u_n\varphi_{\varepsilon,j})\right\rangle dx + \int_\Omega \phi_{u_n}u_n^2\varphi_{\varepsilon,j}dx\right.\\
&\left.-\int_\Omega |u_n|^6\varphi_{\varepsilon,j}dx - \lambda\left(\int_\Omega F(x,u_n)dx\right)^s\int_\Omega f(x,u_n)u_n\varphi_{\varepsilon,j}dx\right\}\\
=& \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{\left(\int_\Omega |\nabla u_n|^2dx\right)^r\int_\Omega |\nabla u_n|^2\varphi_{\varepsilon,j}dx + \left(\int_\Omega |\nabla u_n|^2dx\right)^r\int_\Omega\langle\nabla u_n, \nabla\varphi_{\varepsilon,j}\rangle u_ndx - \int_\Omega |u_n|^6\varphi_{\varepsilon,j}dx\right\}\\
\ge& \lim_{\varepsilon\to 0}\left\{\left(\int_\Omega |\nabla u|^2dx + \mu_j\right)^r\left(\int_\Omega |\nabla u|^2\varphi_{\varepsilon,j}dx + \mu_j\right) - \nu_j\right\}\\
\ge& \mu_j^{r+1} - \nu_j,
\end{aligned}$$

that is, $\nu_j \ge \mu_j^{r+1}$. If $\nu_j > 0$, by (2.7), we obtain

$$\nu_j \ge S^{\frac{3(r+1)}{2-r}}, \quad \mu_j \ge S^{\frac{3}{2-r}}. \tag{2.14}$$

Now, we show that (2.14) is impossible. Assume that there exists $j_0 \in J$, such that $\mu_{j_0} \geq S^{\frac{3}{2-r}}$ and $x_{j_0} \in \Omega$. It follows from (2.2)–(2.5) that

$$\begin{aligned}
c_\lambda &= \lim_{n\to\infty}\left\{I_\lambda(u_n) - \frac{1}{6}\langle I'_\lambda(u_n), u_n\rangle\right\}\\
&= \lim_{n\to\infty}\left\{\left(\frac{1}{2(r+1)} - \frac{1}{6}\right)\left(\int_\Omega |\nabla u_n|^2 dx\right)^{r+1} + \left(\frac{1}{4} - \frac{1}{6}\right)\int_\Omega \phi_{u_n} u_n^2 dx\right.\\
&\quad \left. - \frac{\lambda}{s+1}\left(\int_\Omega F(x,u_n)dx\right)^{s+1} + \frac{\lambda}{6}\left(\int_\Omega F(x,u_n)dx\right)^{s}\int_\Omega f(x,u_n)u_n dx\right\}\\
&\geq \frac{2-r}{6(r+1)}\left(\int_\Omega |\nabla u|^2 dx + \sum_{j\in J}\mu_j\right)^{r}\left(\int_\Omega |\nabla u|^2 dx + \sum_{j\in J}\mu_j\right)\\
&\quad - \lambda\left[\frac{1}{s+1}\left(\frac{a_2}{q}\right)^{s+1} - \frac{q}{6}\left(\frac{a_1}{q}\right)^{s+1}\right]\left(\int_\Omega |u|^q dx\right)^{s+1}\\
&\geq \frac{2-r}{6(r+1)}\mu_{j_0}^{r+1} + \frac{2-r}{6(r+1)}\|u\|^{2(r+1)} - \lambda\frac{6a_2^{s+1} - q(s+1)a_1^{s+1}}{6(s+1)q^{s+1}}|\Omega|^{\frac{(6-q)(s+1)}{6}}S^{-\frac{q(s+1)}{2}}\|u\|^{q(s+1)}.
\end{aligned}$$

Let

$$G(t) = \frac{2-r}{6(r+1)}t^{2(r+1)} - \lambda\frac{6a_2^{s+1} - q(s+1)a_1^{s+1}}{6(s+1)q^{s+1}}|\Omega|^{\frac{(6-q)(s+1)}{6}}S^{-\frac{q(s+1)}{2}}t^{q(s+1)}.$$

It is clear that $\lim_{t\to 0} G(t) = 0$, and $\lim_{t\to+\infty} G(t) = +\infty$. Therefore, there exists $T > 0$ such that $G(T) = \min_{t\geq 0} G(t)$, that is,

$$\begin{aligned}
G'(t)|_T &= \frac{2-r}{3}T^{2r+1} - \lambda\frac{6a_2^{s+1} - q(s+1)a_1^{s+1}}{6q^s}|\Omega|^{\frac{(6-q)(s+1)}{6}}S^{-\frac{q(s+1)}{2}}T^{q(s+1)-1}\\
&= 0.
\end{aligned} \tag{2.15}$$

From (2.15) we obtain

$$T = \left(\frac{\lambda(6a_2^{s+1} - q(s+1)a_1^{s+1})}{2q^s(2-r)}|\Omega|^{\frac{(6-q)(s+1)}{6}}S^{-\frac{q(s+1)}{2}}\right)^{\frac{1}{2(r+1)-q(s+1)}},$$

and by simple calculation, we have

$$\begin{aligned}
G(T) &= \frac{2-r}{6(r+1)}\left[\lambda\frac{6a_2^{s+1} - q(s+1)a_1^{s+1}}{2q^s(2-r)}|\Omega|^{\frac{(6-q)(s+1)}{6}}S^{-\frac{q(s+1)}{2}}\right]^{\frac{2(r+1)}{2(r+1)-q(s+1)}}\\
&\quad - \lambda\frac{6a_2^{s+1} - q(s+1)a_1^{s+1}}{6(s+1)q^{s+1}}\left(|\Omega|^{\frac{6-q}{6}}S^{-\frac{q}{2}}\right)^{s+1}\left[\lambda\frac{6a_2^{s+1} - q(s+1)a_1^{s+1}}{2q^s(2-r)}|\Omega|^{\frac{(6-q)(s+1)}{6}}S^{-\frac{q(s+1)}{2}}\right]^{\frac{q(s+1)}{2(r+1)-q(s+1)}}\\
&= \frac{(2-r)[q(s+1) - 2(r+1)]}{6q(r+1)(s+1)}\left[\lambda\frac{6a_2^{s+1} - q(s+1)a_1^{s+1}}{2q^s(2-r)}|\Omega|^{\frac{(6-q)(s+1)}{6}}S^{-\frac{q(s+1)}{2}}\right]^{\frac{2(r+1)}{2(r+1)-q(s+1)}}.
\end{aligned}$$

Hence, we can see that

$$c_\lambda \geq \frac{2-r}{6(r+1)}S^{\frac{3(r+1)}{2-r}} - D\lambda^{\frac{2(r+1)}{2(r+1)-q(s+1)}},$$

where

$$D = \frac{(2-r)[2(r+1)-q(s+1)]}{6q(r+1)(s+1)} \left[\frac{6a_2^{s+1} - q(s+1)a_1^{s+1}}{2q^s(2-r)} |\Omega|^{\frac{(6-q)(s+1)}{6}} S^{-\frac{q(s+1)}{2}} \right]^{\frac{2(r+1)}{2(r+1)-q(s+1)}}.$$

We obtain that $c_\lambda \geq c_*$. This is a contradiction, which indicates that $\nu_j = \mu_j = 0$ for every $j \in J$, which implies that $u_n \to u$ in $L^6(\Omega)$. We may assume that

$$\int_\Omega |\nabla u_n|^2 dx \to A^2, \int_\Omega |\nabla u|^2 dx \leq A^2.$$

Combining with (2.5) and (2.6), we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \langle I'_\lambda(u_n), u_n - u \rangle \\
&= \lim_{n\to\infty} \Bigg[\left(\int_\Omega |\nabla u_n|^2 dx \right)^r \left(\int_\Omega |\nabla u_n|^2 dx - \int_\Omega \nabla u_n \nabla u dx \right) + \int_\Omega \phi_{u_n} u_n (u_n - u) dx - \int_\Omega |u_n|^4 u_n (u_n - u) dx \\
&\quad - \lambda \left(\int_\Omega F(x, u_n) dx \right)^s \int_\Omega f(x, u_n)(u_n - u) dx \Bigg] \\
&= A^{2r} \left(A^2 - \int_\Omega |\nabla u|^2 dx \right).
\end{aligned}$$

Hence, we obtain

$$\int_\Omega |\nabla u_n|^2 dx \to \int_\Omega |\nabla u|^2 dx \ \text{ as } n \to \infty,$$

which implies $u_n \to u$ in $H_0^1(\Omega)$. The proof is complete. □

Choose the extremal function

$$U_\varepsilon(x) = \frac{(3\varepsilon^2)^{\frac{1}{4}}}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}, \ \varepsilon > 0,$$

satisfying

$$-\Delta U_\varepsilon = U_\varepsilon^5 \ \text{ in } \mathbb{R}^3.$$

Let $\Psi \in C^1(\mathbb{R}^3)$ such that $\Psi(x) = 1$ on $B_{\frac{R}{2}}(0)$, $\Psi(x) = 0$ on $\mathbb{R}^3 \setminus B_R(0)$, and $0 \leq \Psi(x) \leq 1$ on $\mathbb{R}^3$. Set $u_\varepsilon(x) = \Psi(x) U_\varepsilon(x)$. From [25], one has

$$\begin{cases} \int_\Omega |\nabla u_\varepsilon|^2 dx = S^{\frac{3}{2}} + O(\varepsilon), \\ \int_\Omega |u_\varepsilon|^6 dx = S^{\frac{3}{2}} + O(\varepsilon^3), \end{cases}$$

and

$$|u_\varepsilon|_\alpha^\alpha = \begin{cases} O(\varepsilon^{\frac{\alpha}{2}}), & \alpha \in [2,3), \\ O(\varepsilon^{\frac{\alpha}{2}} |\ln \varepsilon|), & \alpha = 3, \\ O(\varepsilon^{\frac{6-\alpha}{2}}), & \alpha \in (3,6). \end{cases} \tag{2.16}$$

Then, we have the following Lemma.

Lemma 2.4. *Suppose that $\lambda > 0$, $q(s+1) < 4 \le 2(r+1) < 6$, and $0 < s < \frac{1-r}{3(r+1)}$. Then,*

$$\sup_{t\ge 0} I_\lambda(tu_\varepsilon) < \frac{2-r}{6(r+1)} S^{\frac{3(r+1)}{2-r}} - D\lambda^{\frac{2(r+1)}{2(r+1)-q(s+1)}}.$$

Proof. According to the definition of u_ε and (2.4), it holds that

$$\begin{aligned}
\frac{\lambda}{s+1}\left(\int_\Omega F(x,tu_\varepsilon)dx\right)^{s+1} &\ge \frac{\lambda}{s+1} t^{q(s+1)} \left(\frac{a_1}{q}\right)^{s+1} \left(\int_\Omega |u_\varepsilon|^q dx\right)^{s+1} \\
&\ge C_3\lambda \left(\int_{B_{R/2}(0)} \frac{\varepsilon^{\frac{q}{2}}}{(\varepsilon^2+|x|^2)^{\frac{q}{2}}} dx\right)^{s+1} \\
&= C_3\lambda\varepsilon^{\frac{(6-q)(s+1)}{2}} \left(\int_0^{R/2\varepsilon} \frac{y^2}{(1+y^2)^{\frac{q}{2}}} dy\right)^{s+1} \\
&\ge C_3\lambda\varepsilon^{\frac{(6-q)(s+1)}{2}} \left(\int_0^1 \frac{y^2}{(1+y^2)^{\frac{q}{2}}} dy\right)^{s+1} \\
&\ge C_4\lambda\varepsilon^{\frac{(6-q)(s+1)}{2}}.
\end{aligned} \tag{2.17}$$

From Lemma 2.1 and (2.16), we have the following estimate:

$$\int_\Omega \phi_{u_\varepsilon} u_\varepsilon^2 dx \le S^{-1} |u_\varepsilon|_{12/5}^4 \le O(\varepsilon^2). \tag{2.18}$$

Since $I_\lambda(tu_\varepsilon) \to -\infty$ as $t \to \infty$, by Lemma 2.2, there exists $t_\varepsilon > 0$ such that

$$I_\lambda(t_\varepsilon u_\varepsilon) = \sup_{t>0} I_\lambda(tu_\varepsilon) \ge \delta > 0.$$

Moreover, by the continuity of I_λ, there exist positive constants t_1 and t_2 such that $0 < t_1 \le t_\varepsilon \le t_2 < +\infty$. As a consequence of the above fact, one has

$$\begin{aligned}
\sup_{t\ge 0} I_\lambda(tu_\varepsilon) &= \sup_{t\ge 0}\left\{ \frac{1}{2(r+1)}\left(\int_\Omega |\nabla tu_\varepsilon|^2 dx\right)^{r+1} + \frac{1}{4}\int_\Omega \phi_{tu_\varepsilon} |tu_\varepsilon|^2 dx \right. \\
&\quad \left. - \frac{1}{6}\int_\Omega |tu_\varepsilon|^6 dx - \frac{\lambda}{s+1}\left(\int_\Omega F(x,tu_\varepsilon)dx\right)^{s+1} \right\} \\
&\le \sup_{t\ge 0}\left\{ \frac{1}{2(r+1)}\left(\int_\Omega |\nabla tu_\varepsilon|^2 dx\right)^{r+1} - \frac{1}{6}\int_\Omega |tu_\varepsilon|^6 dx \right\} + O(\varepsilon^2) - \lambda C_4 \varepsilon^{\frac{(6-q)(s+1)}{2}} \\
&\le \frac{2-r}{6(r+1)} \left[\frac{\left(\int_\Omega |\nabla u_\varepsilon|^2 dx\right)^{r+1}}{\int_\Omega |u_\varepsilon|^6 dx} \right]^{\frac{2(r+1)}{6-2(r+1)}} \left(\int_\Omega |\nabla u_\varepsilon|^2 dx\right)^{r+1} + O(\varepsilon^2) - \lambda C_4 \varepsilon^{\frac{(6-q)(s+1)}{2}} \\
&\le \frac{2-r}{6(r+1)} S^{\frac{3(r+1)}{2-r}} + C_5\varepsilon - \lambda C_4 \varepsilon^{\frac{(6-q)(s+1)}{2}} \\
&< \frac{2-r}{6(r+1)} S^{\frac{3(r+1)}{2-r}} - D\lambda^{\frac{2(r+1)}{2(r+1)-q(s+1)}}.
\end{aligned}$$

We have used the fact that $\frac{6(r+1)}{r+2} < q < \frac{4}{s+1}$ and let $\varepsilon = \lambda^{\frac{2(r+1)}{2(r+1)-q(s+1)}}$.

$$0 < \lambda < \Lambda_1 = \min\left\{ \left[\frac{2-r}{6(r+1)D} S^{\frac{3(r+1)}{2-r}}\right]^{\frac{2(r+1)-q(s+1)}{2(r+1)}}, \left(\frac{C_5 + D}{C_4}\right)^{\frac{2(r+1)-q(s+1)}{(s+1)[(r+1)(6-q)-q]}} \right\},$$

and then

$$\begin{aligned} C_5\varepsilon - C_4\lambda\varepsilon^{\frac{(6-q)(s+1)}{2}} &= C_5\lambda^{\frac{2(r+1)}{2(r+1)-q(s+1)}} - C_4\lambda^{\frac{(r+1)(6-q)(s+1)}{2(r+1)-q(s+1)}+1} \\ &= \lambda^{\frac{2(r+1)}{2(r+1)-q(s+1)}}\left(C_5 - C_4\lambda^{\frac{(s+1)[(r+1)(6-q)-q]}{2(r+1)-q(s+1)}}\right) \\ &< -D\lambda^{\frac{2(r+1)}{2(r+1)-q(s+1)}}. \end{aligned}$$

The proof is complete. □

Lemma 2.5. *Suppose that $0 < \lambda < \Lambda_0$ (Λ_0 is as in Lemma 2.2). Then, system (1.1) has a positive solution u_λ satisfying $I_\lambda(u_\lambda) < 0$.*

Proof. It follows from Lemma 2.2 that

$$m = \inf_{u\in\overline{B_\rho(0)}} I_\lambda(u) < 0.$$

By the Ekeland variational principle [26], there exists a minimizing sequence $\{u_n\} \subset \overline{B_\rho(0)}$ such that

$$I_\lambda(u_n) \leq \inf_{u\in\overline{B_\rho(0)}} I_\lambda(u) + \frac{1}{n}, \quad I_\lambda(v) \geq I_\lambda(u_n) - \frac{1}{n}\|v - u_n\|, \quad v \in \overline{B_\rho(0)}.$$

Therefore, we obtain that $I_\lambda(u_n) \to m$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. Since $\{u_n\}$ is a bounded sequence, and $\overline{B_\rho(0)}$ is a closed convex set, we may assume up to a subsequence, still denoted by $\{u_n\}$, there exists $u_\lambda \in \overline{B_\rho(0)} \subset H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_\lambda, & \text{weakly in } H_0^1(\Omega), \\ u_n \to u_\lambda, & \text{strongly in } L^q(\Omega)\ (1 \leq p < 6), \\ u_n(x) \to u_\lambda(x), & \text{a.e. in } \Omega. \end{cases}$$

By the lower semi-continuity of the norm with respect to weak convergence, we get

$$\begin{aligned} m &\geq \liminf_{n\to\infty}\left[I_\lambda(u_n) - \frac{1}{6}\langle I'_\lambda(u_n), u_n\rangle\right] \\ &= \liminf_{n\to\infty}\Bigg[\left(\frac{1}{2(r+1)} - \frac{1}{6}\right)\left(\int_\Omega |\nabla u_n|^2 dx\right)^{r+1} + \left(\frac{1}{4} - \frac{1}{6}\right)\int_\Omega \phi_{u_n} u_n^2 dx \\ &\quad - \lambda\frac{1}{s+1}\left(\int_\Omega F(x,u_n)dx\right)^{s+1} + \frac{\lambda}{6}\left(\int_\Omega F(x,u_n)dx\right)^s \int_\Omega f(x,u_n)u_n dx\Bigg] \\ &\geq \frac{2-r}{6(r+1)}\left(\int_\Omega |\nabla u_\lambda|^2 dx\right)^{r+1} + \frac{1}{12}\int_\Omega \phi_{u_\lambda} u_\lambda^2 dx \\ &\quad - \lambda\frac{1}{s+1}\left(\int_\Omega F(x,u_\lambda)dx\right)^{s+1} + \frac{\lambda}{6}\left(\int_\Omega F(x,u_\lambda)dx\right)^s \int_\Omega f(x,u_\lambda)u_\lambda dx \\ &= I_\lambda(u_\lambda) - \frac{1}{6}\langle I'_\lambda(u_\lambda), u_\lambda\rangle = I_\lambda(u_\lambda) \geq m. \end{aligned}$$

Thus, $I_\lambda(u_\lambda) = m < 0$, and we can see that $u_\lambda \not\equiv 0$. $I_\lambda(|u_\lambda|) = I_\lambda(u_\lambda)$, which suggests that $u_\lambda \geq 0$. Therefore, by the strong maximum principle, we obtain that u_λ is a positive solution of system (1.1). The proof is complete. □

Lemma 2.6. *Assume that* $0 < \lambda < \Lambda_*$ $(\Lambda_* = \min\{\Lambda_0, \Lambda_1\})$. *Then, the system (1.1) has a positive solution* $u_* \in H_0^1(\Omega)$ *with* $I_\lambda(u_*) > 0$.

Proof. From the mountain pass lemma and Lemma 2.2, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$I_\lambda(u_n) \to c_\lambda > 0, \quad \text{and} \quad I'_\lambda(u_n) \to 0 \text{ as } n \to \infty,$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \left\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\right\}.$$

According to Lemma 2.3, we know that $\{u_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$, such that $u_n \to u_*$ in $H_0^1(\Omega)$ as $n \to \infty$.

$$I_\lambda(u_*) = \lim_{n\to\infty} I_\lambda(u_n) = c_\lambda > \delta > 0,$$

which implies that $u_* \not\equiv 0$. It is similar to Lemma 2.5 that $u_* > 0$, that is, u_* is a positive solution of system (1.1) such that $I_\lambda(u_*) > 0$. The proof is complete. □

3. Conclusions

In this paper, we considered a class of bi-nonlocal Kirchhoff-Schrödinger-Poisson system with critical growth. Under some suitable assumptions, by using the concentration compactness principle, we obtained the multiplicity of positive solutions.

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Conflict of interest

The authors declare there are no conflicts of interest.

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